

An example of Newton's method for an equation in Gevrey series

Alexander Getmanenko ^{a,b,*}

^a *Kavli IPMU, University of Tokyo, Japan*

^b *Mathematics Institute of Jussieu, UPMC, Paris, France*

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Abstract

In the context of complex WKB analysis, we discuss a one-dimensional Schrödinger equation

$$-h^2 \partial_x^2 f(x, h) + [Q(x) + hQ_1(x, h)]f(x, h) = 0, \quad h \rightarrow 0,$$

where $Q(x)$, $Q_1(x, h)$ are analytic near the origin $x = 0$, $Q(0) = 0$, and $Q_1(x, h)$ is a factorially divergent power series in h . We show that there is a change of independent variable $y = y(x, h)$, analytic near $x = 0$ and factorially divergent with respect to h , that transforms the above Schrödinger equation to a canonical form. The proof goes by reduction to a mildly nonlinear equation on $y(x, h)$ and by solving it using an appropriately modified Newton's method of tangents.

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1. Introduction

Context. The stationary Schrödinger equations for an anharmonic oscillator

$$-h^2 \partial_x^2 f(x, h) + Q(x)f(x, h) = 0, \tag{1}$$

* Correspondence to: Mathematics Institute of Jussieu, UPMC, Paris, France.

E-mail address: getmanenko@math.jussieu.fr.

where $h \rightarrow 0+$, $f(x, h)$ is an unknown function, and $Q(x)$ is, say, a polynomial with

$$Q(0) = Q'(0) = 0 \neq Q''(0),$$

is one of the basic problems in Mathematical Physics. It is easy to find formal WKB solutions of (1) by substituting an ansatz

$$f(x, h) \sim e^{S(x)/h} (a_0(x) + a_1(x)h + a_2(x)h^2 + \dots) \quad (2)$$

and recursively solving for new unknown functions $S(x), a_0(x), a_1(x), \dots$, yet analytic properties of this expansion are quite subtle. Even in the example of the harmonic oscillator (e.g., [10, p. 152]), the terms of the series $a_0(x) + a_1(x)h + a_2(x)h^2 + \dots$ have singularity at the origin and diverge factorially (i.e., are *Gevrey series*, Section 3) away from zeros of $Q(x)$. A lot of effort is being put into proving Borel summability divergent series of this origin, see [5] as one of the recent developments.

The singular behavior at the origin of the coefficients $a_j(x)$ should not be all that uncontrollable: the closer x gets to 0, the more Eq. (1) resembles the Schrödinger equation of a harmonic oscillator whose eigenfunctions can be expressed in terms of well-known cylindric-parabolic functions.

And in fact, in [2] it is shown that one can find a change of coordinates $y = y(x, h)$, where y is no more than factorially divergent power series in h with coefficients holomorphic functions of x in a full neighborhood of the origin that reduces (1) to a harmonic oscillator. The proof goes by reducing the problem to a mildly nonlinear equation (case $M = 2$ and $Q_1(x, h) = 0$ of our (13), see also (14)), recursively solving it for $E_{0,n}, T_n(z)$ where $E_0 = \sum_{n \geq 0} E_{0,n}h^n$, $T = \sum_{n \geq 0} T_n(z)h^n$ and analyzing the growth of $|E_{0,n}|, |T_n(z)|$. Much more is contained in the rich and beautiful paper [2], but here we will concentrate just on this aspect.

Motivation. Our interest in the Witten Laplacian [8] led us to study the paper [2] with the task to extend its results to equations of slightly more general type than (1), namely:

$$-h^2 \partial_x^2 f(x, h) + (Q(x) + hQ_1(x, h))f(x, h) = 0, \quad (3)$$

where $Q_1(x, h) = \sum_{n \geq 0} Q_{1,n}(x)h^n$ is a factorially divergent series in h , $Q(0) = Q'(0) = 0 \neq Q''(0)$; in particular, to construct a coordinate change $y = y(x, h)$ in a full neighborhood of $x = 0$ that would deform (3) to a harmonic oscillator. By the time we asked all *why* and *why not* questions about the original proof of [2], we ended up with their proof repackaged in the form of Newton's method of tangents in the spaces of factorially divergent series. This version gives us the desired generalization automatically. Just as automatically we have obtained a generalization to an arbitrary order of zero of $Q(x)$ thus deforming the equation to the case studied (for even order zeros) in [18, §7]. Apart from a benign situation $Q_1(x, h) = Q_1(x)$ in the case of the Witten Laplacian, examples of type (3) have been looked at e.g. in [16].

A word of caution: unlike e.g. [11], we do not attempt here to prove Borel summability of the resulting series $y(x, h)$. It is understood that factorial growth estimates for series coefficients are just the first concept check for stronger results to be proven in the future.

For example, for $Q_1 = 0$, via a Borel transformation from power series in h to their majors in the sense of [6], Eq. (3) becomes a PDE in the class of ramified analytic functions of two complex variables (x, ξ)

$$-\partial_x^2 F(x, \xi) + Q(x) \partial_\xi^2 F(x, \xi) = 0$$

which we studied in [9]. There, we took x to be an element of the universal cover of the complex plane of x minus zeros of $Q(x)$. The paper you are reading now suggests that it was unnecessary and that solutions $F(x, \xi)$ are not ramified around the complex lines $\{(x, \xi) \in \mathbb{C}^2 : Q(x) = 0\}$.

We finally stress that for us the Newton method is a way to organize and conceptualize the treatment in [2] so as to avoid mistakes in generalizing their result.

The **main result** is:

Theorem 1.1. *Suppose that in Eq. (3):*

- a) *all functions $Q(x)$, $Q_{1,n}(x)$, $n \geq 0$, are analytic on a common neighborhood $U \subset \mathbb{C}$ of $x = 0$;*
- b) *$Q(0) = Q'(0) = \dots = Q^{(M-1)}(0) = 0 \neq Q^{(M)}(0)$, $M \geq 1$;*
- c) *there are constants τ , C_0 such that $\sup_{x \in U} |Q_{1,n}(x)| \leq C_0 \tau^n n!$.*

Then there exist:

- 1) *series $y = y(x, h) = \sum_{n \geq 0} y_n(x) h^n$, where $y_n(x)$ are holomorphic functions on a common neighborhood $U' \subset \mathbb{C}$ of $x = 0$;*
- 2) *series $E_j(h) = \sum_{n \geq 0} E_{j,n} h^n$, $E_{j,n} \in \mathbb{C}$, for $0 \leq j \leq M - 2$;*
- 3) *series $\psi = \psi(x, h) = \sum_{n \geq 0} \psi_n(x) h^n$, $\psi_n(x)$ analytic on U' ;*
- 4) *constants C'_0 , τ'*

so that

$$\sup_{x \in U'} |y_n(x)| \leq C'_0 \tau'^n n!, \quad \sup_{x \in U'} |\psi_n(x)| \leq C'_0 \tau'^n n!, \quad |E_{j,n}| \leq C'_0 \tau'^n n!,$$

and the change of independent variable $y = y(x, h)$ followed by rescaling of the unknown function $f(x, h) = \psi(x, h)g(y(x), h)$ in (3) transforms this equation to

$$-h^2 \partial_y^2 g(y, h) + \left(h E_0(h) + h E_1(h) y + \dots + h E_{M-2}(h) y^{M-2} - \frac{y^M}{4} \right) g(y, h) = 0.$$

Plan of the paper. Section 2 is devoted to performing the Langer transform that reformulates our problem in terms of a mildly nonlinear equation (13). Sections 3 and 4 introduce definitions of functional spaces used in the rest of the paper. Section 5 discusses inversion of the linear operator which is the dominant member of Eq. (13). Section 6 contains no precise definitions or results but explains the intuition based on the Newton's method behind our argument. Section 7 contains quite general results about analytic operations with factorially divergent expansions. The main result is proven in Section 8.

The **contribution** of this paper is, in our understanding, as follows. First of all, Theorem 1.1 is a new generalization of [2, Theorem A.1].

The Newton method that we use to organize our argument figures prominently in the KAM theory for problems of reduction to the normal form, see [7, Ch. 3 and Ch. 4] and references

therein. Note in particular that the Hard Implicit Function theorem [7, Th. 4.2], or the Nash–Moser theorem are close cousins of our Lemmas 7.4 and 7.5. Some applications of these ideas to WKB analysis appear in [1], but, in our understanding, without addressing the questions of Gevrey growth of WKB expansions. We thus think that our application of the Newton’s method to Gevrey growth of WKB series is a new approach.

The intermediate results included in Section 7 should be known to experts, at least implicitly, but as of this writing we do not have a specific reference. This may be related to the fact that the mainstream way in the literature to control Gevrey growth of series solutions of differential equations is the method of majorant functions as in [14, §3] or [17, §2] and references therein. Majorant functions is essentially a convenient notation for scales of Banach spaces and related contracting principle types of arguments. However, in our problem we aim to control the growth of coefficients in the h -expansion of the unknown function $g(y, h)$ as well as of the additional y -independent parameters $E_j(h)$; extending the method of majorant functions appropriately would be routine but verbose. We hope that the reader will enjoy our alternative perspective instead.

Note in passing that [14, Th. 1.7] as stated does not cover the case $M = 1$ of our equation (13) as its linear part does not satisfy [14, Condition 2, p. 210], where $(z_0)_{\text{ref.}[14]} = h_{\text{here}}$; we think however that a necessary modification should not be hard.

2. Formal reduction to the normal form

The idea of reducing a Schrödinger type equation to a normal form by changing a dependent variable is classical and known as a Langer transform, [13, §7.2.5]. The normal form itself may vary depending on the circumstances, e.g. it is the Airy equation in [13], or a Bessel equation in [4]. The algebraic manipulations are always very similar, but the analytic setup varies.

Following the setup of [2, §1 and Appendix A] whose treatment corresponds to ours in case $M = 2$, $Q_1 = 0$, we consider an equation

$$-h^2 \frac{\partial^2 f(x, h)}{\partial x^2} + (Q(x) + hQ_1(x, h))f(x, h) = 0, \quad (4)$$

where $Q(x)$ is an analytic function in a neighborhood $U \subset \mathbb{C}$ of the origin, $Q(x) = cx^M + O(x^{M+1})$ as $x \rightarrow 0$ for some $c \neq 0$, and $Q_1(x, h) = \sum_{n \geq 0} h^n Q_{1,n}(x)$ is a formal (for now) power series in h with coefficients $Q_{1,n}(x)$ analytic functions on a common neighborhood of $x = 0$.

We seek to simplify (4) by an invertible change of independent variable $y = y(x, h) = \sum y_j(x)h^j$, which is (for now) a formal power series in h with coefficients $y_j(x)$ holomorphic functions in a common neighborhood of 0. Then (4) becomes

$$-h^2 \left(\frac{d^2 f}{dy^2} - \frac{d^2 x}{dy^2} \left(\frac{dx}{dy} \right)^{-1} \frac{df}{dy} \right) + \left(\frac{dx}{dy} \right)^2 (Q(x) + hQ_1(x, h))f(x) = 0. \quad (5)$$

Let us further replace the unknown function $f(x, h)$ with a product $\psi(x, h)g(x, h)$, where $g(x, h)$ will be the new unknown function and ψ will be conveniently chosen:

$$\begin{aligned}
& -h^2 \left(-\frac{d^2x}{dy^2} \left(\frac{dx}{dy} \right)^{-1} \left[\psi \frac{dg}{dy} + g \frac{d\psi}{dy} \right] + \left[\psi \frac{d^2g}{dy^2} + 2 \frac{d\psi}{dy} \frac{dg}{dy} + g \frac{d^2\psi}{dy^2} \right] \right) \\
& + \left(\frac{dx}{dy} \right)^2 (Q(x) + hQ_1(x, h)) \psi(x) g(x) = 0.
\end{aligned}$$

The following choice eliminates the summands containing $\frac{\partial g}{\partial y}$:

$$\psi(x, h) = \exp \left\{ \frac{1}{2} \int \frac{d^2x}{dy^2} \left(\frac{dx}{dy} \right)^{-1} dy \right\}. \quad (6)$$

Eq. (4) is thus reduced to

$$\begin{aligned}
& -h^2 \frac{d^2g}{dy^2} + \left\{ \left(\frac{dx}{dy} \right)^2 (Q(x) + hQ_1(x, h)) + h^2 \frac{d^2x}{dy^2} \left(\frac{dx}{dy} \right)^{-1} \frac{(\partial\psi/\partial y)}{\psi} \right. \\
& \left. - h^2 \frac{(\partial^2\psi/\partial y^2)}{\psi} \right\} g = 0.
\end{aligned} \quad (7)$$

We will now achieve that the expression in braces in (7) takes a form $\sum_{j=0}^{M-2} hE_j(h)y^j - \frac{y^M}{4}$ for power series $E_0(h), \dots, E_{M-2}(h)$. Let us spell out this condition:

$$\begin{aligned}
& (Q(x) + hQ_1(x, h)) + h^2 \left(\frac{dy}{dx} \right)^2 \left[\frac{d^2x}{dy^2} \left(\frac{dx}{dy} \right)^{-1} \frac{(\partial\psi/\partial y)}{\psi} - \frac{(\partial^2\psi/\partial y^2)}{\psi} \right] \\
& = \left(\frac{dy}{dx} \right)^2 \left(\sum_{j=0}^{M-2} hE_j(h)y^j - \frac{y^2}{4} \right).
\end{aligned}$$

A one-page elementary calculation allows us to rewrite the h^2 -term on the left and obtain an equation

$$\begin{aligned}
& Q(x) + hQ_1(x, h) \\
& = \left(\frac{dy}{dx} \right)^2 \left(\sum_{j=0}^{M-2} hE_j(h)y^j - \frac{y^M}{4} \right) - \frac{h^2}{2} \left[\frac{d^3y}{dx^3} \left(\frac{dy}{dx} \right)^{-1} - \frac{3}{2} \left(\frac{d^2y}{dx^2} \right)^2 \left(\frac{dy}{dx} \right)^{-2} \right]. \quad (8)
\end{aligned}$$

Following [2], in Eq. (8) we replace the independent variable x by z , where $z = A \left(\int_0^x \sqrt{-Q(t)} dt \right)^{\frac{M}{M+2}}$. It will be convenient to choose $A = (M+2)^{\frac{2}{M+2}}$.

