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Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion

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ABSTRACT

We establish global existence and uniqueness theorems for the two-dimensional non-diffusive Boussinesq system with anisotropic viscosity acting only in the horizontal direction, which arises in ocean dynamics models. Global well-posedness for this system was proven by Danchin and Paicu; however, an additional smoothness assumption on the initial density was needed to prove uniqueness. They stated that it is not clear whether uniqueness holds without this additional assumption. The present work resolves this question and we establish uniqueness without this additional assumption. Furthermore, the proof provided here is more elementary; we use only tools available in the standard theory of Sobolev spaces, and without resorting to para-product calculus. We use a new approach by defining an auxiliary “stream-function” associated with the density, analogous to the stream-function associated with the vorticity in 2D incompressible Euler equations, then we adapt some of the ideas of Yudovich for proving uniqueness for 2D Euler equations.

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1. Introduction

The two-dimensional Boussinesq system of ocean and atmosphere dynamics (without rotation), in a domain $\Omega \subset \mathbb{R}^2$ over the time interval $[0, T]$ is given by

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$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \theta \mathbf{e}_2 + \nu \Delta \mathbf{u}, \quad \text{in } \Omega \times [0, T], \quad (1.1a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega \times [0, T], \quad (1.1b)$$

$$\partial_t \theta + (\mathbf{u} \cdot \nabla) \theta = \kappa \Delta \theta, \quad \text{in } \Omega \times [0, T], \quad (1.1c)$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \quad \text{in } \Omega, \quad (1.1d)$$

with appropriate boundary conditions (discussed below). Here $\nu \geq 0$ is the fluid viscosity, $\kappa \geq 0$ is the diffusion coefficient. We denote problem (1.1) by $P_{\nu, \kappa}$ when $\kappa > 0$ and $\nu > 0$. The spatial variable is denoted by $\mathbf{x} = (x^1, x^2) \in \Omega$. The unknowns are the fluid velocity field $\mathbf{u} \equiv \mathbf{u}(\mathbf{x}, t) \equiv (u^1(\mathbf{x}, t), u^2(\mathbf{x}, t))$, the fluid pressure $p(\mathbf{x}, t)$, and the scalar function $\theta \equiv \theta(\mathbf{x}, t)$, which may be interpreted physically as (the fluctuation of) a density variable (e.g., when $\kappa = 0$), or a thermal variable (e.g., when $\kappa > 0$). We write $\mathbf{e}_2 = (0, 1)$ for the second standard basis vector in \mathbb{R}^2 . It is worth mentioning that all the results reported here are also valid in the presence of the Coriolis rotation term.

The main purpose of this study is to remove the additional smoothness condition on the initial density, which was crucial to the uniqueness proof in [9]. Indeed, the authors of [9] state:

“...it is not clear that those global solutions are unique if there is no additional regularity assumption on θ .” [9, p. 425]

In this work, we resolve this difficulty, and develop a new technique which uses a “stream-like function,” adapting Yudovich methods to handle the present case. This new approach allows us to not only overcome the difficulty mentioned in [9], but also to do so using only elementary tools from Sobolev spaces, avoiding the use of highly-sophisticated tools from harmonic analysis.

In two dimensions, the global regularity in time of the problem $P_{\nu, \kappa}$ is well-known (see, e.g., [5, 22]), and follows essentially from the classical methods for Navier–Stokes equations (NSE). However, in the case $\nu = 0$, $\kappa = 0$, $(P_{0,0})$, global existence and uniqueness still remain an open problem (see, e.g., [6,7] for studies in this direction). The local existence and uniqueness of classical solutions to $P_{0,0}$ were established in [7], assuming the initial data $(\mathbf{u}_0, \theta_0) \in H^3 \times H^3$. In particular, an analogous Beale–Kato–Majda criterion for blow-up of smooth solutions is established in [7] for the inviscid, non-diffusive Boussinesq system; namely, that the smooth solution exists on $[0, T]$ if and only if $\int_0^T \|\nabla \theta(t)\|_{L^\infty} dt < \infty$.

It has been shown in [6,17] that the system $P_{\nu,0}$, in the case of whole space \mathbb{R}^2 , admits a unique global solution provided the initial data $(\mathbf{u}_0, \theta_0) \in H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)$ with $m \geq 3$, m an integer. In fact, in [17], the authors only required $(\mathbf{u}_0, \theta_0) \in H^m(\mathbb{R}^2) \times H^{m-1}(\mathbb{R}^2)$ with $m \geq 3$. In [6], it is also shown that the problem $P_{0,\kappa}$ admits a unique global solution provided the initial data $(\mathbf{u}_0, \theta_0) \in H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)$ with $m \geq 3$. Global well-posedness results for somewhat rougher initial data (in Besov spaces) are established in [15]. The requirements on the initial data were weakened further in [10], where global well-posedness results were established in the whole space without any smoothness assumption, namely they only require $\mathbf{u}_0, \theta_0 \in L^2(\mathbb{R}^2)$ for both existence and uniqueness. The proof of their main results arise under the Besov and Lorentz space setting and involves the use of Littlewood–Paley decomposition and paradifferential calculus introduced by J.-M. Bony [4].

In Section 3, we establish the global well-posedness of the anisotropic case in a periodic domain $\mathbb{T}^2 = [0, 1]^2 = \mathbb{R}^2/\mathbb{Z}^2$. More precisely, assuming initial vorticity $\omega_0 \in \sqrt{L}$ (defined below in (1.3)), initial density (or temperature) fluctuation $\theta_0 \in L^\infty(\mathbb{T}^2)$, and $\int_{\mathbb{T}^2} \omega_0 d\mathbf{x} = \int_{\mathbb{T}^2} \theta_0 d\mathbf{x} = 0$, we establish global well-posedness for the following system (with the advection terms written in divergence form), which we denote as $P_{\nu, \kappa, 0}$:

$$\partial_t \mathbf{u} + \sum_{j=1}^2 \partial_j (u^j \mathbf{u}) = \nu \partial_1^2 \mathbf{u} - \nabla p + \theta \mathbf{e}_2, \quad \text{in } \mathbb{T}^2 \times [0, T], \quad (1.2a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \mathbb{T}^2 \times [0, T], \quad (1.2b)$$

$$\partial_t \theta + \nabla \cdot (\mathbf{u}\theta) = 0, \quad \text{in } \mathbb{T}^2 \times [0, T], \quad (1.2c)$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \quad \text{in } \mathbb{T}^2. \quad (1.2d)$$

Our key idea in proving the uniqueness result is by writing $\theta = \Delta \xi$, with $\int_{\mathbb{T}^2} \xi \, dx = 0$, for some ξ , and then adapting the techniques of Yudovich in [25] (see also [20]).

Recently, in [9], a global well-posedness result for the system $P_{v_k,0}$ (in the whole space \mathbb{R}^2), under various regularity conditions on initial data, was successfully established. More precisely, it is proven in [9] that, given $\theta_0 \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, with $s \in (1/2, 1]$, $\mathbf{u}_0 \in H^1(\mathbb{R}^2)$ and $\omega_0 \in L^p(\mathbb{R}^2)$ for all $2 \leq p < \infty$, and such that ω_0 satisfies

$$\|\omega_0\|_{\sqrt{L}} := \sup_{p \geq 2} \frac{\|\omega_0\|_{L^p(\mathbb{R}^2)}}{\sqrt{p-1}} < \infty, \quad (1.3)$$

the Boussinesq system (1.2) in the whole space with anisotropic viscosity admits a unique globally regular solution. The condition $\theta_0 \in H^s$ with $s \in (\frac{1}{2}, 1]$ was needed for establishing uniqueness in [9]. We relax this condition in our current contribution. We remark again that the main idea is to write $\theta = \Delta \xi$, and then proceed using the techniques of Yudovich [25] for the 2D incompressible Euler equations to prove uniqueness. Furthermore, our method uses more elementary tools than those used in [9].

During the peer review process, it has been brought to our attention that a similar approach to proving stability estimates for equations in spaces other than the energy space has been utilized in two fairly recent papers. One is [16] in the context of compressible fluids, where estimates for H^{-1} differences in densities and L^2 differences in velocities are obtained by duality from bounds for the corresponding adjoint system. The second, more recent paper is [14], where the 2D isotropic Boussinesq system without heat diffusion but with viscosity in all directions of the velocity field has been studied in bounded domains. In this regard, it is worth mentioning however, that already in the 2010 arXiv version [18] of this manuscript, we have announced and posted our current results on the 2D Boussinesq equations without heat diffusion and with anisotropic viscosity, and where we also proposed an inviscid α -regularization for the two-dimensional inviscid, non-diffusive Boussinesq system of equations, which we call the Boussinesq-Voigt equations.

It is also worthwhile to mention that recently, in [2], the global regularity of classical solutions to the two-dimensional Boussinesq system in the case of vertical viscosity and vertical thermal diffusion was established.

2. Preliminaries

In this section, we introduce some preliminary material and notations which are commonly used in the mathematical study of fluids, in particular in the study of the NSE. For a more detailed discussion of these topics, we refer to [8,12,21,23].

