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Algorithmic framework for group analysis of differential equations and its application to generalized Zakharov–Kuznetsov equations

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Abstract

In this paper, we explain in more details the modern treatment of the problem of group classification of (systems of) partial differential equations (PDEs) from the algorithmic point of view. More precisely, we revise the classical Lie algorithm of construction of symmetries of differential equations, describe the group classification algorithm and discuss the process of reduction of (systems of) PDEs to (systems of) equations with smaller number of independent variables in order to construct invariant solutions. The group classification algorithm and reduction process are illustrated by the example of the generalized Zakharov–Kuznetsov (GZK) equations of form $u_t + (F(u))_{xxx} + (G(u))_{xyy} + (H(u))_x = 0$. As a result, a complete group classification of the GZK equations is performed and a number of new interesting nonlinear invariant models which have non-trivial invariance algebras are obtained. Lie symmetry reductions and exact solutions for two important invariant models, i.e., the classical and modified Zakharov–Kuznetsov equations, are constructed. The algorithmic framework for group analysis of differential equations presented in this paper can also be applied to other nonlinear PDEs.

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1. Introduction

One of the most famous two-dimensional generalizations (together with Kadomtsev–Petviashvili equation) of the Korteweg–de Vries (KdV) equation is given by the Zakharov–Kuznetsov (ZK) equation

$$u_t + au_{xxx} + bu_{xyy} + cuu_x = 0. \tag{1}$$

It was first derived by Zakharov and Kuznetsov [41] to describe nonlinear ion-acoustic waves in a magnetized plasma. More precisely, they considered a plasma in a strong magnetic field, $\mathbf{B} = \boldsymbol{B}\hat{z}$, with cold ions and hot electrons ($T_e \gg T_i$). The ion motions are described by the following equations

$$n_t + \nabla \cdot (n\mathbf{u}) = 0, \quad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{e}{m_i}\nabla\phi + \mathbf{u} \times \Omega_i,$$
$$\nabla^2\phi = -4\pi e(n - n_e),$$

where n is number density of ions, $\mathbf{u} = (u, v, w)$ is ion velocity, m_i is ion mass, ϕ is electric potential, $\Omega_i = \frac{e\mathbf{B}}{m_i c}$ is scaled magnetic field and $n_e = \exp(\frac{e\phi}{KT_e})$.

After introducing the dimensionless variables and approximating the x-component of $\mathbf{u}$ by the polarization shift these equations look like

$$n_t - (n\phi_{tx})_x + (nw)_z = 0, \quad w_t = \phi_{tx}w_x + ww_z - \phi_z,$$
$$\alpha\phi_{xx} + \phi_{zz} = e^\phi - n.$$

Now, after a change of independent variables $\xi = \varepsilon^{1/2}(z - t)$, $\eta = \varepsilon^{1/2}x$, $\tau = \varepsilon^{3/2}t$ assuming a solution of the latter equations of the form

$$n = 1 + \sum_{j=1}^{\infty} \varepsilon^j n_j, \quad \phi = \sum_{j=1}^{\infty} \varepsilon^j \phi_j, \quad w = \sum_{j=1}^{\infty} \varepsilon^j w_j,$$

one gets that $n_1 = \phi_1 = w_1$ with ϕ_1 being solution of

$$\phi_{1,t} + \phi_1\phi_{1,\xi} + \frac{1}{2}(\phi_{1,\xi\xi\xi} + (1 + \alpha)\phi_{1,\xi\eta\eta}) = 0$$

which has the form (1).

In the more realistic situation in which the electrons are non-isothermal, Munro and Parkes [25,26] showed that, with an appropriate modified form of the electron number density proposed by Schamel [33], a reductive perturbation procedure leads to a modified form of the Zakharov–Kuznetsov (mZK1) equation, namely

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$$u_t + au^{1/2}u_x + bu_{xxx} + cu_{xyy} = 0.$$

One more modification of the Zakharov–Kuznetsov equation is given by Kakutani and Ono [21] who have shown that the modified KdV equation governs the propagation of Alfvén waves at a critical angle to the undisturbed magnetic field. The presence of the transverse dispersion has been physically attributed to the finite Larmor radius effects [14]. The resulting two-dimensional equation in this physical context is known as the modified Zakharov–Kuznetsov equation (mZK2) [4]

$$u_t + au^2u_x + u_{xxx} + u_{xyy} = 0.$$

Here, the sign of a cannot be forced to be definite by scaling considerations and the two signs of a correspond to different physical phenomenon. For example, the focusing equation can be derived as a model for the evolution of ion acoustic perturbations with a negative ion component, while the defocusing equation models the evolution of ion acoustic perturbations in a plasma with two negative ion components of different temperature.

In order to encompass as many physical applications as possible, many researchers consider dispersive models of the Zakharov–Kuznetsov (dZK) type equations of form [39]

$$u_t + a(u^n)_{xxx} + b(u^m)_{xyy} + c(u^k)_x = 0,$$

or even the generalized Zakharov–Kuznetsov (GZK) equations

$$u_t + (F(u))_{xxx} + (G(u))_{xyy} + (H(u))_x = 0 \tag{2}$$

with enough smooth functions $F(u)$, $G(u)$ and $H(u)$.

Many mathematical properties such as the stability or transverse instability of solitary-wave solutions, initial-boundary value problems, generalized Painlevé formulation, compactons and solitons and so on for the special cases of class (2) have been investigated exhaustively by many authors [2,4,6,7,10,15,19–21,25,26,32,34,35,37,39]. However, despite of great interest of researchers and importance of class (2), very few facts of its Lie symmetry structure and related topics are known. Therefore, for the sake of providing more information to understand the mathematical structures of the ZK-like equations, in this paper we will perform detailed group analysis for the class of GZK equations (2), where $F(u)$, $G(u)$ and $H(u)$ are arbitrary smooth (analytic) functions, $F_uG_uH_{uu} \neq 0$.

It is known that the Lie group analysis is a systematic and powerful method for handling partial differential equations (PDEs) [5,12,28,29]. Moreover, it forms a basis for many useful techniques in both pure and applied areas of mathematics, physics, mechanics, etc. For the PDEs, admitting symmetry is an essential part of their intrinsic nature. Based on the symmetries of a PDE, one can successively consider many other important properties of the equation such as integrability, conservation laws and linearizations, reducing equations and invariant solutions, fundamental solution and invariant numerical integrators and so on [3,5,9,12,13,23,28,29,36,38]. In general, for a single PDE one can directly implement the classical Lie method to compute the symmetries. However, for parameterized classes of PDEs (namely, equations containing arbitrary constants or functions), one cannot derive all the symmetries by direct usage of this method. One will face the so called group classification problem of PDEs, which is the keystone of group analysis of differential equations. Although this problem has been widely investigated for different

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subclasses of (2), many of the existing classifications are incomplete. In fact, one can find a huge number of recent papers on symmetry analysis of PDEs, including different generalizations of the Zakharov–Kuznetsov equation, where the group classification problem is solved incompletely or incorrectly and papers studying symmetries of some fixed equations with low physical motivation. There are also many papers on "preliminary group classification" where authors list some cases with new symmetries but do not claim that the general classification problem is solved. In many respects this can be explained to two main reasons: (i) many researchers do not incorporate the equivalence transformation theory to the classification problem; (ii) overdetermined systems of PDEs derived from the invariance criterion of parameterized PDEs under consideration often cannot be solved completely.

That is why, in this paper we will describe an algorithmic framework for group classification of (systems of) partial differential equations. More precisely, we revise the classical Lie method of construction of symmetries of differential equations in more details and write down the precise formulation of Ovsiannikov's algorithm [29] of group classification of a class (of systems) of differential equations by extending equivalence transformation theory and introducing a compatibility method for solving the overdetermined system of PDEs. Moreover, we describe a systematic way of reduction of (systems of) partial differential equations to (systems of) equations with smaller number of independent variables so that we can find all possible invariant solutions of (systems of) differential equations. We will illustrate this well-known theoretical background and algorithmic framework by the running example of the GZK equations (2).

Therefore, the purpose of this paper is two fold. On the one hand, we explain the modern treatment of the problem of group classification of (systems of) PDEs from the algorithmic point of view. On the other hand, we perform systematically the complete group classification and construct invariant reductions for the GZK equations (2). The rest of this paper is organized as follows. In Section 2, we describe the algorithmic framework for group classification of (systems of) PDEs and give an exhaustive algorithm of solving such problems. An efficient algorithm of constructing optimal systems of subalgebras of Lie symmetry algebras and invariant solutions of differential equations is also given. In Section 3, we investigate the equivalence transformations of the GZK equations (2). The complete group classification of class (2) is presented in Section 4 by using a compatibility method. Section 5 contains results on optimal systems of subalgebras of Lie symmetry algebras of two equations from class (2). Invariant solutions of the equations under consideration are also constructed. Finally, some conclusion and discussion are given in Section 6.

2. Algorithmic framework for group analysis of differential equations

2.1. *Computation of Lie symmetries of differential equations*

For construction of symmetries of differential equations we use Lie infinitesimal criterion of invariance [28,29].

Consider the system $\mathcal{L}$: $L(x, u_{(p)}) = 0$ of l differential equations for m unknown functions $u = (u^1, \dots, u^m)$ of n independent variables $x = (x_1, \dots, x_n)$. Here $u_{(p)}$ denotes the set of all the derivatives of u with respect to x of order not greater than p, including u as the derivatives of the zero order. $L = (L^1, \dots, L^l)$ is a tuple of l fixed functions depending on x and $u_{(p)}$.

Let $\mathcal{L}_{(k)}$ denote the set of all algebraically independent differential consequences of the system $\mathcal{L}$ that have, as differential equations, orders not greater than k. Under the local approach, the system $\mathcal{L}_{(k)}$ is identified with the manifold determined by $\mathcal{L}_{(k)}$ in the jet space $J^{(k)}$.

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Each one-parameter group of point transformations that leaves the system $\mathcal{L}$ invariant corresponds to an infinitesimal symmetry operator of the form

$$Q = \xi^i(x,u)\partial_{x_i} + \eta^a(x,u)\partial_{u^a}.$$

Here and below the summation over the repeated indices is assumed. The indices i and a run from 1 to n and from 1 to m, respectively.