Quite generally, change of independent variable can be performed using formulas

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dz}{dx} \frac{dy}{dz}, \\
\frac{d^2y}{dx^2} &= -\frac{dy}{dz} \frac{d^2x}{dz^2} \left(\frac{dx}{dz} \right)^{-3} + \frac{d^2y}{dz^2} \left(\frac{dx}{dz} \right)^{-2}, \\
\frac{d^3y}{dx^3} &= -3 \frac{d^2y}{dz^2} \frac{d^2x}{dz^2} \left(\frac{dx}{dz} \right)^{-4} - \frac{dy}{dz} \frac{d^3x}{dz^3} \left(\frac{dx}{dz} \right)^{-4} + 3 \frac{dy}{dz} \left(\frac{d^2x}{dz^2} \right)^2 \left(\frac{dx}{dz} \right)^{-5} + \frac{d^3y}{dz^3} \left(\frac{dx}{dz} \right)^{-3}.
\end{aligned}$$

Eq. (8) becomes:

$$\begin{aligned} Q(x) + hQ_1(x, h) &= \left(\frac{dy}{dz}\right)^2 \left(\frac{dz}{dx}\right)^2 \left(\sum_{j=0}^{M-2} E_j(h)y^j - \frac{y^M}{4}\right) \\ &\quad - \frac{h^2}{2} \left[\left(\left\{ -3 \frac{d^2y}{dz^2} \frac{d^2x}{dz^2} \left(\frac{dx}{dz}\right)^{-3} - \frac{dy}{dz} \frac{d^3x}{dz^3} \left(\frac{dx}{dz}\right)^{-3} \right. \right. \right. \\ &\quad \left. \left. \left. + 3 \frac{dy}{dz} \left(\frac{d^2x}{dz^2}\right)^2 \left(\frac{dx}{dz}\right)^{-4} + \frac{d^3y}{dz^3} \left(\frac{dx}{dz}\right)^{-2} \right\} / \frac{dy}{dz} \right) \right. \\ &\quad \left. - \frac{3}{2} \left(\left\{ -\frac{dy}{dz} \frac{d^2x}{dz^2} \left(\frac{dx}{dz}\right)^{-2} + \frac{d^2y}{dz^2} \left(\frac{dx}{dz}\right)^{-1} \right\} / \frac{dy}{dz} \right)^2 \right]. \end{aligned} \quad (9)$$

Multiplying both sides by $\left(\frac{dx}{dz}\right)^2$ and using

$$\left(\frac{dx}{dz}\right)^2 = -A^{-M-2} \left(\frac{M+2}{2}\right)^2 \frac{z^M}{Q(x)}$$

we get

$$\begin{aligned} &-A^{-M-2} \left(\frac{M+2}{2}\right)^2 z^M + h \left(\frac{dz}{dx}\right)^{-2} Q_1(x, h) \\ &= \left(\frac{dy}{dz}\right)^2 \left(h \sum_{j=0}^{M-2} E_j(h)y^j - \frac{y^M}{4}\right) - \frac{h^2}{2} \left[-\frac{d^3x}{dz^3} \frac{dz}{dx} + \frac{3}{2} \left(\frac{d^2x}{dz^2}\right)^2 \left(\frac{dz}{dx}\right)^2 \right. \\ &\quad \left. + \frac{d^3y}{dz^3} \left(\frac{dy}{dz}\right)^{-1} - \frac{3}{2} \left(\frac{d^2y}{dz^2}\right)^2 \left(\frac{dy}{dz}\right)^{-2} \right]. \end{aligned} \quad (10)$$

To make the structure even more transparent, put

$$\tilde{Q}_1(z, h) := \sum_{n \geq 0} h^n \tilde{Q}_{1,n}(z) := \left(\frac{dz}{dx}\right)^{-2} Q_1(x(z), h) - h \left[-\frac{d^3x}{dz^3} \frac{dz}{dx} + \frac{3}{2} \left(\frac{d^2x}{dz^2}\right)^2 \left(\frac{dz}{dx}\right)^2 \right].$$

Then (10) becomes (recall that $A = (M+2)^{\frac{2}{M+2}}$)

$$\begin{aligned} &-\frac{z^M}{4} + h\tilde{Q}_1(z, h) \\ &= \left(\frac{dy}{dz}\right)^2 \left(\sum_{j=0}^{M-2} E_j(h)y^j - \frac{y^M}{4}\right) - \frac{h^2}{2} \left[\frac{d^3y}{dz^3} \left(\frac{dy}{dz}\right)^{-1} - \frac{3}{2} \left(\frac{d^2y}{dz^2}\right)^2 \left(\frac{dy}{dz}\right)^{-2} \right]. \end{aligned} \quad (11)$$

From (11) we can see directly that the h^0 term in $y(z, h)$ equals to z . Thus, we can replace the unknown y of (11) by T as follows:

$$y(z, h) = z + hT(z, h).$$

In terms of $T(z, h)$ Eq. (11) becomes (with primes denoting $\frac{d}{dz}$):

$$-\frac{z^M}{4} + h\tilde{Q}_1(z, h) = (1 + hT')^2 \left(\sum_{j=0}^{M-2} E_j(h)(z + hT)^j - \frac{(z + hT)^M}{4} \right) - \frac{h^2}{2} \left[\frac{hT'''}{1 + hT'} - \frac{3}{2} \frac{(hT'')^2}{(1 + hT')^2} \right], \quad (12)$$

which after cancellation of h^0 terms becomes

$$\begin{aligned} F(T, E_0, \dots, E_{M-2}) := & \sum_{j=0}^{M-2} E_j(h)(z + hT)^j - \frac{\sum_{\mu=1}^M \binom{M}{\mu} h^{\mu-1} z^{M-\mu} T^\mu}{4} \\ & + T'(2 + hT') \left(-\frac{(z + hT)^M}{4} + h \sum_{j=0}^{M-2} E_j(h)(z + hT)^j \right) \\ & - \frac{h^2}{2} \frac{T'''}{1 + hT'} + \frac{3h^3}{4} \frac{T''^2}{(1 + hT')^2} - \tilde{Q}_1(z, h) = 0. \end{aligned} \quad (13)$$

Proposition 2.1. *The relation*

$$F(T, E_0, \dots, E_{M-2}) = 0$$

seen as an equation on formal power series in h $E_j(h) = \sum_{n \geq 0} h^n E_{j,n}$, $E_{j,n} \in \mathbb{C}$, and $T(z, h) = \sum_{n \geq 0} T_n(z)h^n$, has a solution with $T_n(z)$ holomorphic in any connected neighborhood of $z = 0$ on which all $\tilde{Q}_{1,j}(z)$ are holomorphic.

Proof. A simple recursion with respect to the power of h can be set up similarly to [2, Th. 1.1, Rmk 1.1] once we notice that

$$\begin{aligned} F(T, E_0, \dots, E_{M-2}) \\ = E_0 + \dots + E_{M-2}z^{M-2} - \frac{1}{2}z^{\frac{M}{2}}(z^{\frac{M}{2}}T)' - \tilde{Q}_1(z, h) + O(h). \end{aligned} \quad \square \quad (14)$$

It is the goal of the rest of the paper to estimate the growth of term of the obtained power series in h .

3. Norms on Gevrey series

The content of this section is classical.

Let U be an open subset of \mathbb{C} , let $P(z, h) = \sum_{k \geq 0} h^k p_k(z)$ be a formal power series in h with holomorphic coefficients $p_k(z) \in \mathcal{O}(U)$.

We say that $P(z, h)$ is Gevrey-type on U if for any compact subset $K \subset U$, there are constants M_K, ρ_K such that

$$\sup_{z \in K} |p_k(z)| \leq M_K \rho_K^k k!.$$

Consider the space of those $P(z, h) = P(h)$ which do not depend on z , endowed with the norms which depend on a parameter $t > 0$:

$$N_0(P, t) = \sum_{k \geq 0} \frac{|p_k|}{k!} t^k. \quad (15)$$

An easy calculation establishes:

Proposition 3.1. Suppose $P(h)$, $Q(h)$, and t are such that $N_0(P, t), N_0(Q, t) < \infty$. Then:

$$\begin{aligned} N_0(PQ, t) &\leq N_0(P, t) \cdot N_0(Q, t), \\ N_0(P + Q, t) &\leq N_0(P, t) + N_0(Q, t). \end{aligned}$$

Let V^t be the subspace of those $P(h)$ for which $N_0(P, t)$ is finite. Analogously to the proof of completeness of $\ell^1(\mathbb{C})$, one verifies that V^t is a Banach space.

Following [15, p. 15], for a Banach space A and a number $\rho > 0$ denote by $A(\rho)$ the space of all formal series $\{g = \sum_{j \geq 0} a_j \tau^j; a_j \in A\}$ such that

$$\|g\|_\rho := \sum_{j \geq 0} \|a_j\| \rho^j \quad (16)$$

is finite. The space $A(\rho)$ endowed with $\|\cdot\|_\rho$ is a Banach space (which can be proven analogously to the proof of completeness of the $\ell^1(\mathbb{C})$ space). If A is a Banach algebra with $\|ab\| \leq \|a\| \|b\|$ (e.g., if $A = V^t$), then given another such series $\tilde{g} = \sum_{j \geq 0} \tilde{a}_j \tau^j$, we have $\|g\tilde{g}\|_\rho \leq \|g\|_\rho \|\tilde{g}\|_\rho$ for the usual product of power series.

Clearly an element of $V^t(\rho)$ for $\rho, t > 0$ gives rise to a Gevrey series $P(z, h)$ for z in a small disc around the origin.

The following lemma replaces the Cauchy integral formula when it comes to estimating the norm of a derivative of $\sum_j a_j \tau^j$:

Lemma 3.2. If $g = \sum_{n \geq 0} a_n \tau^n \in A(\rho)$ and we let $g^{(k)} := \sum_{n \geq 0} a_n n(n-1)\dots(n-k+1) \tau^{n-k}$ for $k \geq 1$, then for any ε , $0 < \varepsilon < \rho$, we have

$$\|g^{(k)}\|_{A(\rho-\varepsilon)} \leq \frac{k!}{\varepsilon^k} \|g\|_{A(\rho)}.$$

Proof. Indeed, by the binomial formula

$$\begin{aligned} & \frac{\varepsilon^k}{k!} \sum_{n \geq 0} \|a_n\| n(n-1)\dots(n-k+1)(\rho - \varepsilon)^{n-k} \\ &= \sum_{n \geq 0} \|a_n\| \binom{n}{k} (\rho - \varepsilon)^{n-k} \varepsilon^k \leq \sum_{n \geq 0} \|a_n\| \rho^n. \quad \square \end{aligned}$$

4. Functional analytic setup

Let us specify the functional spaces between which the nonlinear functional F given in (13) will define a continuous map.

Notation 4.1. Let us fix for the rest of the paper $\tau_0 > 0$, $\rho_0 > 0$ in such a way that $\tilde{Q}_1(z, h)$ defines an element of $V^{\tau_0}(\rho_0)$.

If necessary, shrink ρ_0 to be $< 1/M$, this will be used on p. 4624.