Let \mathcal{F} be the set of all trigonometric polynomials with periodic domain $\mathbb{T}^2 := [0, 1]^2$. We define the space of smooth functions which incorporates the divergence-free and zero-average condition to be

$$\mathcal{V} := \left\{ \varphi \in \mathcal{F}^2 : \nabla \cdot \varphi = 0 \text{ and } \int_{\mathbb{T}^2} \varphi \, dx = 0 \right\}.$$

We denote by L^p , $W^{s,p}$, $H^s \equiv W^{s,2}$, $C^{0,\gamma}$ the usual Lebesgue, Sobolev, and Hölder spaces, and define H and V to be the closures of \mathcal{V} in L^2 and H^1 respectively. We restrict ourselves to finding solutions whose average over the periodic box \mathbb{T}^2 is zero. Observe from (1.1b) and (1.1c), if we assume that $\int_{\mathbb{T}^2} \theta_0(x) \, dx = 0$, then $\int_{\mathbb{T}^2} \theta(x, t) \, dx = 0$ for all $t \geq 0$, and also $\int_{\mathbb{T}^2} \mathbf{u}(x, t) \, dx = 0$ for all $t \geq 0$ provided $\int_{\mathbb{T}^2} \mathbf{u}_0(x) \, dx = 0$. Therefore, we can work in the spaces defined above consistently. The notation

$V^s := H^s(\mathbb{T}^2) \cap V$ will be convenient. When necessary, we write the components of a vector \mathbf{y} as y^j , $j = 1, 2$. We define the inner products on H and V respectively by

$$(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^2 \int_{\mathbb{T}^2} u^i v^i dx \quad \text{and} \quad ((\mathbf{u}, \mathbf{v})) = \sum_{i,j=1}^2 \int_{\mathbb{T}^2} \partial_j u^i \partial_j v^i dx,$$

and the associated norms $|\mathbf{u}| = (\mathbf{u}, \mathbf{u})^{1/2}$, $\|\mathbf{u}\| = ((\mathbf{u}, \mathbf{u}))^{1/2}$. (We use these notations indiscriminately for both scalars and vectors, which should not be a source of confusion.) Note that $((\cdot, \cdot))$ is a norm due to the Poincaré inequality, (2.16), below. We denote by V' the dual space of V . The action of V' on V is denoted by $\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_{V'}$. Note that we have the continuous embeddings

$$V \hookrightarrow H \hookrightarrow V'. \quad (2.1)$$

Moreover, by the Rellich–Kondrachov Compactness Theorem (see, e.g., [1,11]), these embeddings are compact.

Following [9], we define the spaces

$$\sqrt{L} := \{w \mid \|w\|_{\sqrt{L}} < \infty\},$$

where $\|\cdot\|_{\sqrt{L}}$ is defined by (1.3). This space arises naturally, due to the following inequality, valid in two dimensions, which can be proven without using the Littlewood–Paley theory (see, e.g., [19] for such a proof, with a sharper dependence on p),

$$\|\mathbf{w}\|_p \leq C\sqrt{p-1}\|\mathbf{w}\|_{H^1}, \quad (2.2)$$

for all $\mathbf{w} \in H^1(\mathbb{T}^2)$, for any $p \in [2, \infty)$, and where we denote by $\|\cdot\|_p$ the usual L^p norm (see [9] for a proof using the Littlewood–Paley theory). Note that clearly $L^\infty \subset \sqrt{L} \subset L^p$ for every $p \in [2, \infty)$. We also recall the following well-known elliptic estimate, which follows from the Calderón–Zygmund theory and the Biot–Savart law for an incompressible vector field \mathbf{u} , satisfying $\nabla \cdot \mathbf{u} = 0$, and $\nabla \times \mathbf{u} = \omega$:

$$\|\nabla \mathbf{u}\|_p \leq Cp\|\omega\|_p \quad (2.3)$$

for any $p \in (1, \infty)$ (see, e.g., [20,25]).

Let Y be a Banach space. We denote by $L^p([0, T], Y)$ (which we also denote as $L_T^p Y_X$) the space of (Bochner) measurable functions $t \mapsto w(t)$, where $w(t) \in Y$ for a.e. $t \in [0, T]$, such that the integral $\int_0^T \|w(t)\|_Y^p dt$ is finite (see, e.g., [1]). A similar convention is used in the notation $C^k([0, T], X)$ for k -times differentiable functions of time on the interval $[0, T]$ with values in Y . Abusing notation slightly, we write $w(\cdot)$ for the map $t \mapsto w(t)$. In the same vein, we often write the vector-valued function $w(\cdot, t)$ as $w(t)$ when w is a function of x and t . We denote by $\dot{C}^\infty(\mathbb{T}^2 \times [0, T])$ the set of infinitely differentiable functions in the variable x and t which are periodic in x with $\int_{\mathbb{T}^2} \varphi(\cdot, t) dx = 0$. Similarly, we denote by $\dot{L}^p(\mathbb{T}^2) = \{\varphi \in L^p(\mathbb{T}^2) : \int_{\mathbb{T}^2} \varphi(x) dx = 0\}$.

We denote by $P_\sigma : \dot{L}^2 \rightarrow H$ the Leray–Helmholtz projection operator and define the Stokes operator $A := -P_\sigma \Delta$ with domain $\mathcal{D}(A) := H^2 \cap V$. For $\varphi \in \mathcal{D}(A)$, we have the norm equivalence $|A\varphi| \cong \|\varphi\|_{H^2}$ (see, e.g., [8,23]). In particular, the Stokes operator A can be extended as a linear operator from V into V' associated with the bilinear form $((\mathbf{u}, \mathbf{v}))$,

$$\langle A\mathbf{u}, \mathbf{v} \rangle = ((\mathbf{u}, \mathbf{v})) \quad \text{for all } \mathbf{v} \in V.$$

It is known that $A^{-1} : H \rightarrow \mathcal{D}(A) \hookrightarrow H$ is a positive-definite, self-adjoint, compact operator from H into itself, and therefore it has an orthonormal basis of positive eigenvectors $\{\mathbf{w}_k\}_{k=1}^\infty$ in H corresponding to a non-increasing sequence of eigenvalues (see, e.g., [8,21]). The vectors $\{\mathbf{w}_k\}_{k=1}^\infty$ are also the eigenvectors of A . Since the corresponding eigenvalues of A^{-1} can be ordered in a decreasing order, we can label the eigenvalues λ_k of A so that $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$. Let $H_n := \text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$, and let $P_n : H \rightarrow H_n$ be the L^2 orthogonal projection onto H_n . Notice that in the case of periodic boundary conditions in the torus \mathbb{T}^2 we have $\lambda_1 = (2\pi)^{-2}$. We will abuse notation slightly and also use P_n in the scalar case for the corresponding projection onto eigenfunctions of $-\Delta$, but this should not be a source of confusion. Furthermore, in our case it is known that $A = -\Delta$ due to the periodic boundary conditions (see, e.g., [8,21]) and the eigenvectors \mathbf{w}_j are of the form $\mathbf{a}_k e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$, with $\mathbf{a}_k \cdot \mathbf{k} = 0$.

It will be convenient to use the following standard notation for the bilinear term

$$B(\mathbf{w}_1, \mathbf{w}_2) := P_\sigma \sum_{j=1}^2 \partial_j (w_1^j \mathbf{w}_2) \quad (2.4)$$

for $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{V}$. We list some important properties of B which can be found for example in [8,12,21,23].

Lemma 2.1. *The operator B defined in (2.4) is a bilinear form which can be extended as a continuous map $B : V \times V \rightarrow V'$ such that*

$$\langle B(\mathbf{w}_1, \mathbf{w}_2), \mathbf{w}_3 \rangle = \int_{\mathbb{T}^2} (\mathbf{w}_1 \cdot \nabla \mathbf{w}_2) \cdot \mathbf{w}_3 \, dx, \quad (2.5)$$

for every $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathcal{V}$, satisfying the following properties:

(i) For $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in V$,

$$\langle B(\mathbf{w}_1, \mathbf{w}_2), \mathbf{w}_3 \rangle_{V'} = -\langle B(\mathbf{w}_1, \mathbf{w}_3), \mathbf{w}_2 \rangle_{V'}, \quad (2.6)$$

and therefore

$$\langle B(\mathbf{w}_1, \mathbf{w}_2), \mathbf{w}_2 \rangle_{V'} = 0. \quad (2.7)$$

(ii) For $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in V$,

$$|\langle B(\mathbf{w}_1, \mathbf{w}_2), \mathbf{w}_3 \rangle_{V'}| \leq C |\mathbf{w}_1|^{1/2} \|\mathbf{w}_1\|^{1/2} \|\mathbf{w}_2\| \|\mathbf{w}_3\|^{1/2} \|\mathbf{w}_3\|^{1/2}, \quad (2.8)$$

$$|\langle B(\mathbf{w}_1, \mathbf{w}_2), \mathbf{w}_3 \rangle_{V'}| \leq C |\mathbf{w}_1|^{1/2} \|\mathbf{w}_1\|^{1/2} |\mathbf{w}_2|^{1/2} \|\mathbf{w}_2\|^{1/2} \|\mathbf{w}_3\|. \quad (2.9)$$

Let us define the scalar analogue of the bilinear operator (2.4), motivated by the transport term in the density equation,

$$\mathcal{B}(\mathbf{w}, \psi) := \sum_{j=1}^2 \partial_j (w^j \psi) \quad (2.10)$$

for $\mathbf{w} \in \mathcal{V}$ and $\psi \in \mathcal{F}$ with $\int_{\mathbb{T}^2} \psi \, dx = 0$. We have the following similar properties for \mathcal{B} which can be proven easily as in the proof of Lemma 2.1.

Lemma 2.2. *The operator \mathcal{B} defined in (2.10) is a bilinear form which can be extended as a continuous map $\mathcal{B} : V \times H^1 \rightarrow H^{-1}$, such that*

$$\langle \mathcal{B}(\mathbf{w}, \psi), \phi \rangle_{H^{-1}} = - \int_{\mathbb{T}^2} \mathbf{w} \cdot \nabla \phi \psi \, dx, \quad (2.11)$$

for every $\mathbf{w} \in \mathcal{V}$ and $\phi, \psi \in \dot{C}^1$. Moreover,

$$\langle \mathcal{B}(\mathbf{w}, \psi), \phi \rangle_{H^{-1}} = - \langle \mathcal{B}(\mathbf{w}, \phi), \psi \rangle_{H^{-1}}, \quad (2.12)$$

and therefore

$$\langle \mathcal{B}(\mathbf{w}, \phi), \phi \rangle_{H^{-1}} = 0. \quad (2.13)$$

Furthermore, \mathcal{B} is also a bilinear form which can be extended as a continuous map $\mathcal{B} : \mathcal{D}(A) \times L^2 \rightarrow H^{-1}$.