The infinitesimal criterion of invariance of the system $\mathcal{L}$ with respect to the Lie symmetry operator Q has the form

$$Q_{(p)}L(x,u_{(p)})\big|_{\mathcal{L}_{(p)}} = 0, \quad \text{where} \quad Q_{(p)} := Q + \sum_{0<|\alpha|\leqslant p} \eta^{a\alpha}\partial_{u^a_\alpha},$$

i.e., the result of acting by $Q_{(p)}$ on L vanishes on the manifold $\mathcal{L}_{(p)}$. Here $Q_{(p)}$ denotes the standard p-th prolongation of the operator Q, coefficient $\eta^{a\alpha} = D_1^{\alpha_1}\dots D_n^{\alpha_n}Q[u^a] + \xi^i u^a_{\alpha,i}$, operator $D_i = \partial_i + u^a_{\alpha,i}\partial_{u^a_\alpha}$ is the operator of total differentiation with respect to the variable x_i, and $Q[u^a] = \eta^a(x,u) - \xi^i(x,u)u^a_i$ is the characteristic of operator Q, associated with u^a. The tuple $\alpha = (\alpha_1,\dots,\alpha_n)$ is a multi-index, $\alpha_i \in \mathbb{N}\cup\{0\}$, $|\alpha| := \alpha_1+\dots+\alpha_n$. The variables u^a_α and $u^a_{\alpha,i}$ of the jet space $J^{(r)}$ correspond to the derivatives

$$\frac{\partial^{|\alpha|}u^a}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}} \quad \text{and} \quad \frac{\partial^{|\alpha|+1}u^a}{\partial x_1^{\alpha_1}\dots\partial x_{i-1}^{\alpha_{i-1}}\partial x_i^{\alpha_i+1}\partial x_{i+1}^{\alpha_{i+1}}\dots\partial x_n^{\alpha_n}} \quad \text{respectively.}$$

Example 1. For equations of form (2) we look for the infinitesimal symmetry generator in form

$$Q = \tau(t,x,y,u)\partial_t + \xi(t,x,y,u)\partial_x + \zeta(t,x,y,u)\partial_y + \eta(t,x,y,u)\partial_u.$$

(In the above notation $(t,x,y) = (x_0,x_1,x_2)$ and $\tau = \xi^0$, $\xi = \xi^1$, $\zeta = \xi^2$.) Application of the Lie infinitesimal criterion to (2) gives

$$\begin{aligned}
&\eta^t + \eta F_{uuuu}u_x^3 + 3\eta^x F_{uuu}u_x^2 + 3\eta F_{uuu}u_x u_{xx} + 3F_{uu}(\eta^x u_{xx} + \eta^{xx}u_x) + \eta F_{uu}u_{xxx}\\
&\quad + \eta^{xxx}F_u + \eta G_{uuuu}u_x u_y^2 + G_{uu}(\eta^x u_y^2 + 2\eta^y u_x u_y) + 2\eta G_{uuu}u_y u_{xy}\\
&\quad + 2G_{uu}(\eta^y u_{xy} + \eta^{xy}u_y) + \eta G_{uuu}u_x u_{yy} + G_{uu}(\eta^x \eta_{yy} + \eta^{yy}u_x)\\
&\quad + \eta G_{uu}u_{xyy} + \eta^{xyy}G_u + \eta H_{uu}u_x + H_u\eta^x = 0.
\end{aligned}$$

One can also verify that the coefficients of the first prolongation of Q look like

$$\begin{aligned}
\eta^t &= \eta_t + \eta_u u_t - u_t(\tau_t + \tau_u u_t) - u_x(\xi_t + \xi_u u_t) - u_y(\zeta_t + \zeta_u u_t),\\
\eta^x &= \eta_x + \eta_u u_x - u_t(\tau_x + \tau_u u_x) - u_x(\xi_x + \xi_u u_x) - u_y(\zeta_x + \zeta_u u_x),\\
\eta^y &= \eta_y + \eta_u u_y - u_t(\tau_y + \tau_u u_y) - u_x(\xi_y + \xi_u u_y) - u_y(\zeta_y + \zeta_u u_y).
\end{aligned}$$

In an analogous way the higher order coefficients can be found.

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Example 2. Consider the Lie symmetries of the ZK equation

$$u_t + au_{xxx} + bu_{xyy} + cuu_x = 0.$$

Applying the Lie infinitesimal criterion to the above equation we get

$$\eta^t + a\eta^{xxx} + b\eta^{xyy} + c\eta u_x + cu\eta^x = 0.$$

Substituting the coefficients of the prolongation of the operator Q into the above equation and splitting it with respect to the unconstrained variables, we obtain the system of determining equations

$$\begin{aligned}
&\tau_t = 3\xi_x, \quad \tau_x = 0, \quad \tau_y = 0, \quad \tau_u = 0;\\
&\xi_t = c\eta + 2cu\xi_x, \quad \xi_{xx} = 0, \quad \xi_y = 0, \quad \xi_u = 0;\\
&\zeta_t = 0, \quad \zeta_x = 0, \quad \zeta_y = \xi_x, \quad \zeta_u = 0;\\
&\eta_{uu} = 0, \quad \eta_t + a\eta_{xxx} + b\eta_{xyy} + cu\eta_x = 0.
\end{aligned}$$

The general solution of this system is

$$\tau = c_1 + c_2 t, \quad \xi = \frac{1}{3}c_2 x + c_3 t + c_4, \quad \zeta = \frac{1}{3}c_2 y + c_5, \quad \eta = -\frac{2}{3}c_2 u + \frac{1}{c}c_3,$$

where $c_1, \dots, c_5$ are arbitrary constants. Therefore the most general infinitesimal symmetry generator has the form

$$Q = (c_1 + c_2 t)\partial_t + \Big(\frac{1}{3}c_2 x + c_3 t + c_4\Big)\partial_x + \Big(\frac{1}{3}c_2 y + c_5\Big)\partial_y + \Big(-\frac{2}{3}c_2 u + \frac{1}{c}c_3\Big)\partial_u$$

As $c_1, \dots, c_5$ are arbitrary constants, we conclude that the maximal Lie invariance algebra of the equation under consideration is 5-dimensional and is spanned by the following generators

$$\langle \partial_t,\ \partial_x,\ \partial_y,\ 3t\partial_t + x\partial_x + y\partial_y - 2u\partial_u,\ ct\partial_x + \partial_u \rangle.$$

Example 3. Consider the Lie symmetries of the modified ZK1 equation:

$$u_t + au^{\frac{1}{2}}u_x + bu_{xxx} + cu_{xyy} = 0.$$

Application of the Lie infinitesimal criterion to the above equation gives

$$\eta^t + \frac{1}{2}a\eta u^{-\frac{1}{2}}u_x + a\eta^x u^{\frac{1}{2}} + b\eta^{xxx} + c\eta^{xyy} = 0,$$

or, multiplying it by two for convenience,

$$2\eta^t u^{\frac{1}{2}} + a\eta u_x + 2a\eta^x u + 2b\eta^{xxx}u^{\frac{1}{2}} + 2c\eta^{xyy}u^{\frac{1}{2}} = 0.$$

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Substituting the coefficients of the prolongation of the operator Q into the above equation, we obtain the system of determining equations

$$\begin{aligned}
&\tau_x = 0, \quad \tau_y = 0, \quad \tau_u = 0, \quad \tau_t = 3\xi_x, \quad \tau_{tt} = 0;\\
&\xi_y = 0, \quad \xi_u = 0, \quad \xi_{xx} = 0, \quad \xi_{xt} = 0;\\
&\zeta_t = 0, \quad \zeta_x = 0, \quad \zeta_u = 0, \quad \zeta_y = \xi_x;\\
&\eta_{xu} = 0, \quad \eta_{yu} = 0, \quad \eta_{uu} = 0, \quad a\eta - 2u^{\frac{1}{2}}\xi_t + 4au\xi_x = 0,\\
&\eta_t + au^{\frac{1}{2}}\eta_x + b\eta_{xxx} + c\eta_{xyy} = 0.
\end{aligned}$$

Its general solution is

$$\tau = c_1 + c_2 t, \quad \xi = \frac{1}{3}c_2 x + c_3, \quad \zeta = \frac{1}{3}c_2 y + c_4, \quad \eta = -\frac{4}{3}c_2 u,$$

where $c_1, \dots, c_4$ are arbitrary constants. Therefore the most general infinitesimal symmetry operator has the form

$$Q = (c_1 + c_2 t)\partial_t + \left(\frac{1}{3}c_2 x + c_3\right)\partial_x + \left(\frac{1}{3}c_2 y + c_4\right)\partial_y - \frac{4}{3}c_2 u\partial_u,$$

that implies that the Lie algebra of infinitesimal symmetry generators can be represented as

$$\langle \partial t,\ \partial x,\ \partial y,\ 3t\partial t + x\partial x + y\partial y - 4u\partial u\rangle.$$

In the completely similar way, for the modified ZK2 equation:

$$u_t + au^2 u_x + u_{xxx} + u_{xyy} = 0$$

application of the Lie infinitesimal criterion gives $\eta^t + \eta^{xxx} + g\eta^{xyy} + 2h\eta u u_x + h\eta^x u^2 = 0$. Substituting the coefficients of the prolongation of the operator Q into the above equation, we obtain the system of determining equation

$$\begin{aligned}
&\tau_x = 0, \quad \tau_y = 0, \quad \tau_u = 0, \quad \tau_t = 3\xi_x, \quad \tau_{tt} = 0;\\
&\xi_y = 0, \quad \xi_u = 0, \quad \xi_{xx} = 0, \quad \xi_{xt} = 0;\\
&\zeta_t = 0, \quad \zeta_x = 0, \quad \zeta_u = 0, \quad \zeta_y = \xi_x;\\
&\eta_{xu} = 0, \quad \eta_{yu} = 0, \quad \eta_{uu} = 0, \quad 2a\eta u + a(\tau_t - \xi_x)u^2 - \xi_t = 0,\\
&\eta_t + au^2\eta_x + \eta_{xxx} + \eta_{xyy} = 0.
\end{aligned}$$

General solution of the above system supply us with the four-dimensional Lie algebra of infinitesimal symmetry generators

$$\langle \partial_t,\ \partial_x,\ \partial_y,\ 3t\partial_t + x\partial_x + y\partial_y - u\partial_u\rangle.$$

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As in the above example it appears that in the most of cases computation of symmetries for a single equation is an algorithmic and simple exercise, which can be easily done by direct computing or by many standard mathematical software such as MAPLE, MATHEMATICA, MuLie and so on. In contrast to this, for parametric classes of differential equations, an exhaustive investigation of symmetries is usually a very difficult task, that requires to solve the so-called group classification problem.

2.2. *Group classification of classes of differential equations*

One of the most famous problems of group analysis of differential equations is the group classification which is one of the symmetry methods used to choose physically relevant models from parametric classes of systems of (partial or ordinary) differential equations [29]. The parameters can be constants or functions of independent variables, unknown functions and their derivatives. Solving this problem is interesting not only from mathematical point of view, but is also important for applications. In physical models there often exist a priori requirements to symmetry groups that follow from physical laws (in particular, from the Galilei or relativistic theories). In such cases the natural choice for the first try of modeling equation is the equation satisfying the following property. The modeling differential equation has to admit a group with certain properties or the richest symmetry group among the possible ones.

Exhaustive consideration of the problem of group classification in (its classical formulation) for a parametric class $\mathcal{L}$ of systems of differential equations includes the following steps:

1. Finding the group $G^{\cap}$ of local transformations that are symmetries for all systems from $\mathcal{L}$.
2. Construction of the group $G^{\sim}$ (the equivalence group) of local transformations which transform $\mathcal{L}$ into itself.
3. Description of all possible $G^{\sim}$-inequivalent values of parameters that admit maximal invariance groups wider than $G^{\cap}$.

Following S. Lie, one usually considers infinitesimal transformations instead of finite ones. In such a way the problem of group classification can be simplified to a problem for Lie algebras of vector fields (infinitesimal generators of symmetry groups). Thus the group classification in a class of differential equations is reduced to integration of an overdetermined system of partial differential equations with respect to both coefficients of infinitesimal symmetry operators and arbitrary elements. That is why it is much more complicated problem than finding the Lie symmetry group of a single differential equation.