We define the following spaces for $0 < s \leq 1$, $0 < t \leq \tau_0$:

$$\begin{aligned} \mathcal{X}_s^t := & \left\{ (E_0, \dots, E_{M-2}, T) \in \underbrace{V^t \times \dots \times V^t}_{M-1 \text{ factors}} \times V^t \left(\frac{\rho_0(1+s)}{2} \right) : \right. \\ & \left. T, T', T'', T''' \in V^t \left(\frac{\rho_0(1+s)}{2} \right) \right\} \end{aligned} \quad (17)$$

endowed with the norm

$$\| (E_0, \dots, E_{M-2}, T) \|_{s,t} := \sum_{j=0}^{M-2} \|E_j\|_{V^t} + \sum_{j=0}^3 \left\| \frac{d^j}{dz^j} T \right\|_{V^t \left(\frac{\rho_0(1+s)}{2} \right)} \quad (18)$$

(the reason we do not want to consider arbitrarily small neighborhoods of $z = 0$ is the appearance of the radius of a neighborhood in the denominator of the estimates in Lemma 5.1);

$$\mathcal{Y}_s^t := V^t \left(\frac{\rho_0(1+s)}{2} \right); \quad (19)$$

$$\begin{aligned} \mathcal{Z}_s^t := & \left\{ (E_0, \dots, E_{M-2}, T) \in \underbrace{V^t \times \dots \times V^t}_{M-1 \text{ factors}} \times V^t \left(\frac{\rho_0(1+s)}{2} \right) : T, T' \in V^t \left(\frac{\rho_0(1+s)}{2} \right) \right\} \\ & (20) \end{aligned}$$

endowed with the norm

$$\| (E_0, \dots, E_{M-2}, T) \|_{s,t} := \sum_{j=0}^{M-2} \|E_j\|_{V^t} + \sum_{j=0}^1 \left\| \frac{d^j}{dz^j} T \right\|_{V^t \left(\frac{\rho_0(1+s)}{2} \right)}. \quad (21)$$

Clearly, for every s, t such that $0 < s \leq 1$, $0 < t \leq \tau_0$, F defines a continuous and even analytic map F_s^t from a neighborhood \mathcal{U}_s^t of the origin in \mathcal{X}_s^t to the space \mathcal{Y}_s^t ; for concreteness, $\mathcal{U}_s^t = \{(E_0, \dots, E_{M-1}, T) : t \|T'\|_{V^t(s)} < 1\}$.

The spaces \mathcal{Z}_s^t are auxiliary and will be used in the proof later on.

For any fixed t , we have the following properties, of which 1) and 2) are trivialities and 3) follows from Lemma 3.2:

- 1) For $s' < s$, there are inclusions of norm ≤ 1 : $\mathcal{X}_s^t \hookrightarrow \mathcal{X}_{s'}^t$, $\mathcal{Y}_s^t \hookrightarrow \mathcal{Y}_{s'}^t$, $\mathcal{Z}_s^t \hookrightarrow \mathcal{Z}_{s'}^t$. This is expressed by saying that $\mathcal{X}_{(\cdot)}^t$, $\mathcal{Y}_{(\cdot)}^t$, $\mathcal{Z}_{(\cdot)}^t$ are *scales of Banach spaces*;
- 2) the obvious map $\mathcal{X}_s^t \xrightarrow{\text{id}} \mathcal{Z}_s^t$ has norm ≤ 1 ;
- 3) for every $s' < s$, there is a map $\mathcal{Z}_s^t \rightarrow \mathcal{X}_{s'}^t$ so that

$$\|\mathcal{Z}_s^t \rightarrow \mathcal{X}_{s'}^t\| \leq \frac{B}{(s - s')^2}, \quad (22)$$

where B depends on ρ_0 but not on t, s, s' .

5. The dominant term of the equation $F(T, E_0, \dots, E_{M-2}) = 0$

In this section we are studying the h^0 component of the equation $F(T, E_0, \dots, E_{M-2}) = 0$, cf. (13). The following lemma is inspired by [2, Lemma A.3].

Lemma 5.1. *Let $M \geq 1$ be an integer, $v(z)$ a holomorphic function on $\Delta = \{z : |z| < r_0\}$ with values in a Banach space B , and consider the following equation for an unknown holomorphic function $u(z) : \Delta \rightarrow B$ and constants $E_0, \dots, E_{M-2} \in B$:*

$$\left(\frac{z^M}{2} \frac{d}{dz} + \frac{M}{4} z^{M-1} \right) u(z) = E_0 + zE_1 + \dots + z^{M-2} E_{M-2} + v(z). \quad (23)$$

Then (23) has a unique solution $(u(z), E_0, \dots, E_{M-2})$ and moreover for any r , $0 < r < r_0$:

$$\|E_j\|_B \leq \frac{1}{r^j} \|v\|_{B(r)}, \quad j = 0, \dots, M-2; \quad (24)$$

$$\|u\|_{B(r)} \leq \frac{4}{r^{M-1}} \|v\|_{B(r)}; \quad (25)$$

$$\left\| \frac{du}{dz} \right\|_{B(r)} \leq \frac{2}{r^M} \|v\|_{B(r)}. \quad (26)$$

Proof. Rewrite Eq. (23) as follows:

$$\frac{1}{2} z^{\frac{M}{2}} \left(z^{\frac{M}{2}} u(z) \right)' = E_0 + zE_1 + \dots + z^{M-2} E_{M-2} + v(z). \quad (27)$$

If $v(z) = v_0 + v_1 z + v_2 z^2 + \dots$, then (27) can be rewritten as

$$\frac{1}{2} \left(z^{\frac{M}{2}} u(z) \right)' = \sum_{j=0}^{M-2} z^{j-\frac{M}{2}} (E_j + v_j) + \sum_{j=M-1}^{\infty} z^{j-\frac{M}{2}} v_j. \quad (28)$$

If $u(z)$ is to be holomorphic, the RHS of (28) should not have any terms with $z^{\leq -1}$, hence

$$v_j = -E_j, \quad \text{if } j - \frac{M}{2} \leq -1.$$

Inserting this, we have

$$\frac{1}{2} \left(z^{\frac{M}{2}} u(z) \right)' = \sum_{j \in \mathbb{N}_0; \frac{M}{2} - 1 \leq j \leq M-2} z^{j-\frac{M}{2}} (E_j + v_j) + \sum_{j=M-1}^{\infty} z^{j-\frac{M}{2}} v_j, \quad (29)$$

or

$$u(z) = \sum_{j \in \mathbb{N}_0; \frac{M}{2} - 1 \leq j \leq M-2} \frac{2}{j - \frac{M}{2} + 1} z^{j-M+1} (E_j + v_j) + \sum_{j=M-1}^{\infty} \frac{2}{j - \frac{M}{2} + 1} z^{j-M+1} v_j. \quad (30)$$

We conclude that the solution is necessarily of the form

$$E_j = -v_j, \quad j = 0, \dots, M-2, \\ u(z) = \sum_{j=M-1}^{\infty} \frac{2}{j - \frac{M}{2} + 1} v_j z^{j-M+1}. \quad (31)$$

As $j - \frac{M}{2} + 1 \geq \frac{1}{2}$ for $j \geq M-1$, we have

$$\|u(z)\|_{B(r)} \leq 4 \sum_{j=M-1}^{\infty} \|v_j\|_{B(r)} r^{j-M+1} \leq 4r^{1-M} \sum_{j=M-1}^{\infty} \|v_j\|_{B(r)} r^j \leq 4r^{1-M} \|v(z)\|_{B(r)}.$$

Further,

$$u'(z) = 2 \sum_{j=M-1}^{\infty} \frac{j - M + 1}{j - \frac{M}{2} + 1} v_j z^{j-M},$$

hence, as the fraction is always ≤ 1 ,

$$\|u'(z)\|_{B(r)} \leq 2 \sum_{j=M}^{\infty} \|v_j\|_{B(r)} r^{j-M} \leq 2r^{-M} \|v(z)\|_{B(r)}.$$

From (31) we obviously get (24). \square

6. Newton's method

This section contains no precise results, it will not be referred to except for purposes of intuition.

6.1. Newton's method in Banach spaces

Newton's method of tangents for solving scalar nonlinear equations appears in almost all elementary calculus textbooks. In this subsection we review the Newton's method in Banach spaces following [12, Ch. XVIII].

Suppose \mathcal{X}, \mathcal{Y} are two Banach spaces, $x^{(0)} \in \mathcal{X}, \mathcal{U} \subset \mathcal{X}$ is an open neighborhood, $\mathcal{F} : \mathcal{U} \rightarrow \mathcal{Y}$ a continuous map admitting two continuous and bounded Fréchet derivatives. Suppose that for all $x \in \mathcal{U}$ the inverse of the Fréchet derivative, $(dF_x)^{-1} : \mathcal{Y} \rightarrow \mathcal{X}$ is uniformly bounded, i.e. $\|(dF_x)^{-1}\| \leq A$.

By Newton's method we mean the following iterative procedure of finding a zero of \mathcal{F} :

- 1) $x^{(0)}$ is given; $y^{(0)} = F(x^{(0)})$;
- 2) for $k \geq 0$, define $w^{(k)} = -(dF_{x^{(k)}})^{-1}(y^{(k)})$;
- 3) $x^{(k+1)} = x^{(k)} + w^{(k)}$; assume or prove that $x^{(k+1)} \in \mathcal{U}$; $y^{(k+1)} = F(x^{(k+1)})$.

Assume that x^* is such that $F(x^*) = 0$. Then

$$y^{(k+1)} = F(x^{(k+1)}) = \underbrace{F(x^{(k)}) + [dF_{x^{(k)}}](w^{(k)})}_{=0} + O(\|w^{(k)}\|^2) = O(\|y^{(k)}\|^2)$$

which is smaller than $\|y^{(k)}\|$ if $y^{(k)}$ was small already. So the method converges if $x^{(0)}$ was sufficiently close to x^* and, loosely, far enough from the boundary of \mathcal{U} . Formalization of these conditions can be found in [12, Ch. XVIII, §1.5].

6.2. Newton's method leads to Gevrey expansions

In this subsection is to motivate that factorially divergent or, for brevity, Gevrey expansions naturally arise when we attempt to apply Newton's method for solving our equation $F(E_0, \dots, E_{M-2}, T) = 0$.

The Newton's method requires inverting the Fréchet derivative of F . Let us for simplicity consider the case $M = 2$ and try to invert the Fréchet derivative of F at the point $(E_0, T) = 0$. If $(\mathcal{E}_0, \mathcal{T})$ is a tangent vector, then

$$dF_0(\mathcal{E}_0, \mathcal{T}) = \mathcal{E}_0 - \frac{z^2}{2}\mathcal{T}' - \frac{z}{2}\mathcal{T} - \frac{1}{2}h^2\mathcal{T}'''.$$

Then dF_0 as an operator $\mathcal{X}'_s \rightarrow \mathcal{Y}'_s$ can be written as $G + h^2H$, where H is bounded and

$$G : (\mathcal{E}_0, \mathcal{T}) \mapsto \mathcal{E}_0 - \frac{z^2}{2}\mathcal{T}' - \frac{z}{2}\mathcal{T} : \mathcal{X}'_s \rightarrow \mathcal{Y}'_s$$

is inverted by an operator $K : \mathcal{Y}_s \rightarrow \mathcal{X}_{s'}$, $s' < s$, of the norm $\leq \frac{c}{(s-s')^2}$, cf. Lemma 5.1 and (22). Let us attempt to write the inverse of dF_0 as

$$(dF_0)^{-1} = \sum_{n \geq 0} h^{2n} (-1)^n (KH)^n K$$

where $(KH)^n K$ is

$$\mathcal{Y}_s^t \xrightarrow{K} \mathcal{X}_{s_1}^t \xrightarrow{H} \mathcal{Y}_{s_1}^t \xrightarrow{K} \mathcal{X}_{s_2}^t \xrightarrow{H} \dots \xrightarrow{H} \mathcal{Y}_{s_n}^t \xrightarrow{K} \mathcal{Y}_{s'}^t \quad (32)$$

and where we can think of $s_n < \dots < s_1$ as arbitrary numbers subject to $s' < s_n$ and $s_1 < s$. Choosing s_1, \dots, s_n to maximize the product for $(s - s_1)(s_1 - s_2) \dots (s_n - s')$ for fixed s, s' , we can prove an estimate

$$\|(KH)^n K : \mathcal{X}_s^t \rightarrow \mathcal{Y}_{s'}^t\| \leq c'' \frac{(2n)!(c')^{2n}}{(1-s)^{2n}}$$

for some new constants c', c'' , but no dramatically better estimates are available. Thus, given $v \in \mathcal{Y}_1^t$, we can hope to represent $(dF_0)^{-1}v \in \mathcal{X}_s^t$ by the following expansion whose convergence needs to be discussed separately:

$$(dF_0)^{-1}v = \sum_{n \geq 0} h^n u_{n,s}, \quad \|u_{n,s}\|_{\mathcal{X}_s^t} \leq C' \frac{n!C^n}{(1-s)^n}, \quad (33)$$

with constants C', C independent of s . This shows that expansions of type (33) appear naturally in our problem and motivates our decision to formulate intermediate results in terms of them.

7. Calculus of Gevrey expansions

7.1. Definitions

By a Gevrey expansion we mean, vaguely, a formal expansion $\sum_n h^n u_n$, where the straightfont h is a formal variable, and u_n are elements of some Banach space such that $\|u\| \leq c_0 c^n n!$ for some constants c_0 and c^n , compare (33). We choose however to avoid this notion in the mathematically precise body of the article. Instead, the following seems to be a more convenient terminology.

In this section, let $\mathcal{X}_s, \mathcal{Y}_s$, $0 < s \leq 1$ be two arbitrary scales of Banach spaces, i.e. we suppose that for $s' < s$ there is an inclusion $\mathcal{X}_s \hookrightarrow \mathcal{X}_{s'}$, $\mathcal{Y}_s \hookrightarrow \mathcal{Y}_{s'}$ of norm ≤ 1 . In this section, $\mathcal{X}_s, \mathcal{Y}_s$ have a priori nothing to do with $\mathcal{X}_s^t, \mathcal{Y}_s^t$ in the rest of the paper.