Here and below, C, C_j , etc. denote generic constants which may change from line to line. $C_\alpha, C(\cdots)$, etc. denote generic constants which depend only upon the indicated parameters. K, K_j , etc. denote constants which depend on norms of initial data, and also may vary from line to line. Next, we recall the Ladyzhenskaya inequality; namely, that for $f \in H^1(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} f \, dx = 0$, we have

$$\|f\|_{L^4} \leq \|f\|^{1/2} \|f\|^{1/2}. \quad (2.14)$$

We also recall Agmon's inequality in two dimensions (see, e.g., [3,8]). For $\mathbf{w} \in \mathcal{D}(A)$ we have

$$\|\mathbf{w}\|_{L^\infty} \leq C \|\mathbf{w}\|^{1/2} \|A\mathbf{w}\|^{1/2}. \quad (2.15)$$

Furthermore, for all $\varphi \in W^{1,p}(\mathbb{T}^2)$, $p \geq 2$, we have the Poincaré inequality

$$\|\varphi\|_{L^p} \leq C \|\nabla \varphi\|_{L^p}, \quad (2.16)$$

with $C = \lambda_1^{-1/2}$ if $p = 2$.

Finally, we note a result of de Rham [23,24], which implies that, if \mathbf{g} is a locally integrable function (or more generally, a distribution), we have

$$\mathbf{g} = \nabla p \quad \text{for some distribution } p \quad \text{iff} \quad \langle \mathbf{g}, \mathbf{w} \rangle = 0 \quad \text{for all } \mathbf{w} \in \mathcal{V}, \quad (2.17)$$

which one uses to recover the pressure.

3. Global well-posedness for the 2D non-diffusive Boussinesq equations with anisotropic viscosity $(P_{\nu_x, 0})$

Before we prove our main result for the case $P_{\nu_x, 0}$, we record some theorems for the fully viscous case, which we will refer to in our proof.

3.1. Known global well-posedness results for the fully viscous case.

Let us first define the weak formulation of problem $P_{\nu,\kappa}$ in $\mathbb{T}^2 \times [0, T]$. By choosing a suitable phase space which incorporates the divergence free condition of the Boussinesq equations, we can eliminate the pressure from the equation, as is standard in the theory of the Navier–Stokes equations. Consider the scalar test functions $\varphi(x, t) \in \dot{C}^\infty(\mathbb{T}^2 \times [0, T])$, such that $\varphi(x, T) = 0$; and the vector test functions $\Phi(x, t) \in [\dot{C}^\infty(\mathbb{T}^2 \times [0, T])]^2$ such that $\nabla \cdot \Phi(\cdot, t) = 0$ and $\Phi(x, T) = 0$. Then the weak formulation of problem $P_{\nu,\kappa}$ in $\mathbb{T}^2 \times [0, T]$ (and similarly of problem $P_{\nu,0}$, when $\kappa = 0$, in $\mathbb{T}^2 \times [0, T]$) is written as follows:

$$\begin{aligned} & - \int_0^T (\mathbf{u}(s), \Phi'(s)) ds + \nu \int_0^T ((\mathbf{u}(s), \Phi(s))) ds + \sum_{j=1}^2 \int_0^T (u_j \mathbf{u}, \partial_j \Phi) ds \\ & = (\mathbf{u}_0(x), \Phi(x, 0)) + \int_0^T (\theta(s) \mathbf{e}_2, \Phi(s)) ds, \end{aligned} \quad (3.1a)$$

$$\begin{aligned} & - \int_0^T (\theta(s), \varphi'(s)) ds + \int_0^T (\mathbf{u} \theta, \nabla \varphi) ds + \kappa \int_0^T ((\theta(s), \varphi(s))) ds \\ & = (\theta_0(x), \varphi(x, 0)). \end{aligned} \quad (3.1b)$$

Remark 3.1. It will become clear later that (3.1) will hold for a larger class of test functions, and consequently it will be sufficient to consider only test functions of the form

$$\Phi(x, t) = \Gamma_{\mathbf{m}}(t) e^{2\pi i \mathbf{m} \cdot \mathbf{x}}, \quad \text{with } \Gamma_{\mathbf{m}} \in [C^\infty([0, T])]^2 \text{ and } \mathbf{m} \cdot \Gamma_{\mathbf{m}}(t) = 0, \quad (3.2a)$$

and

$$\varphi(x, t) = \chi_{\mathbf{m}}(t) e^{2\pi i \mathbf{m} \cdot \mathbf{x}}, \quad \text{with } \chi_{\mathbf{m}} \in C^\infty([0, T]), \quad (3.2b)$$

for $\mathbf{m} \in (\mathbb{Z} \setminus \{0\})^2$, since such functions form a basis for the corresponding larger spaces of test functions.

From here on, we only work on spaces of functions which are periodic and with spatial average zero. Therefore, to simplify notation, we write \dot{L}^2 as L^2 , \dot{C}^k as C^k , etc.

In the two-dimensional case, the global well-posedness of system $P_{\nu,\kappa}$ in (1.1), that is, in the case $\kappa > 0$, $\nu > 0$, is well-known, and can be proven in a similar manner following the work of [13] (see also [5,22]). We have the following existence and uniqueness results for the system $P_{\nu,\kappa}$, which will be used to prove the existence of weak solutions for the system $P_{\nu,0}$.

Theorem 3.2. *Let $T > 0$, $\nu > 0$ be fixed but arbitrary. Then, the following results hold:*

- (i) *If $\mathbf{u}_0 \in H$, $\theta_0 \in L^2$ then for each $\kappa > 0$, (1.1) has a unique solution $(\mathbf{u}_\kappa, \theta_\kappa)$ in the sense of (3.1) such that $\mathbf{u}_\kappa \in C([0, T], H) \cap L^2([0, T], V)$, $\theta_\kappa \in C_w([0, T], L^2)$. Furthermore, there exists a constant $K_0 > 0$ independent of κ such that the following bounds hold: $\|\mathbf{u}_\kappa\|_{L^2([0, T], V)} \leq K_0$, $\|\mathbf{u}_\kappa\|_{L^\infty([0, T], H)} \leq K_0$, $\|\frac{d}{dt} \mathbf{u}_\kappa\|_{L^2([0, T], V')} \leq K_0$, $\|\theta_\kappa\|_{L^\infty([0, T], L^2)} \leq |\theta_0|$, $\|\frac{d}{dt} \theta_\kappa\|_{L^2([0, T], H^{-2})} \leq K_0$ and $\sqrt{\kappa} \|\theta_\kappa\|_{L^2([0, T], H^1)} \leq K_0$.*
- (ii) *If the initial data $\mathbf{u}_0 \in V$ and $\theta_0 \in L^2$, then $\mathbf{u}_\kappa \in C([0, T], V) \cap L^2([0, T], \mathcal{D}(A))$ and we also have the bounds: $\|\mathbf{u}_\kappa\|_{L^2([0, T], \mathcal{D}(A))} \leq K_0$, $\|\mathbf{u}_\kappa\|_{L^\infty([0, T], V)} \leq K_0$, $\|\frac{d}{dt} \mathbf{u}_\kappa\|_{L^2([0, T], H)} \leq K_0$ and $\|\frac{d}{dt} \theta_\kappa\|_{L^2([0, T], H^{-1})} \leq K_0$.*

- (iii) If $\theta_0 \in L^\infty$ and $\mathbf{u}_0 \in H$, then $\|\theta_\kappa\|_{L^\infty([0,T],L^\infty)} \leq \|\theta_0\|_\infty$.
 (iv) If $\mathbf{u}_0 \in H^3$ and $\theta_0 \in H^2$ then for each $\kappa > 0$, (1.1) has a unique solution $\mathbf{u}_\kappa \in C([0, T], H^3) \cap L^2([0, T], H^4)$ and $\theta_\kappa \in C([0, T], H^2) \cap L^2([0, T], H^3)$.

Proof. Parts (i) and (ii) are essentially proven in [5,13,22] following the classical theory of the Navier–Stokes equations. The uniform bounds in part (ii) will be established explicitly in the later proofs when called for. Part (iii) can be proven using a maximum principle and is proven for example in [5, 22]. Part (iv) can be proven using basic energy estimates and Grönwall's inequality again following the classical theory of the Navier–Stokes equations. \square

3.2. Global existence for the case of anisotropic viscosity

We now consider the Boussinesq equations with anisotropic viscosity as given in (1.2). We will first define what we mean by weak solution to system (1.2) and then show its existence. Setting additional notation, we denote the vorticity $\omega := \partial_1 u^2 - \partial_2 u^1$, which satisfies the following equation

$$\partial_t \omega + \nabla \cdot (\omega \mathbf{u}) - \nu \partial_1^2 \omega = \partial_1 \theta. \quad (3.3)$$

The best global well-posedness result we are aware of for problem (1.2), prior to the present work, in the case of the whole plane \mathbb{R}^2 is stated in the following theorem, established in [9].

Theorem 3.3. (See Danchin and Paicu [9].) Let $\Omega = \mathbb{R}^2$. Suppose $\theta_0 \in L^2 \cap L^\infty$, and $\mathbf{u}_0 \in V$ with $\omega_0 \in \sqrt{L}$. Then system (1.2) admits a global solution (\mathbf{u}, θ) such that $\theta \in C_B([0, \infty); L^2) \cap C_w([0, \infty); L^\infty) \cap L^\infty([0, \infty), L^\infty)$ and $\mathbf{u} \in C_w([0, \infty); H^1)$, $\mathbf{u} \cdot \mathbf{e}_2 \in L_{loc}^2([0, \infty); H^2)$, $\omega \in L_{loc}^\infty([0, \infty), \sqrt{L})$, $\nabla \mathbf{u} \in L_{loc}^2([0, \infty), \sqrt{L})$. If in addition $\theta_0 \in H^s$ for some $s \in (0, 1]$, then $\theta \in C([0, \infty); H^{s-\epsilon})$ for all $\epsilon > 0$. Finally, if $s > 1/2$, then the solution is unique.