Below we present the classical algorithm of group classification restricting ourselves, for simplicity, to the case of one differential equation of the form

$$L^{\theta}(x, u_{(n)}) = L(x, u_{(n)}, \theta(x, u_{(n)})) = 0. \tag{3}$$

Here $x = (x_1, \ldots, x_l)$ denotes independent variables, u is a dependent variable, $u_{(n)}$ is the set of all the partial derivatives of the function u with respect to x of order no greater than n, including u as the derivative of zero order. L is a fixed function of x, $u_{(n)}$ and θ. θ denotes the set of arbitrary (parametric) functions $\theta(x, u_{(p)}) = (\theta^1(x, u_{(p)}), \ldots, \theta^k(x, u_{(p)}))$ satisfying the conditions

$$S(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) = 0, \quad S = (S_1, \ldots, S_r). \tag{4}$$

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These conditions consist of r differential equations on θ, where x and $u_{(p)}$ play the role of independent variables and $\theta_{(q)}$ stands for the set of all the partial derivatives of θ of order no greater than q. In what follows we call the functions $\theta(x, u_{(p)})$ arbitrary elements. Sometimes this set is additionally constrained by the non-vanish condition $S'(x, u(p), \theta_{(q)}(x, u(p))) \neq 0$ with another tuple S' of differential functions. Denote the class of equations of form (3) with the arbitrary elements θ satisfying the constraint (4) as $L|_S$.

Example 4. For the class (2) of the generalized Zakharov–Kuznetsov equation, in the above notation $p = 0$, the set of arbitrary elements $\theta(x, u_{(p)})$ consists of functions F, G and H satisfying system of differential equations

$$F_t = 0, \quad F_x = 0, \quad G_t = 0, \quad G_x = 0, \quad H_t = 0, \quad H_x = 0, \tag{5}$$

and system of inequalities

$$F_u \neq 0, \quad G_u \neq 0, \quad H_{uu} \neq 0. \tag{6}$$

Note 1. The main idea of group classification process is very straightforward. Technically we work with class (3) like it is an equation. To find its symmetries, one just applies Lie's criterion and writes down the determining equations. This will give conditions on θ that must be satisfied to guarantee that the equation admits non-trivial symmetries. These conditions are usually formulated as a system of differential equations on θ. In the simplest cases these equations can be easily solved directly. In more difficult situation, as it was proposed by Ovsiannikov [29], one is forced to use equivalence transformations to analyze, simplify and solve. Below we concentrate on a rigorous formulation of group classification algorithm for such less trivial case.

Let the functions θ be fixed. Each one-parameter group of local point transformations that leaves equation (3) invariant corresponds to an infinitesimal symmetry operator of form

$$Q = \xi^a(x, u)\partial_{x_a} + \eta(x, u)\partial_u.$$

The complete set of such groups generates the principal group $G^{\max} = G^{\max}(L, \theta)$ of equation (3). The principal group $G^{\max}$ has a corresponding Lie algebra $A^{\max} = A^{\max}(L, \theta)$ of infinitesimal symmetry operators of equation (3). The group $G^{\cap}$ of local transformations that are symmetries for all systems is

$$G^{\cap} = G^{\cap}(L, S) = \bigcap_{\theta: S(\theta)=0} G^{\max}(L, \theta)$$

with the corresponding Lie algebra of form

$$A^{\cap} = A^{\cap}(L, S) = \bigcap_{\theta: S(\theta)=0} A^{\max}(L, \theta).$$

Let $G^{\sim} = G^{\sim}(L, S)$ denote the point transformations group preserving the form of equations from $L|_S$ (group of equivalence transformations). In other words, $G^{\sim}$ maps any equation from class $L|_S$ to an equation (possibly, another one) from the same class.

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Note 2. Sometimes one considers a subgroup instead of the complete equivalence group, e.g., subgroup of scalar transformations or of continuous transformations of the complete equivalence group. This is because the group of continuous transformations can be easily found using infinitesimal method, and to find the complete equivalence group one needs to apply more cumbersome and sophisticated direct method of construction of equivalence transformations.

Note 3. In the simplest cases it is possible to solve the group classification problem without explicit construction of equivalence group. Indeed, one can simply apply Lie's method to the general equation and obtain arbitrary elements for which the symmetry group is non-trivial. However, in cases when equations under consideration have more or less difficult nonlinear structure, it is impossible to solve determining equations (that often are very cumbersome and complicated) without extensive use of equivalence transformations (see [18] and references therein for more details and examples).

Example 5. To find the connected component of the identity of the equivalence group $G^{\sim}$ of class (2), i.e., the subgroup of continuous equivalence transformations, we have to investigate Lie symmetries of the system that consists of equation (2) and additional conditions (5) subject to (6). In other words we must seek for an operator from $G^{\sim}$ in the form

$$X = \tau\partial_t + \xi\partial_x + \zeta\partial_y + \eta\partial_u + \pi\partial_F + \rho\partial_G + \theta\partial_H \tag{7}$$

using the infinitesimal invariance criterion applied to the system

$$\begin{aligned} &u_t + (F(u))_{xxx} + (G(u))_{xyy} + (H(u))_x = 0, \\ &F_t = F_x = F_y = 0, \quad G_t = G_x = G_y = 0, \quad H_t = H_x = H_y = 0, \end{aligned} \tag{8}$$

subject to conditions (6). Here u, F, G and H are considered as differential variables: u on the space (t, x, y) and F, G, H on the extended space (t, x, y, u). The coordinates τ, ξ, ζ and η of the operator (7) are sought as functions of t, x, y and u while the coordinates π, ρ and θ are sought as functions of t, x, y, u, F, G, H.

The problem of group classification consists in finding all possible inequivalent cases of extensions of $A^{\max}$, i.e. in a listing all $G^{\sim}$-inequivalent values of θ that satisfy equation (4) and the condition $A^{\max}(L,\theta) \neq A^{\cap}$.

In the approach used here group classification is application of the following algorithm due to Ovsiannikov [29]:

1. From the infinitesimal Lie invariance criterion [28,29] we find the system of determining equations for the coefficients of the infinitesimal generator Q. It is possible that some of the determining equations do not contain arbitrary elements and therefore can be integrated immediately. The equations containing arbitrary elements explicitly are called classifying equations.
2. We decompose the determining equations with respect to all unconstrained derivatives of arbitrary elements. This gives a system of partial differential equations for coefficients of the infinitesimal operator Q only. Solving this system yields the algebra $A^{\cap}$ of point transformations that are symmetries for all equations from $L|_S$.

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3. To construct the equivalence group $G^{\sim}$ of the class $L|_S$ one has to investigate the point symmetry transformations of system (3), (4), considering it as a system of partial differential equations with respect to θ with the independent variables $x,\ u_{(n)}$. If we restrict ourselves to studying the connected component of the identity in $G^{\sim}$, (i.e., finding continuous equivalence transformations only), the Lie infinitesimal method of finding symmetries of this system can be applied. To find the complete equivalence group (including discrete transformations) one has to use the direct method.
4. If $A^{\max}$ is an extension of $A^{\cap}$ (i.e. in the case $A^{\max}(L,\theta) \neq A^{\cap}$), then the classifying equations define a system of non-trivial equations for arbitrary elements θ. Depending on their form and number, we obtain different cases of extensions of $A^{\cap}$.

Note 4. To integrate completely the determining equations often it is necessary to investigate a large number of different cases of extensions of $A^{\cap}$. There exist different methods allowing to avoid cumbersome enumeration of possibilities in solving the determining equations. To solve determining equations for the coefficients of symmetry generators of Zakharov–Kuznetsov equations we use a method which involves the investigation of compatibility of the classifying equations [27,31].

The result of application of the above algorithm is a list of equations with their Lie invariance algebras. The problem of group classification is assumed to be completely solved if

1. the list contains all possible inequivalent cases of extensions;
2. all equations from the list are mutually inequivalent with respect to the transformations from $G^{\sim}$;
3. the obtained algebras are the maximal invariance algebras of the respective equations.

Such list may include equations being mutually equivalent with respect to point transformations which do not belong to $G^{\sim}$. Knowing such additional equivalences allows to simplify essentially further investigation of $L|_S$. Constructing them sometimes is considered as the fifth step of the algorithm of group classification. Then, the above enumeration of requirements to the resulting list of classification can be completed by the following step:

4. all possible additional equivalences between the listed equations are constructed in explicit form.

See, e.g., [31] for more details.

When the symmetry group is known, a wide range of applications becomes available, e.g., construction of invariant solutions of nonlinear equations. Indeed, group analysis is one of very few systematic methods known for deducing exact solutions of nonlinear partial differential equations.

2.3. *Invariant solutions of differential equations*

Although Lie symmetry analysis does not help to construct general solutions of systems of nonlinear PDEs it often gives an approach to deduct wide classes of solutions being invariant with respect to different subgroups of the Lie symmetry group. Roughly speaking, the main theorem on invariant solutions [28,29] claims that all solutions invariant with respect to

r-parametric group of symmetries (with some restrictions on the form of the algebra) of the given n-dimensional system can be obtained by solving a system of differential equation with $n-r$ independent variables. In particular, if $r=n-1$, invariant solutions can be constructed via solving a system of ordinary differential equations.

Example 6. Generalized Zakharov–Kuznetsov equations (2) are three-dimensional. Therefore, to reduce an equation of form (2) to ordinary differential equation we have to use two-dimensional subalgebras of its symmetry algebra. Reduction of (2) with respect to one-dimensional symmetry algebras provides us with the two-dimensional reduced equations. Note, that the reduction with respect to the three-dimensional subalgebras leads to algebraic equations.

To construct solutions of a system of partial differential equations invariant with respect to r-dimensional symmetry algebra spanned by symmetry generators $v_i = \xi_j^i \partial_i + \eta_j \partial_{u^j}$, $j = 1, \dots, r$, we need to solve a system of r first-order PDEs:

$$\xi_j^i u_i = \eta_j, \quad j = 1, \dots, n.$$

Solution of this system provides us with expressions for $n-r$ new independent variables and Ansatz for the dependent variables. Substituting this to the initial system we obtain a system of differential equations with $n-r$ independent variables. A detailed example of implementation of this method will be given in Section 5.

Note 5. The above mentioned procedure works only if the symmetry algebra satisfies the property of transversality. For more details see [28,29].

As we have already noticed, in general “almost every” subgroup of a Lie symmetry group corresponds to a class of invariant solutions. Since almost always there exist an infinite number of such subgroups, often it is practically impossible to list all invariant solutions. Therefore one needs an effective systematic tool of their classification that gives an “optimal system” of such solutions, from where one can find all possible invariant solutions.

Any two conjugate subgroups of a Lie symmetry group of a system of PDEs give rise to reduced equations that are related by a conjugacy transformation in the point symmetry group of the system acting on the invariant solutions determined by each subgroup. Hence, up to the action of the point symmetry transformations, all invariant solutions for a given system can be obtained by selecting a subgroup in each conjugacy class of all admitted point symmetry subgroups. Such a selection is called an optimal set of subgroups [29]. A set of subalgebras of the Lie symmetry algebra corresponding to the optimal set of subgroups consists of subalgebras inequivalent with respect to the action of adjoint representation of the Lie symmetry group on its Lie algebra.

An effective algorithm of construction of optimal systems of subalgebras of Lie algebras is given in [29] (see also a simpler explanation and examples for one-dimensional subalgebras in [28]). In Section 5 we illustrate this algorithm by examples of the usual and modified Zakharov–Kuznetsov equations.

3. Equivalence transformations of the GZK equations

We start the group classification procedure of class (2) of the GZK equations from investigating its group of equivalence transformations.

