ARTICLE IN PRESS YJDEQ:8306

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Journal of Differential Equations

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Global existence for small data of the viscous Green–Naghdi type equations

Dena Kazerani [a,b,c,*]

[a] Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France
[b] CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France
[c] INRIA-Paris-Rocquencourt, EPC ANGE, Domaine de Voluceau, BP105, 78153 Le Chesnay Cedex, France

Received 11 October 2015; revised 22 February 2016

Abstract

We consider the Cauchy problem for the Green–Naghdi equations with viscosity, for small initial data. It is well-known that adding a second order dissipative term to a hyperbolic system leads to the existence of global smooth solutions, once the hyperbolic system is symmetrizable and the so-called Kawashima–Shizuta condition is satisfied. In a previous work, we have proved that the Green–Naghdi equations can be written in a symmetric form, using the associated Hamiltonian. This system being dispersive, in the sense that it involves third order derivatives, the symmetric form is based on symmetric differential operators. In this paper, we use this structure for an appropriate change of variable to prove that adding viscosity effects through a second order term leads to global existence of smooth solutions, for small data. We also deduce that constant solutions are asymptotically stable.

Keywords: Green–Naghdi equations; Viscosity; Small solutions; Symmetric structure; Energy equality; Global existence

1. Introduction

The Green–Naghdi system is a shallow water approximation of the water waves problem which models incompressible flows. The vertical and horizontal speeds are averaged vertically.

* Correspondence to: Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France.

E-mail address: kazerani@ljll.math.upmc.fr.

http://dx.doi.org/10.1016/j.jde.2016.03.022

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Moreover, vertical acceleration is supposed too small to be considered [13]. In other words, Green–Naghdi equations is one order higher in approximation compared to the Saint-Venant (called also isentropic Euler) system [3]. To obtain the latter system, not only the vertical acceleration but also the vertical speed are neglected. This leads to a hyperbolic system of equations whereas the Green–Naghdi equation is dispersive due to the term $\alpha h^2 \ddot{h}$ defined below. In this work, we focus on the Green–Naghdi type equation with a second order viscosity:

$$\begin{cases} \partial_t h + \partial_x (hu) = 0, \\ \partial_t (hu) + \partial_x (hu^2) + \partial_x (gh^2/2 + \alpha h^2 \ddot{h}) = \mu \partial_x (h \partial_x u) \end{cases} \tag{1}$$

We assume that $h(x,t) > 0$, α and μ are strictly positive and g is the gravity constant. The unknown h represents the fluid height and u its average horizontal speed. Moreover, the material derivative $\dot{()}$ is defined by $\dot{()} = \partial_t () + u \partial_x ()$.

Remark 1.1. Let us note that the $\alpha = 0$ case gives us the Saint-Venant system. We can also learn more about the derivation of the system in [21,1,15] for $(\mu, \alpha) = (0, \frac{1}{3})$, and in [7] for $(\mu, \alpha) = (0, \frac{1}{4})$.

It is worth remarking that (1) admits the following energy equality [11,7],

$$\partial_t E + \partial_x (u(E + p)) = \mu u \partial_x (h \partial_x u), \tag{2}$$

where

$$E(h,u) = gh^2/2 + hu^2/2 + \alpha h^3 (\partial_x u)^2/2,$$

and

$$p(h,u) = gh^2/2 + \alpha h^2 \ddot{h}.$$

Then, we can check that (1) admits a family of relative energy conservation equalities given by

$$\partial_t E_{h_e,u_e} + \partial_x P_{h_e,u_e} = \mu (u - u_e) \partial_x (h \partial_x u), \tag{3}$$

where

$$E_{h_e,u_e}(h,u) = g(h - h_e)^2/2 + h(u - u_e)^2/2 + \alpha h^3 (\partial_x u)^2/2, \tag{4a}$$

and

$$P_{h_e,u_e}(h,u) = u E_{h_e,u_e}(h,u) + (u - u_e) p(h,u) - \frac{g h_e^2}{2} u. \tag{4b}$$

This family is parametrized by $(h_e, u_e) \in \mathbb{R}^2$ with $h_e > 0$.

Remark 1.2. Let us assume that $\alpha = 0$. Then, $E(h,u)$ and $E_{h_e,u_e}(h,u)$ are convex entropies for Saint-Venant system.

The dissipative term $\mu \partial_x (h \partial_x u)$ considered here in the right hand side of (1), is presented in [12] and some other references, as the viscosity for Saint-Venant system. Indeed, Saint-

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Venant system with this viscosity is derived in [12] from the Navier–Stokes equations under the shallow water assumption. On the one hand, this term is stabilizing for the hyperbolic Saint-Venant system. On the other hand, Green–Naghdi equation is a higher order approximation of the water waves problem and contains Saint-Venant system in addition to some dispersive terms. Therefore, we are interested to learn more about the role this viscosity plays on Green–Naghdi equations. Following the result of this work, we see that the dispersion does not cancel the stabilizing effect of the viscosity.

The aim of this paper is to study the stability of equilibriums based on the symmetric structure of the system. The intuition comes from the Kawashima–Shizuta works on hyperbolic–parabolic systems [24,17] and Hanouzet–Natalini and Yong [14,26] on entropy dissipative symmetric hyperbolic equations. All these results have been proved using the symmetric structure of hyperbolic systems. In particular, Saint-Venant system with friction can be treated by the general result obtained in [14,26] whereas Saint-Venant system with viscosity fits the general frame considered in [24,17].

The notion of symmetric structure and of Godunov systems has been extended to some dispersive systems in [18]. In particular, the Green–Naghdi equations enter in this framework and then can be written under a symmetric structure which is recalled in Subsection 1.1. We show in this work how this structure enables us to extend the techniques used in [17,14,26] for symmetric hyperbolic equations to the dispersive Green–Naghdi equations.

Remark 1.3. The order of the dissipative term $\mu\partial_x(h\partial_x u)$ plays a very important role in this work. Indeed, we can prove the global existence for small initial data only if the dissipative term, considered in the right hand side of (1), is a second order term with respect to u. For instance, we are not able to generalize the results presented in Section 2, if we replace the dissipative term $\mu\partial_x(h\partial_x u)$ with a friction type term such as $-\kappa u$ for some $\kappa > 0$. Likewise, if we consider a fourth order dissipation such as $-\mu\partial_x^2(h\partial_x(h\partial_x u))/4$ (suggested in [7]) instead of the second order $\mu\partial_x(h\partial_x u)$, the estimates we find are not sufficient to conclude the global existence.

In all this work, partial derivatives with respect to x of any differentiable function f are presented by $\partial_x f$. The differential of the application F at U is symbolized by $D_U F(U)$. The adjoint of the operator $\mathcal{A}$ is denoted by $\mathcal{A}^\star$.

1.1. Symmetric structure

Following Li's notations in [21], we use the unknown $U = (h, m)$ defined by a Sturm–Liouville operator called $\mathcal{L}$:

$$m = hu - \alpha\partial_x(h^3\partial_x u) = \mathcal{L}_h(u).$$

Let us note that $\mathcal{L}_h : \mathbb{H}^s(\mathbb{R}) \to \mathbb{H}^{s-2}(\mathbb{R})$ is an isomorphism if h is strictly positively bounded by below and $s \geq 2$. Therefore, System (1) can be written under

$$\partial_t U + \partial_x F(U) = Q(U),$$

where

$$F(U) = \begin{pmatrix} h\mathcal{L}_h^{-1}(m) \\ m\mathcal{L}_h^{-1}(m) - 2\alpha h^3(\partial_x\mathcal{L}_h^{-1}(m))^2 + \frac{g}{2}h^2 - \frac{g}{2}h_e^2 \end{pmatrix}, \tag{5}$$

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and

$$Q(U) = \begin{pmatrix} 0 \\ \mu\partial_x(h\partial_x u) \end{pmatrix} \tag{6}$$

Based on the structure presented in [21], it is easy to check that the unknown U enables us to write (1) under a Hamiltonian structure where the Hamiltonian $\mathcal{H}_{h_e,u_e}$ is defined by the integral of the relative energy i.e. by

$$\mathcal{H}_{h_e,u_e} := \int_{\mathbb{R}} E_{h_e,u_e}.$$

This unknown presents also another advantage. In fact, we can recover the physical variable $V = (h, u)$ from U using the interesting change of variable $V = (h, \delta_m \mathcal{H}_{h_e,u_e}(U))$, where δ_m denotes the variational derivative with respect to m.[1] This consideration, as suggested in the following theorem, enables us to symmetrize the system in the physical variable with a diagonal locally definite positive operator (see Appendix A for more details).

Theorem 1.4. *(See [18].) Let $V_e = (h_e, u_e)$ be a constant solution of (1) with $h_e > 0$. Let also $s \geq 2$ be an integer. Then, as long as the solution $V = (h, u)$ remains close to V_e for the usual norm of $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$, the system is equivalent to the following symmetric form:*

$$\mathcal{A}_0(V)\partial_t V + \mathcal{A}_1(V)\partial_x V = \begin{pmatrix} 0 \\ \mu\partial_x(h\partial_x u) \end{pmatrix} \tag{7}$$

where

$$\begin{aligned} \mathcal{A}_0(V) &= D_V U^\star(V) \left(\delta_U^2 \mathcal{H}\right) D_V U(V) \\ &= \begin{pmatrix} g - 3\alpha h(\partial_x u)^2 & 0 \\ 0 & \mathcal{L}_h \end{pmatrix} \end{aligned} \tag{8}$$

is a positive definite operator and

$$\begin{aligned} \mathcal{A}_1(V) &= D_V U^\star(V) \left(\delta_U^2 \mathcal{H}\right) (D_U F(U)) D_V U(V) \\ &= \begin{pmatrix} gu - 3\alpha hu(\partial_x u)^2 & gh - 3\alpha h^2(\partial_x u)^2 \\ gh - 3\alpha h^2(\partial_x u)^2 & hu + 2\alpha\partial_x(h^3\partial_x u) - \alpha\partial_x(h^3 u)\partial_x - \alpha h^3 u \partial_x^2 \end{pmatrix} \end{aligned} \tag{9}$$

is a symmetric one.

Proof. Let us consider the conservative form

$$\partial_t U + \partial_x F(U) = Q(U).$$

[1] We have

$$\delta_h \mathcal{H}_{h_e,u_e}(U) = g(h - h_e) - \frac{u^2 - u_e^2}{2} - \frac{3}{2}\alpha h^2(\partial_x u)^2,$$

and

$$\delta_m \mathcal{H}_{h_e,u_e}(U) = u - u_e.$$

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Obviously, we have

$$D_V U(V)\partial_t V + D_U F(U) D_V U(V)\partial_x V = Q(U).$$

Then, acting $D_V U^\star(V)\left(\delta_U^2 \mathcal{H}_{h_e,u_e}\right)$ on the system and considering the fact that $Q(U)$ is an invariant vector of $D_V U^\star(V)\left(\delta_U^2 \mathcal{H}_{h_e,u_e}\right)$, we get the result (See Appendix A for more details). □

Let us note that $\mathcal{A}_0(V)$ and $\mathcal{A}_1(V)$ are linear second order differential operators. Therefore, they can be decomposed as

$$\mathcal{A}_0(V) = \mathcal{A}_0^0(V) + \mathcal{A}_0^1(V)\partial_x + \mathcal{A}_0^2(V)\partial_x^2 \tag{10}$$

$$\mathcal{A}_1(V) = \mathcal{A}_1^0(V) + \mathcal{A}_1^1(V)\partial_x + \mathcal{A}_1^2(V)\partial_x^2 \tag{11}$$

where the expressions of symmetric matrix $\mathcal{A}_i^j(V)$ for $i, j \in \{0, 1, 2\}$ are given by

$$\mathcal{A}_0^0(V) = \begin{pmatrix} g - 3\alpha h(\partial_x u)^2 & 0 \\ 0 & h \end{pmatrix}, \quad \mathcal{A}_0^1(V) = \begin{pmatrix} 0 & 0 \\ 0 & -3\alpha h^2 \partial_x h \end{pmatrix}, \quad \mathcal{A}_0^2(V) = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha h^3 \end{pmatrix},$$

$$\mathcal{A}_1^0(V) = \begin{pmatrix} gu - 3\alpha hu(\partial_x u)^2 & gh - 3\alpha h^2(\partial_x u)^2 \\ gh - 3\alpha h^2(\partial_x u)^2 & hu + 2\alpha\partial_x(h^3\partial_x u) \end{pmatrix}, \quad \mathcal{A}_1^1(V) = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha\partial_x(h^3 u) \end{pmatrix},$$

$$\mathcal{A}_1^2(V) = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha h^3 u \end{pmatrix}.$$

Remark 1.5. The definite positivity of a real matrix is equivalent to its coercivity. However, this fact does not necessary hold true for definite positive operators i.e. some definite positive operators are not coercive. It is important to point out that, as illustrated in Section 3, one of the keys which lets us generalize the hyperbolic methods to our symmetric system is actually the coercivity of $\mathcal{A}_0(V)$ for the convenient norm. This means that we would not be able to generalize the method if $\mathcal{A}_0(V)$ was definite positive but not coercive.