In the present work, we remove the assumption that $s > 1/2$ on the initial data, and require only that $\theta \in L^\infty$. To begin with, we weaken the notion of solution by making the following definition.

Definition 3.4 (Weak solutions for the anisotropic case). Let $T > 0$. Let $\theta_0 \in L^2$, $\omega_0 := \nabla^\perp \cdot \mathbf{u}_0 \in L^2$. We say that (\mathbf{u}, θ) is a weak solution to (1.2) on the interval $[0, T]$ if $\omega \in L^\infty([0, T]; L^2) \cap C_w([0, T]; L^2)$ and $\theta \in L^\infty([0, T]; L^2) \cap C_w([0, T]; L^2)$, $u^2 \in L^2([0, T], H^2)$, $\frac{d\mathbf{u}}{dt} \in L^1([0, T], V')$, $\frac{d\theta}{dt} \in L^1([0, T], H^{-2})$ and also (\mathbf{u}, θ) satisfies (1.2) in the weak sense; that is, for any Φ, φ , chosen as in (3.2), it holds that

$$\begin{aligned} & - \int_0^T (\mathbf{u}(s), \Phi'(s)) ds + \nu \int_0^T (\partial_1 \mathbf{u}(s), \partial_1 \Phi(s)) ds + \sum_{j=1}^2 \int_0^T (u^j \mathbf{u}, \partial_j \Phi) ds \\ & = (\mathbf{u}_0, \Phi(0)) + \int_0^T (\theta(s) \mathbf{e}_2, \Phi(s)) ds, \end{aligned} \quad (3.4a)$$

$$- \int_0^T (\theta(s), \varphi'(s)) ds + \int_0^T (\theta \mathbf{u}, \nabla \varphi) ds = (\theta_0, \varphi(0)), \quad (3.4b)$$

where $' \equiv \frac{d}{ds}$.

Remark 3.5. Following standard arguments as in the theory of the NSE (see, e.g., [8,23]) one can show that the above system is equivalent to the functional form

$$\frac{d\mathbf{u}}{dt} + \nu \partial_1^2 \mathbf{u} + B(\mathbf{u}, \mathbf{u}) = P_\sigma(\theta \mathbf{e}_2) \quad \text{in } L^2([0, T], V') \quad \text{and} \quad (3.5a)$$

$$\frac{d\theta}{dt} + B(\mathbf{u}, \theta) = 0 \quad \text{in } L^2([0, T], H^{-2}). \quad (3.5b)$$

We now state and prove our main results for the system (1.2) ($P_{\nu_x, 0}$). The global existence and regularity results will be stated in the theorem below and the uniqueness theorem will follow.

Theorem 3.6 (Global existence and regularity). *Let $T > 0$ be given. Let $\theta_0 \in L^2$ and $\omega_0 \in L^2$. Then, the following hold:*

- (1) *There exists a weak solution to (1.2) in the sense of Definition 3.4.*
- (2) *If $\omega_0 \in L^p$, and $\theta_0 \in L^p$, with $p \in [2, \infty)$ fixed, then there exists a weak solution satisfying the additional regularity properties that $\omega \in L^\infty([0, T], L^p)$ and $\theta \in L^\infty([0, T], L^p)$.*
- (3) *Furthermore, if $\omega_0 \in \sqrt{L}$ and $\theta_0 \in L^\infty$, then there exists a solution (\mathbf{u}, θ) such that $\omega \in L^\infty([0, T], \sqrt{L}) \cap C_w([0, T], L^2)$, $\frac{d\omega}{dt} \in L^2([0, T], V')$ and $\theta \in L^\infty([0, T], L^\infty) \cap C([0, T], w^*-L^\infty)$ (where w^*-L^∞ denotes the weak-* topology on L^∞) with $\frac{d\theta}{dt} \in L^\infty([0, T], H^{-1})$.*

Proof. The outline of our proof is as follows. We begin by generating an approximate sequence of solutions $(\mathbf{u}^{(n)}, \theta^{(n)})$ to $P_{\nu_x, 0}$ by adding an artificial vertical viscosity $\nu_y^{(n)} > 0$, and an artificial diffusion $\kappa^{(n)} > 0$, where $\kappa^{(n)}, \nu_y^{(n)} \rightarrow 0$ as $n \rightarrow \infty$, and also by smoothing the initial data. Global existence of solutions to the fully viscous system $P_{\nu, \kappa}$, given smoothed initial condition is guaranteed (see, Theorem 3.2, part (iv)). Next, we establish uniform bounds, for the relevant norms of the approximate sequence of solutions which are independent of the index n using basic energy estimates. We then employ the Aubin Compactness Theorem (see, e.g., [8,23]) to show that the sequence of approximate solutions has a subsequence converging in appropriate function spaces. This limit will serve as a candidate weak solution. We then show that one can pass to the limit to show that the candidate functions satisfy the weak formulation (3.4). Finally, we establish some regularity results.

Step 1: *Generating solutions to the regularized system given smoothed initial data.*

Let $\nu_x > 0$ be fixed and let $\kappa^{(n)}, \nu_y^{(n)}$ be a sequence of positive numbers, converging to zero. Without loss of generality, one can assume that both $\kappa^{(n)} \leq \nu_x$ and $\nu_y^{(n)} \leq \nu_x$. Let $(\mathbf{u}_0^{(n)}, \theta_0^{(n)})$ be a sequence of smooth initial data such that $\mathbf{u}_0^{(n)} \rightarrow \mathbf{u}_0$ in V and $\theta_0^{(n)} \rightarrow \theta_0$ in L^2 , chosen in such a way that for each $n \in \mathbb{N}$, $\|\mathbf{u}_0^{(n)}\| \leq \|\mathbf{u}_0\| + \frac{1}{n}$ and $|\theta_0^{(n)}| \leq |\theta_0| + \frac{1}{n}$. Notice, since $\mathbf{u}_0^{(n)}$ are smooth it follows that $\omega_0^{(n)} := \nabla^\perp \cdot \mathbf{u}_0^{(n)}$ are also smooth functions, bounded in L^2 . Using Theorem 3.2, part (iii) (with a trivial modification to account for values of the viscosity which differ in the horizontal and vertical directions), we have that for each n , there exist $(\mathbf{u}^{(n)}, \theta^{(n)})$ satisfying the following equations:

$$\begin{aligned} & - \int_0^T (\mathbf{u}^{(n)}(s), \Phi'(s)) ds + \nu_x \int_0^T (\partial_1 \mathbf{u}^{(n)}(s), \partial_1 \Phi(s)) ds \\ & + \nu_y^{(n)} \int_0^T (\partial_2 \mathbf{u}^{(n)}(s), \partial_2 \Phi(s)) ds + \sum_{j=1}^2 \int_0^T (u^{j, (n)} \mathbf{u}^{(n)}, \partial_j \Phi) ds \\ & = (\mathbf{u}_0^{(n)}, \Phi(0)) + \int_0^T (\theta^{(n)}(s) \mathbf{e}_2, \Phi(s)) ds, \end{aligned} \quad (3.6a)$$

$$\begin{aligned}
& \int_0^T (\theta^{(n)}(s), \varphi'(s)) ds - \int_0^T (\theta^{(n)} \mathbf{u}^{(n)}, \nabla \varphi) ds \\
&= \kappa^{(n)} \int_0^T (\nabla \theta^{(n)}(s), \nabla \Phi(s)) ds - (\theta_0^{(n)}, \varphi(0)).
\end{aligned} \tag{3.6b}$$

Step 2: *A priori estimates and using compactness arguments to prove convergence of a subsequence.*

We next establish *a priori* estimates on $(\mathbf{u}^{(n)}, \theta^{(n)})$ uniformly in n (independent of $\nu_y^{(n)}$ and $\kappa^{(n)}$). From the above smoothness properties of $(\mathbf{u}^{(n)}, \theta^{(n)})$, we can now derive *a priori* estimates using basic energy estimates in which the derivatives and integrations are well defined. Using the fact that $\operatorname{div} \mathbf{u}^{(n)} = 0$, one easily obtains

$$|\theta^{(n)}(t)| \leq |\theta_0^{(n)}| \leq |\theta_0| + \frac{1}{n}, \tag{3.7}$$

and

$$\begin{aligned}
& |\mathbf{u}^{(n)}(t)|^2 + 2\nu_x \int_0^t |\partial_1 \mathbf{u}^{(n)}(\tau)|^2 d\tau + 2\nu_y^{(n)} \int_0^t |\partial_2 \mathbf{u}^{(n)}(\tau)|^2 d\tau \\
& \leq \left(|\mathbf{u}_0| + \frac{1}{n} + t \left(|\theta_0| + \frac{1}{n} \right) \right)^2.
\end{aligned}$$

The calculations above follow by a valid replacement of the test functions by $\theta^{(n)}$ and $\mathbf{u}^{(n)}$ in (3.6) and then integrating by parts.