Suppose that for every s there is a map $F_s : \mathcal{X}_s \supset \mathcal{U}_s \rightarrow \mathcal{Y}_s$. We will say that the family $F = (F_s)$ is *compatible with inclusions* if:

- a) for $s' < s$, the map $\mathcal{X}_s \hookrightarrow \mathcal{X}_{s'}$ maps \mathcal{U}_s into $\mathcal{U}_{s'}$;
- b) for $s' < s$, $(\mathcal{Y}_s \hookrightarrow \mathcal{Y}_{s'}) \circ F_s = F_{s'} \circ (\mathcal{X}_s \hookrightarrow \mathcal{X}_{s'})$.

For a family of linear maps F_s we always think of \mathcal{U}_s as equal to \mathcal{X}_s .

Analogously we define what it means for a family of maps $(G_{s,s'} : \mathcal{X}_s \rightarrow \mathcal{Y}_{s'})_{s' < s}$ to be compatible with inclusions.

We denote by $\mathring{\mathcal{X}}_1$ the inductive limit of \mathcal{X}_s for $s < 1$:

$$\mathring{\mathcal{X}}_1 := \{(g_s)_{0 < s < 1} : g_s \in \mathcal{X}_s, \text{ and for all } s < s' \text{ } (\mathcal{X}_{s'} \hookrightarrow \mathcal{X}_s)(g_{s'}) = g_s\}.$$

We assume that on every \mathcal{X}_s there is an action of a linear operator $h : \mathcal{X}_s \rightarrow \mathcal{X}_s$, compatible with inclusions, satisfying $\|h^n : \mathcal{X}_s \rightarrow \mathcal{X}_s\| \leq \frac{\tau}{n!}$, and similar for \mathcal{Y}_s , for some fixed number $\tau > 0$ independent of s . We will abuse notation by writing $|h|$ instead of τ .

We call a linear operator $G_s : \mathcal{X}_s \rightarrow \mathcal{Y}_s$ (or a family of such operators) h -linear if $G_s \circ h = h \circ G_s$.

We finish this subsection by stating the following combinatorial inequality:

Lemma 7.1. (See [2, Lemma A.4].) *The following inequality holds for all positive integers j, k satisfying $k \leq j$:*

$$\sum_{\substack{j_1 + \dots + j_k = j \\ j_1, \dots, j_k \geq 1}} j_1! j_2! \dots j_k! \leq 4^{k-1} (j - k + 1)!.$$

7.2. Function evaluated on a Gevrey expansion gives a Gevrey expansion

For $k \in \mathbb{Z}_{\geq 0}$, we call $\Omega_k : \mathcal{X}_s \rightarrow \mathcal{Y}_s$ an h - k -linear map of norm ≤ 1 if $\Omega_k(v) = \tilde{\Omega}_k(v, v, \dots, v)$ where $\tilde{\Omega}_k : \mathcal{X}_s \times \mathcal{X}_s \times \dots \times \mathcal{X}_s \rightarrow \mathcal{Y}_s$ is symmetric, satisfies $\|\tilde{\Omega}(v_1, \dots, v_k)\|_{\mathcal{Y}_s} \leq \|v_1\|_{\mathcal{X}_s} \|v_2\|_{\mathcal{X}_s} \dots \|v_k\|_{\mathcal{X}_s}$ and, moreover, $\tilde{\Omega}_k(\dots, v_{j-1}, h v_j, v_{j+1}, \dots) = h \tilde{\Omega}_k(\dots, v_{j-1}, v_j, v_{j+1}, \dots)$.

Lemma 7.2. *Let $f : \mathcal{X}_s \rightarrow \mathcal{Y}_s$ be a map compatible with inclusions, $x_0 \in \mathcal{X}_1$; $\alpha \in \mathbb{R}_{>0}$. Suppose that for any $w \in \mathcal{X}_s$ such that $\|w\|_s < \frac{1}{\alpha}$*

$$\mathcal{Y}_s \ni f(x_0 + w) = f(x_0) + \sum_{j \geq 1} \alpha^j \Omega_j(w), \quad (34)$$

where $\Omega_{j,s} : \mathcal{X}_s \rightarrow \mathcal{Y}_s$ is an h - j -linear map of norm ≤ 1 , compatible with inclusions.

Let for all $n \geq 1$, $g_n = (g_{n,s}) \in \mathring{\mathcal{X}}_1$ be such that

$$\|g_{n,s}\|_{\mathcal{X}_s} \leq C_0 \frac{C^n n!}{(1-s)^n}. \quad (35)$$

Then there are $H_n = (H_{n,s}) \in \mathring{\mathcal{Y}}_1$, $n \geq 1$ satisfying

$$\|H_{n,s}\|_{\mathcal{Y}_s} \leq \beta(C_0 \alpha) \frac{C^n n!}{(1-s)^n}$$

for $\beta(t) = t e^{4t}$, such that if

$$\frac{|h|C}{1-s} < 1 \quad \text{and} \quad \sum_{n \geq 1} \frac{|h|^n C^n}{(1-s)^n} < \frac{1}{C_0 \alpha}, \quad (36)$$

then the series

$$W_s = \sum_{n \geq 1} h^n g_{n,s} \quad (37)$$

converges in \mathcal{X}_s and

$$f(x_0 + W_s) - f(x_0) = \sum_{n \geq 1} h^n H_{n,s}. \quad (38)$$

Proof. Motivated by a “formula”

$$\begin{aligned} f(x_0 + W_s) - f(x_0) &= \sum_{k \geq 1} a_k \tilde{\Omega}_k \left(\sum_{n_1 \geq 1} h^{n_1} g_{n_1,s}, \dots, \sum_{n_k \geq 1} h^{n_k} g_{n_k,s} \right) \\ &= \sum_{k \geq 1} a_k \sum_{m \geq 1} h^m \sum_{\substack{n_1 + \dots + n_k = m \\ n_1, \dots, n_k \geq 1}} \tilde{\Omega}_k(g_{n_1,s}, \dots, g_{n_k,s}) \end{aligned} \quad (39)$$

to which we will give an analytic meaning in a moment, consider

$$H_{m,s} := \sum_{k \geq 1} a_k \sum_{\substack{n_1 + \dots + n_k = m \\ n_1, \dots, n_k \geq 1}} \tilde{\Omega}_k(g_{n_1,s}, \dots, g_{n_k,s}).$$

We have, with help of [Lemma 7.1](#),

$$\begin{aligned} \|H_{m,s}\|_{\mathcal{Y}_s} &\leq \sum_{k \geq 1} (C_0 \alpha)^k \frac{C^m}{(1-s)^m} \sum_{\substack{n_1 + \dots + n_k = m \\ n_1, \dots, n_k \geq 1}} n_1! \dots n_k! \\ &\leq \frac{C^m}{(1-s)^m} \sum_{k \geq 1} (C_0 \alpha)^k 4^{k-1} (m-k+1)! \\ &\leq (C_0 \alpha) \frac{C^m}{m!(1-s)^m} \sum_{k \geq 0} (C_0 \alpha)^k 4^k \frac{1}{k!} \\ &\leq (C_0 \alpha) e^{4C_0 \alpha} \frac{C^m m!}{(1-s)^m}, \end{aligned}$$

compare [\[2, \(A.50\)\]](#).

Then, assuming [\(36\)](#) and hence $\| \frac{h^n C^n}{(1-s)^n} \| \leq \frac{1}{n!}$, $\sum_{m \geq 1} h^m H_{m,s}$ as well as [\(37\)](#) are absolutely convergent, so $f(x_0 + W_s)$ can be written in terms of Taylor series [\(34\)](#), and hence [\(38\)](#) holds by the algebraic manipulations of [\(39\)](#). \square

7.3. Increment of a function is well approximated by the first derivative

Let us now give a meaning to the formula $f(x + h^b v) = f(x) + [f'(x)](h^b v) + h^{2n} O(v^2)$ for $x = x_0 + g$ in the context of Gevrey expansions.

$$\text{Let } \beta_1(t) = \sum_{k \geq 0} \frac{(k+1)4^k}{k!} t^{k+1}, \beta_2(t) = \sum_{k \geq 1} \frac{t^{k+1} 4^k}{k!}.$$

Lemma 7.3. Suppose $f : \mathcal{X}_s \rightarrow \mathcal{Y}_s$, x_0 , α , Ω_j are as in [\(34\)](#), and $g_n \in \hat{\mathcal{X}}_1$, $n \geq 1$, satisfy [\(35\)](#). Assume $C \geq 1$.

Then for any integer $b \geq 0$ and any sequence $v_k \in \mathcal{X}_1$ with $\|v_{k,s}\|_{\mathcal{X}_s} \leq A \frac{k!C^k}{(1-s)^k}$, there are:

a) a sequence of elements $w_j \in \mathcal{Y}_1$ with $\|w_{j,s}\| \leq \frac{A}{C_0} \beta_1(C_0\alpha) \frac{j!C^j}{(1-s)^j}$ satisfying the property: under conditions

$$\frac{|h|C}{(1-s)} < 1, \quad \sum_{n=1}^{\infty} \frac{h^n C^n}{(1-s)^n} < \frac{1}{C_0\alpha}, \quad (40)$$

we have two vectors defined by an absolutely convergent sums

$$v_s := h^b \sum_{k=1}^{\infty} h^k v_{k,s} \in \mathcal{X}_s, \quad g_s := \sum_{k \geq 1} h^k g_{k,s} \in \mathcal{X}_s$$

and such that

$$f'(x_0 + g_s)(v_s) = \sum_{j \geq 1} j\alpha^j \tilde{\mathcal{Q}}_j(g_s, \dots, g_s, v_s) = \sum_j h^j w_{j,s} \quad \text{in } \mathcal{Y}_s; \quad (41)$$

b) a sequence of elements $u_j \in \mathcal{Y}_1$, $j \geq 1$, with

$$\|u_{j,s}\|_{\mathcal{Y}_s} \leq \beta_2(\alpha(C_0 + A)) \frac{j!C^j}{(1-s)^j}$$

satisfying the following property: under conditions

$$\frac{|h|C}{(1-s)} < 1, \quad \sum_{n=1}^{\infty} \frac{h^n C^n}{(1-s)^n} < \frac{1}{(C_0 + A)\alpha} \quad (42)$$

we have

$$f(x_0 + g_s + v_s) - f(x_0 + g_s) - [f'(x_0 + g_s)](v_s) = h^{2b} \sum_{j \geq 1} h^j u_{j,s} \quad \text{in } \mathcal{Y}_s. \quad (43)$$

Proof. *Preliminary remark.* If v is a formal in h written as $v = h^b \sum_{n \geq 1} h^n v_n$ as in the statement of the lemma, then it can also be written $v = \sum_{m \geq b+1} h^m \tilde{v}_m$ with $\tilde{v}_m = v_{m-b}$ and hence satisfying $\|\tilde{v}_{m,s}\| \leq A \frac{C^{m-b}(m-b)!}{(1-s)^{m-b}} \leq A \frac{C^m m!}{(1-s)^m}$ if $C \geq 1$.

Proof of a). Rewrite (41) as the following “formula” whose analytic meaning will be clarified in a moment:

$$\sum_{j \geq 1} \sum_{k_1, \dots, k_{j-1}, k_j \geq 1} j\alpha^j h^{k_1 + \dots + k_{j-1}} \tilde{\mathcal{Q}}_j(g_{k_1}, \dots, g_{k_{j-1}}, \tilde{v}_{k_j}) = \sum_{n \geq 1} h^n w_n. \quad (44)$$

Thus we define

$$w_n = \sum_{j \geq 1} \sum_{\substack{k_1, \dots, k_{j-1}, k_j \geq 1 \\ k_1 + \dots + k_{j-1} + k_j = n}} j C_0^{j-1} A \alpha^j \Omega_j(g_{k_1}, \dots, g_{k_{j-1}}, \tilde{v}_{k_j}),$$

and with help of [Lemma 7.1](#)

$$\begin{aligned} \|w_{n,s}\|_{\mathcal{Y}_s} &\leq \sum_{j \geq 1} \sum_{\substack{k_1, \dots, k_j \geq 1 \\ k_1 + \dots + k_j = n}} j \alpha^j C_0^{j-1} A \frac{C^n k_1! \dots k_j!}{(1-s)^n} \\ &\leq \sum_{j \geq 1} j \alpha^j C_0^{j-1} A \frac{C^n 4^{j-1} (n-j+1)!}{(1-s)^n} \\ &\leq \frac{A}{C_0} \left\{ \sum_{j' \geq 0} (j'+1) 4^{j'} (C_0 \alpha)^{j'+1} \frac{1}{j'!} \right\} \frac{C^n n!}{(1-s)^n}. \end{aligned}$$

Under condition (40), Eq. (44) is literally true in \mathcal{Y}_s , both sides being absolutely convergent series in which we can change the order of summation, hence (41).