We can also remark that the symmetric structure suggested in this section is similar to the structure used in [16] to study the local well-posedness of the Green–Naghdi equations without viscosity.

1.2. Outline

We are going to study the global existence of solutions of the viscous Green–Naghdi type equations for smooth initial data close to equilibriums. A local well-posedness result is proved in Appendix B. Let us also mention that some similar writings as (7) have been used to study the local well-posedness of some dispersive systems (see [23] and [9] for instance). Then, we use the dissipative character of the viscosity as well as the symmetric structure of the system to prove the global existence of the local solution. In fact, the first step of the proof contains some initial estimates obtained by taking the scalar product of the s^{th} derivative of the equation with the s^{th} derivative of the solution. As it is exposed in Subsections 3.1 and 3.2, these estimates are obtained by almost the same approach as in the hyperbolic case ([14] and [26]). Then, the second step is to estimate the time integral of the norm of the solution. In the case of hyperbolic systems, this

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estimate is found by using the Kawashima–Shizuta condition. This condition has been introduced in [24] as a stability condition for constant solutions. It is based on the existence of a constant real matrix such that its product with the definite positive matrix (the one equivalent to $\mathcal{A}_0$) is skew-symmetric at equilibrium while the symmetric part of its product with the symmetric matrix (the one equivalent to $\mathcal{A}_1$), added to the right hand side term matrix, gives a definite positive matrix. However, in the case of Green–Naghdi system, we have not been able to find any operator generalization of the Kawashima–Shizuta condition. Hence, we must use a slightly different approach to find a convenient estimate. Indeed, we can find a null diagonal real matrix K such that $K\mathcal{A}_1(V_e)$ is a symmetric definite positive matrix for all equilibriums V_e with $u_e = 0$. However, $K\mathcal{A}_0(V_e)$ is not a skew-symmetric operator. Nevertheless, we are able to put some non-straightforwardly controllable term under a time integral of a time derivative[2] and estimate the remaining terms in a convenient manner (see Subsection 3.3). Then, using a symmetry group of the system, we can generalize the result to the case of equilibriums V_e with $u_e \neq 0$.

This paper is organized on 4 sections. The global existence theorem and its corollaries are presented in Section 2. Section 3 contains the steps of the proof. Some perspectives are suggested in Section 4. The advantages of the symmetric structure used in this study are explained in Appendix A. So we can see why this symmetric structure is more appropriate than others. Appendix B contains the proof of the local well-posedness Theorem 2.1. Appendix C highlights one of the other utilities of the symmetric structure. In fact, linear stability of equilibrium of non-viscous Green–Naghdi can be proved using this structure.

2. Main results

The local well-posedness of (1) has been studied in [16] and [21] for the case $\mu = 0$. We see here that we can prove the local well-posedness of (1), around constant solutions, based on the idea used for symmetric hyperbolic systems. To do so, we first note that the set of constant solutions of (1) is

$$\{V_e = (h_e, u_e);\; h_e > 0, u_e \in \mathbb{R}\}.$$

We may also call these solutions the equilibriums of the system.

We denote the norm associated with the affine space $\mathbb{X}^s(\mathbb{R}) = (\mathbb{H}^s(\mathbb{R}) + h_e) \times (\mathbb{H}^{s+1}(\mathbb{R}) + u_e)$ by

$$\| (f, g) \|_{\mathbb{X}^s}^2 = \| f \|_{\mathbb{H}^s}^2 + \| g \|_{\mathbb{H}^{s+1}}^2 .$$

Moreover, the s-neighborhood of radius δ and center V_e is presented by $B_s(V_e, \delta) = \{V \in \mathbb{X}^s(\mathbb{R}), \| V - V_e \|_{\mathbb{X}^s} \leq \delta\}$ for all integer $s \in \mathbb{R}$.

Let us also denote by C the universal constant of the following Gagliardo–Nirenberg inequality:

$$\| f \|_{\mathbb{L}^\infty} \leq C \| \partial_x f \|_{\mathbb{L}^2}^{\frac{1}{2}} \| f \|_{\mathbb{L}^2}^{\frac{1}{2}} \quad \forall f \in \mathbb{H}^1(\mathbb{R}). \tag{12}$$

[2] The skew-symmetry of $K\mathcal{A}_0$ for hyperbolic systems lets us put the non-straightforwardly controllable terms under a time derivative. Therefore, we can deal with them by taking the time integral. Although, we are not able here to obtain a skew-symmetry $K\mathcal{A}_0$, we try to deal with non-straightforwardly controllable terms by a similar idea.

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We are now able to announce the local well-posedness theorem,

Theorem 2.1. *Let $s \geq 2$ be an integer and consider a constant solution V_e of System (1). Then, there exists $0 < \delta < h_e$ such that for all initial data $V_0 \in B_s(V_e, \delta)$, there exists $T > 0$ such that the system admits a unique solution which belongs to $C([0,T), \mathbb{X}^s(\mathbb{R}))$.*

The proof of the theorem is given in Appendix B. The steps of the proof are the same as for hyperbolic systems (see [8,5] for instance). However, the necessary estimate to reach the final result of each step, is obtained by the same technique used in Section 3.2. In fact, we can see again in this part, how the generalized symmetric structure (7) of the system enables us to generalize the techniques used for symmetric hyperbolic systems.

An immediate corollary for Theorem 2.1 is the following. It states the positivity of the water height for small data and for short times.

Corollary 2.2. *Let $s \geq 2$ be an integer and consider a constant solution V_e of System (1). Let us also consider $\delta \in (0, \frac{h_e}{C})$ and $0 < T$ both conveniently small, and $V_0 \in B_s(V_e, \delta)$ such that (1) admits a unique solution $(h,u) \in C([0,T), \mathbb{X}^s(\mathbb{R}))$. Then, for all $\eta_0 \in (0, \inf\limits_{x\in\mathbb{R}} h_0(x))$, there exists a time $\tilde{T} \in (0,T)$ such that*

$$\inf_{x\in\mathbb{R}} h(t,x) \geq \eta_0 \quad \forall t \in [0, \tilde{T}]. \tag{13}$$

Proof. Let us first note that $\inf\limits_{x\in\mathbb{R}} h_0(x) > 0$. This is a consequence of the Gagliardo–Nirenberg inequality. Indeed,

$$\| h_0 - h_e \|_{\mathbb{L}^\infty} \leq C \| \partial_x h_0 \|_{\mathbb{L}^2}^{\frac{1}{2}} \| h_0 - h_e \|_{\mathbb{L}^2}^{\frac{1}{2}} .$$

Considering the fact that $V_0 \in B_s(V_e, \delta)$ with $s \geq 2$ and $\delta < \frac{h_e}{C}$, the inequality becomes

$$\| h_0 - h_e \|_{\mathbb{L}^\infty} \leq C\delta < h_e.$$

Therefore,

$$0 < h_e - C\delta \leq h_0(x) \leq h_e + C\delta < 2h_e \quad \forall x \in \mathbb{R}.$$

Then, we conclude that

$$\inf_{x\in\mathbb{R}} h_0(x) \geq h_e - C\delta > 0.$$

Let us now fix $\eta_0 \in (0, \inf_{x\in\mathbb{R}} h_0(x))$. The unique solution of (1) belongs to $C([0,T), \mathbb{X}^s(\mathbb{R}))$. Hence, there exists $\tilde{T} \in (0,T)$ such that

$$\| h(t) - h_0 \|_{\mathbb{X}^s} \leq \frac{\inf_{x\in\mathbb{R}} h_0(x) - \eta_0}{C} \quad \forall t \in [0, \tilde{T}].$$

Again, Gagliardo–Nirenberg inequality leads us to

$$\| h(t) - h_0 \|_{\mathbb{L}^\infty} \leq \inf_{x\in\mathbb{R}} h_0(x) - \eta_0 \quad \forall t \in [0, \tilde{T}].$$

Then, we have

$$\eta_0 - \inf_{x\in\mathbb{R}} h_0(x) \le h(t,x) - h_0(x) \quad \forall (x,t) \in \mathbb{R}\times[0,\tilde{T}],$$

and finally

$$\eta_0 \le \eta_0 + h_0(x) - \inf_{x\in\mathbb{R}} h_0(x) \le h(t,x) \quad \forall (x,t) \in \mathbb{R}\times[0,\tilde{T}]. \quad \square$$

The main result of this study is the following theorem on the asymptotic stability of equilibriums.

Theorem 2.3. *Let us consider an equilibrium $V_e = (h_e, u_e)$ of (1) and $s \ge 2$ an integer. Then, there exists $\delta > 0$ such that for all initial data $V_0 = (h_0, u_0) \in B_s(V_e, \delta)$, the solution V exists for all time and converges asymptotically to V_e.*

In other words, every constant solution $V_e = (h_e, u_e)$ of (1) is asymptotically stable.

Let us remark that we can prove Theorem 2.3 by considering $u_e = 0$. This is due to the fact that $\upsilon = t\partial_x + \partial_u$ is a infinitesimal generator of a symmetry group of (1). This means that

$$V_\beta = (h(x-\beta t, t), u(x-\beta t, t) + \beta)$$

is also a solution of (1) for all solution $V = (h, u)$ and all $\beta \in \mathbb{R}$. This fact has been mentioned in [20,2] for the case $\mu = 0$. It is easy to check that the second order viscosity right hand side does not change this symmetry group. Hence, from now on, all the equilibriums considered in this work are of the form

$$V_e = (h_e, 0).$$

The key of this study is the following proposition which is a consequence of the primitive estimates in $\mathbb{X}^s$ and the estimation of the time integral of the $\mathbb{H}^{s-1}$ norm of h_x obtained in Section 3. In order to understand this study, let us mention that symbol $C_S(\delta)$ stands for a function of δ, defined by the elements of the set S, which converges to a strictly positive limit while δ goes to 0. On the other hand, $\Theta_S(\delta)$ stands for a function, defined by the elements of the set S, which converges to zero while δ goes to 0. Let us also mention that the estimate suggested in Proposition 2.4 has a similar structure to the estimate given in Theorem 3.1 of [26].

Proposition 2.4. *Let us consider an equilibrium $V_e = (h_e, 0)$ of System (1), an integer $s \ge 2$ and $\delta > 0$ such that the system is locally well-posed for all initial data $V_0 \in B_s(V_e, \delta)$. Assume also that there exists $T > 0$ such that the unique local solution V satisfies $V(t) \in B_s(V_e, \delta)$ for all $0 \le t < T$. Then, the following estimate holds true for all $t \in [0, T)$,*

$$(1 - \Theta_{\{h_e,\alpha\}}(\delta)) \parallel V(t) - V_e \parallel^2_{\mathbb{X}^s} + C_{\{h_e,\mu\}}(\delta) \int_0^t \parallel \partial_x u \parallel^2_{\mathbb{H}^s} \le C_{\{h_e,\alpha\}}(\delta) \parallel V(0) - V_e \parallel^2_{\mathbb{X}^s}$$
$$+ \Theta_{\{h_e,\mu,\alpha\}}(\delta) \int_0^t \parallel \partial_x u \parallel^2_{\mathbb{H}^s}$$

Besides, if δ is conveniently small, this inequality leads to

$$\| V(t) - V_e \|_{\mathbb{X}^s}^2 + C_{\{h_e,\mu\}}(\delta) \int_0^t \| \partial_x u \|_{\mathbb{H}^s}^2 \leq C_{\{h_e,\alpha\}}(\delta) \| V(0) - V_e \|_{\mathbb{X}^s}^2 .$$

Now, we get the global existence theorem as a result. In fact, we have

Theorem 2.5. *Let us consider an equilibrium $V_e = (h_e, 0)$ of (1) and an integer $s \geq 2$. Then, there exists $\nu > 0$ such that for all initial data $V_0 = (h_0, u_0) \in B_s(V_e, \nu)$, the solution V exists for all time.*

In other words, the equilibrium solutions $V_e = (h_e, 0)$ of (1) are stable.