Next, from the evolution equation of the vorticity, namely the equation

$$\partial_t \omega^{(n)} + \mathbf{u}^{(n)} \cdot \nabla \omega^{(n)} - \nu_x \partial_1^2 \omega^{(n)} - \nu_y^{(n)} \partial_2^2 \omega^{(n)} = \partial_1 \theta^{(n)}, \tag{3.8}$$

it follows similarly that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} |\omega^{(n)}|^2 + \nu_x |\partial_1 \omega^{(n)}|^2 + \nu_y^{(n)} |\partial_2 \omega^{(n)}|^2 = -(\theta^{(n)}, \partial_1 \omega^{(n)}) \\
& \leq \frac{\nu_x}{2} |\partial_1 \omega^{(n)}|^2 + \frac{1}{2\nu_x} |\theta^{(n)}|^2.
\end{aligned}$$

Integrating in time, we have

$$|\omega^{(n)}|^2 + \nu_x \int_0^t |\partial_1 \omega^{(n)}|^2 d\tau + 2\nu_y^{(n)} \int_0^t |\partial_2 \omega^{(n)}|^2 d\tau \tag{3.9}$$

$$\leq \left(|\omega_0| + \frac{1}{n} \right)^2 + \frac{t}{2\nu_x} \left(|\theta_0| + \frac{1}{n} \right)^2, \tag{3.10}$$

which implies that $\omega^{(n)}$ is uniformly bounded in $L^\infty([0, T], L^2)$ with respect to n , and therefore $\mathbf{u}^{(n)}$ is uniformly bounded in $L^\infty([0, T], V)$ with respect to n . Furthermore, (3.9) shows that $\partial_1 \omega^{(n)}$ is uniformly bounded in $L^2([0, T], L^2)$ with respect to n . We also observe that

$$\partial_1 \omega^{(n)} = \partial_1^2 u^{2,(n)} - \partial_1 \partial_2 u^{1,(n)} = \partial_1^2 u^{2,(n)} + \partial_2^2 u^{2,(n)} = \Delta u^{2,(n)}.$$

Therefore, $\Delta u^{2,(n)}$ is uniformly bounded in $L^2([0, T], L^2)$, so that $u^{2,(n)}$ is uniformly bounded in $L^2([0, T], H^2)$ by elliptic regularity, and thus $\nabla u^{2,(n)}$ is uniformly bounded in $L^2([0, T], H^1)$, all with respect to n . Next we derive uniform bounds on the derivatives $(\frac{d\mathbf{u}^{(n)}}{dt})_{n \in \mathbb{N}}$. Note that

$$\frac{d\omega^{(n)}}{dt} = -\mathcal{B}(\omega^{(n)}, \mathbf{u}^{(n)}) + \nu_x \partial_1^2 \omega^{(n)} + \nu_y^{(n)} \partial_2^2 \omega^{(n)} + \partial_1 \theta^{(n)}.$$

Thus,

$$\begin{aligned} \left\| \frac{d\omega^{(n)}}{dt} \right\|_{H^{-2}} &\leq \sup_{\|\mathbf{w}\|_{\dot{H}^2}=1} |\langle \mathcal{B}(\omega^{(n)}, \mathbf{u}^{(n)}), \mathbf{w} \rangle| + \nu_x \sup_{\|\mathbf{w}\|_{\dot{H}^2}=1} |\langle \partial_1^2 \omega^{(n)}, \mathbf{w} \rangle| \\ &\quad + \nu_y^{(n)} \sup_{\|\mathbf{w}\|_{\dot{H}^2}=1} |\langle \partial_2^2 \omega^{(n)}, \mathbf{w} \rangle| + \sup_{\|\mathbf{w}\|_{\dot{H}^2}=1} |\langle \partial_1 \theta^{(n)}, \mathbf{w} \rangle| \\ &= \sup_{\|\mathbf{w}\|_{\dot{H}^2}=1} |\langle \omega^{(n)} \mathbf{u}^{(n)}, \nabla \mathbf{w} \rangle| + \nu_x \sup_{\|\mathbf{w}\|_{\dot{H}^2}=1} |\langle \omega^{(n)}, \partial_1^2 \mathbf{w} \rangle| \\ &\quad + \nu_y^{(n)} \sup_{\|\mathbf{w}\|_{\dot{H}^2}=1} |\langle \omega^{(n)}, \partial_2^2 \mathbf{w} \rangle| + \sup_{\|\mathbf{w}\|_{\dot{H}^2}=1} |\langle \theta^{(n)}, \partial_1 \mathbf{w} \rangle| \\ &\leq C |\omega^{(n)}| |\mathbf{u}^{(n)}|^{1/2} \|\mathbf{u}^{(n)}\|^{1/2} + \nu_x |\omega^{(n)}| + \nu_x |\omega^{(n)}| + |\theta^{(n)}|. \end{aligned} \quad (3.11)$$

Since each of the terms on the right-hand side of the inequality above is bounded independently of n , we deduce by the Calderón–Zygmund elliptic estimate (2.3) that $\partial_t \mathbf{u}^{(n)}$ is bounded in $L^\infty([0, T], V')$ independently of n . Similarly, one can show easily that

$$\left\| \frac{d\theta^{(n)}}{dt} \right\|_{H^{-2}} \leq |\theta^{(n)}| |\mathbf{u}^{(n)}|^{1/2} \|\mathbf{u}^{(n)}\|^{1/2}, \quad (3.12)$$

which implies also that $\frac{d\theta^{(n)}}{dt}$ is bounded in $L^\infty([0, T], H^{-2})$ independently of n . To summarize, we have from the above results that

$$(\theta^{(n)})_{n \in \mathbb{N}} \text{ is bounded in } L^\infty([0, T], L^2), \quad (3.13a)$$

$$(\mathbf{u}^{(n)})_{n \in \mathbb{N}} \text{ is bounded in } L^\infty([0, T], V), \quad (3.13b)$$

$$(u^{2,(n)})_{n \in \mathbb{N}} \text{ is bounded in } L^2([0, T], H^2), \quad (3.13c)$$

$$\left(\frac{d\mathbf{u}^{(n)}}{dt} \right)_{n \in \mathbb{N}} \text{ is bounded in } L^\infty([0, T], V'), \quad (3.13d)$$

$$\left(\frac{d\theta^{(n)}}{dt} \right)_{n \in \mathbb{N}} \text{ is bounded in } L^\infty([0, T], H^{-2}). \quad (3.13e)$$

Using the Banach–Alaoglu and Aubin Compactness Theorems (see, e.g., [8,23]), the uniform bounds with respect to n , as stated in (3.13) imply that one can extract a further subsequence (which we relabel with the index n if necessary) such that

$$\theta^{(n)} \rightharpoonup \theta \quad \text{weakly in } L^2([0, T], L^2) \text{ and weak-* in } L^\infty([0, T], L^2), \quad (3.14a)$$

$$\mathbf{u}^{(n)} \rightarrow \mathbf{u} \quad \text{strongly in } L^2([0, T], H), \quad (3.14b)$$

$$\mathbf{u}^{(n)} \rightharpoonup \mathbf{u} \quad \text{weakly in } L^2([0, T], V) \text{ and weak-* in } L^\infty([0, T], V), \quad (3.14c)$$

$$u^{2,(n)} \rightharpoonup u^2 \quad \text{weakly in } L^2([0, T], H^2), \quad (3.14d)$$

$$\frac{d\mathbf{u}^{(n)}}{dt} \rightharpoonup \frac{d\mathbf{u}}{dt} \quad \text{weakly in } L^2([0, T], V') \text{ and weak-* in } L^\infty([0, T], V'), \quad (3.14e)$$

$$\frac{d\theta^{(n)}}{dt} \rightharpoonup \frac{d\theta}{dt} \quad \text{weakly in } L^2([0, T], H^{-2}) \text{ and weak-* in } L^\infty([0, T], H^{-2}). \quad (3.14f)$$

Step 3: *Passing to the limit in the system.*

It remains to show that (3.14) is enough to pass to the limit in (3.6) to show that (\mathbf{u}, θ) satisfies (3.4). To do this, in accordance with Remark 3.1 and Definition 3.4, we only consider test functions of the form (3.2), which we note is sufficient for showing that (\mathbf{u}, θ) satisfies (3.4). For the linear terms, it is straightforward to pass to the limit $(\kappa^{(n)}, v_y^{(n)} \rightarrow 0)$ in (3.6), by the weak convergence in (3.14c) and (3.14a)

It remains to show the convergence of the remaining non-linear terms. Let

$$I(n) := \sum_{j=1}^2 \int_0^T (u^{j,(n)} \mathbf{u}^{(n)}, \Gamma_{\mathbf{m}}(s) \partial_j e^{2\pi i \mathbf{m} \cdot \mathbf{x}}) ds - \sum_{j=1}^2 \int_0^T (u^j \mathbf{u}, \Gamma_{\mathbf{m}}(s) \partial_j e^{2\pi i \mathbf{m} \cdot \mathbf{x}}) ds,$$

$$J(n) := \int_0^T (\mathbf{u}^{(n)}(s) \theta^{(n)}(s), \chi_{\mathbf{m}}(s) \nabla e^{2\pi i \mathbf{m} \cdot \mathbf{x}}) ds - \int_0^T (\mathbf{u}(s) \theta(s), \chi_{\mathbf{m}}(s) \nabla e^{2\pi i \mathbf{m} \cdot \mathbf{x}}) ds.$$

To show $I(n) \rightarrow 0$ as $n \rightarrow \infty$, we write $I(n) = I_1(n) + I_2(n)$, the definitions of which are given below. We have

$$\begin{aligned} |I_1(n)| &:= \left| \sum_{j=1}^2 \int_0^T ((u^{j,(n)}(s) - u^j(s)) \mathbf{u}^{(n)}(s), \partial_j e^{2\pi i \mathbf{m} \cdot \mathbf{x}}) \Gamma_{\mathbf{m}}(s) ds \right| \\ &\leq \int_0^T |\mathbf{u}^{(n)}(s) - \mathbf{u}(s)| |\mathbf{u}^{(n)}(s)| |\nabla e^{2\pi i \mathbf{m} \cdot \mathbf{x}} \Gamma_{\mathbf{m}}(s)| ds \\ &\leq \|\mathbf{u}^{(n)} - \mathbf{u}\|_{L_T^2 H_x} \|\mathbf{u}^{(n)}\|_{L_T^\infty H_x} \|\nabla e^{2\pi i \mathbf{m} \cdot \mathbf{x}} \Gamma_{\mathbf{m}}\|_{L_T^2 L_x^\infty} \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, since $\mathbf{u}^{(n)} \rightarrow \mathbf{u}$ strongly in $L^2([0, T], H)$ and $\mathbf{u}^{(n)}$ is uniformly bounded in $L^\infty([0, T], V)$ and hence in $L^\infty([0, T], H)$. Similarly, for I_2 , we have that as $n \rightarrow \infty$

$$I_2(n) := \sum_{j=1}^2 \int_0^T (u^j(s) (\mathbf{u}^{(n)}(s) - \mathbf{u}(s)), \partial_j e^{2\pi i \mathbf{m} \cdot \mathbf{x}}) \Gamma_{\mathbf{m}}(s) ds \rightarrow 0.$$