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First, we describe construction of the continuous equivalence transformations. As it is stated in Example 5, to reach this goal one has to investigate Lie symmetries of system (8). Application of the infinitesimal invariance criterion to system (8) yields the following determining equations for τ, ξ, ζ, η, π, ρ and θ:

$$\begin{aligned}
&\tau_x=\tau_y=\tau_u=0, \quad \xi_y=\xi_u=\xi_{xx}=0,\\
&\zeta_t=\zeta_x=\zeta_u=\zeta_{yy}=0, \quad \eta_t=\eta_x=\eta_y=0,\\
&\pi_t=\pi_x=\pi_y=\pi_u=\pi_G=\pi_H=\pi_{FF}=0, \quad \pi_F-\eta_u+\tau_t-3\xi_x=0,\\
&\rho_t=\rho_x=\rho_y=\rho_u=\rho_F=\rho_H=\rho_{GG}=0, \quad \rho_G-\eta_u+\tau_t-\xi_x-2\zeta_y=0,\\
&\theta_t=\theta_x=\theta_y=\theta_F=\theta_G=0, \quad \theta_u-\xi_t=0, \quad \theta_H-\eta_u+\tau_t-\xi_x=0.
\end{aligned} \tag{9}$$

After easy calculations from (9), we find the coefficients of the infinitesimal operators of continuous equivalent transformations of class (2) have the form

$$\begin{aligned}
&\tau=c_1+c_5t, \quad \xi=c_2+c_6x+c_9t, \quad \zeta=c_3+c_7y, \quad \eta=c_4+c_8u,\\
&\pi=(c_5-3c_6-c_8)F+c_{11}, \quad \rho=(c_5-c_6-2c_7-c_8)G+c_{12},\\
&\theta=(c_5-c_6-c_8)H+c_9u+c_{10}.
\end{aligned}$$

Then the Lie algebra of the equivalence group $G^{\sim}$ for class (2) is

$$\begin{gathered}
A^{\sim}=\langle\partial_t,\,\partial_x,\,\partial_y,\,\partial_u,\,t\partial_t+F\partial_F+G\partial_G+H\partial_H,\,x\partial_x-3F\partial_F-G\partial_G-H\partial_H,\\
y\partial_y-2G\partial_G,\,u\partial_u-F\partial_F-G\partial_G-H\partial_H,\,t\partial_x+u\partial_H,\,\partial_F,\,\partial_G,\,\partial_H\rangle.
\end{gathered}$$

Now, to recover the formulas for the transformations of variables one needs to solve the so-called Lie equations (see, e.g., [28,29]). Namely, for any infinitesimal operator $X=\tau\partial_t+\xi\partial_x+\zeta\partial_y+\eta\partial_u+\pi\partial_F+\rho\partial_G+\theta\partial_H$ the corresponding finite transformations of variables can be found from solving the system

$$\begin{aligned}
&\frac{d\tilde t}{d\varepsilon}=\tau, \quad \tilde t|_{\varepsilon=0}=t,\\
&\frac{d\tilde x}{d\varepsilon}=\xi, \quad \tilde x|_{\varepsilon=0}=x,\\
&\dots\\
&\frac{d\tilde H}{d\varepsilon}=\theta, \quad \tilde H|_{\varepsilon=0}=H.
\end{aligned}$$

Solving this system for every infinitesimal equivalence generator and gathering all 12 results together, we get that the group $G^{\sim}_{\rm cont}$ of *continuous* equivalence transformations of class (2) is as follows:

$$\tilde t = e^{\varepsilon_1} t + \varepsilon_6, \quad \tilde x = e^{\varepsilon_2} x + e^{\varepsilon_5} t + \varepsilon_7, \quad \tilde y = e^{\varepsilon_3} y + \varepsilon_8, \quad \tilde u = e^{\varepsilon_4} u + \varepsilon_9,$$
$$\tilde F = e^{-\varepsilon_1 + 3\varepsilon_2 + \varepsilon_4} F + \varepsilon_{10}, \quad \tilde G = e^{-\varepsilon_1+\varepsilon_2+2\varepsilon_3+\varepsilon_4} G + \varepsilon_{11},$$
$$\tilde H = e^{-\varepsilon_1+\varepsilon_2+\varepsilon_4} H + e^{-\varepsilon_1+\varepsilon_4+\varepsilon_5} u + \varepsilon_{12}.$$

To find the *complete* equivalence group of class (2) we use the direct method. More precisely, we look for transformations of form

$$\tilde t = \tilde t(t,x,y,u), \quad \tilde x = \tilde x(t,x,y,u), \quad \tilde y = \tilde y(t,x,y,u), \quad \tilde u = \tilde u(t,x,y,u)$$

that relates equations $u_t + (F(u))_{xxx} + (G(u))_{xyy} + (H(u))_x = 0$ and $\tilde u_{\tilde t} + (\tilde F(\tilde u))_{\tilde x\tilde x\tilde x} + (\tilde G(\tilde u))_{\tilde x\tilde y\tilde y} + (\tilde H(\tilde u))_{\tilde x} = 0$.

Now we have to express the non-transformed variables in terms of the "tilded" ones and substitute them into equation (2). Requiring that the obtained equation belongs to class (2) we get a polynomial equation with respect to the derivatives of u. Setting to zero its coefficients with respect to the unconstrained variables we obtain a system of overdetermined partial differential equations, general solution of which gives that the most general form of the transformation from the complete equivalence group $G^\sim$ of class (2) is

$$\tilde t = \varepsilon_1 t + \varepsilon_6, \quad \tilde x = \varepsilon_2 x + \varepsilon_5 t + \varepsilon_7, \quad \tilde y = \varepsilon_3 y + \varepsilon_8, \quad \tilde u = \varepsilon_4 u + \varepsilon_9,$$
$$\tilde F = \varepsilon_1^{-1}\varepsilon_2^3\varepsilon_4 F + \varepsilon_{10}, \quad \tilde G = \varepsilon_1^{-1}\varepsilon_2\varepsilon_3^2\varepsilon_4 G + \varepsilon_{11}, \quad \tilde H = \varepsilon_1^{-1}\varepsilon_2\varepsilon_4 H + \varepsilon_1^{-1}\varepsilon_4\varepsilon_5 u + \varepsilon_{12},$$

where $\varepsilon_1, \dots, \varepsilon_{12}$ are arbitrary constants, $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 \neq 0$.

For more details and examples of application of the direct method of finding of equivalence transformations see, e.g., [22].

Note 6. As one can see, the only discrete equivalence transformations for class (2) are alternating of signs of the dependent and independent variables (extended to the arbitrary elements). However, for many classes this is not the case, i.e., there can exist non-trivial complicated discrete equivalence transformations that can essentially simplify classification problem (see, e.g., [16–18] and references therein).

4. Lie symmetries of the GZK equations

We search for infinitesimal generators of Lie symmetries of equations of class (2) in form

$$Q = \tau(t,x,y,u)\partial_t + \xi(t,x,y,u)\partial_x + \zeta(t,x,y,u)\partial_y + \eta(t,x,y,u)\partial_u.$$

From the infinitesimal invariance criterion we obtain the following system of the determining equations for the coefficients of the infinitesimal generators:

$$\tau_x = \tau_y = \tau_u = \xi_y = \xi_u = \zeta_t = \zeta_x = \zeta_u = \eta_{uu} = 0,$$
$$(\eta_x G_u)_u = 0,$$
$$\eta_t + F_u\eta_{xxx} + G_u\eta_{xyy} + \eta_x H_u = 0,$$
$$(F_u\eta_x)_u - F_u\xi_{xx} = 0,$$

Table 1
Group classification of class (2).

N	$F(u)$	$G(u)$	$H(u)$	$A^{\max}$
1.	$\forall$	$\forall$	$\forall$	$A^\cap=\langle\partial_t,\,\partial_x,\,\partial_y\rangle$
2.	u^k	gu^m	hu^n	$A^\cap+\langle(k+2-3n)t\partial_t+(k-n)x\partial_x+(m-n)y\partial_y+2u\partial_u\rangle$
3.	u^k	gu^m	$hu\ln u$	$A^\cap+\langle(k-1)t\partial_t+(2ht+kx-x)\partial_x+(m-1)y\partial_y+2u\partial_u\rangle$
4.	u^k	gu^m	$h\ln u$	$A^\cap+\langle(k+2)t\partial_t+kx\partial_x+my\partial_y+2u\partial_u\rangle$
5.	$\ln u$	gu^m	hu^n	$A^\cap+\langle(3n-2)t\partial_t+nx\partial_x+(n-m)y\partial_y-2u\partial_u\rangle$
6.	$\ln u$	gu^m	$hu\ln u$	$A^\cap+\langle t\partial_t+(x-2ht)\partial_x-(m-1)y\partial_y-2u\partial_u\rangle$
7.	$\ln u$	gu^m	$h\ln u$	$A^\cap+\langle 2t\partial_t+my\partial_y+2u\partial_u\rangle$
8.	e^{ku}	ge^{mu}	he^{nu}	$A^\cap+\langle(k-3n)t\partial_t+(k-n)x\partial_x+(m-n)y\partial_y+2\partial_u\rangle$
9.	e^{ku}	ge^{mu}	hu^2	$A^\cap+\langle kt\partial_t+(kx+4ht)\partial_x+my\partial_y+2\partial_u\rangle$
10.	u	ge^{mu}	he^{nu}	$A^\cap+\langle 3nt\partial_t+nx\partial_x+(n-m)y\partial_y-2\partial_u\rangle$
11.	u	ge^{mu}	hu^2	$A^\cap+\langle 4ht\partial_x+my\partial_y+2\partial_u\rangle$
12.	u^k	$g\ln u$	hu^n	$A^\cap+\langle(k-3n+2)t\partial_t+(k-n)x\partial_x-ny\partial_y+2u\partial_u\rangle$
13.	u^k	$g\ln u$	$h\ln u$	$A^\cap+\langle(k+2)t\partial_t+kx\partial_x+2u\partial_u\rangle$
14.	u^k	$g\ln u$	$hu\ln u$	$A^\cap+\langle(k-1)t\partial_t+(2ht+kx-x)\partial_x-y\partial_y+2u\partial_u\rangle$
15.	$\ln u$	$g\ln u$	hu^n	$A^\cap+\langle(3n-2)t\partial_t+nx\partial_x+ny\partial_y-2u\partial_u\rangle$
16.	$\ln u$	$g\ln u$	$h\ln u$	$A^\cap+\langle t\partial_t+u\partial_u\rangle$
17.	$\ln u$	$g\ln u$	$hu\ln u$	$A^\cap+\langle t\partial_t+(x-2ht)\partial_x+y\partial_y-2u\partial_u\rangle$
18.	e^{ku}	gu	he^{nu}	$A^\cap+\langle(k-3n)t\partial_t+(k-n)x\partial_x-ny\partial_y+2\partial_u\rangle$
19.	e^{ku}	gu	hu^2	$A^\cap+\langle kt\partial_t+(kx+4ht)x\partial_x+2\partial_u\rangle$
20.	u	gu	hu^2	$A^\cap+\langle 3t\partial_t+x\partial_x+y\partial_y-2u\partial_u,\,2ht\partial_x+\partial_u\rangle$
21.	u	$u^{-1/3}$	$u^{-1/3}$	$A^\cap+\langle 6t\partial_t+2x\partial_x+3u\partial_u,\,\cos 2y\partial_y+3u\sin 2y\partial_u,\,\sin 2y\partial_y-3u\cos 2y\partial_u\rangle$
22.	u	$-u^{-1/3}$	$-u^{-1/3}$	$A^\cap+\langle 6t\partial_t+2x\partial_x+3u\partial_u,\,\cos 2y\partial_y+3u\sin 2y\partial_u,\,\sin 2y\partial_y-3u\cos 2y\partial_u\rangle$
23.	u	$-u^{-1/3}$	$u^{-1/3}$	$A^\cap+\langle 6t\partial_t+2x\partial_x+3u\partial_u,\,e^{2y}(\partial_y-3u\partial_u),\,e^{-2y}(\partial_y+3u\partial_u)\rangle$
24.	u	$u^{-1/3}$	$-u^{-1/3}$	$A^\cap+\langle 6t\partial_t+2x\partial_x+3u\partial_u,\,e^{2y}(\partial_y-3u\partial_u),\,e^{-2y}(\partial_y+3u\partial_u)\rangle$

Here $g=\pm1$, $h=\pm1$.

$$
\begin{aligned}
&2(\eta_y G_u)_u-G_u\zeta_{yy}=0,\\
&\eta F_{uu}+(\tau_t-3\xi_x)F_u=0,\\
&\eta G_{uu}+(\tau_t-\xi_x-2\zeta_y)G_u=0,\\
&3(\eta_{xx}F_u)_u+(\eta_{yy}G_u)_u+\eta H_{uu}-\xi_{xxx}F_u+(\tau_t-\xi_x)H_u-\xi_t=0.
\end{aligned}\tag{10}
$$

Solving this system up to the equivalence group $G^\sim$ we obtain the complete group classification of class (2) that presented in Table 1.