Proof. Let us first remark that if $\delta > 0$ is small enough, we have

$$1 - \Theta_{\{h_e,\alpha\}}(\delta) > \frac{1}{2} \quad \text{and} \quad \frac{C_{\{h_e,\mu\}}(\delta) - \Theta_{\{h_e,\mu,\alpha\}}(\delta)}{1 - \Theta_{\{h_e,\alpha\}}(\delta)} > 0.$$

Let us also assume that δ satisfies the assumptions of Proposition 2.4. Then, as long as $V \in B_s(V_e, \delta)$, it satisfies

$$\| V(t) - V_e \|_{\mathbb{X}^s}^2 + \frac{C_{\{h_e,\mu\}}(\delta) - \Theta_{\{h_e,\mu,\alpha\}}(\delta)}{1 - \Theta_{\{h_e,\alpha\}}(\delta)} \int_0^t \| \partial_x u \|_{\mathbb{H}^s}^2 \leq C_{\{g,h_e,\alpha\}}(\delta) \| V_0 - V_e \|_{\mathbb{X}^s}^2$$

Therefore, while $V \in B_s(V_e, \delta)$,

$$\| V(t) - V_e \|_{\mathbb{X}^s}^2 \leq L(\delta) \| V_0 - V_e \|_{\mathbb{X}^s}^2$$

where L is a function of δ such that $\lim_{\delta \to 0} L(\delta) = l > 0$. Setting $\nu \leq \delta$ such that $L(\delta)\nu \leq \delta/2$, we have

$$\| V(t) - V_e \|_{\mathbb{X}^s}^2 \leq \delta/2, \quad \text{while} \quad V(t) \in B_s(V_e, \delta).$$

Then, considering the uniqueness of the local solution as well as its continuity for the norm $\mathbb{X}^s$ we have the following conclusion: For $V(0) \in B_s(V_e, \nu)$, the local solution can not go out from $B_s(V_e, \delta/2)$ for any time. Therefore, the norm of the local solution does not blow up. Hence, the unique local solution exists for all time. □

Corollary 2.6 *(Asymptotic stability of equilibriums). Let $s \geq 2$ be an integer and consider the equilibrium $V_e = (h_e, 0)$ of (1). Then, there exists $\delta > 0$ such that for all initial data $V_0 = (h_0, u_0)$ in $B_s(V_e, \delta)$, the global solution $V(x,t)$ in $\mathbb{X}^s(\mathbb{R})$ of (1) converges asymptotically to V_e. In other words, $\lim_{t \to \infty} V(x,t) = V_e$ for all $x \in \mathbb{R}$.*

Proof. We use a similar logic to the one used in [26] for symmetric entropy dissipative hyperbolic systems satisfying the stability condition. We first take the x derivative of the first equation of (1), the time integral on $[t_1, t_2]$ and consider the $\mathbb{L}^2$ norm. This leads us to

$$\| \partial_x h(t_2) - \partial_x h(t_1) \|_{\mathbb{L}^2} = \| \int_{t_1}^{t_2} \partial_{xx}(hu) \|_{\mathbb{L}^2} . \tag{14}$$

Therefore,

$$\| \partial_x h(t_2) - \partial_x h(t_1) \|_{\mathbb{L}^2} \leq |t_2 - t_1| \left(\sup_{t_1 \leq t \leq t_2} \| \partial_{xx}(hu) \|_{\mathbb{L}^2} \right).$$

Considering the fact that $\| \partial_{xx}(hu) \|_{\mathbb{L}^2}$ is bounded by Proposition 2.4, there exists $\tilde{C} > 0$ such that we have for all t_1, t_2 positive,

$$| \| \partial_x h(t_1) \|_{\mathbb{H}^1 \times \mathbb{L}^2} - \| \partial_x h(t_2) \|_{\mathbb{L}^2} | \leq \| \partial_x h(t_2) - \partial_x h(t_1) \|_{\mathbb{L}^2} \leq \tilde{C} |t_2 - t_1|.$$

This means that $t \mapsto \| \partial_x h(t) \|_{\mathbb{L}^2}$ is Lipschitz continuous. On the other hand, it is $\mathbb{L}^2([0, \infty))$ according to the estimate of the same proposition together with Proposition 3.12 of Subsection 3.3. Therefore, $\| \partial_x h(t) \|_{\mathbb{L}^2}$ converges to 0 at the limit $t \to \infty$.

Let us now consider the second equation of (1) which writes [9] also

$$u_t = -u\partial_x u - \mathcal{L}_h^{-1} \partial_x \left(gh^2/2 + 2\alpha h^3 (\partial_x u)^2 - \mu h \partial_x u \right).$$

Again, we derivate with respect to x, take the $[t_1, t_2]$ time integral and consider its $\mathbb{L}^2$ norm:

$$\| \partial_x u(t_2) - \partial_x u(t_1) \|_{\mathbb{L}^2} = \| \int_{t_1}^{t_2} \partial_x \left(-u\partial_x u - \mathcal{L}_h^{-1} \partial_x \left(gh^2/2 + 2\alpha h^3 (\partial_x u)^2 - \mu h \partial_x u \right) \right) \|_{\mathbb{L}^2} .$$

Therefore,

$$\begin{aligned} &\| \partial_x u(t_2) - \partial_x u(t_1) \|_{\mathbb{L}^2} \\ &\leq |t_2 - t_1| \left(\sup_{t_1 \leq t \leq t_2} \| \partial_x \left(-u\partial_x u - \mathcal{L}_h^{-1} \partial_x \left(gh^2/2 + 2\alpha h^3 (\partial_x u)^2 - \mu h \partial_x u \right) \right) \|_{\mathbb{L}^2} \right). \end{aligned}$$

Considering the fact that $\| \partial_x \left(m\mathcal{L}_h^{-1}(m) - 2\alpha h^3 (\partial_x \mathcal{L}_h^{-1}(m))^2 + \frac{g}{2} h^2 - \mu h \partial_x u \right) \|_{\mathbb{L}^2}$ is bounded, the Lipschitz continuity of $t \mapsto \| \partial_x u(t) \|_{\mathbb{L}^2}$ is concluded. This together with the fact that $t \mapsto \| u_x(t) \|_{\mathbb{L}^2}$ is square integrable (according to the estimate of Proposition 2.4), leads to

$$\lim_{t \to \infty} \| \partial_x u(t) \|_{\mathbb{L}^2} = 0.$$

We just now need to consider Gagliardo–Nirenberg inequality

$$\| V(t) - V_e \|_{\mathbb{L}^\infty \times \mathbb{L}^\infty} \leq C \| \partial_x V(t) \|_{\mathbb{L}^2 \times \mathbb{L}^2}^{\frac{1}{2}} \| V(t) - V_e \|_{\mathbb{L}^2 \times \mathbb{L}^2}^{\frac{1}{2}} .$$

Then, considering the facts that $\| V(t) - V_e \|_{\mathbb{L}^2 \times \mathbb{L}^2}^{\frac{1}{2}}$ is bounded by $\sqrt{\delta}$ and $\| \partial_x V(t) \|_{\mathbb{L}^2 \times \mathbb{L}^2}^{\frac{1}{2}}$ converges to 0, the uniform convergence of $V(x, t)$ to V_e is concluded. $\square$

Remark 2.7. In addition to the asymptotic stability of constant solutions, the question of decay rates naturally arises. This point has been studied in [25] for linear symmetric systems of hyperbolic–parabolic type, by means of Fourier techniques in the frame of an energy method. Then, the result is used in [17] for the linearized symmetric hyperbolic–parabolic system to obtain a polynomial decay rate for the non-linear equation. The study of decay rates of linearized Green–Naghdi equations with viscosity, seems to be necessary to obtain a decay rate for the non-linear system and beyonds the scope of this work.

3. A priori estimates

The goal of this part is to obtain some a priori estimates of (1) similar to the estimate obtained in [14,26,24,17] for hyperbolic systems. To do so, we use the Hamiltonian dissipation to find a 0^{th} order estimate. We then take the ℓ^{th} order derivative of the symmetric equation and consider the scalar product with the ℓ^{th} order spatial derivative of the solution for all $1 \leq \ell \leq s$. Then, using the properties of the operators $\mathcal{A}_0(V)$ and $\mathcal{A}_1(V)$, especially the coercivity of $\mathcal{A}_0(V)$ and their symmetry, we get a ℓ^{th} order estimate for the solution $V \in B_s(V_e, \delta)$. Then, in Subsection 3.3, we get an estimation of $\int_0^T \| \partial_x^\ell h \|_{\mathbb{L}^2}^2$ for all $1 \leq \ell \leq s$ which together with the first estimates leads us to Proposition 2.4. These estimates are obtained by acting a hollow real matrix on the system. The equilibrium V_e we consider in all this section is of the form $V_e = (h_e, 0)$ and s is an integer equal or greater than 2.

3.1. Estimate in $\mathbb{X}^0$

System (1) admits a $\mathbb{X}^0$ estimation which is obtained by using the dissipation of the integral $\mathcal{H}_{h_e,0}$ of the relative energy $E_{h_e,0}$ defined in Section 1. In fact, the following proposition holds true.

Proposition 3.1. *Let $\delta, t > 0$ be small and $V_0 \in B_s(V_e, \delta)$ such that System (1) admits a unique solution $(h, u) \in C([0,t], \mathbb{X}^s(\mathbb{R}))$, with h uniformly in time, strictly positively bounded by below.*[3] *Then,*

$$\| u(t) \|_{\mathbb{H}^1}^2 \leq \frac{\mathcal{H}_{h_e,0}(h_0, u_0)}{\min\{\inf_{x\in\mathbb{R}} h(t)/2, \alpha \inf_{x\in\mathbb{R}} h^3(t)/2\}}, \tag{15}$$

and

$$\| h(t) - h_e \|_{\mathbb{L}^2}^2 \leq \frac{2}{g} \mathcal{H}_{h_e,0}(h_0, u_0) \tag{16}$$

Proof. We take the spatial integral of the both sides of the relative energy equality (3) with $u_e = 0$. On the other hand, $(h, u) \in (\mathbb{H}^s(\mathbb{R}) + h_e) \times \mathbb{H}^{s+1}(\mathbb{R})$ and $s \geq 2$. Therefore, an integration by part leads us to the dissipation of the Hamiltonian $\mathcal{H}_{h_e,0}$:

$$\frac{d}{dt} \mathcal{H}_{h_e,0}(h, u) = -\mu \int_{\mathbb{R}} h(\partial_x u)^2 \leq 0.$$

[3] The existence of such δ and t is guaranteed by Theorem 2.1 and Corollary 2.2.

In other words,

$$\mathcal{H}_{h_e,0}(h(t),u(t)) - \mathcal{H}_{h_e,0}(h(0),u(0)) = -\mu \int_0^t \int_{\mathbb{R}} h(\partial_x u)^2 \leq 0. \tag{17}$$

Thus,

$$\mathcal{H}_{h_e,0}(h(t),u(t)) \leq \mathcal{H}_{h_e,0}(h(0),u(0)). \tag{18}$$

On the other hand, $\mathcal{H}_{h_e,0}$ is defined by

$$\mathcal{H}_{h_e,0}(h,u) = \int_{\mathbb{R}} g(h-h_e)^2/2 + hu^2/2 + \alpha h^3(\partial_x u)^2/2,$$

and h is strictly positively bounded by below. Therefore,

$$\frac{g}{2} \parallel h(t) - h_e \parallel_{\mathbb{L}^2}^2 + \left(\inf_{x\in\mathbb{R}} h(t)\right) \parallel u \parallel_{\mathbb{L}^2}^2 + \alpha \left(\inf_{x\in\mathbb{R}} h(t)\right)^3 \parallel \partial_x u \parallel_{\mathbb{L}^2}^2 \leq \mathcal{H}_{h_e,0}(h(t),u(t)).$$

This together with (18) gives us the inequalities of the proposition. □

Let us also remark that the Hamiltonian $\mathcal{H}_{h_e,0}$ is locally $\mathbb{X}^0$-quadratic on V_e, in the sense that the following relation is satisfied for $s \geq 2$ and $\delta > 0$ small:

$$C_{\{h_e,\alpha\}}(\delta) \parallel V - V_e \parallel_{\mathbb{X}^0}^2 \leq \mathcal{H}_{h_e,0}(h,u) \leq C_{\{h_e,\alpha\}}(\delta) \parallel V - V_e \parallel_{\mathbb{X}^0}^2 \quad \forall V \in B_s(V_e,\delta).$$

This together with the dissipation equality (17) of $\mathcal{H}_{h_e,0}$ gives us the following 0^{th} order estimate around equilibriums.

Proposition 3.2. *Let $s \geq 2$ be an integer and V_e be an equilibrium of (1). Let us also assume that there exist $\delta, T > 0$ such that the solution V of the system satisfies*

$$V(t) \in B_s(V_e,\delta) \quad \forall t \in [0,T).$$

Then, the following estimate holds true for such time:

$$\parallel V(t) - V_e \parallel_{\mathbb{X}^0}^2 + C_{\{h_e,\mu,\alpha\}}(\delta) \int_0^t \parallel \partial_x u \parallel_{\mathbb{L}^2}^2 \leq C_{\{h_e,\alpha\}}(\delta) \parallel V(0) - V_e \parallel_{\mathbb{X}^0}^2 . \tag{19}$$

3.2. Estimate in $\mathbb{X}^s$

The main objective of this part is to obtain a convenient a priori estimate of ℓ^{th} order, for all integer $\ell \in [1,s]$. This is done by a similar strategy as for hyperbolic systems. This analogy

Please cite this article in press as: D. Kazerani, Global existence for small data of the viscous Green–Naghdi type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.03.022

works here due to the structure of differential operators $\mathcal{A}_0$ and $\mathcal{A}_1$. More precisely, the operator $\mathcal{A}_0$ writes

$$\mathcal{A}_0(V) = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} + \begin{pmatrix} -3\alpha h(\partial_x u)^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\alpha\partial_x(h^3\partial_x \bullet) \end{pmatrix}. \tag{20}$$

Likewise, $\mathcal{A}_1(V)$ writes

$$\mathcal{A}_1(V) = \begin{pmatrix} gu & gh \\ gh & hu \end{pmatrix} + \begin{pmatrix} -3\alpha hu(\partial_x u)^2 & -3\alpha h^2(\partial_x u)^2 \\ -3\alpha h^2(\partial_x u)^2 & 2\alpha\partial_x(h^3\partial_x u) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\alpha\partial_x(h^3 u\partial_x \bullet) \end{pmatrix}. \tag{21}$$

Indeed, the first term of the right-hand sides of (20) and (21) gives the hyperbolic part of the system i.e. the part which corresponds to Saint-Venant system. Therefore, it can be treated exactly as in [17]. The other terms need a specific treatment but they are not an obstacle to the result. On the one hand, this is due to the fact that the space of local well-posedness for u, is one order higher in regularity compared to the case of the hyperbolic Saint-Venant system. On the other hand, the conservative structure of the last term of (20) and (21) plays an important role in the treatment of the third order terms of (1), responsible for dispersion. For this reason, all along this subsection, different terms of operators $\mathcal{A}_0^1\partial_x$ and $\mathcal{A}_0^2\partial_x^2$ (resp. $\mathcal{A}_1^1\partial_x$ and $\mathcal{A}_1^2\partial_x^2$), introduced by (10) (resp. by (11)), are matched together to form the conservative term presented in the last part of the right hand side of (20) (resp. (21)).