To show $J(n) \rightarrow 0$ as $n \rightarrow \infty$, we write $J(n) = J_1(n) + J_2(n)$. We have

$$J_1(n) := \int_0^T ((\mathbf{u}^{(n)}(s) - \mathbf{u}(s))\theta^{(n)}(s), \nabla e^{2\pi i \mathbf{m} \cdot \mathbf{x}}) \chi_{\mathbf{m}}(s) ds \rightarrow 0,$$

as $n \rightarrow \infty$, since $\mathbf{u}^{(n)} \rightarrow \mathbf{u}$ strongly in $L^2([0, T], H)$, and also $\theta^{(n)} \rightarrow \theta$ weakly in $L^2([0, T], H)$. For J_2 , we have

$$J_2(n) := \int_0^T (\mathbf{u}(s)(\theta^{(n)}(s) - \theta(s)), \nabla e^{2\pi i \mathbf{m} \cdot \mathbf{x}}) \chi_{\mathbf{m}}(s) ds \rightarrow 0,$$

by the weak convergence in (3.14a) and the fact that $\mathbf{u} \in L^2([0, T], H)$. This establishes the existence of a weak solution to the system $P_{\nu_x, 0}$ when $\mathbf{u}_0 \in H^1$ and $\theta_0 \in L^2$.

Step 4: Showing that $\omega \in C_w([0, T]; L^2)$.

By the Arzela-Ascoli Theorem, it suffices to show that (a) $\{\omega^{(n)}(t)\}_{n \in \mathbb{N}}$ is a relatively weakly compact set in $L^2(\mathbb{T}^2)$ for a.e. $t \geq 0$ and (b) for every $\phi \in L^2(\mathbb{T}^2)$ the sequence $\{(\omega^{(n)}(t), \phi)\}_{n \in \mathbb{N}}$ is equicontinuous in $C([0, T])$. Condition (a) follows from the uniform boundedness of $\omega^{(n)}$ in $L^2(\mathbb{T}^2)$ for a.e. $t \geq 0$ given in (3.9). Next, we show that condition (b) is satisfied. We start by assuming that $\phi \in \mathcal{F}$. Integrating (3.8) in time, we estimate

$$\begin{aligned} & |(\omega^{(n)}(t_2), \phi) - (\omega^{(n)}(t_1), \phi)| \\ & \leq \nu_x \int_{t_1}^{t_2} |\partial_1 \omega^{(n)}| |\partial_1 \phi| dt + \nu_x \int_{t_1}^{t_2} |\omega^{(n)}| |\partial_2^2 \phi| dt + \|\nabla \phi\|_\infty \int_{t_1}^{t_2} |\mathbf{u}^{(n)}| |\omega^{(n)}| dt + \int_{t_1}^{t_2} |\theta^{(n)}| |\nabla \phi| dt \\ & \leq \|\nabla \phi\|_\infty |t_2 - t_1|^{1/2} \nu_x \|\partial_1 \omega^{(n)}\|_{L_T^2 L_x^2} + \nu_x |\partial_2^2 \phi|_\infty |t_2 - t_1| \|\omega^{(n)}\|_{L_T^\infty L_x^2} \\ & \quad + \|\nabla \phi\|_\infty |t_2 - t_1| (\|\mathbf{u}^{(n)}\|_{L_T^\infty L_x^2} \|\omega^{(n)}\|_{L_T^\infty L_x^2} + \|\theta^{(n)}\|_{L_T^\infty L_x^2}), \end{aligned}$$

where we recall we have assumed without loss of generality that $\nu_y^{(n)} \leq \nu_x$. Due to the uniform boundedness of $\omega^{(n)}$ (3.9) and $\theta^{(n)}$ (3.7), the right-hand side can be made small when $|t_2 - t_1|$ is small enough. Thus, the set $\{(\omega^{(n)}, \phi)\}$ is equicontinuous in $C([0, T])$ for $\phi \in \mathcal{F}$. One can then extend this result for all test functions ϕ in $L^2(\mathbb{T}^2)$ using a simple density argument as before. This completes the proof of part (1) of Theorem 3.6.

Step 5: Proof of part (2) of Theorem 3.6.

As in Step 1, we choose a sequence of smooth initial data $\omega_0^{(n)} \rightarrow \omega_0$ and similarly $\theta_0^{(n)} \rightarrow \theta_0$ in every L^p with $p \geq 2$ chosen in such a way that for each $n \in \mathbb{N}$, $\|\omega_0^{(n)}\|_p \leq \|\omega_0\|_p + \frac{1}{n}$ and $\|\theta_0^{(n)}\|_p \leq \|\theta_0\|_p + \frac{1}{n}$. From Theorem 3.2, we obtain for each n , a solution $u^{(n)}$ such that $u^{(n)}(t) \in H^3$ for a.e. t , which then gives us $\omega^{(n)}(t) \in H^2$, which is a Banach algebra in two dimensions, hence $|\omega^{(n)}(t)|^{p-2} \omega^{(n)}(t) \in H^2$ for a.e. t . We take the inner product of (3.8) with $|\omega^{(n)}|^{p-2} \omega^{(n)}$. Integrating

by parts, we obtain

$$\begin{aligned}
 & \frac{1}{p} \frac{d}{dt} \|\omega^{(n)}\|_p^p + \nu_x(p-1) \int_{\mathbb{T}^2} |\partial_1 \omega^{(n)}|^2 |\omega^{(n)}|^{p-2} dx + \nu_y^{(n)}(p-1) \int_{\mathbb{T}^2} |\partial_2 \omega^{(n)}|^2 |\omega^{(n)}|^{p-2} dx \\
 & \leq (p-1) \int_{\mathbb{T}^2} |\theta^{(n)}| |\partial_1 \omega^{(n)}| |\omega^{(n)}|^{p-2} dx \\
 & \leq \nu_x(p-1) \int_{\mathbb{T}^2} |\partial_1 \omega^{(n)}|^2 |\omega^{(n)}|^{p-2} dx + \frac{p-1}{4\nu_x} \int_{\mathbb{T}^2} |\theta^{(n)}|^2 |\omega^{(n)}|^{p-2} dx \\
 & \leq \nu_x(p-1) \int_{\mathbb{T}^2} |\partial_1 \omega^{(n)}|^2 |\omega^{(n)}|^{p-2} dx + \frac{p-1}{4\nu_x} \|\theta^{(n)}\|_p^2 \|\omega^{(n)}\|_p^{p-2}.
 \end{aligned}$$

Therefore,

$$\frac{1}{p} \frac{d}{dt} \|\omega^{(n)}\|_p^p \leq \frac{p-1}{4\nu_x} \|\theta^{(n)}\|_p^2 \|\omega^{(n)}\|_p^{p-2} \leq \frac{p-1}{4\nu_x} \left(\|\theta_0\|_p + \frac{1}{n} \right)^2 \|\omega^{(n)}\|_p^{p-2}.$$

It follows that

$$\frac{d}{dt} \|\omega^{(n)}\|_p^2 \leq \frac{p-1}{2\nu_x} \|\theta_0^{(n)}\|_p^2 \leq \frac{p-1}{2\nu_x} \left(\|\theta_0\|_p + \frac{1}{n} \right)^2.$$

Integrating in time, we obtain

$$\begin{aligned}
 \|\omega^{(n)}(t)\|_p^2 & \leq \|\omega_0^{(n)}\|_p^2 + \frac{p-1}{2\nu_x} \left(\|\theta_0\|_p + \frac{1}{n} \right)^2 t \\
 & \leq \left(\|\omega_0\|_p + \frac{1}{n} \right)^2 + \frac{p-1}{2\nu_x} \left(\|\theta_0\|_p + \frac{1}{n} \right)^2 t.
 \end{aligned} \tag{3.15}$$

Hence, $\omega^{(n)}$ is uniformly bounded in $L^\infty([0, T], L^p)$ for each $p \in [2, \infty)$, independent of n . It follows from the Banach–Alaoglu Theorem and a standard diagonalization argument that there exists a further subsequence which we also denote as $\omega^{(n)}$ converging weak-* in $L^\infty([0, T], L^p)$ to some limit which we denote as ω . ω also enjoys the limit of the upper bound in (3.15), that is

$$\|\omega\|_p^2 \leq \left(\|\omega_0\|_p + \frac{1}{n} \right)^2 + \frac{p-1}{2\nu_x} \left(\|\theta_0\|_p + \frac{1}{n} \right)^2 t. \tag{3.16}$$

This implies that $\omega \in L^\infty([0, T]; L^p)$ for all $p \in [2, \infty)$. Similarly, one finds that

$$\|\theta^{(n)}(t)\|_p \leq \|\theta_0^{(n)}\|_p \leq \|\theta_0\|_p + \frac{1}{n}, \tag{3.17}$$

which implies that $\theta^{(n)}$ converges weak-* in $L^\infty([0, T]; L^p)$ to $\theta \in L^\infty([0, T]; L^p)$ for all $p \in [2, \infty)$, and $\|\theta\|_{L^\infty([0, T], L^p)} \leq \|\theta_0\|_p$.

Step 6: Proof of part (3) of Theorem 3.6.