Proof. To obtain the classification result we need to solve the system of determining equations (10). Integrating the equations that do not contain arbitrary elements we get

$$
\tau=\tau(t),\quad \xi=\xi(t,x),\quad \zeta=\zeta(y),\quad \eta=\eta^1(t,x,y)u+\eta^0(t,x,y).\tag{11}
$$

Splitting the rest of the system (10) with respect to the arbitrary elements and their nonvanishing derivatives gives the equations $\tau_t=0$, $\xi_t=\xi_x=0$, $\zeta_y=0$ and $\eta=0$ for the coefficients of the operators from $A^\cap$ of (2). As a result, we get case 1 of Table 1.

As one can see, for any symmetry operator, equations $\eta F_{uu} + (\tau_t - 3\xi_x)F_u = 0$ and $\eta G_{uu} + (\tau_t - \xi_x - 2\zeta_y)G_u = 0$ give some equations (not greater than 3 each) on F and G of the general form

$$(au+b)F_{uu} + cF_u = 0, \quad (au+b)G_{uu} + fG_u = 0,$$

where a, b, c and f are constants. Solving this system up to $G^\sim$, we obtain the following different values of $F(u)$ and $G(u)$: $F(u) = u^k$, $F(u) = e^{ku}$ or $F(u) = \ln u \bmod G^\sim$ and $G(u) = gu^m$, or $G(u) = ge^{mu}$ or $G(u) = g\ln u \bmod G^\sim$. Therefore, to complete the classification we have to consider all possible combinations of the values of $F(u)$ and $G(u)$. We attempted to present our calculations in reasonable detail so that verification would be feasible.

(I) $F = u^k$ and $G = gu^m \bmod G^\sim$, where $m \neq 0$, $g = \pm 1$. Substituting F, G and (11) into the rest of the equations (10) yields

$$\begin{gathered}\eta^1_x = \xi_{xx} = 0, \quad (m-1)\eta^0_x = (m-1)\eta^0_y = (k-1)\eta^0_x = 0, \quad (m-1)\eta^0 = (k-1)\eta^0 = 0,\\ 2m\eta^1_y - \zeta_{yy} = 0, \quad (k-1)\eta^1 + (\tau_t - 3\xi_x) = 0, \quad (m-1)\eta^1 + (\tau_t - \xi_x - 2\zeta_y) = 0. \end{gathered} \tag{12}$$

From the last three equations of this system we can get two classifying conditions:

$$(k-1)\eta^1_y = 0, \quad (3m+1)\eta^1_y = 0, \tag{13}$$

which can be decomposed into four cases:

$$(i)\ k \neq 1, m \neq -\frac{1}{3}; \quad (ii)\ k \neq 1, m = -\frac{1}{3}; \quad (iii)\ k = 1, m \neq -\frac{1}{3}; \quad (iv)\ k = 1, m = -\frac{1}{3}.$$

(i) For this case equations (12) and (13) imply $\eta^1_x = \eta^1_y = \xi_{xx} = \eta^0 = 0$. Hence, the equation for H in (10) looks like $\mu u H_{uu} + \nu H_u + \omega = 0$, where $\mu, \nu, \omega = \text{const}$. Therefore, $H(u)$ must take one of the following three values: $H(u) = hu^n$ or $H(u) = hu\ln u$ or $H(u) = h\ln u \bmod G^\sim$, where $h = \pm 1$. Substituting these three values in (10) and solving the obtained equations, we get cases 2, 3 and 4 respectively.

(ii) From $k \neq 1$ and (12) and (13) we obtain $\eta^1_x = \eta^1_y = \xi_{xx} = \eta^0 = 0$ that leads to the subcases of the previously obtained cases 2, 3 and 4.

(iii) With the restriction $m \neq 1$, we have again $\eta^1_x = \eta^1_y = \xi_{xx} = \eta^0 = 0$. This also leads to the subcases of cases 2, 3 and 4.

If $m = 1$, then from (12) and (13) we have $\eta^1_x = \eta^1_y = \xi_{xx} = 0$. Thus, from the equation for $H(u)$ we deduce that it takes one of the following three values: $H(u) = hu^n$ or $H(u) = hu\ln u$ or $H(u) = h\ln u \bmod G^\sim$, where $h = \pm 1$. Solving the rest of equations (10) and (12) with $H(u) = hu^n$, we can find that the only extension case corresponds to the values $n = 2$ and is tabulated as case 20. The values $H(u) = hu\ln u$ and $H(u) = h\ln u$ give the classification cases 3 and 4 with $k = m = 1$ respectively.

(iv) After obvious simple computations, we get cases 21–24.

(II) $F = u^k$ and $G = ge^{mu} \bmod G^\sim$, where $m \neq 0$, $g = \pm 1$. Substituting F, G and (11) into (10) we derive

$$\eta^1 = \xi_{xx} = 0, \quad \eta^0_x = \eta^0_t = \eta^0_{yy} = 0, \quad k(k-1)\eta^0 = 0,$$
$$\eta^0_y - \zeta_{yy} = 0, \quad \tau_t - 3\xi_x = 0, \quad m\eta^0 + (\tau_t - \xi_x - 2\zeta_y) = 0,$$
$$\eta^0 H_{uu} + (\tau_t - \xi_x)H_u - \xi_t = 0. \tag{14}$$

If $k \neq 1$, then $\eta^0 = 0$. Therefore, $H = \mathrm{const}$ that contradicts with the assumption $H_u \neq 0$. If $k = 1$, then the last equation of system (14) looks like $\mu H_{uu} + \nu H_u + \omega = 0$ with respect to H, where $\mu, \nu, \omega = \mathrm{const}$. Thus, $H(u)$ must take one of the following two values: $H(u) = he^{nu}$ or $H(u) = hu^2 \bmod G^\sim$, where $h = \pm 1$. Substituting these two values into system (14) and solving the obtained equations we get cases 10 and 11 respectively.

(III) $F = u^k$ and $G = g \ln u \bmod G^\sim$, where $g = \pm 1$. System (10) takes now the form

$$\eta^1_t = \eta^1_x = \eta^1_y = \xi_{xx} = 0, \quad \eta^0 = 0, \quad \zeta_{yy} = 0,$$
$$(k-1)\eta^1 + \tau_t - 3\xi_x = 0, \quad -\eta^1 + (\tau_t - \xi_x - 2\zeta_y) = 0,$$
$$\eta^1 u H_{uu} + (\tau_t - \xi_x)H_u - \xi_t = 0. \tag{15}$$

The last equation of system (15) looks like $\mu u H_{uu} + \nu H_u + \omega = 0$ with respect to H, where $\mu, \nu, \omega = \mathrm{const}$. Thus, $H(u)$ must be of one of the following three values: $H(u) = hu^n$ or $H(u) = h \ln u$ or $H(u) = hu \ln u \bmod G^\sim$, where $h = \pm 1$. Solving system (15) for these three values of H, we can obtain cases 12, 13 and 14 respectively.

(IV) $F = e^{ku}$ and $G = gu^m \bmod G^\sim$, where $m \neq 0$, $g = \pm 1$. From (10) we derive

$$\eta^1 = 0, \quad \xi_{xx} = \eta^0_t = \eta^0_x = \eta^0_y = 0, \quad \zeta_{yy} = 0, \quad m(m-1)\eta^0 = 0,$$
$$k\eta^0 + \tau_t - 3\xi_x = 0, \quad \tau_t - \xi_x - 2\zeta_y = 0,$$
$$\eta^0 H_{uu} + (\tau_t - \xi_x)H_u - \xi_t = 0. \tag{16}$$

The fourth and the last equations of system (16) imply $m = 1$, or $H = \mathrm{const} \bmod G^\sim$ which contradicts with the assumption that $H_u \neq 0$. Thus, $H(u)$ must take one of the following two values: $H(u) = he^{nu}$ or $H(u) = hu^2 \bmod G^\sim$, where $h = \pm 1$. Substituting these values of H into system (16) and solving it, we obtain cases 18 and 19 respectively.

(V) $F = e^{ku}$ and $G = ge^{mu} \bmod G^\sim$, $m \neq 0$, $g = \pm 1$. From (10) we derive

$$\eta^1 = 0, \quad \xi_{xx} = \eta^0_t = \eta^0_x = \eta^0_y = 0, \quad \zeta_{yy} = 0,$$
$$k\eta^0 + \tau_t - 3\xi_x = 0, \quad m\eta^0 + \tau_t - \xi_x - 2\zeta_y = 0,$$
$$\eta^0 H_{uu} + (\tau_t - \xi_x)H_u - \xi_t = 0. \tag{17}$$

The last equation of system (17) looks like $\mu H_{uu} + \nu H_u + \omega = 0$ with respect to H, where $\mu, \nu, \omega = \mathrm{const}$. Thus, up to $G^\sim$, $H(u)$ must take one of the following two values: $H(u) = he^{nu}$ or $H(u) = hu^2$, where $h = \pm 1$. Now from system (17) we obtain easily cases 8 and 9.

(VI) $F = \ln u$ and $G = ge^{mu}$ mod $G^{\sim}$, $m \neq 0$, $g = \pm 1$. From (10) we derive

$$\begin{aligned}
&\eta^0 = 0, \quad \xi_{xx} = \eta^1_t = \eta^1_x = \eta^1_y = 0, \quad \zeta_{yy} = 0, \\
&-\eta^1 + \tau_t - 3\xi_x = 0, \quad (m-1)\eta^1 + \tau_t - \xi_x - 2\zeta_y = 0, \\
&\eta^1 u H_{uu} + (\tau_t - \xi_x)H_u - \xi_t = 0.
\end{aligned} \tag{18}$$

The last equation of system (18) looks like $\mu u H_{uu} + \nu H_u + \omega = 0$ with respect to H, where $\mu, \nu, \omega = \text{const}$. Thus, $H(u)$ must take one of the following three values: $H(u) = hu^n$ or $H(u) = hu \ln u$ or $H(u) = h \ln u \bmod G^{\sim}$, where $h = \pm 1$. Solving now system (18) we obtain cases 5, 6 and 7 respectively.