We start the computations by taking the ℓ^{th} derivative of (7) with respect to the spatial variable, taking the scalar product with $\partial_x^\ell V$ and integrating on $[0, T) \times \mathbb{R}$:

$$\int_0^T\int_{\mathbb{R}} \partial_x^\ell(\mathcal{A}_0(V)\partial_t V)\cdot\partial_x^\ell V + \int_0^T\int_{\mathbb{R}} \partial_x^\ell(\mathcal{A}_1(V)\partial_x V)\cdot\partial_x^\ell V = \mu\int_0^T\int_{\mathbb{R}} \partial_x^{\ell+1}(h\partial_x u)\partial_x^\ell u \tag{22}$$

Then, using basic computations and the Leibniz formula, we remark that[4]

$$\begin{aligned}\int_{\mathbb{R}} \partial_x^\ell(\mathcal{A}_0(V)\partial_t V)\cdot\partial_x^\ell V = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \mathcal{A}_0(V)\partial_x^\ell V\cdot\partial_x^\ell V - \frac{1}{2}\int_{\mathbb{R}} \left(\mathcal{A}_{0t}^0 + \mathcal{A}_{0t}^1\partial_x + \mathcal{A}_{0t}^2\partial_x^2\right)\partial_x^\ell V\cdot\partial_x^\ell V \\ + \sum_{i=1}^{\ell}\binom{\ell}{i}\int_{\mathbb{R}} \left(\mathcal{A}_{0i}^0 + \mathcal{A}_{0i}^1\partial_x + \mathcal{A}_{0i}^2\partial_x^2\right)\partial_t\partial_x^{\ell-i}V\cdot\partial_x^\ell V,\end{aligned}$$

where, $\mathcal{A}_{0i}^j$ is another notation for $\partial_x^i(\mathcal{A}_0^j(V))$, the i^{th} spatial derivative of $\mathcal{A}_0^j(V)$, for all $j \in \{0, 1, 2\}$ and for any $i \in \mathbb{N}$.

On the other hand, the integration by part and the symmetry of $\mathcal{A}_1$ imply that

$$\begin{aligned}\int_{\mathbb{R}} \partial_x^\ell(\mathcal{A}_1(V)\partial_x V)\cdot\partial_x^\ell V = \left(\ell - \frac{1}{2}\right)\int_{\mathbb{R}} \left(\mathcal{A}_{1x}^0 + \mathcal{A}_{1x}^1\partial_x + \mathcal{A}_{1x}^2\partial_x^2\right)\partial_x^\ell V\cdot\partial_x^\ell V \\ + \sum_{i=2}^{\ell}\binom{\ell}{i}\int_{\mathbb{R}} \left(\mathcal{A}_{1i}^0 + \mathcal{A}_{1i}^1\partial_x + \mathcal{A}_{1i}^2\partial_x^2\right)\partial_x^{\ell-i+1}V\cdot\partial_x^\ell V.\end{aligned}$$

[4] For sake of simplicity, we use sometimes $\mathcal{A}_1$ or $\mathcal{A}_0$ instead of $\mathcal{A}_1(V)$ or $\mathcal{A}_0(V)$.

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We have also

$$\int_{\mathbb{R}} \partial_x^{\ell+1}(h\partial_x u)\partial_x^\ell u = -\int_{\mathbb{R}} h(\partial_x^{\ell+1}u)^2 - \sum_{i=1}^{\ell}\binom{\ell}{i}\int_{\mathbb{R}} (\partial_x^i h)(\partial_x^{\ell-i+1}u)(\partial_x^{\ell+1}u).$$

Hence, (22) becomes

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{A}_0(V)\partial_x^\ell V(T)\cdot\partial_x^\ell V(T) + 2\mu\int_0^T\int_{\mathbb{R}} h(\partial_x^{\ell+1}u)^2 &= \int_{\mathbb{R}} \mathcal{A}_0(V)\partial_x^\ell V(0)\cdot\partial_x^\ell V(0)\\
&- 2\sum_{i=1}^{\ell}\binom{\ell}{i}\int_0^T\int_{\mathbb{R}}\left(\mathcal{A}_{0i}^0 + \mathcal{A}_{0i}^1\partial_x + \mathcal{A}_{0i}^2\partial_x^2\right)\partial_t\partial_x^{\ell-i}V\cdot\partial_x^\ell V\\
&+ \int_0^T\int_{\mathbb{R}}\left(\mathcal{A}_{0t}^0 + \mathcal{A}_{0t}^1\partial_x + \mathcal{A}_{0t}^2\partial_x^2\right)\partial_x^\ell V\cdot\partial_x^\ell V\\
&+ (1-2\ell)\int_0^T\int_{\mathbb{R}}\left(\mathcal{A}_{1x}^0 + \mathcal{A}_{1x}^1\partial_x + \mathcal{A}_{1x}^2\partial_x^2\right)\partial_x^\ell V\cdot\partial_x^\ell V\\
&- 2\sum_{i=2}^{\ell}\binom{\ell}{i}\int_0^T\int_{\mathbb{R}}\left(\mathcal{A}_{1i}^0 + \mathcal{A}_{1i}^1\partial_x + \mathcal{A}_{1i}^2\partial_x^2\right)\partial_x^{\ell-i+1}V\cdot\partial_x^\ell V\\
&- 2\mu\sum_{i=1}^{\ell}\binom{\ell}{i}\int_{\mathbb{R}}(\partial_x^i h)(\partial_x^{\ell-i+1}u)(\partial_x^{\ell+1}u). \qquad (23)\end{aligned}$$

The two following lemmas present two results which are used several times in the rest of this section. The first one is on the $\mathbb{X}^0$-quadraticity of $\mathcal{A}_0(V)$:

Lemma 3.3. *There exists $\delta > 0$ such that $\mathcal{A}_0(V)$ is quadratic on $B_s(V_e, \delta)$. In other words, we have for all $V = (h, u) \in B_s(V_e, \delta)$ and all $f = (f_1, f_2) \in \mathbb{X}^0(\mathbb{R})$,*

$$C_{\{h_e\}}(\delta) \parallel f \parallel_{\mathbb{X}^0}^2 \leq \int_{\mathbb{R}} \mathcal{A}_0(V)f\cdot f \leq C_{\{h_e\}}(\delta) \parallel f \parallel_{\mathbb{X}^0}^2 .$$

Proof. The expression (8) of $\mathcal{A}_0(V)$ leads to

$$\int_{\mathbb{R}} \mathcal{A}_0(V)f\cdot f = \int_{\mathbb{R}} (g - 3\alpha h(\partial_x u)^2)f_1^2 + f_2\mathcal{L}_h f_2.$$

On the other hand, Gagliardo–Nirenberg inequality (12) leads to

$$\parallel h - h_e \parallel_{\mathbb{L}^\infty} \leq C\delta,$$

or equivalently to

$$h_e - C\delta \leq h(x) \leq h_e + C\delta \quad \forall x \in \mathbb{R}.$$

We also apply this inequality to $\partial_x u$ to get

$$\| \partial_x u \|_{\mathbb{L}^\infty} \leq C\delta,$$

or equivalently

$$-C\delta \leq \partial_x u(x) \leq C\delta \quad \forall x \in \mathbb{R}.$$

Thus,

$$-3\alpha h(\partial_x u)^2 \geq -3\alpha\,(h_e + C\delta)\,(C^2\delta^2),$$

and, if δ is conveniently small,

$$-3\alpha h(\partial_x u)^2 \geq -\frac{g}{2}.$$

Consequently

$$\frac{g \| f_1 \|^2_{\mathbb{L}^2}}{2} + \min\{h_e - \delta, \alpha(h_e - \delta)^3\} \| f_2 \|^2_{\mathbb{H}^1} \leq \int_{\mathbb{R}} \mathcal{A}_0(V) f \cdot f$$
$$\leq g \| f_1 \|^2_{\mathbb{L}^2} + \max\{h_e + \delta, \alpha(h_e + \delta)^3\} \| f_2 \|^2_{\mathbb{H}^1} . \quad \square$$

The second lemma is on the smallness of the $\mathbb{L}^\infty$ norm (in time and space) of $\partial_t h$ and $\partial_t u$ and some of their spatial derivatives as long as $V \in B_s(V_e, \delta)$. Actually, the following lemma holds true.

Lemma 3.4. *Let us assume that the solution $V(t)$ of (1) belongs to $B_s(V_e, \delta)$ for all $t \in [0, T)$. Then, we have for all $0 \leq j \leq s-2$ and all $0 \leq l \leq s-1$,*

$$\lim_{\substack{\delta \to 0 \\ V \in B(V_e,\delta)}} \| \partial_x^j \partial_t h \|_{\mathbb{L}^\infty} = 0, \quad \lim_{\substack{\delta \to 0 \\ V \in B(V_e,\delta)}} \| \partial_x^l \partial_t u \|_{\mathbb{L}^\infty} = 0. \tag{24}$$

Moreover, we have for all $2 \leq k \leq s$,

$$\| \partial_x^k \partial_t u \|_{\mathbb{L}^2} \leq C_{\{h_e,\mu,\alpha\}}(\delta) \left(\| \partial_x u \|_{\mathbb{H}^k} + \| \partial_x h \|_{\mathbb{H}^{k-2}} \right). \tag{25}$$

Proof. The first equation of System (1) gives us $\partial_t h = -h\partial_x u - u\partial_x h$. Therefore,

$$\| \partial_x^j \partial_t h \|_{\mathbb{L}^\infty} \leq \| \sum_{i=0}^{j} \partial_x^i h \partial_x^{j-i+1} u \|_{\mathbb{L}^\infty} + \| \sum_{i=0}^{j} \partial_x^i u \partial_x^{j-i+1} h \|_{\mathbb{L}^\infty} \leq \Theta_{\{h_e\}}(\delta)$$

Likewise, the second equation of the system can be written under the following form,

$$\partial_t u = -u\partial_x u - \mathcal{L}_h^{-1}\partial_x\left(gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h\partial_x u\right). \tag{26}$$

This form can be obtained by applying $\mathcal{A}_0(V)^{-1}$ to (7) and coincides with the form suggested in [16,19].