We divide both sides of (3.16) by $p - 1$ and then take the supremum over all $p > 2$ of both sides to find that $\omega \in L^\infty([0, T], \sqrt{L})$ provided that $\omega_0 \in \sqrt{L}$ and $\theta_0 \in L^\infty$. Next, we want to show $\theta \in C([0, T]; w^*-L^\infty)$. We will use the Arzela-Ascoli Theorem as in Step 4. Notice that if $\theta_0 \in L^\infty$ then (3.17) holds uniformly for all $p \in [2, \infty)$ and hence

$$\|\theta^{(n)}(t)\|_\infty \leq \|\theta_0\|_\infty + \frac{1}{n}. \quad (3.18)$$

This implies that the sequence $\theta^{(n)}(t)$ is a relatively compact set in the weak-* topology of $L^\infty([0, T] \times \mathbb{T}^2)$. It suffices to show that the sequence $\{(\theta^{(n)}, \phi)\}$ is equicontinuous in $C([0, T])$ for every $\phi \in L^1$. It follows automatically from the previous result and the density of $L^2(\mathbb{T}^2)$ in $L^1(\mathbb{T}^2)$ that $\theta \in C_w([0, T], L^2)$. Finally, we would like to show that $\frac{d\theta}{dt} \in L^\infty([0, T], H^{-1})$ and hence $\frac{d\theta}{dt} \in L^2([0, T], H^{-1})$. Since $\omega \in L^\infty([0, T], \sqrt{L})$, we have in particular that $\omega \in L^\infty([0, T], L^3)$, and hence $\mathbf{u} \in L^\infty([0, T], W^{1,3}) \subset L^\infty([0, T], L^\infty)$ by (2.3), (2.16), and the Sobolev Embedding Theorem. From Eq. (3.5b), using the fact that $\theta \in L^\infty([0, T], L^2)$, we obtain

$$\left\| \frac{d\theta}{dt} \right\|_{H^{-1}} = \sup_{\|w\|=1} |\langle \mathcal{B}(\mathbf{u}, \theta), w \rangle| \leq \|\mathbf{u}\|_\infty \|\theta\| < \infty \quad \text{a.e. } t \in [0, T]. \quad (3.19)$$

This completes the proof of part (3) of Theorem 3.6. \square

3.3. Uniqueness for the case of anisotropic viscosity

Theorem 3.7 (Uniqueness for the anisotropic case). Let $\theta_0 \in L^\infty$, $\omega_0 \in \sqrt{L}$. Then, for every $T > 0$, there exists a unique solution (ω, θ) to (1.2), such that $\omega \in L^\infty([0, T], \sqrt{L}) \cap C_w([0, T]; L^2)$ and $\theta \in L^\infty([0, T], L^\infty) \cap C([0, T], w^*-L^\infty)$.

Proof. Let $T > 0$ be arbitrary. The existence of solution on the interval $[0, T]$ is established above, therefore it suffices to show uniqueness. We note that some very important *a priori* estimates that we need in the beginning of this proof were first elegantly derived in [9]. We recall those estimates that we have borrowed from [9]. We have derived them rigorously in the previous theorem and we derive them here again formally to make the proof of uniqueness self-contained. First, one may easily show that for any $p \in [2, \infty]$, we have

$$\|\theta(t)\|_p \leq \|\theta_0\|_p, \quad (3.20)$$

so $\theta \in L^\infty([0, T], L^p)$, $p \in [2, \infty]$. Given that $\omega_0 \in \sqrt{L}$, and hence $\omega_0 \in L^2$, we have

$$\frac{1}{2} \frac{d}{dt} |\omega|^2 + \nu |\partial_1^2 \omega| = -(\theta, \partial_1 \omega) \leq \frac{\nu}{2} |\partial_1 \omega|^2 + \frac{1}{2\nu} |\theta|^2.$$

Integrating in time yields

$$|\omega(t)|^2 + \nu \int_0^t |\partial_1 \omega|^2 d\tau \leq |\omega_0|^2 + \frac{t}{\nu} |\theta_0|^2.$$

This implies that $\omega \in L^\infty([0, T], L^2)$, and therefore $\mathbf{u} \in L^\infty([0, T], V)$. Furthermore, $\partial_1 \omega \in L^2([0, T], L^2)$. Using the divergence free condition (1.2b), we observe that

$$\partial_1 \omega = \partial_1^2 u^2 - \partial_1 \partial_2 u^1 = \partial_1^2 u^2 + \partial_2^2 u^2 = \Delta u^2.$$

Therefore, $\Delta u^2 \in L^2([0, T], L^2)$, so that $u^2 \in L^2([0, T], H^2)$ by elliptic regularity, and thus $\nabla u^2 \in L^2([0, T], H^1)$. By inequality (2.2), we have

$$\|\nabla u^2\|_p \leq C\sqrt{p-1}\|\nabla u^2\|_{H^1}, \quad (3.21)$$

so that $\nabla u^2 \in L^2([0, T], \sqrt{L})$.

Next, we recall that we have global in time control over the $\|\omega\|_{\sqrt{L}}$. Taking the inner product of (3.3) with $|\omega|^{p-2}\omega$ for some $p > 2$ and integrating by parts, and integrating in time, we have

$$\|\omega(t)\|_p^2 \leq \|\omega_0\|_p^2 + \frac{p-1}{2\nu}\|\theta_0\|_p^2 t. \quad (3.22)$$

This shows that $\omega \in L^\infty([0, T], \sqrt{L})$. Using this, and the facts that $\partial_1 u^1 = -\partial_2 u^2$ (by (1.2b)) and $\partial_2 u^1 = \partial_1 u^2 - \omega$, we have thanks to (3.21) that $\nabla u^1 \in L^2([0, T], \sqrt{L})$. Combining this with (3.21) shows that

$$\nabla \mathbf{u} \in L^2([0, T], \sqrt{L}). \quad (3.23)$$

We recall again that all the estimates above were first derived in [9] for the case where $\Omega = \mathbb{R}^2$.

Let $T > 0$ be arbitrary. The existence of solutions on the interval $[0, T]$ is established above; therefore, it suffices to show uniqueness.

Suppose (\mathbf{u}_1, θ_1) and (\mathbf{u}_2, θ_2) are two solutions to (3.4) on the interval $[0, T]$, with the same initial data (\mathbf{u}_0, θ_0) , then they must be equal. Define $\tilde{\mathbf{u}} := \mathbf{u}_1 - \mathbf{u}_2$, $\tilde{\theta} := \theta_1 - \theta_2$, and $\tilde{\xi}_\ell := \Delta^{-1}\theta_\ell$, $\ell = 1, 2$, $\int_{\mathbb{T}^2} \tilde{\xi}_\ell d\mathbf{x} = 0$, and $\tilde{\xi} := \xi_1 - \xi_2$. Based on Remark 3.5, these quantities satisfy the following functional equations:

$$\frac{d\tilde{\mathbf{u}}}{dt} + \nu\partial_1^2\tilde{\mathbf{u}} + B(\tilde{\mathbf{u}}, \mathbf{u}_1) + B(\mathbf{u}_2, \tilde{\mathbf{u}}) = P_\sigma(\Delta\tilde{\xi}\mathbf{e}_2) \quad \text{in } L^2([0, T], V') \quad \text{and} \quad (3.24a)$$

$$\frac{d\Delta\tilde{\xi}}{dt} + B(\tilde{\mathbf{u}}, \Delta\xi_1) + \mathcal{B}(\mathbf{u}_2, \Delta\tilde{\xi}) = 0 \quad \text{in } L^2([0, T], H^{-1}). \quad (3.24b)$$

Taking the action of (3.24a) on $\tilde{\mathbf{u}}$ in $L^2([0, T], V)$ and of (3.24b) in $L^2([0, T], H^{-1})$ on $\tilde{\xi} \in L^2([0, T], H^2)$, thanks to the properties of the operator B in Lemma 2.1 and the operator \mathcal{B} in Lemma 2.2, we obtain the following:

$$\frac{1}{2} \frac{d}{dt} |\tilde{\mathbf{u}}(t)|^2 + \nu \|\partial_1 \tilde{\mathbf{u}}\|^2 = \sum_{j=1}^2 (\tilde{u}^j \tilde{\mathbf{u}}, \partial_j \mathbf{u}_1) + (\Delta\tilde{\xi} \mathbf{e}_2, \tilde{\mathbf{u}}),$$

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\xi}(t)\|^2 = -(\tilde{\mathbf{u}} \Delta \xi_1, \nabla \tilde{\xi}) - (\mathbf{u}_2 \Delta \tilde{\xi}, \nabla \tilde{\xi}),$$

where we have used the Lions–Magenes Lemma (see, e.g., [23]) to have $\langle \frac{d\tilde{\mathbf{u}}}{dt}, \tilde{\mathbf{u}} \rangle = \frac{1}{2} \frac{d}{dt} |\tilde{\mathbf{u}}(t)|^2$ and $\langle \frac{d\Delta\tilde{\xi}}{dt}, \tilde{\xi} \rangle = \frac{1}{2} \frac{d}{dt} \|\tilde{\xi}(t)\|^2$. Next, observe that, due to the divergence free condition, $\mathbf{e}_1 \cdot \partial_1 \tilde{\mathbf{u}} = -\mathbf{e}_2 \cdot \partial_2 \tilde{\mathbf{u}}$, we have

$$\begin{aligned}
|(\Delta \tilde{\xi} \mathbf{e}_2, \tilde{\mathbf{u}})| &\leq \int_{\mathbb{T}^2} (|\partial_1 \tilde{\xi} \mathbf{e}_2 \cdot \partial_1 \tilde{\mathbf{u}}| + |\partial_2 \tilde{\xi} \mathbf{e}_2 \cdot \partial_2 \tilde{\mathbf{u}}|) d\mathbf{x} \\
&= \int_{\mathbb{T}^2} (|\partial_1 \tilde{\xi} \mathbf{e}_2 \cdot \partial_1 \tilde{\mathbf{u}}| + |\partial_2 \tilde{\xi} \mathbf{e}_1 \cdot \partial_1 \tilde{\mathbf{u}}|) d\mathbf{x} \\
&\leq \frac{1}{\nu} |\partial_1 \tilde{\xi}|^2 + \frac{\nu}{4} |\mathbf{e}_2 \cdot \partial_1 \tilde{\mathbf{u}}|^2 + \frac{1}{\nu} |\partial_2 \tilde{\xi}|^2 + \frac{\nu}{4} |\mathbf{e}_1 \cdot \partial_1 \tilde{\mathbf{u}}|^2.
\end{aligned}$$