(VII) $F = \ln u$ and $G = g \ln u$ mod $G^{\sim}$, where $g = \pm 1$. Substituting F, G and (11) into (10) we derive

$$\begin{aligned}
&\eta^0 = 0, \quad \xi_{xx} = \eta^1_t = \eta^1_x = \eta^1_y = 0, \quad \zeta_{yy} = 0, \\
&-\eta^1 + \tau_t - 3\xi_x = 0, \quad -\eta^1 + \tau_t - \xi_x - 2\zeta_y = 0, \\
&\eta^1 u H_{uu} + (\tau_t - \xi_x)H_u - \xi_t = 0.
\end{aligned} \tag{19}$$

Similar to (VI), from the last equation of system (19) we obtain that $H(u)$ must take one of the following three values: $H(u) = hu^n$ or $H(u) = h \ln u$ or $H(u) = hu \ln u \bmod G^{\sim}$, where $h = \pm 1$. These values correspond to cases 15, 16 and 17 respectively.

The problem of group classification of class (2) is completely solved. □

Note 7. Although equations (2) with $H_{uu} = 0$ are of low physical interest, below for completeness we adduce the result of their symmetry classification.

If $H_{uu} = 0$, equation (2) is equivalent to

$$u_t + (F(u))_{xxx} + (G(u))_{xyy} = 0. \tag{20}$$

Result of group classification of class (20) (up to the restriction of the equivalence group $G^{\sim}$ of class (2) to $H(u) = 0$) is presented in Table 2.

5. Exact solutions

In this section we construct exact solutions for some special forms of the GZK equations (2) by means of the classical Lie method and Ovsiannikov's method. We first give a detailed classification of optimal systems of subalgebras of the Lie algebra of the equations under consideration. Then we reduce the $(2+1)$-dimensional equations to $(1+1)$-dimensional equations or ordinary differentiate equation or algebraic equation according to the subalgebras system and the corresponding ansatz. Solving the reduced equations, we can finally obtain exact invariant solutions

Table 2
Group classification of class (20).

N	$F(u)$	$G(u)$	$A^{\max}$
1.	$\forall$	$\forall$	$A_0^{\cap} = \langle \partial_t,\ \partial_x,\ \partial_y,\ 3t\partial_t + x\partial_x + y\partial_y \rangle$
2.	u^k	gu^m	$A_0^{\cap} + \langle (k+2)t\partial_t + kx\partial_x + my\partial_y + 2u\partial_u \rangle$
3.	$\ln u$	gu^m	$A_0^{\cap} + \langle 2t\partial_t + my\partial_y + 2u\partial_u \rangle$
4.	e^{ku}	ge^{mu}	$A_0^{\cap} + \langle kt\partial_t + kx\partial_x + my\partial_y + 2\partial_u \rangle$
5.	u	ge^{mu}	$A_0^{\cap} + \langle my\partial_y + 2\partial_u \rangle$
6.	u^k	$g\ln u$	$A_0^{\cap} + \langle (k+2)t\partial_t + kx\partial_x + 2u\partial_u \rangle$
7.	$\ln u$	$g\ln u$	$A_0^{\cap} + \langle t\partial_t + u\partial_u \rangle$
8.	e^{ku}	gu	$A_0^{\cap} + \langle kt\partial_t + kx\partial_x + 2\partial_u \rangle$
9.	u	$gu^{-1/3}$	$A_0^{\cap} + \langle 2y\partial_y - 3u\partial_u,\ y^2\partial_y - 3yu\partial_u \rangle$
10.	u	gu	$A_0^{\cap} + \langle u\partial_u,\ \varphi\partial_u \rangle$

Here $g = \pm 1$, $\varphi = \varphi(t,x,y)$ is an arbitrary solution of $\varphi_t + \varphi_{xxx} + g\varphi_{xyy} = 0$.

of the original equation. Below, we illustrate this algorithm by two examples: the modified and classical Zakharov–Kuznetsov equations.

5.1. Modified Zakharov–Kuznetsov equation

Consider the modified Zakharov–Kuznetsov equation of form

$$u_t + u_{xxx} + gu_{xyy} + hu^2u_x = 0. \tag{21}$$

It is shown above that equation (21) is invariant with respect to a realization of four-dimensional symmetry algebra $A_{4,5}^{\frac{1}{3},\frac{1}{3}}$ spanned by the following operators

$$v_1 = \partial_t, \quad v_2 = \partial_x, \quad v_3 = \partial_y, \quad v_4 = t\partial_t + \frac{1}{3}x\partial_x + \frac{1}{3}y\partial_y - \frac{1}{3}u\partial_u.$$

The commutation relations of the symmetry operators are given by

$[v_i, v_j]$	v_1	v_2	v_3	v_4
v_1	0	0	0	v_1
v_2	0	0	0	$\frac{1}{3}v_2$
v_3	0	0	0	$\frac{1}{3}v_3$
v_4	$-v_1$	$-\frac{1}{3}v_2$	$-\frac{1}{3}v_3$	0

Since L_1 has zero center, we can directly apply Ovsiannikov's method of classification of subalgebras [29]. Namely, construction of optimal system of one-dimensional subalgebras we start from taking a non-zero vector

$$a_4v_4 + a_3v_3 + a_2v_2 + a_1v_1$$

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and considering its image under the action of adjoint representations adduced in the following table.

Ad	v_1	v_2	v_3	v_4
v_1	v_1	v_2	v_3	$v_4 - \epsilon v_1$
v_2	v_1	v_2	v_3	$v_4 - \frac{1}{3}\epsilon v_2$
v_3	v_1	v_2	v_3	$v_4 - \frac{1}{3}\epsilon v_3$
v_4	$e^{\epsilon} v_1$	$e^{\frac{1}{3}\epsilon} v_2$	$e^{\frac{1}{3}\epsilon} v_3$	v_4

Note 8. Without going to the theoretical details we recall that the adjoint representation of a Lie group G on its Lie algebra L can be reconstructed from the infinitesimal adjoint action ad L on itself by means of the following formula

$$\mathrm{Ad}(\exp(\epsilon v))w = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}(adv)^n(w) = w - \epsilon[v, w] + \frac{\epsilon^2}{2}[v, [v, w]] - \cdots .$$

Here v and w are elements of the Lie algebra L, $\exp(\epsilon v)$ is the one-parameter subgroup of G corresponding to the infinitesimal v. For more details we refer the reader to [28].

Then we try to choose the values of parameters in the adjoint actions in order to simplify possible forms of the class of subalgebras that our vector belongs to. Different possibilities arising under this procedure give us the classes of inequivalent one-dimensional subalgebras. In such a way we find an optimal system of 1-dimensional subalgebras:

$$\langle v_1 + \varepsilon v_2 + \varepsilon v_3\rangle, \quad \langle v_2\rangle, \quad \langle v_3 + a v_2\rangle, \quad \langle v_4\rangle, \quad \langle v_1\rangle.$$

Here and below $\varepsilon = 0, \pm 1$, $a, b = \mathrm{const}$.

An excellent detailed explanation and examples of classification of one-dimensional subalgebras of Lie algebras can be found in the textbooks [28,29]. (Inequivalent subalgebras of all real 2-, 3- and 4-dimensional Lie algebras were classified in [30].)

Constructing optimal system of two-dimensional subalgebras, we can suppose immediately that one of the basis vectors of two-dimensional subalgebra is taken from the previously obtained optimal system of one-dimensional subalgebras. Then we try to choose the parameters in the adjoint actions in order to simplify possible forms of the second basis and not to "spoil" the first one. It is possible that some of the basis vectors of the already classified one-dimensional subalgebras do not belong to any of the two-dimensional subalgebras.

After construction of all two-dimensional subalgebras for all basis vectors of the optimal system of one-dimensional subalgebras, we have to consider the action of inner automorphisms to order and simplify them, similarly to what we have already done for one-dimensional subalgebras. As a result we get an optimal system of two-dimensional subalgebras of the Lie symmetry algebra:

$$\langle v_1, v_2\rangle, \quad \langle v_1, v_3 + a v_2\rangle, \quad \langle v_2, v_3\rangle \quad \langle v_3, v_1 + \varepsilon v_2\rangle, \quad \langle v_1 + \varepsilon v_3, v_2 + a v_3\rangle,$$
$$\langle v_1, v_4\rangle, \quad \langle v_4, v_3 + a v_2\rangle, \quad \langle v_4, v_2\rangle.$$

An optimal system of three-dimensional subalgebras can be constructed by means of extension of the two-dimensional subalgebras completely similarly to the above extension of the one-dimensional subalgebras to the two-dimensional ones. Note, that over $\mathbb{R}$ there exist three-dimensional algebras (all isomorphic to $\langle e_1, e_2, e_3\rangle$, where $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_1, e_3] = -e_2$) that do not contain two-dimensional subalgebras. However, the symmetry algebra L_1 is solvable, therefore, all its three-dimensional subalgebras contain two-dimensional subalgebras and we can directly extend the above classification to construction of the three-dimensional subalgebras.

In such a way we get an optimal system of 3-dimensional subalgebras of L_1 that consists of

$$\langle v_1, v_2, v_3\rangle, \quad \langle v_2, v_3, v_4\rangle, \quad \langle v_1, v_4, v_3 + av_2\rangle, \quad \langle v_1, v_2, v_4\rangle.$$

As one can easily check, the obtained optimal systems coincide with those adduced in [30].

Now we consider reductions of equation (21) with respect to the obtained inequivalent subalgebras.

Reductions with respect to 1-dimensional subalgebras

We try to reduce equation (21) with respect to a subalgebra generated by a Lie symmetry of the form

$$\tau(t, x, y, u)\partial_t + \xi(t, x, y, u)\partial_x + \zeta(t, x, y, u)\partial_y + \eta(t, x, y, u)\partial_u,$$

where τ, ξ, ζ and η are known functions. In order to derive the desired similarity reductions, we need to solve a partial differential equation of form

$$\tau u_t + \xi u_x + \zeta u_y = \eta.$$

The solution of this equation contains three independent integrals which provide the general solution in the form

$$u = \mu(t, x, y, w(p, z)), \quad p = p(t, x, y), \; z = z(t, x, y), \tag{22}$$

where μ is known function and v, p, z are arbitrary functions in their arguments. Formula (22) defines the Ansatz that maps (21) into a partial differential equation with two independent variables p and z, and w being the dependent variable.

Below in case 1 of the reductions with respect to 1-dimensional subalgebras we consider this procedure in details. In the rest of the cases we adduce only the Ansatz, new independent variables, the reduced equations and, in some cases, corresponding invariant solutions.