Proof of b). Let us now formally write

$$\begin{aligned} &f(x_0 + g + v) - f(x_0 + g) - [f'(x_0 + g)](v) \\ &= \sum_{j=1}^{\infty} \alpha^j [\Omega_j(g+v) - \Omega_j(g) - j \tilde{\Omega}_j(g, \dots, g, v)] \end{aligned}$$

(use symmetry and multilinearity of $\tilde{\Omega}_j$)

$$= \sum_{j=2}^{\infty} \alpha^j \sum_{k_1, \dots, k_j \geq 1} h^{k_1 + \dots + k_j} \left[\sum_{\sigma=2}^j \binom{j}{\sigma} \tilde{\Omega}_j(g_{k_1}, \dots, g_{k_\sigma}, \tilde{v}_{k_{\sigma+1}}, \dots, \tilde{v}_{k_j}) \right]$$

(since $\tilde{v}_k = 0$ for $k = 0, \dots, b$)

$$\begin{aligned} &= \sum_{j=2}^{\infty} \alpha^j \sum_{\substack{k_1, \dots, k_{j-2} \geq 1 \\ k_{j-1}, k_j \geq b+1}} h^{k_1 + \dots + k_j} \left[\sum_{\sigma=2}^j \binom{j}{\sigma} \tilde{\Omega}_j(g_{k_1}, \dots, g_{k_\sigma}, \tilde{v}_{k_{\sigma+1}}, \dots, \tilde{v}_{k_j}) \right] \\ &= \sum_{j=2}^{\infty} \alpha^j \sum_{\substack{k_1, \dots, k_{j-2} \geq 1 \\ k_{j-1}, k_j \geq b+1}} h^{k_1 + \dots + k_j} \left[\sum_{\sigma=2}^j \binom{j}{\sigma} \tilde{\Omega}_j(g_{k_1}, \dots, g_{k_\sigma}, \tilde{v}_{k_{\sigma+1}}, \dots, \tilde{v}_{k_{j-2}}, v_{k_{j-1}-b}, v_{k_j-b}) \right] \\ &= \sum_{j=2}^{\infty} \alpha^j \sum_{k_1, \dots, k_j \geq 1} h^{k_1 + \dots + k_j + 2b} \left[\sum_{\sigma=2}^j \binom{j}{\sigma} \tilde{\Omega}_j(g_{k_1}, \dots, g_{k_\sigma}, \tilde{v}_{k_{\sigma+1}}, \dots, \tilde{v}_{k_{j-2}}, v_{k_{j-1}}, v_{k_j}) \right]. \end{aligned}$$

This motivates the definition:

$$u_n = \sum_{j=2}^{\infty} \alpha^j \sum_{\substack{k_1, \dots, k_j \geq 1 \\ k_1 + \dots + k_j = n}} \left[\sum_{\sigma=2}^j \binom{j}{\sigma} \tilde{\mathcal{Q}}_j(g_{k_1}, \dots, g_{k_\sigma}, \tilde{v}_{k_{\sigma+1}}, \dots, \tilde{v}_{k_{j-2}}, v_{k_{j-1}}, v_{k_j}) \right].$$

We have:

$$\begin{aligned} \|u_{n,s}\|_{\mathcal{Y}_s} &\leq \sum_{j=2}^{\infty} \alpha^j \sum_{\substack{k_1, \dots, k_j \geq 1 \\ k_1 + \dots + k_j = n}} \left[\sum_{\sigma=2}^j \binom{j}{\sigma} \frac{C_0^\sigma A^{j-\sigma} C^n}{(1-s)^n} k_1! \dots k_j! \right] \\ &\leq \sum_{j=2}^{\infty} \alpha^j (C_0 + A)^j 4^{j-1} (n-j+1)! \frac{C^n}{(1-s)^n} \\ &\leq \left\{ \sum_{j=2}^{\infty} \alpha^j (C_0 + A)^j 4^{j-1} \frac{1}{(j-1)!} \right\} \frac{C^n n!}{(1-s)^n}. \end{aligned}$$

We conclude that (43) holds under condition (42) in the same way as we did in part a). \square

7.4. Inverse of a regularly perturbed linear operator

In this subsection we will generalize the familiar statement that if an operator \mathbf{A} has a bounded inverse, then its small perturbation $\mathbf{A} + \mathbf{B}$ has a bounded inverse $\mathbf{A}^{-1}(1 + \mathbf{B}\mathbf{A}^{-1} + \mathbf{B}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} + \dots)$ if $\|\mathbf{B}\|\|\mathbf{A}^{-1}\| < 1$. We show that a similar statement holds if \mathbf{B} is replaced by a factorially divergent series in h with operator coefficients.

Lemma 7.4. Suppose given a sequence of h -linear compatible with inclusions operators:

$$\mathcal{H}_{n,s} : \mathcal{X}_s \rightarrow \mathcal{Y}_s : \|\mathcal{H}_{n,s}\| \leq C_1 \frac{n! C^n}{(1-s)^n}, \quad n \geq 1, \quad 0 < s < 1. \quad (45)$$

Suppose that we also have h -linear compatible with inclusions operators $\mathcal{F}_s : \mathcal{X}_s \rightarrow \mathcal{Y}_s$, with inverse $\mathcal{G}_s : \mathcal{Y}_s \rightarrow \mathcal{X}_s$, $\|\mathcal{G}_s\| \leq A$, which is also h -linear and compatible with inclusions.

Then there exist operators, h -linear and compatible with inclusions,

$$\mathcal{L}_{n,s} : \mathcal{Y}_s \rightarrow \mathcal{X}_s : \|\mathcal{L}_{n,s}\| \leq \max\{A, (C_1 A)^2 e^{4(C_1 A)}\} \frac{C^n n!}{(1-s)^n}, \quad n \geq 0, \quad 0 < s < 1$$

satisfying the following property: with notation $\mathcal{H}_0 := \mathcal{F}$,

$$\sum_{m,j \geq 0; j+m=N} \mathcal{H}_{j,s} \mathcal{L}_{m,s} = \begin{cases} 0, & \text{if } N \geq 1; \\ \text{id}, & \text{if } N = 0. \end{cases} \quad (46)$$

Proof. Motivated by the “formulas”

$$“(\mathcal{F} + \mathcal{H})^{-1} = ((1 + \mathcal{H}\mathcal{F}^{-1}) \circ \mathcal{F})^{-1} = \mathcal{G} \circ (1 + \mathcal{H}\mathcal{G})^{-1} = \sum_{k \geq 0} (-1)^k \mathcal{G}(\mathcal{H}\mathcal{G})^k” \quad (47)$$

and, for $k \geq 1$

$$(\mathcal{H}\mathcal{G})^k = \sum_{m \geq 1} h^m \sum_{\substack{n_1 + \dots + n_k = m \\ n_1, \dots, n_k \geq 1}} \mathcal{H}_{n_1, s} \mathcal{G} \mathcal{H}_{n_2, s} \mathcal{G} \dots \mathcal{H}_{n_k, s} \mathcal{G},$$

whose analytic meaning will be clarified shortly, we put

$$\mathcal{K}_{k, m, s} := \sum_{\substack{n_1 + \dots + n_k = m \\ n_1, \dots, n_k \geq 1}} \mathcal{H}_{n_1, s} \mathcal{G} \mathcal{H}_{n_2, s} \mathcal{G} \dots \mathcal{H}_{n_k, s} \mathcal{G}.$$

We have

$$\begin{aligned} \|\mathcal{K}_{k, m, s}\| &\leq \sum_{\substack{n_1 + \dots + n_k = m \\ n_1, \dots, n_k \geq 1}} \frac{C_1 n_1! C^{n_1}}{(1-s)^{n_1}} A \frac{C_1 n_2! C^{n_2}}{(1-s)^{n_2}} A \dots \frac{C_1 n_k! C^{n_k}}{(1-s)^{n_k}} A \\ &\leq (C_1 A)^k \frac{C^m}{(1-s)^m} \sum_{\substack{n_1 + \dots + n_k = m \\ n_1, \dots, n_k \geq 1}} n_1! \dots n_k! \end{aligned}$$

(use [Lemma 7.1](#))

$$\leq (C_1 A)^k \frac{C^m}{(1-s)^m} 4^{k-1} (m-k+1)!.$$

Put for $m \geq 1$:

$$\mathcal{L}_{m, s} := \sum_{k \geq 1} (-1)^k \mathcal{G} \mathcal{K}_{k, m, s}; \quad (48)$$

then

$$\begin{aligned} \|\mathcal{L}_{m, s}\| &\leq (C_1 A) \sum_{k \geq 1} (C_1 A)^k \frac{C^m}{(1-s)^m} 4^{k-1} (m-k+1)! \\ &\leq (C_1 A)^2 \frac{C^m}{(1-s)^m} \sum_{k' \geq 0} (C_1 A)^{k'} 4^{k'} (m-k')! \leq (C_1 A)^2 \frac{C^m m!}{(1-s)^m} e^{4(C_1 A)}. \end{aligned}$$

Put further $\mathcal{L}_{0, s} = \mathcal{G}$.

One proves (46) by writing out the LHS in terms of \mathcal{H}_j and \mathcal{G} and canceling; absolute convergence in (48) justifies manipulations with infinite sums. \square

7.5. The inverse of a singularly perturbed linear operator

We will use the following immediate corollary of [2, (A.31)]

$$\sum_{j_1+j_2=j; j_1, j_2 \geq 0} j_1! j_2! \leq 2j! + 4(j-1)! \leq 6j!, \quad \text{if } j \geq 0. \quad (49)$$

Remark. Let us intuitively motivate the setup of Lemma 7.5 below. We start with an operator-valued Gevrey expansion $\mathcal{F} = \sum_{n \geq 0} h^n \mathcal{F}_n$ which has a one-sided inverse in the form of the operator-valued Gevrey expansion $\mathcal{G} = \sum_{n \geq 0} h^n \mathcal{G}_n$. We consider a third operator-valued Gevrey expansion $\mathcal{H} = \sum_{n \geq 0} h^n \mathcal{H}_n$. We are solving a linear equation $(\mathcal{F} + h^a \mathcal{H})u = v$, where v is a known vector-valued Gevrey expansion and u is the Gevrey expansion to be found. We speak, in the title of this subsection, of a singularly perturbed linear operator because \mathcal{G} does not act between \mathcal{Y}_s and \mathcal{X}_s with the same s .

Lemma 7.5. Let $a \in \{1, 2, \dots\}$, $C'' \geq 1$, and $C \geq \max\{1, 36a^{a-2}BC''e^a\}$.

Suppose we are given compatible with inclusions h -linear operators

$$\mathcal{F}_{n,s} : \mathcal{X}_s \rightarrow \mathcal{Y}_s : \|\mathcal{F}_{n,s}\|_{\mathcal{X}_s, \mathcal{Y}_s} \leq C' \frac{n!C^n}{(1-s)^n}, \quad 0 < s < 1, \quad n \geq 0 \quad (50)$$

and compatible with inclusions h -linear operators

$$\text{for } n \geq 0, \text{ for } s' > s, \quad \mathcal{G}_{n,s',s} : \mathcal{Y}_{s'} \rightarrow \mathcal{X}_s, \quad \|\mathcal{G}_{n,s',s}\| \leq \frac{B}{(s' - s)^a} \frac{n!C^n}{(1-s')^n},$$

so that

$$\sum_{m, j \geq 0; j+m=N} \mathcal{F}_{j,s} \mathcal{G}_{m,s',s} = \begin{cases} 0, & \text{if } N \geq 1; \\ \mathcal{Y}_{s'} \xrightarrow{\text{id}} \mathcal{Y}_s, & \text{if } N = 0. \end{cases} \quad (51)$$

Suppose given another family of h -linear operators compatible inclusions

$$\mathcal{H}_{n,s} : \mathcal{X}_s \rightarrow \mathcal{Y}_s, \quad \|\mathcal{H}_{n,s}\|_{\mathcal{X}_s, \mathcal{Y}_s} \leq C'' \frac{n!C^n}{(1-s)^n}$$

and elements $v_n = (v_{n,s}) \in \mathring{\mathcal{X}}_1$, $n \geq 0$, satisfying

$$\|v_{n,s}\| \leq \frac{n!C^n}{(1-s)^n}.$$

Then there are $u_n = (u_{n,s}) \in \mathring{\mathcal{X}}_1$ such that

$$\|u_{n,s}\|_{\mathcal{X}_s} \leq \frac{C^{n+a+1}(n+a+1)!}{(1-s)^{n+a+1}}, \quad n \geq 0$$

so that: if

$$\frac{|h|C}{1-s} < 1 \quad (52)$$

then $\mathcal{F}_s = (\sum_{n \geq 0} h^n \mathcal{F}_{n,s})$, $\mathcal{H}_s = (\sum_{n \geq 0} h^n \mathcal{H}_{n,s}) : \mathcal{X}_s \rightarrow \mathcal{Y}_s$ are bounded operators, $v = \sum_{n \geq 0} h^n v_{n,s} \in \mathcal{Y}_s$ and $u = \sum_{n \geq 0} h^n u_{n,s} \in \mathcal{X}_s$ are absolutely convergent series, and

$$(\mathcal{F} + h^a \mathcal{H})u = v \quad \text{in } \mathcal{Y}_s. \quad (53)$$