On the other hand, $\mathcal{L}_h : \mathbb{H}^m(\mathbb{R}) \to \mathbb{H}^{m-2}(\mathbb{R})$ is bounded for all $2 \le m \le s$. This is due to the facts that $\| h - h_e \|_{\mathbb{H}^s} \le \delta$ and δ is small. Indeed,

$$\| \mathcal{L}_h(u) \|_{\mathbb{H}^{m-2}} = \| hu - 3\alpha h^2\partial_x h\partial_x u - \alpha h^3\partial_x^2 u \|_{\mathbb{H}^{m-2}} \le C_{\{h_e,\alpha\}}(\delta) \| u \|_{\mathbb{H}^m} .$$

Therefore, $\mathcal{L}_h$ is a linear bijective bounded application from the Banach space $\mathbb{H}^m(\mathbb{R})$ to the Banach space $\mathbb{H}^{m-2}(\mathbb{R})$. We now use the Banach theorem (see [6] for instance) to conclude that $\mathcal{L}_h^{-1} : \mathbb{H}^{m-2}(\mathbb{R}) \to \mathbb{H}^m(\mathbb{R})$ is bounded. Thus, there exists $C > 0$ such that

$$\begin{aligned}
\| \partial_x^\ell\partial_t u \|_{\mathbb{L}^\infty} &\le \| \partial_x^\ell(u\partial_x u) \|_{\mathbb{L}^\infty} + \| \partial_x^\ell\mathcal{L}_h^{-1}\partial_x\left(gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h\partial_x u\right) \|_{\mathbb{L}^\infty} \\
&\le \Theta(\delta) + C \| \partial_x^l\mathcal{L}_h^{-1}\partial_x\left(gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h\partial_x u\right) \|_{\mathbb{H}^1} \\
&\le \Theta(\delta) + C \| \mathcal{L}_h^{-1}\partial_x\left(gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h\partial_x u\right) \|_{\mathbb{H}^{\ell+1}} \\
&\le \Theta(\delta) + C_{\{h_e,\alpha\}}(\delta) \| \partial_x\left(gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h\partial_x u\right) \|_{\mathbb{H}^{\ell-1}} \\
&\le \Theta(\delta) + C_{\{h_e,\alpha\}}(\delta) \| \partial_x\left(gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h\partial_x u\right) \|_{\mathbb{H}^{s-2}} \\
&\le \Theta(\delta) + C_{\{h_e,\alpha\}}(\delta)\Theta_{\{h_e,\alpha,\mu\}}(\delta) \le \Theta_{\{h_e,\alpha,\mu\}}(\delta).
\end{aligned}$$

To prove (25), we use similar computations. Indeed,

$$\begin{aligned}
\| \partial_x^k\partial_t u \|_{\mathbb{L}^2} &\le \| \partial_x^k(u\partial_x u) \|_{\mathbb{L}^2} + \| \partial_x^k\mathcal{L}_h^{-1}\partial_x\left(gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h\partial_x u\right) \|_{\mathbb{L}^2} \\
&\le \sum_{i=0}^{k} \| \partial_x^{k-i}u\ \partial_x^{i+1}u \|_{\mathbb{L}^2} + \| \mathcal{L}_h^{-1}\partial_x\left(gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h\partial_x u\right) \|_{\mathbb{H}^k} \\
&\le \Theta(\delta) \| \partial_x u \|_{\mathbb{H}^k} + \| \partial_x\left(gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h\partial_x u\right) \|_{\mathbb{H}^{k-2}} \\
&\le \Theta(\delta) \| \partial_x u \|_{\mathbb{H}^k} + C_{\{h_e\}}(\delta) \| \partial_x h \|_{\mathbb{H}^{k-2}} + \left(\Theta_{\{\alpha,h_e\}}(\delta) + C_{\{h_e,\mu\}}(\delta)\right) \| \partial_x u \|_{\mathbb{H}^{k-2}} \\
&\le C_{\{h_e,\mu,\alpha\}}(\delta)\left(\| \partial_x u \|_{\mathbb{H}^k} + \| \partial_x h \|_{\mathbb{H}^{k-2}}\right). \quad \square
\end{aligned}$$

We are now able to prove the following lemma which is the key step to achieve the appropriate ℓ^{th} order estimate.

Lemma 3.5. *Let us consider the solution V of (7) and assume that it belongs to $B_s(V_e, \delta)$ for some $\delta > 0$. Then, the following estimates hold true for all integer $1 \le \ell \le s$,*

$$\left|\sum_{i=1}^{\ell}\binom{\ell}{i}\int_{\mathbb{R}} \mathcal{A}^0_{0i}\partial_x^{\ell-i}\partial_t V\cdot\partial_x^{\ell}V\right| \le \Theta_{\{h_e,\alpha,\mu\}}(\delta)\left(\sum_{j=1}^{\ell+1}\|\partial_x^j u\|^2_{\mathbb{L}^2}+\sum_{j=1}^{\ell}\|\partial_x^j h\|^2_{\mathbb{L}^2}\right). \quad (27)$$

$$\left|\int_{\mathbb{R}} \mathcal{A}^0_{0t}\partial_x^{\ell}V\cdot\partial_x^{\ell}V\right| \le \Theta_{\{h_e,\alpha,\mu\}}(\delta)\left(\|\partial_x^{\ell}u\|^2_{\mathbb{L}^2}+\|\partial_x^{\ell}h\|^2_{\mathbb{L}^2}\right). \quad (28)$$

$$\left|\sum_{i=1}^{\ell}\binom{\ell}{i}\int_{\mathbb{R}}\left(\mathcal{A}^1_{0i}\partial_x+A^2_{0i}\partial_x^2\right)\partial_x^{\ell-i}\partial_t V\cdot\partial_x^{\ell}V\right| \le \Theta_{\{h_e,\alpha,\mu\}}(\delta)\left(\sum_{i=1}^{\ell}\|\partial_x^i h\|^2_{\mathbb{L}^2}+\|\partial_x u\|^2_{\mathbb{H}^{\ell}}\right). \quad (29)$$

$$\left|\int_{\mathbb{R}}\left(\mathcal{A}^1_{0t}\partial_x+A^2_{0t}\partial_x^2\right)\partial_x^{\ell}V\cdot\partial_x^{\ell}V\right| \le \Theta_{\{h_e,\alpha,\mu\}}(\delta)\left(\|\partial_x^{\ell+1}u\|^2_{\mathbb{L}^2}\right). \quad (30)$$

$$\left|\int_{\mathbb{R}}\left(\mathcal{A}^0_{1x}+\mathcal{A}^1_{1x}\partial_x+\mathcal{A}^2_{1x}\partial_x^2\right)\partial_x^{\ell}V\cdot\partial_x^{\ell}V\right| \le \Theta_{\{h_e,\alpha\}}(\delta)\left(\|\partial_x^{\ell}h\|^2_{\mathbb{L}^2}+\|\partial_x^{\ell}u\|^2_{\mathbb{H}^1}\right). \quad (31)$$

$$\left|\sum_{i=2}^{\ell}\binom{\ell}{i}\int_{\mathbb{R}}\left(\mathcal{A}^0_{1i}+\mathcal{A}^1_{1i}\partial_x+\mathcal{A}^2_{1i}\partial_x^2\right)\partial_x^{\ell-i+1}V\cdot\partial_x^{\ell}V\right| \le \Theta_{\{h_e,\alpha\}}(\delta)\left(\|\partial_x h\|^2_{\mathbb{H}^{\ell-1}}+\|\partial_x u\|^2_{\mathbb{H}^{\ell}}\right). \quad (32)$$

Proof. Let us first prove (27). The expression of $\mathcal{A}^0_0$ gives us the following equality for all $1 \le i \le \ell$,

$$\int_{\mathbb{R}} \mathcal{A}^0_{0i}\partial_x^{\ell-i}\partial_t V\cdot\partial_x^{\ell}V = -3\alpha\partial_x^i(h(\partial_x u)^2)\,\partial_x^{\ell-i}\,\partial_t h\,\partial_x^{\ell}h+\partial_x^i h\,\partial_x^{\ell-i}\partial_t u\,\partial_x^{\ell}u.$$

Therefore,

$$\begin{aligned}\left|\int_{\mathbb{R}} \mathcal{A}^0_{0i}\partial_x^{\ell-i}\partial_t V\cdot\partial_x^{\ell}V\right| &\le \frac{\|\partial_x^{\ell-i}\partial_t u\|_{\mathbb{L}^\infty}}{2}\left(\|\partial_x^i h\|^2_{\mathbb{L}^2}+\|\partial_x^{\ell}u\|^2_{\mathbb{L}^2}\right)\\ &+\|\partial_x^{\ell-i}\partial_t h\|_{\mathbb{L}^\infty}\left(\|\partial_x^{\ell}h\|^2_{\mathbb{L}^2}+\|3\alpha\partial_x^i h\,(\partial_x u)^2\|^2_{\mathbb{L}^2}+\|6\alpha\sum_{j=0}^{i-1}\partial_x^j h\,\partial_x u\,\partial_x^{i-j+1}u\|^2_{\mathbb{L}^2}\right)\\ &\le \max\left\{\frac{\|\partial_x^{\ell-i}\partial_t u\|_{\mathbb{L}^\infty}}{2}, C_{h_e,\alpha}(\delta)\|\partial_x^{\ell-i}\partial_t h\|_{\mathbb{L}^\infty}\right\}\left(\|\partial_x h\|^2_{\mathbb{H}^{\ell-1}}+\|\partial_x u\|^2_{\mathbb{H}^{\ell}}\right).\end{aligned}$$

Then, considering (24), the proof of (27) is complete.

We are now going to prove (28). To do so, we should first remark that

$$\int_{\mathbb{R}} \mathcal{A}^0_{0t}\partial_x^{\ell}V\cdot\partial_x^{\ell}V = -3\alpha\partial_t\left(h(\partial_x u)^2\right)(\partial_x^{\ell}h)^2+h_t(\partial_x^{\ell}u)^2.$$

Then,

$$\left|\int_{\mathbb{R}} \mathcal{A}^0_{0t}\partial_x^\ell V \cdot \partial_x^\ell V\right| \leq \| 3\alpha\partial_t\left(h(\partial_x u)^2\right)\|_{\mathbb{L}^\infty}\| \partial_x^\ell h \|^2_{\mathbb{L}^2} + \| h_t \|_{\mathbb{L}^\infty}\| \partial_x^\ell u \|^2_{\mathbb{L}^2}.$$

Now, we use (24) to get the result.

The first step to prove (29) is to notice that we have for all $1 \leq i \leq \ell$

$$\int_{\mathbb{R}} \left(\mathcal{A}^1_{0i}\partial_x + \mathcal{A}^2_{0i}\partial_x^2\right)\partial_x^{\ell-i}\partial_t V \cdot \partial_x^\ell V = \alpha\int_{\mathbb{R}} (\partial_x^i h^3)\ \partial_x^{\ell+1}u\ (\partial_x^{\ell-i+1}\partial_t u).$$

Hence we have for all $2 \leq i \leq \ell$,

$$\left|\int_{\mathbb{R}} \left(\mathcal{A}^1_{0i}\partial_x + \mathcal{A}^2_{0i}\partial_x^2\right)\partial_x^{\ell-i}\partial_t V \cdot \partial_x^\ell V\right| \leq \frac{\| \alpha\partial_x^{\ell-i+1}\partial_t u \|_{\mathbb{L}^\infty}}{2}\left(\| \partial_x^i h^3 \|^2_{\mathbb{L}^2} + \| \partial_x^{\ell+1}u\ \|^2_{\mathbb{L}^2}\right)$$

$$\leq \frac{\| \alpha\partial_x^{\ell-i+1}\partial_t u \|_{\mathbb{L}^\infty}}{2}\left(C_{\{h_e\}}(\delta)\sum_{j=1}^{i}\| \partial_x^i h \|^2_{\mathbb{L}^2} + \| \partial_x^{\ell+1}u\ \|^2_{\mathbb{L}^2}\right).$$

Considering (24), we obtain the estimate on the terms where $2 \leq i \leq \ell$. It remains to consider the case $i = 1$. This leads to

$$\int_{\mathbb{R}} \left(\mathcal{A}^1_{01}\partial_x + \mathcal{A}^2_{01}\partial_x^2\right)\partial_x^{\ell-1}\partial_t V \cdot \partial_x^\ell V = \alpha\int_{\mathbb{R}} (\partial_x h^3)\ \partial_x^{\ell+1}u\ (\partial_x^{\ell}\partial_t u).$$

Therefore,

$$\left|\int_{\mathbb{R}} \left(\mathcal{A}^1_{01}\partial_x + \mathcal{A}^2_{01}\partial_x^2\right)\partial_x^{\ell-1}\partial_t V \cdot \partial_x^\ell V\right| \leq \frac{\| \alpha\partial_x(h^3) \|_{\mathbb{L}^\infty}}{2}\left(\| \partial_x^\ell\partial_t u \|^2_{\mathbb{L}^2} + \| \partial_x^{\ell+1}u\ \|^2_{\mathbb{L}^2}\right)$$

$$\leq \Theta_{\{\alpha,h_e\}}(\delta)\left(\| \partial_x^\ell\partial_t u \|^2_{\mathbb{L}^2} + \| \partial_x^{\ell+1}u\ \|^2_{\mathbb{L}^2}\right).$$

We now use (25) and find

$$\left|\int_{\mathbb{R}} \left(\mathcal{A}^1_{01}\partial_x + \mathcal{A}^2_{01}\partial_x^2\right)\partial_x^{\ell-1}\partial_t V \cdot \partial_x^\ell V\right| \leq \Theta_{\{h_e,\alpha,\mu\}}(\delta)\left(\| \partial_x h \|^2_{\mathbb{H}^{\ell-2}} + \| \partial_x u\ \|^2_{\mathbb{H}^\ell}\right).$$

In order to prove (30), we first remark that

$$\int_{\mathbb{R}} \left(\mathcal{A}^1_{0t}\partial_x + \mathcal{A}^2_{0t}\partial_x^2\right)\partial_x^{\ell} V \cdot \partial_x^\ell V = \int_{\mathbb{R}} 3\alpha h^2\partial_t h(\partial_x^{\ell+1}u)^2.$$

Again, using (24), we find

$$\left| \int_{\mathbb{R}} \left(\mathcal{A}_{0t}^{1} \partial_x + \mathcal{A}_{0t}^{2} \partial_x^2 \right) \partial_x^{\ell} V \cdot \partial_x^{\ell} V \right| \leq \Theta_{\{h_e,\alpha\}}(\delta) \parallel \partial_x^{\ell+1} u \parallel_{\mathbb{L}^2}^2 .$$