1. $\langle v_1 + \varepsilon v_2 + \varepsilon v_3\rangle$: We try to construct invariant reduction with respect to the operator $Q = \partial_t + \varepsilon\partial_x + a\partial_y$. In the framework of the above procedure we look for the first integrals of the equation $u_t + \varepsilon u_x + au_y = 0$. They can be chosen as $I_1 = u$, $I_2 = y - at$ and $I_3 = x - \varepsilon t$. Then, the general solution of the above first-order equation has the form $F(I_1, I_2, I_3) = 0$, where F is an arbitrary function of its variables. Expressing now u from the above equation, we get the Ansatz $u = w(p, z)$, $p = y - at$, $z = x - \varepsilon t$ for the reduction. Substituting this Ansatz to the initial equation yields the reduced equation with two independent variables that looks like $-aw_p - \varepsilon w_z + w_{zzz} + gw_{zpp} + hw^2 w_z = 0$.

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The reduced equation is a $(1+1)$-dimensional nonlinear partial differential equation which is difficult to solve. Thus, we turn to find a kind of special solutions, i.e., traveling wave solutions of the reduced equation. In order to arrive at this, we perform the traveling wave transformations for the reduced equation and employ the standard tanh-function method [11], then we succeeded to find some partial solutions and this provides us with the following solutions of the modified ZK equation (21) by using the above-mentioned ansatz.

$$u = A\csc\phi,\quad u = A\sec\phi,\quad u = \frac{A}{\sqrt{2}}\tanh\frac{\phi}{\sqrt{2}},\quad u = \frac{A}{\sqrt{2}}\coth\frac{\phi}{\sqrt{2}},$$
$$u = A\,\mathrm{sech}\,\phi,\quad u = A\,\mathrm{csch}\,\phi,\quad u = \frac{A}{\sqrt{2}}\tan\frac{\phi}{\sqrt{2}},\quad u = \frac{A}{\sqrt{2}}\cot\frac{\phi}{\sqrt{2}},$$

where $A=\sqrt{\left|\frac{6(ac_1+\varepsilon c_2)}{hc_2}\right|}$, $\phi=\sqrt{\left|\frac{ac_1+\varepsilon c_2}{(c_2^3+gc_2c_1^2)}\right|}(-(ac_1+\varepsilon c_2)t+c_2x+c_1y)$, c_1, c_2 are arbitrary constants which satisfy the constrains $\frac{6(ac_1+\varepsilon c_2)}{hc_2}>0$, $\frac{ac_1+\varepsilon c_2}{c_2^3+gc_2c_1^2}<0$ for the first five solutions and $\frac{6(ac_1+\varepsilon c_2)}{hc_2}<0$, $\frac{ac_1+\varepsilon c_2}{c_2^3+gc_2c_1^2}>0$ for the last three solutions. It should be noted that the third to the sixth solutions are solitary wave solutions, while the other four solutions are periodic wave solutions.

2. $\langle v_2\rangle$: $u=w(t,y)$, $w_t=0$ that gives trivial solution $u=u(y)$.

3. $\langle v_3+av_2\rangle$: $u=w(p,z)$, $p=t$, $z=x-ay$, $w_p+w_{zzz}+a^2gw_{zzz}+hw^2w_z=0$. This reduced equation is the well-known modified KdV equations [1], we can construct its solutions by inverse scattering method [1] or the tanh function method [11]. Some of its known invariant solutions [40] provide us with the following solutions for the modified ZK equation (21):

$$u = A\csc\phi,\quad u = A\sec\phi,\quad u = \frac{A}{\sqrt{2}}\tanh\frac{\phi}{\sqrt{2}},\quad u = \frac{A}{\sqrt{2}}\coth\frac{\phi}{\sqrt{2}},$$
$$u = A\,\mathrm{sech}\,\phi,\quad u = A\,\mathrm{csch}\,\phi,\quad u = \frac{A}{\sqrt{2}}\tan\frac{\phi}{\sqrt{2}},\quad u = \frac{A}{\sqrt{2}}\cot\frac{\phi}{\sqrt{2}},$$

where $A=\sqrt{\left|\frac{6c_1}{hc_2}\right|}$, $\phi=\sqrt{\left|\frac{c_1}{c_2^3+a^2gc_2^3}\right|}(c_1t+c_2x-ac_2y)$, c_1, c_2 are arbitrary constants which satisfy the constrains $\frac{6c_1}{hc_2}>0$, $\frac{c_1}{c_2^3+a^2gc_2^3}<0$ for the first five solutions and $\frac{6c_1}{hc_2}<0$, $\frac{c_1}{c_2^3+a^2gc_2^3}>0$ for the last three solutions.

4. $\langle v_4\rangle$: $u=t^{-1/3}w(p,z)$, $p=t^{-1/3}y$, $z=t^{-1/3}x$, $-\frac{1}{3}w-\frac{1}{3}pw_p-\frac{1}{3}zw_z+w_{zzz}+gw_{ppz}+hw^2w_z=0$.

5. $\langle v_1\rangle$: $u=w(x,y)$, $w_{xxx}+gw_{xyy}+hw^2w_x=0$. By performing traveling wave transformation and integrating the reduced equation, we can obtain stationary solutions of the modified ZK equation (21) of form:

$$u=w(kx+ly),$$

where w is implicitly determined by

$$\int \sqrt{\frac{6(k^2+gl^2)}{6c_2w+6c_1-hw^4}}dw = \pm(kx+ly)+c_0,$$

where $c_2,\ c_1,\ c_0,\ k,\ l$ are arbitrary constants.

Reductions with respect to 2-dimensional subalgebras

1. $\langle v_2, v_1\rangle$: leads to the solution $u = v(y)$

2. $\langle v_1, v_3 + av_2\rangle$: $u = v(x-ay)$, $(1+g^2)v''' + hv^2v' = 0$. Integrating this reduced equation, we obtain stationary solutions of the modified ZK equation (21) in implicit form:

$$u = v(\xi), \quad \int \sqrt{\frac{6(1+g^2)}{12c_2v+12c_1-hv^4}}dv = \pm\xi + c_0, \quad \xi = x-ay,$$

where c_2, c_1, c_0 are arbitrary constants.

3. $\langle v_2, v_3\rangle$: leads to the solution $u = \text{const}$.

4. $\langle v_3, v_1 + \varepsilon v_2\rangle$: $u = v(x-\varepsilon t)$, $v''' - \varepsilon v' + hv^2v' = 0$. Solving this reduced equation by direct integrating, we get an y-independent general solutions of the modified ZK equation (21) in implicit form:

$$u = v(\xi), \quad \int \frac{1}{\sqrt{\varepsilon v^2 + 2c_2v + 2c_1 - \frac{1}{6}hv^4}}dv = \pm\xi + c_0, \quad \xi = x-\varepsilon t,$$

where c_2, c_1, c_0 are arbitrary constants.

5. $\langle v_1 + \varepsilon v_3, v_2 + av_3\rangle$: $u = v(y-ax-\varepsilon t)$, $(a^3+ga)v''' + \varepsilon v' + ahv^2v' = 0$. Integrating this reduced equation, we find solutions of the modified ZK equation (21) in implicit form:

$$u = v(\xi), \quad \int \sqrt{\frac{a(a^2+g)}{2c_2v + 2c_1 - \varepsilon v^2 - \frac{1}{6}ahv^4}}dv = \pm\xi + c_0, \quad \xi = y-ax-\varepsilon t,$$

where c_2, c_1, c_0 are arbitrary constants.

6. $\langle v_1, v_4\rangle$: $u = \frac{v(\omega)}{y}$, $\omega = \frac{x}{y}$, $(1+g\omega^2)v''' + 6g\omega v'' + 6gv' + hv^2v' = 0$. Integrating the equation one time, we obtain

$$[(1+gw^2)v]'' + \frac{1}{3}hv^3 = c_0.$$

Using symbolic computation software MAPLE to solve it for $c_0 = 0$, we obtain a stationary solution of the modified ZK equation (21)

$$u(x,y) = \frac{c_2}{y\sqrt{1+g(x/y)^2}} m\,\mathrm{sn}\left[m\left(\frac{\sqrt{36g+6h}\arctan(\sqrt{g}(x/y))}{6\sqrt{g}} + c_1\right), \frac{c_2\sqrt{-6gh-h^2}}{6g+h}\right],$$

where $m = \sqrt{\frac{6g}{6g+h-hc_2^2}}$, c_1, c_2 are arbitrary constants and $\mathrm{sn}(\cdot,\cdot)$ is Jacobi elliptic sine function.

7. $\langle v_4, v_3 + av_2\rangle$: $u = t^{-1/3}v(\omega)$, $\omega = t^{-1/3}(x - ay)$, $(1 + a^2g)v''' - \frac{1}{3}\omega v' - \frac{1}{3}v + hv^2v' = 0$. Solving this equation, we obtain stationary solution of the modified ZK equation (21):

$$u = \pm\sqrt{-\frac{6(1 + a^2g)}{h}}\frac{1}{x - ay},$$

where $\frac{6(1+a^2g)}{h} < 0$.

8. $\langle v_4, v_2\rangle$: leads to the solution of form $u = v(y)$.

Reductions with respect to 3-dimensional subalgebras

Since equation (21) is 3-dimensional, its reductions with respect to 3-dimensional subalgebras L_1 lead to algebraic equations. From invariance of (21) with respect to $\langle v_1, v_4, v_3 + av_2\rangle$ we obtain that

$$u = \frac{c}{x - ay}.$$

Substituting this expression to (21) we get an algebraic equation on c of form $c(6 + 6a^2 + hc^2) = 0$. Its solutions are

$$c = 0, \quad c = \pm\frac{\sqrt{-h(a^2 + 1)}}{h}.$$

It is not difficult to show that all other reductions with respect to 3-dimensional subalgebras lead, at most, to the trivial solution

$$u = u(y).$$

5.2. *Zakharov–Kuznetsov equation*

In this subsection we perform reductions of the Zakharov–Kuznetsov equation

$$u_t + u_{xxx} + gu_{xyy} + huu_x = 0. \tag{23}$$

with respect to 1- and 2-dimensional subalgebras of its maximal Lie symmetry algebra

$$\langle v_1 = \partial_t,\ v_2 = \partial_x,\ v_3 = \partial_y,\ v_4 = ht\partial_x + \partial_u,\ v_5 = 3t\partial_t + x\partial_x + y\partial_y - 2u\partial_u\rangle.$$

As the computations are very similar to the ones from the previous subsection, we skip all technical details and summarize the results only. (Some of the reduced with respect to 1-dimensional subalgebras equations together with their solutions can be found also in [24].) The table of commutation relations is

$[v_i, v_j]$	v_1	v_2	v_3	v_4	v_5
v_1	0	0	0	hv_2	$3v_1$
v_2	0	0	0	0	v_2
v_3	0	0	0	0	v_3
v_4	$-hv_2$	0	0	0	$-2v_4$
v_5	$-3v_1$	$-v_2$	$-v_3$	$2v_4$	0

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Then, the corresponding adjoint representations look like

Ad	v_1	v_2	v_3	v_4	v_5
v_1	v_1	v_2	v_3	$v_4 - h\epsilon v_2$	$v_5 - 3\epsilon v_1$
v_2	v_1	v_2	v_3	v_4	$v_5 - \epsilon v_2$
v_3	v_1	v_2	v_3	v_4	$v_5 - \epsilon v_3$
v_4	$v_1 + h\epsilon v_2$	v_2	v_3	v_4	$v_5 + 2\epsilon v_4$
v_5	$e^{3\epsilon} v_1$	$e^{\epsilon} v_2$	$e^{\epsilon} v_3$	$e^{-2\epsilon} v_4$	v_5

An optimal system of 1-dimensional subalgebras can be chosen as

$$\langle v_1\rangle, \quad \langle v_2\rangle, \quad \langle v_3 + a_2 v_2\rangle, \quad \langle v_3 + \varepsilon_1 v_1\rangle, \quad \langle v_4 + a_3 v_3 + a_1 v_1\rangle, \quad \langle v_5\rangle.$$

Here and below $\varepsilon_i = 0, \pm 1$, $a_i, b_i = \text{const}$.