Proof. *Step 1 (Algebraic construction of the solution).*

Guided by the “formula”

$$“(\mathcal{F} + h^a \mathcal{H})^{-1} = [(1 + h^a \mathcal{H} \mathcal{F}^{-1}) \mathcal{F}]^{-1} = \mathcal{F}^{-1} (1 - h^a \mathcal{H} \mathcal{F}^{-1} + h^{2a} (\mathcal{H} \mathcal{F}^{-1})^2 + \dots)”$$

which however does not make literal sense, we take

$$\begin{aligned} & \mathcal{G}(\mathcal{H}\mathcal{G})^n v \\ “=” & \sum_{\substack{k_0, \dots, k_n \geq 0 \\ \ell_1, \dots, \ell_n, m \geq 0}} h^{k_0 + \dots + k_n + \ell_1 + \dots + \ell_n + m} \mathcal{G}_{k_0, s_1, s_0} \mathcal{H}_{\ell_1, s_1} \mathcal{G}_{k_1, s_2, s_1} \dots \mathcal{H}_{\ell_n, s_n} \mathcal{G}_{k_n, s_{n+1}, s_n} v_{m, s_{n+1}} \end{aligned} \quad (54)$$

with $s_0 = s$ and we reserve the right to choose other s_j ’s differently in each summand subject only to the condition $s_0 < s_1 < \dots < s_{n+1} < 1$ (compositions will not depend on these choices). We have

$$\begin{aligned} & \| \mathcal{G}_{k_0, s_1, s_0} \mathcal{H}_{\ell_1, s_1} \mathcal{G}_{k_1, s_2, s_1} \dots \mathcal{H}_{\ell_n, s_n} \mathcal{G}_{k_n, s_{n+1}, s_n} v_{m, s_{n+1}} \| \\ & \leq \frac{B}{(s_1 - s_0)^a} \frac{k_0! C^{k_0}}{(1 - s_1)^{k_0}} \frac{C'' \ell_1! C^{\ell_1}}{(1 - s_1)^{\ell_1}} \\ & \dots \frac{B}{(s_n - s_{n-1})^a} \frac{k_{n-1}! C^{k_{n-1}}}{(1 - s_n)^{k_{n-1}}} \frac{C'' \ell_n! C^{\ell_n}}{(1 - s_n)^{\ell_n}} \frac{B}{(s_{n+1} - s_n)^a} \frac{k_n! C^{k_n}}{(1 - s_{n+1})^{k_n}} \frac{m! C^m}{(1 - s_{n+1})^m}. \end{aligned} \quad (55)$$

Choose s_{n+1} so that

$$s_{n+1} - s_n = \frac{a}{m + k_n + a} (1 - s_n), \quad (56)$$

then

$$\frac{1}{(s_{n+1} - s_n)^a (1 - s_{n+1})^{k_n + m}} = \frac{\left(\frac{m + k_n + a}{a}\right)^a \left(\frac{m + k_n + a}{m + k_n}\right)^{m + k_n}}{(1 - s_n)^{k_n + m + a}} \leq \frac{(m + k_n + a)^a e^a}{a^a (1 - s_n)^{k_n + m + a}}.$$

Then

$$\begin{aligned} \text{RHS of (55)} & \leq \frac{(C'')^n B^{n+1} e^{(n+1)a}}{a^{(n+1)a} (1 - s_0)^{(n+1)a}} \left(\frac{C}{1 - s_0} \right)^{k_0 + \ell_1 + \dots + \ell_n + k_n + m} k_0! \ell_1! k_1! \dots \ell_n! k_n! m! \\ & \quad \times (k_n + m + a)^a (k_n + m + \ell_n + k_{n-1} + 2a)^a \dots (k_0 + \dots + m + [n + 1]a)^a. \end{aligned}$$

But

$$\begin{aligned} & \sum_{\substack{k_0+\dots+k_n+\ell_1+\dots+\ell_n+m=N \\ k_0,\dots,k_n,\ell_1,\dots,\ell_n,m\geq 0}} k_0!\ell_1!k_1!\dots\ell_n!k_n!m!(k_n+m+a)^a(k_n+m+\ell_n+k_{n-1}+2a)^a \\ & \quad \dots(k_0+\dots+m+[n+1]a)^a \\ & \leq (36a^{a-1})^{n+1}(N+[n+1]a)!. \end{aligned} \quad (57)$$

Indeed, on the LHS of (57) let us introduce new summation indices: $m+k_n=:K_n\geq 0$, $\ell_n+k_{n-1}=:K_{n-1}\geq 0$, ..., $\ell_1+k_0=:K_0\geq 0$ and use (49); let us also use that $\frac{(p+a)^a}{(p+1)\dots(p+a)}\leq a^{a-1}$ for any $p\geq 0$. Then

LHS of (57)

$$\begin{aligned} & \leq (6a^{a-1})^{n+1} \sum_{\substack{K_0+\dots+K_n=N \\ K_0,\dots,K_{n-1},K_n\geq 0}} K_0!\dots K_{n-1}!K_n! \frac{(K_n+a)!}{K!} \cdot \frac{(K_{n-1}+K_n+2a)!}{(K_{n-1}+K_n+a)!} \\ & \quad \dots \frac{(K_0+\dots+K_n+[n+1]a)!}{(K_0+\dots+K_n+na)!} \\ & \leq (6a^{a-1})^{n+1} \cdot \sum_{\substack{K_0+\dots+K_n=N \\ K_0,\dots,K_n\geq 0}} K_0!\dots K_{n-2}!K_{n-1}!(K_n+a)! \frac{(K_{n-1}+K_n+2a)!}{(K_{n-1}+K_n+a)!} \\ & \quad \dots \frac{(K_0+\dots+K_n+[n+1]a)!}{(K_0+\dots+K_n+na)!} \end{aligned}$$

(put $K'_{n-1}:=K_{n-1}+K_n$ and use (49))

$$\begin{aligned} & \leq (6a^{a-1})^{n+1} \cdot 6 \sum_{\substack{K_0+\dots+K_{n-2}+K'_{n-1}=N \\ K_0,\dots,K_{n-2},K'_{n-1}\geq 0}} K_0!\dots K_{n-2}!(K'_{n-1}+2a)! \frac{(K'_{n-1}+K_{n-2}+3a)!}{(K'_{n-1}+K_{n-2}+2a)!} \\ & \quad \dots \frac{(K_0+\dots+K'_{n-2}+[n+1]a)!}{(K_0+\dots+K'_{n-2}+na)!} \end{aligned}$$

(etc.)

$$\leq (6a^{a-1})^{n+1} \cdot 6^n (N+[n+1]a)! < (36a^{a-1})^{n+1} (N+[n+1]a)!,$$

hence (57).

Let us go back to interpreting (54). Formally writing

$$“\sum_{n\geq 0} h^{na} \mathcal{G}(\mathcal{H}\mathcal{G})^n v = \sum_{J\geq 0} h^J u_J”$$

motivates the definition

$$u_J := \sum_{na+N=J, n, N \geq 0} \sum_{k_0+\dots+\ell_n+m=N} \mathcal{G}_{k_0, s_1, s_0} \mathcal{H}_{\ell_1, s_1} \mathcal{G}_{k_1, s_2, s_1} \dots \mathcal{H}_{\ell_n, s_n} \mathcal{G}_{k_n, s_{n+1}, s_n} v_{m, s_{n+1}},$$

where, using (57)

$$\|u_J\| \leq \sum_{na+N=J, n, N \geq 0} \frac{(C'')^n B^{n+1} e^{(n+1)a}}{a^{(n+1)a}} \frac{C^N}{(1-s_0)^{N+(n+1)a}} (36a^{a-1})^{n+1} (N + (n+1)a)!$$

(assuming $C'' \geq 1$, $C \geq 36a^{a-2} B C'' e^a$, $C \geq 1$ and taking into account that we have $\leq J+1 < J+a+1$ summands)

$$< \frac{(J+a+1)! C^{J+1}}{(1-s_0)^{J+a}} < \frac{(J+a+1)! C^{J+a+1}}{(1-s_0)^{J+a+1}}.$$

Step 2 (Proof of (53)).

Note that estimates on $v_{n,s}$ and $u_{n,s}$ for s outside of the range prescribed by $\frac{|h|C}{1-s} < 1$ will be used in an essential way.

In the expression

$$(\mathcal{F} + h^a \mathcal{H}) \left[\sum_{n \geq 0} (-1)^n h^{an} \times \sum_{\substack{k_0, \dots, k_n \geq 0 \\ \ell_1, \dots, \ell_n, m \geq 0}} h^{k_0+\dots+k_n+\ell_1+\dots+\ell_n+m} \mathcal{G}_{k_0, s_1, s_0} \mathcal{H}_{\ell_1, s_1} \mathcal{G}_{k_1, s_2, s_1} \dots \mathcal{H}_{\ell_n, s_n} \mathcal{G}_{k_n, s_{n+1}, s_n} v_{m, s_{n+1}} \right] \quad (58)$$

a well-defined operator is applied to an absolutely convergent sum, once s_j 's, $j \geq 1$, are chosen separately in each summand as in (56) while $s_0 = s$; this absolute convergence justifies the manipulations with infinite sums below.

Let \mathcal{I} denote the set of all indices n, k_0, \dots, m as in the sum (58); to stress dependence of choices of s_1, \dots, s_{n+1} on $t \in \mathcal{I}$ we will write $s_1(t), \dots, s_{n+1}(t)$. The entries of t will be denoted $n(t), k_0(t), \dots, m(t)$; $\sigma(t)$ will denote $k_0 + \dots + \ell_n + m$.

Rewrite (58):

$$(58) = I + II,$$

where

$$\begin{aligned} I &= \sum_{\substack{n \geq 0; j \geq 0; k_0, \dots, k_n \geq 0 \\ \ell_1, \dots, \ell_n, m \geq 0}} (-1)^n h^{an+j+\sigma(t)} \mathcal{F}_{j,s} \mathcal{G}_{k_0, s_1(t), s_0} \mathcal{H}_{\ell_1, s_1(t)} \mathcal{G}_{k_1, s_2(t), s_1(t)} \\ &\quad \circ \dots \circ \mathcal{H}_{\ell_n, s_n(t)} \mathcal{G}_{k_n, s_{n+1}(t), s_n(t)} v_{m, s_{n+1}(t)}; \\ II &= \sum_{\substack{n \geq 0; j \geq 0; k_0, \dots, k_n \geq 0 \\ \ell_1, \dots, \ell_n, m \geq 0}} (-1)^n h^{a(n+1)+j+\sigma(t)} \mathcal{H}_{j,s} \mathcal{G}_{k_0, s_1(t), s_0} \mathcal{H}_{\ell_1, s_1(t)} \mathcal{G}_{k_1, s_2(t), s_1(t)} \\ &\quad \circ \dots \circ \mathcal{H}_{\ell_n, s_n(t)} \mathcal{G}_{k_n, s_{n+1}(t), s_n(t)} v_{m, s_{n+1}(t)}. \end{aligned}$$

In I introduce a new index $p = j + k_0$:

$$I = \sum_{\substack{n \geq 0; p \geq 0; k_1, \dots, k_n \geq 0 \\ \ell_1, \dots, \ell_n, m \geq 0}} (-1)^n h^{an+p+k_1+\dots+\ell_n+m} \left(\sum_{j, k_0 \geq 0; j+k_0=p} \mathcal{F}_{j,s} \mathcal{G}_{k_0, S_1, s_0} \right) \\ \circ \mathcal{H}_{\ell_1, S_1} \mathcal{G}_{k_1, s_2(t), S_1} \dots \mathcal{H}_{\ell_n, s_n(t)} \mathcal{G}_{k_n, s_{n+1}(t), s_n(t)} v_{m, s_{n+1}(t)},$$

where S_1 depends on $n, k_1, \dots, k_n, \ell_1, \dots, \ell_n, m, p$ and is defined as $\min\{s_1(t)\}$ over all ι s with prescribed $n, k_1, \dots, k_n, \ell_1, \dots, \ell_n, m$ and $k_0 \leq p$ (so it is a minimum over a set of $p+1$ elements).

By (51) the sum in parentheses is zero unless $p = 0$ in which case it is the inclusion $\mathcal{Y}_{S_1} \xrightarrow{\text{id}} \mathcal{Y}_{s_0}$. Hence

$$I = \sum_{\substack{n \geq 0; k_1, \dots, k_n \geq 0 \\ \ell_1, \dots, \ell_n, m \geq 0}} (-1)^n h^{an+k_1+\dots+\ell_n+m} \mathcal{H}_{\ell_1, s} \mathcal{G}_{k_1, s_2(t), s} \dots \mathcal{H}_{\ell_n, s_n(t)} \mathcal{G}_{k_n, s_{n+1}(t), s_n(t)} v_{m, s_{n+1}(t)}. \quad (59)$$

We see that most terms of I are canceled by terms of II , and only the $n = 0$ term of (59) remains, which is just v which implies (53). \square

8. Proof of the main result

In this section we prove our main result [Theorem 1.1](#).

The main step in the proof of [Theorem 1.1](#) is, by [Section 2](#), the solution of equation $F_s(E_0, \dots, E_{M-2}, T) = 0$, cf. (13), where $F_s : \mathcal{X}_s^\tau \rightarrow \mathcal{Y}_s^\tau$, $s \leq 1$, $\tau < \tau_0$, see [Section 4](#). With respect to s , F_s form a family of maps compatible with inclusions, in the sense of [Section 7.1](#) – for various maps below we will keep in mind and use compatibility with inclusions without always writing these words. The operator h of [Section 7.1](#) will be just the multiplication by h in the spaces $\mathcal{X}_s^\tau, \mathcal{Y}_s^\tau, \mathcal{Z}_s^\tau$, with $|h| = \tau$. We will use h -linearity properties of various operators without explicitly mentioning it.