To prove (31), we use an integration by part:

$$\begin{aligned} \int_{\mathbb{R}} \left(\mathcal{A}_{1x}^{1} \partial_x + \mathcal{A}_{1x}^{2} \partial_x^2 \right) \partial_x^{\ell} V \cdot \partial_x^{\ell} V &= \int_{\mathbb{R}} -\alpha \partial_x^2 (h^3 u) \partial_x^{\ell+1} u \partial_x^{\ell} u - \alpha \partial_x (h^3 u) \partial_x^{\ell+2} u \partial_x^{\ell} u \\ &= \int_{\mathbb{R}} -\alpha \partial_x \left(\partial_x (h^3 u) \partial_x^{\ell+1} u \right) \partial_x^{\ell+1} u = \int_{\mathbb{R}} \alpha \partial_x (h^3 u) (\partial_x^{\ell+1} u)^2. \end{aligned}$$

Hence,

$$\left| \int_{\mathbb{R}} \left(\mathcal{A}_{1x}^{1} \partial_x + \mathcal{A}_{1x}^{2} \partial_x^2 \right) \partial_x^{\ell} V \cdot \partial_x^{\ell} V \right| \leq \Theta_{\{h_e,\alpha\}}(\delta) \parallel \partial_x^{\ell+1} u \parallel_{\mathbb{L}^2}^2 .$$

We have also

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{A}_{1x}^{0} \partial_x^{\ell} V \cdot \partial_x^{\ell} V = &\int_{\mathbb{R}} \partial_x \left(gu - 3\alpha h u (\partial_x u)^2 \right) \, (\partial_x^{\ell} h)^2 + 2 \int_{\mathbb{R}} \partial_x \left(gh - 3\alpha h^2 (\partial_x u)^2 \right) \, \partial_x^{\ell} h \, \partial_x^{\ell} u \\ &+ \int_{\mathbb{R}} \partial_x \left(hu + 2\alpha \partial_x (h^3 \partial_x u) \right) \, (\partial_x^{\ell} u)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{A}_{1x}^{0} \partial_x^{\ell} V \cdot \partial_x^{\ell} V = &\int_{\mathbb{R}} \partial_x \left(gu - 3\alpha h u (\partial_x u)^2 \right) \, (\partial_x^{\ell} h)^2 + 2 \int_{\mathbb{R}} \partial_x \left(gh - 3\alpha h^2 (\partial_x u)^2 \right) \, \partial_x^{\ell} h \, \partial_x^{\ell} u \\ &- 2 \int_{\mathbb{R}} \left(hu + 2\alpha \partial_x (h^3 \partial_x u) \right) \, \partial_x^{\ell} u \, \partial_x^{\ell+1} u. \end{aligned}$$

Then,

$$\left| \int_{\mathbb{R}} \mathcal{A}_{1x}^{0} \partial_x^{\ell} V \cdot \partial_x^{\ell} V \right| \leq \Theta_{\{h_e,\alpha\}}(\delta) \left(\parallel \partial_x^{\ell} h \parallel_{\mathbb{L}^2}^2 + \parallel \partial_x^{\ell} h \parallel_{\mathbb{H}^1}^2 \right).$$

The last estimate (32) is just a consequence of the following fact which holds true for all $2 \leq i \leq \ell$. It is due to the structure of $\mathcal{A}_1^1(V)$ and $\mathcal{A}_1^2(V)$ together with an integration by part:

$$\int_{\mathbb{R}} \left(\mathcal{A}_{1i}^{1} \partial_x + \mathcal{A}_{1i}^{2} \partial_x^2 \right) \partial_x^{\ell-i+1} V \cdot \partial_x^{\ell} V = \alpha \int_{\mathbb{R}} (\partial_x^{i} (h^3 u)) \, \partial_x^{\ell-i+2} u \, (\partial_x^{\ell+1} u).$$

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Hence, as long as $V \in B_s(V_e, \delta)$ and $2 \le i \le \ell - 1$,

$$\left| \int_{\mathbb{R}} \left(\mathcal{A}^1_{1i} \partial_x + \mathcal{A}^2_{1i} \partial_x^2 \right) \partial_x^{\ell-i+1} V \cdot \partial_x^\ell V \right| \le \frac{\| \alpha \partial_x^i (h^3 u) \|_{\mathbb{L}^\infty}}{2} \left(\| \partial_x^{\ell-i+2} u \|^2_{\mathbb{L}^2} + \| \partial_x^{\ell+1} u \ \|^2_{\mathbb{L}^2} \right)$$
$$\le \Theta_{\{h_e,\alpha\}}(\delta) \left(\| \partial_x^{\ell-i+2} u \|^2_{\mathbb{L}^2} + \| \partial_x^{\ell+1} u \ \|^2_{\mathbb{L}^2} \right).$$

On the other hand,

$$\int_{\mathbb{R}} \left(\mathcal{A}^1_{1\ell} \partial_x + \mathcal{A}^2_{1\ell} \partial_x^2 \right) \partial_x V \cdot \partial_x^\ell V = \alpha \int_{\mathbb{R}} (\partial_x^\ell (h^3 u))\ \partial_x^2 u\ (\partial_x^{\ell+1} u).$$

Therefore,

$$\left| \int_{\mathbb{R}} \left(\mathcal{A}^1_{1\ell} \partial_x + \mathcal{A}^2_{1\ell} \partial_x^2 \right) \partial_x V \cdot \partial_x^\ell V \right| \le \Theta_{\{\alpha\}}(\delta) \left(\| \partial_x^\ell (h^3 u) \|^2_{\mathbb{L}^2} + \| \partial_x^{\ell+1} u \|^2_{\mathbb{L}^2} \right)$$
$$\le \Theta_{\{h_e,\alpha\}}(\delta) \left(\| \partial_x h \|^2_{\mathbb{H}^{\ell-1}} + \| \partial_x u \|^2_{\mathbb{H}^\ell} \right).$$

Let us now treat the remaining terms of the left hand side of the estimate. In fact, we have for all $2 \le i \le \ell - 2$,

$$\left| \int_{\mathbb{R}} \mathcal{A}^0_{1i} \partial_x^{\ell-i+1} V \cdot \partial_x^\ell V \right| \le 2 \| \mathcal{A}^0_{1i} \|_{\mathbb{L}^\infty} \left(\| \partial_x^{\ell-i+1} u \|^2_{\mathbb{L}^2} + \| \partial_x^\ell u \|^2_{\mathbb{L}^2} + \| \partial_x^{\ell-i+1} h \|^2_{\mathbb{L}^2} + \| \partial_x^\ell h \|^2_{\mathbb{L}^2} \right)$$
$$\le \Theta_{\{h_e,\alpha\}}(\delta) \left(\| \partial_x^{\ell-i+1} u \|^2_{\mathbb{L}^2} + \| \partial_x^\ell u \|^2_{\mathbb{L}^2} + \| \partial_x^{\ell-i+1} h \|^2_{\mathbb{L}^2} + \| \partial_x^\ell h \|^2_{\mathbb{L}^2} \right),$$

since the structure of $\mathcal{A}^0_1$ gives us for all integer $i \in [2, \ell - 2]$,

$$\lim_{\substack{\delta \to 0 \\ V \in B_s(V_e,\delta)}} \| \mathcal{A}^0_{1i} \|_{\mathbb{L}^\infty} = 0.$$

On the other hand,

$$\left| \int_{\mathbb{R}} \mathcal{A}^0_{1(\ell-1)} \partial_x^2 V \cdot \partial_x^\ell V + \mathcal{A}^0_{1\ell} \partial_x V \cdot \partial_x^\ell V \right|$$
$$\le \max\{\| \partial_x^\ell (gu - 3\alpha h u (\partial_x u)^2) \|_{\mathbb{L}^\infty}, \| \partial_x^{\ell-1} (gu - 3\alpha h u (\partial_x u)^2) \|_{\mathbb{L}^\infty}\} \left(\| \partial_x h \|^2_{\mathbb{H}^1} + \| \partial_x^\ell h \|^2_{\mathbb{L}^2} \right)$$
$$+ \max\{\| \partial_x^\ell (gh - 3\alpha h^2 (\partial_x u)^2) \|_{\mathbb{L}^\infty}, \| \partial_x^{\ell-1} (gh - 3\alpha h^2 (\partial_x u)^2) \|_{\mathbb{L}^\infty}\} \left(\| \partial_x h \|^2_{\mathbb{H}^{\ell-1}} + \| \partial_x u \|^2_{\mathbb{H}^{\ell-1}} \right)$$
$$+ \left| \int_{\mathbb{R}} \partial_x \left(\partial_x^{\ell-1} (hu + 2\alpha \partial_x (h^3 \partial_x u))\ \partial_x u \right)\ \partial_x^\ell u \right|.$$

Therefore,

$$\left|\int_{\mathbb{R}} \mathcal{A}^0_{1(\ell-1)}\partial_x^2 V\cdot\partial_x^\ell V+\mathcal{A}^0_{1\ell}\partial_x V\cdot\partial_x^\ell V\right|\le\Theta_{\{h_e,\alpha\}}(\delta)\left(\|\,\partial_x h\,\|^2_{\mathbb{H}^{\ell-1}}+\|\,\partial_x u\,\|^2_{\mathbb{H}^{\ell-1}}\right)$$
$$+\left|\int_{\mathbb{R}}\partial_x^{\ell-1}(hu+2\alpha\partial_x(h^3\partial_x u))\,\partial_x u\,\partial_x^{\ell+1}u\right|$$
$$\le\Theta_{\{h_e,\alpha\}}(\delta)\left(\|\,\partial_x h\,\|^2_{\mathbb{H}^{\ell-1}}+\|\,\partial_x u\,\|^2_{\mathbb{H}^{\ell}}\right).$$

Hence, estimate (32) is totally proved. □

This lemma together with the coercivity of $\mathcal{A}_0$ and relation (23) leads us to the following propositions.

Proposition 3.6. *Let us assume that there exists $\delta>0$, $T>0$ such that the solution V of (1) satisfies $V(t)\in B_s(V_e,\delta)$ for all $t\in[0,T)$. Then, we have for all $1\le\ell\le s$,*

$$\|\,\partial_x^\ell(V(t)-V_e)\,\|^2_{\mathbb{X}^0}+C_{\{h_e,\mu\}}(\delta)\int_0^t\|\,\partial_x^{\ell+1}u\,\|^2_{\mathbb{L}^2}\le C_{\{h_e,\alpha\}}(\delta)\,\|\,\partial_x^\ell(V(0)-V_e)\,\|^2_{\mathbb{X}^0}$$
$$+\Theta_{\{h_e,\alpha,\mu\}}(\delta)\int_0^t\|\,\partial_x V\,\|^2_{\mathbb{X}^{\ell-1}}.$$

Then, considering this proposition together with the 0^{th} order estimate of Subsection 3.1, we reach the final primary estimate which is given in the following proposition. This estimate together with the result of the next part enables us to prove the main theorem.

Proposition 3.7. *Let us assume that there exists $\delta>0$, $T>0$ such that the solution V of (1) satisfies $V(t)\in B_s(V_e,\delta)$ for all $t\in[0,T)$. Then, we have for such T,*

$$\|\,V(t)-V_e\,\|^2_{\mathbb{X}^s}+C_{\{h_e,\mu\}}(\delta)\int_0^t\|\,\partial_x u\,\|^2_{\mathbb{H}^s}\le C_{\{h_e,\alpha\}}(\delta)\,\|\,V(0)-V_e\,\|^2_{\mathbb{X}^s}$$
$$+\Theta_{\{h_e,\mu\}}(\delta)\int_0^t\|\,\partial_x V\,\|^2_{\mathbb{X}^{s-1}}. \tag{33}$$

3.3. Estimate on $\int_0^t\|\,\partial_x^s h\,\|^2_{\mathbb{L}^2}$

This part is the final step to prove Proposition 2.4. In fact, we need to find a convenient estimate on $\int_0^t\|\,\partial_x V\,\|^2_{\mathbb{X}^{s-1}}$ to be able to control the right hand side of (33). This idea has been used in [26,14] and [24]. Actually, Estimate (33) has a similar appearance as the estimate found in these references for symmetric hyperbolic systems with dissipative terms. Then, they use the Kawashima stability condition to control the norm of spatial derivatives of first components of the solution. Let us note that, as in the case of hyperbolic system, we do not need to control the norm of second components. This is due to the presence of the second term of the left hand side of inequality (33). Therefore, what we need to control in the case of Green–Naghdi equation, is the time integral of the norm of the spatial derivative of h. Nevertheless, the main difficulty is the generalization of the Kawashima–Shizuta condition. Actually, we have not been able to find any operator version of the Kawashima–Shizuta condition for Green–Naghdi equation. However, we are going to see that it is possible to find an appropriate upper bound for $\int_0^t \| \partial_x^s h \|_{\mathbb{L}^2}^2$ by using a slightly different technique from the hyperbolic case. To do so, we consider the 2×2 hollow real matrix $K(V_e)$ defined by

$$K(V_e) = \begin{pmatrix} 0 & 1 \\ -\frac{h_e}{g} & 0 \end{pmatrix}. \tag{34}$$

As we will see further, the reason why we consider this matrix, is the fact that $K(V_e)\mathcal{A}_1(V_e)$ is a diagonal real matrix with a strictly positive first component. In other words, there exists a matrix of the form $B = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix}$ with $L \geq 0$ such that $K(V_e)\mathcal{A}_1(V_e) + B$ is definite positive. This enables us, as in [26,14], to get an upper bound for $\int_0^t \| \partial_x^s h \|_{\mathbb{L}^2}^2$. This upper bound is convenient even though, unlike the case of hyperbolic systems, $K(V_e)\mathcal{A}_0(V_e)$ is not a skew-symmetric operator. This is due to the fact that we can extract from $K(V_e)\mathcal{A}_0(V)$, a part which plays a quite similar role to a skew-symmetric operator such that the norm of the remaining part is controllable in a suitable manner. So, let us write (7) under the form

$$\mathcal{A}_0(V)\partial_t V + \mathcal{A}_1(V_e)\partial_x V = H(V), \tag{35}$$

where $H(V)$ is defined by

$$H(V) = [\mathcal{A}_1(V_e) - \mathcal{A}_1(V)]\, \partial_x V + \begin{pmatrix} 0 \\ \mu \partial_x (h \partial_x u) \end{pmatrix}. \tag{36}$$