An optimal system of 2-dimensional subalgebras is

$$\langle v_2, v_1\rangle, \quad \langle v_3, v_2\rangle, \quad \langle v_3 + \varepsilon_2 v_2, v_1\rangle, \quad \langle v_4 + \varepsilon_3 v_3, v_1\rangle, \quad \langle v_4 + a_3 v_3, v_2\rangle,$$
$$\langle v_4 + a_1 v_1, v_3\rangle, \quad \langle v_5, v_2\rangle, \quad \langle v_5, v_1\rangle, \quad \langle v_5, v_3 - b_2 v_2\rangle, \quad \langle v_5, v_4\rangle$$

Reductions with respect to 1-dimensional subalgebras

1. $\langle v_1\rangle$: $u = w(x, y)$, $w_{xxx} + g w_{xyy} + h w w_x = 0$. With the aid of symbolic computation software MAPLE, an invariant solution arising from this case is

$$w(x, y) = \frac{8}{h}(c_2^2 + g c_3^2) - \frac{12}{h}(c_2^2 + g c_3^2)\tanh^2(c_1 + c_2 x + c_3 y),$$

where $c_1,\ c_2,\ c_3$ are arbitrary constants.

2. $\langle v_2\rangle$: $u = w(t, y)$, $w_t = 0$, $u = w(y)$

3. $\langle v_3 + a_2 v_2\rangle$: $u = w(t, z)$, $z = x - a_2 y$, $w_t + (1 + a_2^2 g) w_{zzz} + h w w_z = 0$. This equation is the famous KdV equation [1], we can construct its solutions by inverse scattering method [1] or the tanh function method [11]. Using some of its invariant solutions [40] we can easily construct some exact solutions for the ZK equation (23):

$$u = A\csc^2\phi, \quad u = A\sec^2\phi, \quad u = -\frac{A}{3}(1 - 3\tanh^2\phi), \quad u = -\frac{A}{3}(1 - 3\coth^2\phi),$$
$$u = A\,\mathrm{sech}^2\phi, \quad u = -A\,\mathrm{csch}^2\phi, \quad u = -\frac{A}{3}(1 + 3\tan^2\phi), \quad u = -\frac{A}{3}(1 + 3\cot^2\phi),$$

where $A = -\frac{3c_1}{hc_2}$, $\phi = \frac{1}{2}\sqrt{\left|\frac{c_1}{c_2^3 + a_2^2 g c_2^3}\right|}(c_1 t + c_2 x - a_2 c_2 y)$, c_1, c_2 are arbitrary constants which satisfy the constrains $\frac{c_1}{c_2^3 + a_2^2 g c_2^3} > 0$ for the first five solutions and $\frac{c_1}{c_2^3 + a_2^2 g c_2^3} < 0$ for the last three solutions. It should be noted that the third to the sixth solutions are solitary wave solutions, while the other four solutions are periodic wave solutions.

4. $\langle v_3 + \epsilon_1 v_1\rangle$: $u = w(p, x)$, $p = t - \varepsilon_1 y$, $w_p + w_{xxx} + \varepsilon_1^2 g w_{ppx} + h w w_x = 0$. Solving this reduced equation by the tanh function method [11], we can obtain the following exact solutions of the ZK equation (23):

$$u = A\csc^2\phi,\quad u = A\sec^2\phi,\quad u = -\frac{A}{3}(1-3\tanh^2\phi),\quad u = -\frac{A}{3}(1-3\coth^2\phi),$$
$$u = A\,\mathrm{sech}^2\phi,\quad u = -A\,\mathrm{csch}^2\phi,\quad u = -\frac{A}{3}(1+3\tan^2\phi),\quad u = -\frac{A}{3}(1+3\cot^2\phi),$$

where $A = -\frac{3c_1}{hc_2}$, $\phi = \frac{1}{2}\sqrt{\left|\frac{c_1}{c_2^3+\varepsilon_1^2 g c_1^2 c_2}\right|}(c_1 t + c_2 x - a_2 c_2 y)$, c_1, c_2 are arbitrary constants which satisfy the constrains $\frac{c_1}{c_2^3+\varepsilon_1^2 g c_1^2 c_2} > 0$ for the first five solutions and $\frac{c_1}{c_2^3+\varepsilon_1^2 g c_1^2 c_2} < 0$ for the last three solutions.

5. $\langle v_4 + a_3 v_3 + a_1 v_1\rangle$: if $a_1 = 0$, then $u = \frac{x}{ht} + w(t,z)$, $z = \frac{a_3 x}{ht} - y$, $h^3 t^3 w_t + w_{zzz}(a_3^3 + g a_3 h^2 t^2) + h^3 t^2 w + h^3 t^2 a_3 w w_z = 0$. If $a_1 \neq 0$, then $u = \frac{t}{a_1} + w(p,z)$, $z = x - \frac{ht^2}{2a_1}$, $p = a_1 y - a_3 t$, $\frac{1}{a_1} - a_3 w_p + w_{zzz} + g a_1^2 w_{zpp} + h w w_z = 0$.

6. $\langle v_5\rangle$: $u = t^{-2/3} w(p,z)$, $p = t^{-1/3} y$, $z = t^{-1/3} x$, $-2w - p w_p - z w_z + 3 w_{zzz} + 3 g w_{zpp} + 3 h w w_z = 0$.

Reductions with respect to 2-dimensional subalgebras

1. $\langle v_2, v_1\rangle$: $u = v(y)$ is a solution.

2. $\langle v_3, v_2\rangle$: $u = v(t)$, $v_t = 0$, therefore, $u = \text{const}$.

3. $\langle v_3 + \varepsilon_2 v_2, v_1\rangle$: $u = v(x - \varepsilon_2 y)$, $(1 + g\varepsilon_2^2)v''' + h v v' = 0$. Solving this reduced equation by direct integrating, we obtain stationary solution of the ZK equation (23) in implicit form:

$$u = v(\xi),\qquad \int\sqrt{\frac{(1+g\varepsilon_2^2)}{2c_2 v + 2c_1 - \frac{1}{3}hv^3}}\,dv = \pm\xi + c_0,\qquad \xi = x - \varepsilon_2 y,$$

where $c_2,\ c_1,\ c_0$ are arbitrary constants.

4. $\langle v_4 + a_3 v_3, v_2\rangle$: if $a_3 = 0$ then there is no reduction, if $a_3 \neq 0$, then $u = \frac{y}{a_3} + v(t)$, $v_t = 0$ that gives $u = \frac{y}{a_3} + c$, where c is an arbitrary constant.

5. $\langle v_4 + a_1 v_1, v_3\rangle$: if $a_1 = 0$, then $u = \frac{x}{ht} + v(t)$, $v' + \frac{v}{t} = 0$ that gives $u = \frac{x}{ht} + \frac{c}{t}$, where c is an arbitrary constant. If $a_1 \neq 0$, then $u = \frac{t}{a_1} + v(x - \frac{h}{2a_1}t^2)$, $v''' + hvv' + \frac{1}{a_1} = 0$. Integrating this equation one time, we obtain $v'' + \frac{1}{2}hv^2 + \frac{1}{a_1}\xi = c$, where $\xi = x - \frac{h}{2a_1}t^2$.

6. $\langle v_5, v_2\rangle$: $u = t^{-2/3} v(\omega)$, $\omega = t^{-1/3} y$, $2v + \omega v' = 0$. Solving this equation, we obtain solution of the ZK equation (23) of form $u = \frac{c_0}{y^2}$.

7. $\langle v_5, v_1\rangle$: $u = \frac{1}{y^2} v(\omega)$, $\omega = \frac{x}{y}$, $(1 + g\omega^2)v''' + 12 g v' + 8 g\omega v'' + h v v' = 0$. Integrating the equation one time, we obtain

$$(1 + g w^2)v'' + 6g(wv)' + \frac{1}{2}hv^2 = c_0.$$

Using symbolic computation software MAPLE for $c_0 = 0$, we obtain stationary solution of the ZK equation (23) in implicit form:

$$u = \frac{1}{y^2}v(\omega),\qquad \frac{1}{\sqrt{g}}\arctan(\sqrt{g}\omega) \pm 3\int\frac{1}{\sqrt{9c_1 - 36gZ^2 - 3hZ^3}}\,dZ - c_2 = 0,\qquad \omega = \frac{x}{y},$$

where $Z = v + gv\omega^2$, c_2, c_1 are arbitrary constants.

8. $\langle v_5, v_3 - b_2 v_2\rangle$: $u = t^{-2/3} v(\omega)$, $\omega = t^{-1/3}(x + b_2 y)$, $3(1 + g b_2^2)v''' - 2v - \omega v' + 3hvv' = 0$. This reduced equation has a solution $v = \frac{1}{h}\omega$, thus we obtain a rational solution of the ZK equation (23): $u = \frac{x+b_2 y}{ht}$.

9. $\langle v_5, v_4\rangle$: $u = \frac{x}{ht} + t^{-2/3} v(\omega)$, $\omega = t^{-1/3} y$, $\omega v' - v = 0$ that gives $u = \frac{x}{ht} + \frac{cy}{t}$, where c is an arbitrary constant.

Note 9. In both cases of the modified Zakharov–Kuznetsov equation and of the classical Zakharov–Kuznetsov equation, we do not solve all the reduced equations. One of the possible ways of constructing their solutions is, again, group analysis. More precisely, one can derive Lie symmetries for the reduced partial differential equations with the ultimate goal to construct similarity reductions that transform these equations into ordinary differential equations or algebraic equations that are easier to solve.

6. Conclusion and discussion

In summary, we have described an algorithmic framework for group classification of (systems of) partial differential equations. More precisely, we have revised the classical Lie method of construction of symmetries (of system) of differential equations in more details and written down the precise formulation of Ovsiannikov's algorithm of group classification of a class (of systems) of differential equations. We also described a systematic way of finding all possible invariant solutions of (of systems) of differential equations. All the theory and algorithms were illustrated by the running example of the GZK equations (2). Specifically, we performed a complete group classification of the class of GZK equations (2) by using the equivalence transformations and the compatibility method. The main results of classification are collected in Tables 1–2 where we list all inequivalent cases of extensions with the corresponding Lie invariance algebras. It is shown that the symmetry algebras of invariant models of form (2), are at most six-dimensional. For the classical Zakharov–Kuznetsov (23) and the modified Zakharov–Kuznetsov (21) we construct optimal systems of inequivalent subalgebras, corresponding Lie ansätze and invariant solutions.

The present paper should be an inspiration for further investigations of other properties of class (2). For example, one can classify the nonclassical (conditional) symmetries. Furthermore, one can perform conservation law classification, then to use these results to systematical calculation of nonlocal symmetries, higher order local and potential conservation laws, nonclassical potential solutions and linearizations, etc. Motivated by Craddock and Dooley's work on the equivalence of Lie symmetries and group representations [8], we can also consider the global action of the Lie symmetries of the GZK equations (2). We will investigate these subjects elsewhere. The algorithmic framework for group analysis of differential equations presented in this paper can also be applied to other nonlinear PDEs in mathematical physics.

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