We will take $x^{(0)} \in \mathcal{X}_1^\tau$ to be the tuple (E_0, \dots, E_{M-2}, T) of *polynomials in h* that solves $F(E_0, \dots, E_{M-2}, T) = 0$ up to order h^{b_0+1} , $b_0 = 8$, see (63), on the disc of radius ρ_0 (see [Notation 4.1](#)); such a tuple exists by [Proposition 2.1](#).

If we write down the expression for $F_s(E_0, \dots, E_{M-2}, T)$ as a power series in $(E_0, \dots, E_{M-2}, T) \in \mathcal{X}_s^\tau$, the series will converge provided $\|hT'\|_{V'(s)} < 1$ which is definitely the case if $\tau \| (E_0, \dots, E_{M-2}, T) \|_{\mathcal{X}_s^\tau} < 1$. Therefore, if we impose on τ the condition

$$\tau \| x^{(0)} \|_{\mathcal{X}_1^\tau} < \frac{1}{2}, \quad (60)$$

then $F_s(x^{(0)} + w)$ can be represented in the form (34) with some finite α independent of s .

Since $F_s(E_0, \dots, E_{M-2}, T)$ is an analytic function of E_0, \dots, E_{M-2}, T , we can write its Fréchet derivative by means of usual Calculus formulas. Since F_s explicitly depends only on the $E_0, \dots, E_{M-2}, T, \dots, T'''$, we can write $dF_{(E_0, \dots, E_{M-2}, T)}(\mathcal{E}_0, \dots, \mathcal{E}_{M-2}, \mathcal{T})$ as a finite sum (index s suppressed)

$$dF_{(E_0, \dots, E_{M-2}, T)}(\mathcal{E}_0, \dots, \mathcal{E}_{M-2}, \mathcal{T}) = \sum_{v=1}^N \mathcal{F}_v(E_0, \dots, E_{M-2}, T) \cdot L_v(E_0, \dots, E_{M-2}, \mathcal{T}), \quad (61)$$

where $(\mathcal{E}_0, \dots, \mathcal{E}_{M-2}, \mathcal{T})$ belong to the tangent space of \mathcal{X}_s^τ at (E_0, \dots, E_{M-2}, T) , and $\mathcal{F}_v : \mathcal{X}_s^\tau \rightarrow \mathcal{Y}_s^\tau$ are analytic on an open subset of \mathcal{X}_s^τ compatible with inclusions $\mathcal{X}_s^\tau \rightarrow \mathcal{X}_{s'}^\tau$ and $\mathcal{Y}_s^\tau \rightarrow \mathcal{Y}_{s'}^\tau$, and $L_v : \mathcal{X}_s^\tau \rightarrow \mathcal{Y}_s^\tau$ are constant (i.e. (E_0, \dots, E_{M-2}, T) -independent) linear maps $\mathcal{X}_s^\tau \rightarrow \mathcal{Y}_s^\tau$ of norm ≤ 1 , also compatible with inclusions. The dot (\cdot) on the RHS of (61) is the pointwise multiplication of elements of \mathcal{Y}_s^τ which has norm 1 in the sense that $\|y_1 \cdot y_2\|_{\mathcal{Y}_s^\tau} \leq \|y_1\|_{\mathcal{Y}_s^\tau} \|y_2\|_{\mathcal{Y}_s^\tau}$.

For concreteness, varying the LHS (13), we pick $N = M + 5$ and make the following choices where specifics of big formulas will be unimportant later on:

$$\begin{aligned} \mathcal{F}_1 &= 1, & L_1 &= \sum_{\mu=0}^{m-2} z^\mu \mathcal{E}_\mu - \frac{z^M}{2} \mathcal{T}' - \frac{Mz^{M-1}}{4} \mathcal{T}, \\ \mathcal{F}_{2+j} &= h\hat{\mathcal{F}}_{2+j} = \sum_{\mu=1}^j \binom{j}{\mu} h^\mu T^\mu z^{j-\mu} + hT'(2 + hT')(z + hT)^j, \\ L_{2+j} &= \mathcal{E}_j, \quad j = 0, \dots, M-2, \\ \mathcal{F}_{M+1} &= h\hat{\mathcal{F}}_{M+1} \quad (\text{see below}), & L_{M+1} &= \mathcal{T}, \\ \mathcal{F}_{M+2} &= h\hat{\mathcal{F}}_{M+2} \quad (\text{see below}), & L_{M+2} &= \mathcal{T}', \\ \mathcal{F}_{M+3} &= h^2 \hat{\mathcal{F}}_{M+3} = \frac{h^3}{2} \frac{T'''}{(1 + hT')^2} - \frac{3h^4}{2} \frac{T''^2}{(1 + hT')^3}, & L_{M+3} &= \mathcal{T}', \\ \mathcal{F}_{M+4} &= h^2 \hat{\mathcal{F}}_{M+4} = \frac{3h^3}{2} \frac{T''}{(1 + hT')^2}, & L_{M+4} &= \mathcal{T}'', \\ \mathcal{F}_{M+5} &= h^2 \hat{\mathcal{F}}_{M+5} = -\frac{h^2}{2} \frac{1}{1 + hT'}, & L_{M+5} &= \mathcal{T}''', \end{aligned} \quad (62)$$

where

$$\begin{aligned} \hat{\mathcal{F}}_{M+1} &= (1 + hT')^2 \sum_{j=0}^{M-2} j E_j (z + hT)^{j-1} - \frac{\sum_{\mu=2}^M \binom{M}{\mu} h^{\mu-2} z^{M-\mu} \mu T^{\mu-1}}{4} \\ &\quad - \frac{MT'(2 + hT')(z + hT)^{M-1}}{4}, \\ \hat{\mathcal{F}}_{M+2} &= 2(1 + hT') \left(\sum_{j=0}^{M-2} E_j (z + hT)^j - \frac{1}{4} \sum_{\mu=1}^M z^{M-\mu} h^{\mu-1} T^\mu \right). \end{aligned}$$

Notice that norms of L_v are all ≤ 1 thanks to the assumption on ρ_0 that we made in [Notation 4.1](#).

Motivated by our considerations in [Section 6.2](#), we have grouped the terms $\mathcal{F}_v L_v$ in such a way that terms $v = 1, \dots, M + 2$ only depend on $E_0, \dots, E_{M-2}, T, T'$ and $\mathcal{E}_0, \dots, \mathcal{E}_{M-2}, \mathcal{T}, \mathcal{T}'$; terms containing higher derivatives are put into summands for $v = M + 3, M + 4, M + 5$;

we notice that all summands in the latter group also contain a factor of h^2 . We will thus treat $\mathcal{F}_{M+3}L_{M+3} + \mathcal{F}_{M+4}L_{M+4} + \mathcal{F}_{M+5}L_{M+5}$ as a perturbation of $\mathcal{F}_1L_1 + \dots + \mathcal{F}_{M+2}L_{M+2}$. In turn, as $\mathcal{F}_2, \dots, \mathcal{F}_{M+2}$ contain a prefactor of h , we treat $\mathcal{F}_2L_2 + \dots + \mathcal{F}_{M+2}L_{M+2}$ as a perturbation of \mathcal{F}_1L_1 .

Modify α in such a way that representations of the form (34) also hold for $\hat{\mathcal{F}}_1(x^{(0)} + w), \dots, \hat{\mathcal{F}}_{M+2}(x^{(0)} + w), \hat{\mathcal{F}}_{M+3}(x^{(0)} + w), \dots, \hat{\mathcal{F}}_{M+5}(x^{(0)} + w)$.

From now on we fix the following notation:

A is the constant such that L_1 understood as a map $\mathcal{Z}_s^\tau \rightarrow \mathcal{Y}_s^\tau$ has an inverse of norm $\leq A$;
 B is the constant such that the restriction map $\mathcal{Z}_s^\tau \rightarrow \mathcal{X}_{s'}^\tau, s' < s$, is of norm $\leq \frac{B}{(s-s')^2}$;

we fix the following functions:

$$\beta(t) = te^{4t}; \quad \beta_2(t) = \sum_{k \geq 1} \frac{t^{k+1}4^k}{k!}.$$

We will need an integer sequence $b_j, j \geq 0$, satisfying the properties:

$$\text{a) } b_{j+1} = 2b_j - 7; \quad \text{b) } b_j \geq 3 + j; \quad (63)$$

clearly the condition a) together with $b_0 = 8$ generates such a sequence.

We assume that τ satisfies (60), (67), (71), (73), (79), and (80); we assume that α is a finite number chosen as above, and we assume that C satisfies (78) and (67).

Step 1: Constructing iterations of the Newton method as formal factorially divergent expansions.

We will construct Gevrey expansions which will constitute iterations of the Newton method. For our intuition, we suggest the following correspondence (which we will not make precise) of the notions of Section 6.1 to the objects introduced below:

$$\begin{aligned} x_{\text{sec.6.1}}^{(j)} &\leftrightarrow x^{(0)} + g^{(j)}, \\ w_{\text{sec.6.1}}^{(j)} &\leftrightarrow \sum_{n \geq 1} h^n \tilde{w}_n^{(j)} \quad \text{and} \quad h^{b_j-3} \sum_{n \geq 1} h^n w_n^{(j)}, \\ y_{\text{sec.6.1}}^{(j)} &\leftrightarrow h^{b_j} \sum_{n \geq 0} h^n y_n^{(j)}. \end{aligned}$$

Speaking rigorously, we will now construct the following objects for all $j \geq 0$:

$$\text{for } n \geq 0 \quad y_n^{(j)} = (y_{n,s}^{(j)}) \in \mathcal{Y}_1^\tau; \quad \|y_{n,s}^{(j)}\| \leq \frac{n!C^n}{(1-s)^n}; \quad (64)$$

$$\text{for } n \geq 1 \quad \tilde{w}_n^{(j)} = (\tilde{w}_{n,s}^{(j)}) \in \mathcal{X}_1^\tau; \quad \|\tilde{w}_{n,s}^{(j)}\| \leq \frac{1}{2^j} \frac{C^n n!}{(1-s)^n}. \quad (65)$$

We will use the notation:

$$\text{for } n \geq 1 \quad g_{n,s}^{(0)} = 0; \quad g_{n,s}^{(j)} = \tilde{w}_{n,s}^{(0)} + \dots + \tilde{w}_{n,s}^{(j-1)}, \quad j \geq 1; \quad (66)$$

clearly $g_n^{(j)} \in \mathring{X}_1^\tau$ and, as soon as (65) is true, $g_{n,s}^{(j)}$ satisfy (35) with $C_0 = 2$, i.e.

$$\|g_{n,s}^{(j)}\|_{\mathcal{X}_s^\tau} \leq 2 \cdot \frac{C^n n!}{(1-s)^n}.$$

By our choice of $x^{(0)}$, we can write $F(x^{(0)})$ as a convergent power series $h^{b_0+1} \sum_{n \geq 0} \tilde{y}_n(z) h^n$, where $\tilde{y}_n(z)$ is an h -independent function and $\|\tilde{y}_n(z)\|_{\mathcal{Y}_1^\tau} \leq c' \cdot c^n$. Put $y_{n,s}^{(0)} := h \tilde{y}_n(z)$; then assuming

$$C \geq c \quad \text{and} \quad \tau \cdot c' \leq 1 \quad (67)$$

we obtain (64) for $j = 0$.

Suppose $y_{n,s}^{(j')}$ are defined for $j' \leq j$ and $w_{n,s}^{(j')}$ are defined for $j' < j - 1$ (vacuously true if $j = 0$). Let us construct $w^{(j)}$ and $y^{(j+1)}$.

Using Lemma 7.2 with $x^{(0)}$ for x_0 , with $g_{n,s}^{(j)}$ for $g_{n,s}$, and with $\hat{\mathcal{F}}_v$ and $\hat{\hat{\mathcal{F}}}_v$ for f , delivers elements

$$G_{v,n} \in \mathring{\mathcal{Y}}_1^\tau, \quad \|G_{v,n,s}\|_{\mathcal{Y}_s^\tau} \leq \beta(C_0 \alpha) \frac{C^n n!}{(1-s)^n}, \quad n \geq 1, \quad v = 2, \dots, M+2; \quad (68)$$

$$\begin{aligned} G'_{v,n} \in \mathring{\mathcal{Y}}_1^\tau, \quad \|G'_{v,n,s}\|_{\mathcal{Y}_s^\tau} &\leq \max\{\|\hat{\hat{\mathcal{F}}}_v(x^{(0)})\|_{\mathcal{Y}_1^\tau}, \beta(C_0 \alpha)\} \frac{C^n n!}{(1-s)^n}, \\ n \geq 0, \quad v &= M+3, M+4, M+5, \end{aligned} \quad (69)$$

such that, intuitively speaking, $\hat{\mathcal{F}}_v(x^{(0)} + \sum_n h^n g_n^{(j)})$ corresponds to $\hat{\mathcal{F}}(x^{(0)}) + \sum_{n \geq 1} h^n G_{v,n}$, and $\hat{\hat{\mathcal{F}}}_v(x^{(0)} + \sum_n h^n g_n^{(j)})$ corresponds to $\sum_{n \geq 0} h^n G'_{v,n}$.