We then take the action of the operator $K(V_e)\partial_x^{\ell-1}$ on (35) and take the scalar product with $\partial_x^\ell V$. This leads us to

$$\int_0^T \int_{\mathbb{R}} K(V_e)\partial_x^{\ell-1} (\mathcal{A}_0(V)\partial_t V) \cdot \partial_x^\ell V + \int_0^T \int_{\mathbb{R}} K(V_e)\mathcal{A}_1(V_e)\partial_x^\ell V \cdot \partial_x^\ell V = \int_0^T \int_{\mathbb{R}} K(V_e)\partial_x^{\ell-1} H(V) \cdot \partial_x^\ell V,$$

or equivalently to

$$\int_0^T \int_{\mathbb{R}} K(V_e)\mathcal{A}_1(V_e)\partial_x^\ell V \cdot \partial_x^\ell V = \int_0^T \int_{\mathbb{R}} K(V_e)\partial_x^{\ell-1} H(V) \cdot \partial_x^\ell V - \int_0^T \int_{\mathbb{R}} K(V_e)\partial_x^{\ell-1} (\mathcal{A}_0(V)\partial_t V) \cdot \partial_x^\ell V. \tag{37}$$

Please cite this article in press as: D. Kazerani, Global existence for small data of the viscous Green–Naghdi type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.03.022

Let us note that

$$K(V_e)\mathcal{A}_1(V_e) = \begin{pmatrix} gh_e & 0 \\ 0 & -h_e^2 \end{pmatrix}. \tag{38}$$

Hence,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} K(V_e)\mathcal{A}_1(V_e)\partial_x^\ell V\cdot\partial_x^\ell V &= \int_0^T\int_{\mathbb{R}}\left(gh_e(\partial_x^\ell h)^2 - h_e^2(\partial_x^\ell u)^2\right)\\ &= gh_e\int_0^T \|\partial_x^\ell h\|_{\mathbb{L}^2}^2 - h_e^2\int_0^T\|\partial_x^\ell u\|_{\mathbb{L}^2}^2.\end{aligned} \tag{39}$$

Gathering (37) and (39), we get

$$g\int_0^T \|\partial_x^\ell h\|_{\mathbb{L}^2}^2 = h_e\int_0^T\|\partial_x^\ell u\|_{\mathbb{L}^2}^2 + \frac{1}{h_e}\int_0^T\int_{\mathbb{R}}\partial_x^{\ell-1}\left(K(V_e)H(V) - K(V_e)\mathcal{A}_0(V)\partial_t V\right)\cdot\partial_x^\ell V. \tag{40}$$

It is now sufficient to give a convenient estimate on the last term of (40). This estimation is given in the following lemma.

Lemma 3.8. *Let $V_e = (h_e, 0)$ be an equilibrium (with $h_e > 0$) and $\delta > 0$ be small such that System (1) admits a local solution $V \in C^0([0,T);\mathbb{X}^s(\mathbb{R}))$ for initial data in $B_s(V_e,\delta)$. Then, as long as V remains in $B_s(V_e,\delta)$, we have for all $1 \le \ell \le s$,*

$$\begin{aligned}\int_{\mathbb{R}} K(V_e)\partial_x^{\ell-1}\left(H(V) - \mathcal{A}_0(V)\partial_t V\right)\cdot\partial_x^\ell V &= \int_{\mathbb{R}}\partial_t\left(\partial_x^{\ell-1}\mathcal{L}_h u\cdot\partial_x^\ell h\right) + \mu\partial_x^\ell(h\partial_x u)\partial_x^\ell h\\ &\quad + R[h,u],\end{aligned} \tag{41}$$

where

$$\left|\int_{\mathbb{R}} R[h,u]\right| \le \Theta_{\{h_e,\alpha\}}(\delta)\,\|\partial_x h\|_{\mathbb{H}^{\ell-1}}^2 + C_{\{h_e,\alpha\}}(\delta)\,\|\partial_x u\|_{\mathbb{H}^\ell}^2. \tag{42}$$

Proof. First of all, we look at the first term of the left hand side of (41). To do so, we first remark that

$$\mathcal{A}_1(V_e) - \mathcal{A}_1(V) = \begin{pmatrix} -gu + 3\alpha hu(\partial_x u)^2 & g(h_e - h) + 3\alpha h^2(\partial_x u)^2 \\ g(h_e - h) + 3\alpha h^2(\partial_x u)^2 & -hu - 2\alpha\partial_x(h^3\partial_x u) + \alpha\partial_x(h^3 u\partial_x ()) \end{pmatrix}. \tag{43}$$

Thus, the definition (36) of $H(V)$ leads to

$$K(V_e)\partial_x^{\ell-1}H(V)\cdot\partial_x^\ell V = \boxed{\mu\partial_x^\ell(h\partial_x u)\partial_x^\ell h} + \partial_x^{\ell-1}\left(g(h_e-h)(\partial_x h)+3\alpha h^2(\partial_x u)^2\partial_x h\right)\partial_x^\ell h$$
$$+\partial_x^{\ell-1}\left(h_e u\partial_x h - \frac{3\alpha h_e}{g}hu\partial_x h(\partial_x u)^2\right)\partial_x^\ell u - \partial_x^{\ell-1}\left(h_e(h_e-h)(\partial_x u)+\frac{3\alpha h_e}{g}h^2(\partial_x u)^3\right)\partial_x^\ell u$$
$$-\partial_x^{\ell-1}\left(hu\partial_x u + 2\alpha\partial_x(h^3\partial_x u)\partial_x u\right)\partial_x^\ell h + \boxed{\alpha\partial_x^\ell\left(h^3 u\partial_x^2 u\right)\partial_x^\ell h}. \tag{44}$$

Let us remark here that all of the non-boxed terms of (44) are straightforwardly controllable as in (42).

We now consider the second term of the left hand side of (41) and observe that

$$K(V_e)\mathcal{A}_0(V) = \begin{pmatrix} 0 & \mathcal{L}_h \\ -h_e + \frac{3\alpha}{g}h_e h(\partial_x u)^2 & 0 \end{pmatrix}. \tag{45}$$

Therefore,

$$K(V_e)\partial_x^{\ell-1}\left(\mathcal{A}_0(V)\partial_t V\right)\cdot\partial_x^\ell V = \partial_x^{\ell-1}\mathcal{L}_h(\partial_t u)\cdot\partial_x^\ell h + \partial_x^{\ell-1}\left(\frac{3\alpha h_e}{g}h(\partial_x u)^2\partial_t h - h_e\partial_t h\right)\partial_x^\ell u. \tag{46}$$

Now, we need the following lemma to deal with non-straightforwardly controllable term of the right hand side of (46).

Lemma 3.9. *Assume that* $(h, f) \in C^0([0,T], \mathbb{X}^s(\mathbb{R}))$ *for some* $T>0$. *Then, we have*

$$\mathcal{L}_h\partial_t f = \partial_t\mathcal{L}_h f - f\partial_t h + 3\alpha\partial_x\left(h^2\partial_t h\partial_x f\right). \tag{47}$$

We now use the lemma to rewrite (46):

$$K(V_e)\partial_x^{\ell-1}\left(\mathcal{A}_0(V)\partial_t V\right)\cdot\partial_x^\ell V = \partial_t\left(\partial_x^{\ell-1}\mathcal{L}_h u\cdot\partial_x^\ell h\right) - \partial_x^{\ell-1}\mathcal{L}_h u\cdot\partial_t\partial_x^\ell h$$
$$-\partial_x^{\ell-1}(u\partial_t h)\cdot\partial_x^\ell h + 3\alpha\partial_x^\ell\left(h^2\partial_t h\partial_x u\right)\partial_x^\ell h + \partial_x^{\ell-1}\left(\frac{3\alpha h_e}{g}h(\partial_x u)^2 h_t - h_e h_t\right)\partial_x^\ell u. \tag{48}$$

We then use the mass conservation equation, $h_t = -\partial_x(hu)$, to find

$$K(V_e)\partial_x^{\ell-1}\left(\mathcal{A}_0(V)\partial_t V\right)\cdot\partial_x^\ell V = \boxed{\partial_t\left(\partial_x^{\ell-1}\mathcal{L}_h u\cdot\partial_x^\ell h\right)} + \boxed{\partial_x^{\ell-1}\mathcal{L}_h u\cdot\partial_x^{\ell+1}(hu)}$$
$$+\partial_x^{\ell-1}(u\partial_x(hu))\cdot\partial_x^\ell h - 3\alpha\partial_x^\ell\left(h^2\partial_x(hu)\partial_x u\right)\partial_x^\ell h$$
$$-\partial_x^{\ell-1}\left(\frac{3\alpha h_e}{g}h(\partial_x u)^2\partial_x(hu) - h_e\partial_x(hu)\right)\partial_x^\ell u. \tag{49}$$

Considering the fact that all of the non-boxed terms of (49) are straightforwardly controllable as in (42), we notice that the form (49) of $K(V_e)\partial_x^{\ell-1}\left(\mathcal{A}_0(V)\partial_t V\right)\cdot\partial_x^\ell V$ is very interesting. This is due on the one hand to the fact the non-desirable term $g\partial_x^\ell(h^2/2)\,\partial_x^\ell h$ is hidden in the boxed time derivative term $\partial_t\left(\partial_x^{\ell-1}\mathcal{L}_h u\cdot\partial_x^\ell h\right)$. Therefore, we can easily deal with this term by a time integration. On the other hand, as detailed in the following lemma, this formulation gathers the other non-straightforwardly controllable term under the boxed term $\partial_x^{\ell-1}\mathcal{L}_h u\cdot\partial_x^{\ell+1}(hu)$ which is cancellable with the boxed term $\alpha\partial_x^\ell\left(h^3u\partial_x^2u\right)\partial_x^\ell h$ of (44).

Lemma 3.10. *Assume that $V\in B_s(V_e,\delta)$. Then, we have for all $1\le\ell\le s$,*

$$\left|\int_{\mathbb{R}}\alpha\partial_x^\ell\left(h^3u\partial_x^2u\right)\partial_x^\ell h-\partial_x^{\ell-1}\mathcal{L}_h u\cdot\partial_x^{\ell+1}(hu)\right|\le\Theta_{\{h_e,\alpha\}}(\delta)\parallel\partial_x h\parallel_{\mathbb{H}^{\ell-1}}^2+C_{\{h_e,\alpha\}}(\delta)\parallel\partial_x u\parallel_{\mathbb{H}^\ell}^2 \tag{50}$$

We just now need to consider (44), (49) together with Lemma 3.10 to complete the proof. □

Proof of Lemma 3.10. We first use an integration by part and the definition of $\mathcal{L}_h$ to write

$$\begin{aligned}&\int_{\mathbb{R}}\alpha\partial_x^\ell\left(h^3u\partial_x^2u\right)\partial_x^\ell h-\partial_x^{\ell-1}\mathcal{L}_h u\cdot\partial_x^{\ell+1}(hu)=\int_{\mathbb{R}}\alpha\partial_x^\ell\left(h^3u\partial_x^2u\right)\partial_x^\ell h+\partial_x^\ell\mathcal{L}_h u\cdot\partial_x^\ell(hu)\\&=\int_{\mathbb{R}}\alpha\partial_x^\ell\left(h^3u\partial_x^2u\right)\partial_x^\ell h+\partial_x^\ell(hu)\cdot\partial_x^\ell(hu)-\int_{\mathbb{R}}\alpha\partial_x^{\ell+1}(h^3\partial_x u)\cdot\partial_x^\ell(hu).\end{aligned}$$