Combining the above estimates, we find

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} |\tilde{\mathbf{u}}|^2 + \nu |\partial_1 \tilde{\mathbf{u}}|^2 &\leq \int_{\mathbb{T}^2} |\nabla \mathbf{u}_1| |\tilde{\mathbf{u}}|^2 d\mathbf{x} + \frac{2}{\nu} \|\tilde{\xi}\|^2 + \frac{\nu}{2} |\partial_1 \tilde{\mathbf{u}}|^2 \\
&\leq \|\tilde{\mathbf{u}}\|_\infty^{2/p} \int_{\mathbb{T}^2} |\nabla \mathbf{u}_1| |\tilde{\mathbf{u}}|^{2-2/p} d\mathbf{x} + \frac{2}{\nu} \|\tilde{\xi}\|^2 + \frac{\nu}{2} |\partial_1 \tilde{\mathbf{u}}|^2 \\
&\leq \|\nabla \mathbf{u}_1\|_p \|\tilde{\mathbf{u}}\|_\infty^{2/p} |\tilde{\mathbf{u}}|^{2-2/p} + \frac{2}{\nu} \|\tilde{\xi}\|^2 + \frac{\nu}{2} |\partial_1 \tilde{\mathbf{u}}|^2
\end{aligned}$$

where we have used Hölder's inequality. Similarly, by [Lemma 2.2](#)

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\tilde{\xi}\|^2 &\leq \left| \int_{\mathbb{T}^2} \tilde{\mathbf{u}} \cdot \nabla \tilde{\xi} \Delta \xi_1 d\mathbf{x} \right| + \int_{\mathbb{T}^2} |\nabla \mathbf{u}_2| |\nabla \tilde{\xi}|^2 d\mathbf{x} \\
&\leq \|\tilde{\mathbf{u}}\| \|\nabla \tilde{\xi}\| \|\Delta \xi_1\|_\infty + \|\nabla \mathbf{u}_2\|_p \|\nabla \tilde{\xi}\|_\infty^{2/p} |\nabla \tilde{\xi}|^{2-2/p}.
\end{aligned}$$

From the estimates above we can now adapt the well-known Yudovich argument for the 2D incompressible Euler equations (see, e.g., [\[25\]](#)) to complete the uniqueness proof. Let $X^2 := |\tilde{\mathbf{u}}(t)|^2 + \|\tilde{\xi}(t)\|^2 + \eta^2$ for some arbitrary $\eta > 0$. Adding the above two inequalities and using Young's inequality gives

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} X^2 + \frac{\nu}{2} |\partial_1 \tilde{\mathbf{u}}|^2 &\leq K_\nu (|\tilde{\mathbf{u}}|^2 + \|\tilde{\xi}\|^2 + \eta^2) \\
&\quad + (\|\nabla \mathbf{u}_2\|_p + \|\nabla \mathbf{u}_1\|_p) (\|\tilde{\mathbf{u}}\|_\infty^{2/p} + \|\nabla \tilde{\xi}\|_\infty^{2/p}) (|\tilde{\mathbf{u}}|^{2-2/p} + |\nabla \tilde{\xi}|^{2-2/p}) \\
&\leq K_\nu X^2 + C (\|\nabla \mathbf{u}_2\|_p + \|\nabla \mathbf{u}_1\|_p) (\|\tilde{\mathbf{u}}\|_\infty^{2/p} + \|\nabla \tilde{\xi}\|_\infty^{2/p}) X^{2-2/p},
\end{aligned}$$

where $K_\nu = 2/\nu + (1/2) \|\Delta \xi_1\|_\infty$. Neglecting the term $\frac{\nu}{2} |\partial_1 \tilde{\mathbf{u}}|^2$, dividing by X , and making the change of variables $Y(t) = e^{-K_\nu t} X(t)$, we have after a simple calculation,

$$\dot{Y} \leq C e^{-2K_\nu t/p} (\|\nabla \mathbf{u}_2\|_p + \|\nabla \mathbf{u}_1\|_p) (\|\tilde{\mathbf{u}}\|_\infty^{2/p} + \|\nabla \tilde{\xi}\|_\infty^{2/p}) Y^{1-2/p}.$$

Integrating this equation and using the fact that $e^{-2K_\nu t/p} \leq 1$, we find

$$Y(t) \leq \left[\eta^{2/p} + C \int_0^t \frac{1}{p} (\|\nabla \mathbf{u}_2(s)\|_p + \|\nabla \mathbf{u}_1(s)\|_p) (\|\tilde{\mathbf{u}}(s)\|_\infty^{2/p} + \|\nabla \tilde{\xi}(s)\|_\infty^{2/p}) ds \right]^{p/2}.$$

Letting $\eta \rightarrow 0$ we discover that for all $t \in [0, T]$,

$$|\tilde{\mathbf{u}}(t)|^2 + \|\tilde{\xi}(t)\|^2 \leq (\|\tilde{\mathbf{u}}\|_{L_T^\infty L_x^\infty} + \|\nabla \tilde{\xi}\|_{L_T^\infty L_x^\infty}) \cdot \left(C \int_0^t \frac{1}{p} (\|\nabla \mathbf{u}_2(s)\|_p + \|\nabla \mathbf{u}_1(s)\|_p) ds \right)^{p/2}. \quad (3.25)$$

Thanks to the fact that $\tilde{\Delta} \tilde{\xi} = \tilde{\theta} \in L^\infty([0, T], L^\infty) \subset L^\infty([0, T], L^4)$, we have by elliptic regularity that $\tilde{\xi} \in L^\infty([0, T], W^{2,4})$, and therefore $\nabla \tilde{\xi} \in L^\infty([0, T], W^{1,4})$. Thus, by the Sobolev Embedding Theorem, we have $\nabla \tilde{\xi} \in L^\infty([0, T], W^{1,4}) \subset L^\infty([0, T], C^{0,\gamma})$, for some $\gamma \in (0, 1)$. Furthermore, $\tilde{\omega} \in L^\infty([0, T], \sqrt{L})$ implies, for instance that $\tilde{\mathbf{u}} \in L^\infty([0, T], W^{1,4})$ by the Calderón–Zygmund elliptic estimate (2.3). Using the Sobolev Embedding Theorem again, we have $\tilde{\mathbf{u}} \in L^\infty([0, T], C^{0,\gamma})$, for some $\gamma \in (0, 1)$. Therefore, the first factor on the right-hand side of (3.25) is bounded. Now, since $\nabla \mathbf{u}_\ell \in L^2([0, T], \sqrt{L})$, $\ell = 1, 2$, by (3.23), we have by Cauchy–Schwarz

$$\int_0^t \frac{\|\nabla \mathbf{u}_\ell(s)\|_p}{p} ds \leq \left(t \int_0^t \sup_{p \geq 2} \frac{\|\nabla \mathbf{u}_\ell(s)\|_p^2}{p-1} ds \right)^{1/2}.$$

Let $M_\ell = \int_0^T \sup_{p \geq 2} \frac{\|\nabla \mathbf{u}_\ell(s)\|_p^2}{p-1} ds$, $\ell = 1, 2$, and $M = \max\{M_1, M_2\}$. Thus, from the above, for every fixed $\tau \in (0, T]$ we have

$$|\tilde{\mathbf{u}}(t)|^2 + \|\tilde{\xi}(t)\|^2 \leq K(2CM\tau)^{p/4}, \quad \text{for all } t \in [0, \tau], \quad (3.26)$$

where

$$K = (\|\tilde{\mathbf{u}}\|_{L_T^\infty L_x^\infty} + \|\nabla \tilde{\xi}\|_{L_T^\infty L_x^\infty}).$$

Now choose $\tau = \tau_0 = \min\{T, \frac{1}{4CM}\}$, and consider (3.26) on $[0, \tau_0]$. Taking the limit as $p \rightarrow \infty$, we get that $|\tilde{\mathbf{u}}(t)|^2 + \|\tilde{\xi}(t)\|^2 \leq 0$ for all $t \in [0, \tau_0]$. Restarting the time at $t = \tau_0$ and noting the fact that

$$\int_{\tau_0}^{t+\tau_0} \frac{\|\nabla \mathbf{u}_\ell(s)\|_p}{p} ds \leq \left(t \int_0^T \sup_{p \geq 2} \frac{\|\nabla \mathbf{u}_\ell(s)\|_p^2}{p-1} ds \right)^{1/2},$$

we have from an analogue of (3.25) on $[\tau_0, T]$ that $|\tilde{\mathbf{u}}(t)|^2 + \|\tilde{\xi}(t)\|^2 \leq K(2CM\tau_0)^{p/4}$ for all $t \in [\tau_0, 2\tau_0]$. Since we defined $\tau_0 \leq \frac{1}{4CM}$, we take the limit $p \rightarrow \infty$ and find that on the interval $[\tau_0, 2\tau_0]$, we also have that $|\tilde{\mathbf{u}}(t)|^2 + \|\tilde{\xi}(t)\|^2 \leq 0$. We can continue this argument on the intervals $[2\tau_0, 3\tau_0]$, $[3\tau_0, 4\tau_0]$, and so on. Thus, we have $|\tilde{\mathbf{u}}(t)|^2 + \|\tilde{\xi}(t)\|^2 \leq 0$ for all $t \in [0, T]$. This implies that, $|\tilde{\mathbf{u}}(t)| = 0$ and $\|\tilde{\xi}(t)\| = 0$ for all $t \in [0, T]$. \square

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