Let us find an infinite series of operators $\mathcal{L}_{n,s} : \mathcal{Y}_s^\tau \rightarrow \mathcal{Z}_s^\tau$ so that the infinite sum $\sum_{n \geq 0} h^n \mathcal{L}_{n,s}$ will play the role of the inverse of $\sum_{v=1}^{M+2} \mathcal{F}_v(x) \cdot L_v$ in the precise sense specified below. An estimate of $\|\mathcal{L}_{n,s}\| / (\frac{C^n n!}{(1-s)^n})$ by $2A$ will be important in (77) which, in turn, lets us preserve the same constant C from one induction step to the other.

With this goal in mind, we apply Lemma 7.4 with operators

$$(\mathcal{F})_{\text{Lemma}} = \sum_{v=1}^4 \mathcal{F}_v(x^{(0)}) \cdot L_v, \quad (\mathcal{H}_{n,s})_{\text{Lemma}} := \sum_{v=2}^{M+2} h G_{v,n,s} \cdot L_v, \quad n \geq 1, \quad (70)$$

and constants

$$(C_1)_{\text{Lemma}} = (M+1)\tau\beta(C_0\alpha); \quad A_{\text{Lemma}} = 2A_{\text{here}}.$$

In order to assure that, as the assumptions of Lemma 7.4 require, \mathcal{F} has inverse of norm $\leq 2A$, we remember that L_1 has an inverse of norm $\leq A$; it is for that reason that we assumed that τ should satisfy

$$\tau \cdot \sum_{v=2}^{M+2} \|\hat{\mathcal{F}}_v(x^{(0)})\|_{\mathcal{Y}_1^\tau} \leq \frac{1}{A}. \quad (71)$$

We have assumed above that τ is so small that

$$((C_1)_{\text{Lemma}} A)^2 e^{4(C_1)_{\text{Lemma}} A} \leq 2A, \quad (72)$$

or, more explicitly,

$$((M+1)\tau A\beta(2\alpha))^2 e^{4(M+1)\tau A\beta(2\alpha)} \leq 2A; \quad (73)$$

therefore, [Lemma 7.4](#) yields operators

$$\mathcal{L}_{n,s} : \mathcal{Y}_s^\tau \rightarrow \mathcal{Z}_s^\tau \quad \text{s.t.} \quad \|\mathcal{L}_{n,s}\| \leq 2A \frac{C^n n!}{(1-s)^n}$$

satisfying [\(46\)](#).

For any $s' < s$ we can compose the operator $\mathcal{L}_{n,s}$ with the restriction map $\mathcal{Z}_s^\tau \rightarrow \mathcal{X}_{s'}^\tau$ and get $\mathcal{L}'_{n,s,s'} : \mathcal{Y}_s^\tau \rightarrow \mathcal{X}_{s'}^\tau$ of norm $\leq \frac{B}{(s-s')^2} \cdot 2A \frac{C^n n!}{(1-s)^n}$.

We are thus in the situation of [Lemma 7.5](#) with $a = 2$, operators (cf. [\(68\)](#), [\(69\)](#))

$$(\mathcal{F}_0)_{\text{Lemma}} = \sum_{v=1}^{M+2} \mathcal{F}_v(x^{(0)}) \cdot L_v, \quad (\mathcal{F}_n)_{\text{Lemma}} = \sum_{v=2}^{M+2} G_{v,n} \cdot L_v, \quad \text{if } n \geq 1; \quad (74)$$

$$(\mathcal{H})_{\text{Lemma}} = \sum_{v=M+3}^{M+5} \hat{\mathcal{F}}_v(x) \cdot L_v, \quad \text{i.e.} \quad (\mathcal{H}_n)_{\text{Lemma}} = \sum_{v=M+3}^{M+5} G'_{v,n} \cdot L_v, \quad (75)$$

vectors (see [\(64\)](#))

$$(v_n)_{\text{Lemma}} = -y_n^{(j)}, \quad n \geq 0; \quad (76)$$

and constants

$$\begin{aligned} (C')_{\text{Lemma}} &= \max \left\{ \sum_{v=1}^{M+2} |\mathcal{F}_v(x^{(0)})|_{\mathcal{X}_1^\tau}, C_1 \right\}; \\ (C'')_{\text{Lemma}} &= C'' := \sum_{v=M+3}^{M+5} \max \{ \|\hat{\mathcal{F}}_v(x^{(0)})\|_{\mathcal{Y}_1^\tau}, \beta(C_0\alpha) \}; \\ B_{\text{Lemma}} &= 2B_{\text{here}} A, \end{aligned} \quad (77)$$

and we must assume

$$C \geq \max \{ 1, 36 \cdot 2BA \cdot C'' \cdot e^2 \}. \quad (78)$$

Then [Lemma 7.5](#) delivers vectors u_n s.t.

$$\|u_{n,s}\|_{\mathcal{X}_s^\tau} \leq \frac{C^{n+3}(n+3)!}{(1-s)^{n+3}}.$$

Put $w_n^{(j)} = u_{n-3}$ for $n \geq 3$, $w_1^{(j)} = w_2^{(j)} = 0$, then $w = \sum_{n \geq 1} h^n w_n$ plays the role of the solution of

$$(dF)_x(w) = h^3 \sum_{n \geq 1} h^n y_n^{(j)}.$$

Define $\tilde{w}^{(j)} = h^{b_j-3} w$. Because of (63)(b) and because we have assumed

$$\tau < \frac{1}{2}, \quad (79)$$

we see that the estimate of (65) is satisfied.

We will now obtain vectors $y_n^{(j+1)}$ such that, intuitively speaking, $h^{b_{j+1}} \sum_{n \geq 1} h^n y_n^{(j+1)}$ plays the role of $f(x^{(0)} + g^{(j)} + w^{(j)}) - f(x^{(0)} + g^{(j)}) - [f'(x^{(0)} + g^{(j)})(w^{(j)})]$ (the cancellation of the second and third summands will be discussed later). Apply Lemma 7.3(b) with the following inputs:

$$\begin{aligned} f_{\text{Lemma}} &= F_{\text{here}}; & (x_0)_{\text{Lemma}} &= x^{(0)}, & g_{\text{Lemma}} &= g^{(j)}, \\ b_{\text{Lemma}} &= b_j - 3, & (v_n)_{\text{Lemma}} &= w_n^{(j)}, \\ \alpha_{\text{Lemma}} &= \alpha_{\text{here}}, & (C_0)_{\text{Lemma}} &= 2, & A_{\text{Lemma}} &= 1. \end{aligned}$$

Then Lemma 7.3(b) yields

$$u_{n,s} \in \mathcal{Y}_s^\tau, \quad n \geq 1$$

satisfying

$$\|u_{n,s}\|_{\mathcal{Y}_s^\tau} \leq \beta_2 \alpha_{\text{Lemma}} ((C_0)_{\text{Lemma}} + A_{\text{Lemma}}) \frac{n! C^n}{(1-s)^n} = \beta_2 (3\alpha) \frac{n! C^n}{(1-s)^n}$$

such that $h^{2b_j-6} \sum_{j \geq 1} h^j u_j$ plays the role of $y^{(j+1)}$.

In (63)(a) we have defined $b_{j+1} = 2b_j - 7$; now take $y_n^{(j+1)} = h u_n$ and assume that

$$\tau \cdot \beta_2 (3\alpha) \leq 1. \quad (80)$$

Then $y_n^{(j+1)}$ satisfy the inductive assumption (64).

The inductive construction of Step 1 is thus complete.

Step 2: Passing from Gevrey expansions to actual vectors.

If condition

$$\tau C < 1 - s \quad (81)$$

is satisfied, then we can define the following vectors by means of absolutely convergent sums:

$$\mathbf{w}^{(j)} = \sum_{n \geq 1} h^n w_{n,s}^{(j)} \stackrel{\text{abs.conv.}}{=} h^{b_j-3} \sum_{n \geq 1} h^n w_{n,s}^{(j)} \in \mathcal{X}_s^\tau; \quad (82)$$

$$\mathbf{g}^{(j)} = \mathbf{w}^{(0)} + \dots + \mathbf{w}^{(j-1)} \stackrel{\text{abs.conv.}}{=} \sum_{n \geq 1} h^n g_{n,s}^{(j)} \in \mathcal{X}_s^\tau; \quad (83)$$

$$\mathbf{y}^{(j)} = h^{b_j} \sum_{n \geq 0} h^n y_n^{(j)} \in \mathcal{Y}_s^\tau; \quad (84)$$

the equalities marked above as (abs.conv.) are justifiable by operations on absolutely convergent sums.

Claim. If (85) and (87) are satisfied, then $F(x^{(0)} + \mathbf{g}^{(j)}) = \mathbf{y}^{(j)}$.

The case $j = 0$ is obvious by definitions.

Suppose the Claim is true for j , let us deduce it for $j + 1$.

Indeed, applying Lemma 7.3 with the same ingredients as in Step 1 but under condition

$$\sum_{n=1}^{\infty} \frac{\tau^n C^n}{(1-s)^n} < \frac{1}{3\alpha} \quad (85)$$

(which we have assumed above) corresponding to (42), we have

$$F(x^{(0)} + \mathbf{g}^{(j)} + w^{(j)}) = \mathbf{y}^{(j+1)} + F(x^{(0)} + \mathbf{g}^{(j)}) + [F'(x^{(0)} + \mathbf{g}^{(j)})](\mathbf{w}^{(j)})$$

so our claim, in view of induction hypothesis, is reduced to showing that

$$\mathbf{y}^{(j)} = -[F'(x^{(0)} + \mathbf{g}^{(j)})](\mathbf{w}^{(j)}). \quad (86)$$

By Lemma 7.5, $\mathbf{w}^{(j)}$ solves the equation

$$(\mathbf{F} + h^2 \mathbf{H})\mathbf{w}^{(j)} = -\mathbf{y}^{(j)}$$

where (cf. (74), (75))

$$\begin{aligned} \mathbf{F} &= \sum_{v=1}^4 \mathcal{F}_v(x^{(0)}) \cdot L_v + \sum_{n \geq 1} h^n \sum_{v=2}^4 G_{v,n,s} \cdot L_v; \\ \mathbf{H} &= \sum_{n \geq 0} h^n \sum_{v=5}^7 G'_{v,n,s} \cdot L_v. \end{aligned}$$

But the definition of G_v and G'_v by means of Lemma 7.2 shows that under condition

$$\sum_{n \geq 1} \frac{\tau^n C^n}{(1-s)^n} < \frac{1}{2\alpha} \quad (87)$$

(which we have assumed) corresponding to (36), we have

$$\mathbf{F} = \sum_{v=1}^4 F(x^{(0)} + \mathbf{g}); \quad \sum_{v=5}^7 \hat{\mathcal{F}}_v(x^{(0)} + \mathbf{g}) \cdot L_v = \mathbf{H}$$

and so by (61) we conclude the proof of the claim.

Step 3: Proving the convergence of the Newton's method.

By the above construction,

$$\begin{aligned} \|\mathbf{y}^{(j)}\|_{\mathcal{Y}_s^\tau} &\leq \tau^{b_j} \sum_{n \geq 0} \frac{\tau^n C^n}{(1-s)^n} < \frac{1}{2j} \left(1 + \frac{1}{3\alpha}\right), \\ \|\mathbf{w}^{(j)}\|_{\mathcal{X}_s^\tau} &\leq \frac{1}{2j} \sum_{n \geq 1} \frac{\tau^n C^n}{(1-s)^n} \leq \frac{1}{3\alpha 2j}. \end{aligned}$$

Thus $x^{(0)} + \mathbf{g}^{(j)}$ has a limit $x^{(0)} + \mathbf{g}^\infty$ in \mathcal{X}_s^τ and $F(x^{(0)} + \mathbf{g}^\infty) = 0$.

Let us review the choices of various constants that we have made. In the beginning of this section we have chosen and fixed α large enough depending on the initial data (namely, on $\tilde{Q}_1(z, h)$). In Step 1 we have assumed that τ is small enough to satisfy (60), (67), (71), (73), (79), and (80) (which depend only on the initial data); then we have chosen and fixed C large enough to satisfy (78) and (67); a choice of C valid for one value of τ is also valid for smaller values of τ . In Step 2, in (81) we choose and fix s ; a choice of s valid for one value of τ will also be valid for smaller values of τ . Then we shrink τ if necessary to satisfy (85) and (87). We conclude that there exist s_* and τ_* such that for any $s < s_*$ and $\tau < \tau_*$, the equation $F(\mathbf{x}) = 0$ has a solution $\mathbf{x} \in \mathcal{X}_s^\tau$.

Finally, in view of Section 2 we have shown that E_0, \dots, E_{M-2}, T , and hence also $y(x, h)$ have the Gevrey growth condition. Recall that by e.g. [3, Exercises to §4.3] algebraic operations, as well as differentiation with respect to a holomorphic parameter, preserve the Gevrey growth condition; exponentiation of a Gevrey series gives a Gevrey series by Lemma 7.2 type of argument, and integration with respect to a parameter preserves Gevrey growth of a series obviously. Thus it follows from (6) that also $\psi(x, h)$ satisfies a Gevrey growth condition. This concludes the proof of Theorem 1.1.

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