Then, we use a simple development to get

$$\begin{aligned}&\int_{\mathbb{R}}\alpha\partial_x^\ell\left(h^3u\partial_x^2u\right)\partial_x^\ell h-\partial_x^{\ell-1}\mathcal{L}_h u\cdot\partial_x^{\ell+1}(hu)=\int_{\mathbb{R}}\alpha\ \partial_x^\ell h\left(\sum_{j=1}^{\ell}\partial_x^j(h^3u)\ \partial_x^{\ell-j+2}u\right)\\&+\left(\partial_x^\ell(hu)\right)^2-\alpha\partial_x^\ell(hu)\left(\sum_{j=1}^{\ell}\partial_x^j(h^3)\partial_x^{\ell-j+2}u\right)-\alpha\boxed{\int_{\mathbb{R}}\partial_x^\ell(hu)\ \partial_x^{\ell+1}(h^3)\partial_x u}.\end{aligned} \tag{51}$$

We now see that the term $\int_{\mathbb{R}}\partial_x^\ell(hu)\ \partial_x^{\ell+1}(h^3)\partial_x u$ may be the only obstacle to the estimate (50). However, we can treat this term as following to get the desired estimate. Indeed, we use the fact that[5]

$$\partial_x^{\ell+1}(h^3)=3h^2\partial_x^{\ell+1}h+[\partial_x^\ell,3h^2]\partial_x h,$$

to write

$$\int_{\mathbb{R}}\partial_x^\ell(hu)\ \partial_x^{\ell+1}(h^3)\partial_x u=3\int_{\mathbb{R}}\partial_x^\ell(hu)\ h^2\ \partial_x^{\ell+1}(h)\partial_x u+\partial_x^\ell(hu)\ \left([\partial_x^\ell,h^2]\partial_x h\right)\ \partial_x u.$$

---

[5] As in [26], symbol $[\partial_x^\ell,A]U$ represents the commutator of $A\in\mathbb{H}^s(\mathbb{R})$ and $U\in\mathbb{H}^{s-1}(\mathbb{R})$. In other words, we have

$$[\partial_x^\ell,A]U=\partial_x^\ell(AU)-A\partial_x^\ell U.$$

Likewise, we have

$$\int_{\mathbb{R}} \partial_x^\ell(hu)\ \partial_x^{\ell+1}(h^3)\partial_x u = 3\int_{\mathbb{R}} uh^2\partial_x u\ \partial_x^\ell h\ \partial_x^{\ell+1} h + h^2\partial_x u\ \left([\partial_x^\ell, u]h\right)\ \partial_x^{\ell+1} h$$
$$+ \partial_x^\ell(hu)\ \left([\partial_x^\ell, h^2]\partial_x h\right)\ \partial_x u.$$

We just now use an integration by part to get

$$\int_{\mathbb{R}} \partial_x^\ell(hu)\ \partial_x^{\ell+1}(h^3)\partial_x u = -3\int_{\mathbb{R}} \partial_x\left(uh^2\partial_x u\right)\ (\partial_x^\ell h)^2 + \partial_x\left((h^2\partial_x u\ \left([\partial_x^\ell, u]h\right)\right)\ \partial_x^\ell h$$
$$+3\int_{\mathbb{R}} \partial_x^\ell(hu)\ \left([\partial_x^\ell, h^2]\partial_x h\right)\ \partial_x u.$$

Therefore,

$$\left|\alpha\int_{\mathbb{R}} \partial_x^\ell(hu)\ \partial_x^{\ell+1}(h^3)\partial_x u\right| \le \Theta_{\{h_e,\alpha\}}(\delta) \parallel \partial_x h \parallel^2_{\mathbb{H}^{\ell-1}} + C_{\{h_e,\alpha\}}(\delta) \parallel \partial_x u \parallel^2_{\mathbb{H}^\ell} .$$

This together with (51) leads to

$$\left|\int_{\mathbb{R}} \alpha\partial_x^\ell\left(h^3 u\partial_x^2 u\right)\partial_x^\ell h - \partial_x^{\ell-1}\mathcal{L}_h u\cdot\partial_x^{\ell+1}(hu)\right|$$
$$\le \Theta_{\{h_e,\alpha\}}(\delta) \parallel \partial_x h \parallel^2_{\mathbb{H}^{\ell-1}} + C_{\{h_e,\alpha\}}(\delta) \parallel \partial_x u \parallel^2_{\mathbb{H}^\ell} . \quad \square$$

The last step to get the estimate of Proposition 2.4 is to give an estimate on the first two terms of the right hand side of (41). This is done in the following lemma.

Lemma 3.11. *Let $V = (h, u)$ be in $C^0([0, T); \mathbb{X}^s(\mathbb{R}))$ and assume that it belongs to $B_s(V_e, \delta)$ for all $t \in [0, T)$. Then, we have for all $1 \le \ell \le s$,*

$$\parallel \mu\partial_x^\ell(h\partial_x u)\partial_x^\ell h \parallel_{\mathbb{L}^1} \le \Theta_{\{\mu\}}(\delta) \parallel \partial_x V \parallel^2_{\mathbb{X}^{\ell-1}} + C_{\{\mu,h_e\}}(\delta) \parallel \partial_x^{\ell+1} u \parallel^2_{\mathbb{L}^2} + \frac{g}{2} \parallel \partial_x^\ell h \parallel^2_{\mathbb{L}^2}, \quad (52)$$

and

$$\int_0^t\int_{\mathbb{R}} \partial_t\left(\partial_x^{\ell-1}\mathcal{L}_h u\cdot\partial_x^\ell h\right) \le C_{\{h_e,\alpha\ \}}(\delta)\left(\parallel u(t) \parallel^2_{\mathbb{H}^{\ell+1}} + \parallel \partial_x^\ell h(t) \parallel^2_{\mathbb{L}^2}\right)$$
$$+ C_{\{h_e,\alpha\ \}}(\delta)\left(\parallel u(0) \parallel^2_{\mathbb{H}^{\ell+1}} + \parallel \partial_x^\ell h(0) \parallel^2_{\mathbb{L}^2}\right). \quad (53)$$

Proof. The first estimate (52) is a consequence of Leibniz formula and the fact that

$$\left|\partial_x^{\ell+1} u \ \partial_x^{\ell} h\right| \leq \frac{2\mu(h_e+\delta)}{g}\left(\partial_x^{\ell+1} u\right)^2 + \frac{g}{2\mu(h_e+\delta)}\left(\partial_x^{\ell} h\right)^2.$$

To prove (53), we use the definition of $\mathcal{L}_h$ to write

$$\left|\partial_x^{\ell-1}\mathcal{L}_h u \cdot \partial_x^{\ell} h\right| = \left|\partial_x^{\ell-1}(hu)\cdot \partial_x^{\ell} h - \alpha \partial_x^{\ell}(h^3 \partial_x u)\cdot \partial_x^{\ell} h\right|.$$

Then, the estimate is obtained by very basic computations. Indeed,

$$\left|\partial_x^{\ell-1}(hu)\cdot \partial_x^{\ell} h - \alpha \partial_x^{\ell}(h^3 \partial_x u)\cdot \partial_x^{\ell} h\right| \leq \left|\partial_x^{\ell-1}(hu)\cdot \partial_x^{\ell} h\right| + \left|\alpha \partial_x^{\ell}(h^3 \partial_x u)\cdot \partial_x^{\ell} h\right|$$

On the other hand, we have

$$\left|\partial_x^{\ell-1}(hu)\cdot \partial_x^{\ell} h\right| \leq C_{\{h_e\}}(\delta)\left(\| u \|^2_{\mathbb{H}^{\ell-1}} + \| \partial_x^{\ell} h \|^2_{\mathbb{L}^2}\right),$$

and

$$\left|\alpha \partial_x^{\ell}(h^3 \partial_x u)\cdot \partial_x^{\ell} h\right| \leq C_{\{h_e\}}(\delta)\left(\| \partial_x u \|^2_{\mathbb{H}^{\ell-1}} + \| \partial_x^{\ell} h \|^2_{\mathbb{L}^2}\right).$$

Hence, the lemma is proved. □

We now sum (40) for $1 \leq \ell \leq s$. This together with (41) and Lemma 3.11 enables us to give an estimation on $\int_0^T \| h_x \|^2_{\mathbb{H}^{s-1}}$:

Proposition 3.12. *Let us assume that there exists $T > 0$ such that the local solution of (1) satisfies $V(t) \in B_s(V_e, \delta)$ for all $t \in [0, T)$. Then, we have,*

$$\int_0^t \| \partial_x h \|^2_{\mathbb{H}^{s-1}} \leq C_{\{h_e,\mu\}}(\delta) \int_0^t \| \partial_x u \|^2_{\mathbb{H}^s} + C_{\{h_e,\alpha\}}(\delta)\left(\| u(t) \|^2_{\mathbb{H}^{s+1}} + \| \partial_x h(t) \|^2_{\mathbb{H}^{s-1}}\right) + C_{\{h_e,\alpha\}}(\delta)\left(\| u(0) \|^2_{\mathbb{H}^{s+1}} + \| \partial_x h(0) \|^2_{\mathbb{H}^{s-1}}\right). \tag{54}$$

This proposition together with Proposition 3.7 gives the a priori estimate of Proposition 2.4.

Remark 3.13. In this work, α and μ are supposed to be strictly positive. However, we can use the same approach and computations for the viscous Saint-Venant system i.e. for $\alpha = 0$. In this case, the system fits the general framework considered in [17] and our approach, as wall as our result, is exactly the same. Indeed, the main difference between the case $\alpha = 0$ (Saint-Venant system) and the case $\alpha > 0$ (Green–Naghdi system) is the space on which the Hamiltonian $\mathcal{H}_{h_e,0}$ and the operator $\mathcal{A}_0(V)$ are quadratic: this space is $(\mathbb{H}^s(\mathbb{R}) + h_e) \times \mathbb{H}^s(\mathbb{R})$ when $\alpha = 0$ whereas it is $\mathbb{X}^s(\mathbb{R})$ when $\alpha > 0$. As a matter of fact, in both cases, the space of quadraticity of $\mathcal{H}_{h_e,0}$ and $\mathcal{A}_0(V)$ is the same as the space on which the system is locally well-posed. For this reason, instead of the estimate of Proposition 2.4, we find the following estimate

$$(1-\Theta_{\{h_e\}}(\delta)) \parallel V(T)-V_e \parallel^2_{\mathbb{H}^s\times\mathbb{H}^s} + C_{\{h_e,\mu\}}(\delta)\int_0^T \parallel \partial_x u \parallel^2_{\mathbb{H}^s}$$

$$\leq C_{\{h_e\}}(\delta) \parallel V(0)-V_e \parallel^2_{\mathbb{H}^s\times\mathbb{H}^s} + \Theta_{\{h_e,\mu\}}(\delta)\int_0^T \parallel u_x \parallel^2_{\mathbb{H}^s},$$

which writes for small $\delta > 0$,

$$\parallel V(T)-V_e \parallel^2_{\mathbb{H}^s\times\mathbb{H}^s} + C_{\{h_e,\mu\}}(\delta)\int_0^T \parallel \partial_x u \parallel^2_{\mathbb{H}^s} \leq C_{\{h_e\}}(\delta) \parallel V(0)-V_e \parallel^2_{\mathbb{H}^s\times\mathbb{H}^s}.$$

Remark 3.14. The dissipative right hand side term, $\mu\partial_x(h\partial_x u)$, plays a very important role to obtain the stability result in both hyperbolic and dispersive cases. Indeed, it is well-known that equilibriums of Saint-Venant system without any dissipative term, are unstable[6] (see [8] for instance). Such an instability result does not exist for the Green–Naghdi equations. However, we are not able to prove the global existence result if the dissipative term is absent, i.e. if $\mu = 0$. More precisely, the presence of the $\int_0^T \parallel \partial_x u \parallel^2_{\mathbb{H}^s}$ term in the left hand side of the estimate of Proposition 2.4 is due to the strict positivity of μ. Therefore, this term disappears if $\mu = 0$. This means that the estimate of Proposition 2.4 becomes

$$(1-\Theta_{\{h_e,\alpha\}}(\delta)) \parallel V(T)-V_e \parallel^2_{\mathbb{X}^s} \leq C_{\{h_e,\alpha\}}(\delta) \parallel V(0)-V_e \parallel^2_{\mathbb{X}^s} + \Theta_{\{h_e,\alpha\}}(\delta)\int_0^T \parallel \partial_x u \parallel^2_{\mathbb{H}^s}.$$

Hence, $\parallel V(T)-V_e \parallel^2_{\mathbb{X}^s}$ is not any longer controlled by the norm of the initial data and the global existence for small data can not be concluded.

4. Conclusion and perspectives

During this study, we proved the global existence for small data and the asymptotic stability of constant solutions of the Green–Naghdi system with a second order viscosity. This result is obtained by generalizing the technique used for symmetric entropy dissipative hyperbolic equations thanks to the generalized symmetric structure of the system. The study of the rate of convergence to equilibrium is one of the perspectives of this work. [17].

Let us however recall that the result found in this study can not be generalized by this method to the Green–Naghdi system with friction $-\kappa u$ (with $\kappa > 0$), without the viscosity $\mu u\partial_x(h\partial_x u)$. In fact, in absence of this term, the first estimations are not coherent with the estimation of $\int_0^t \parallel \partial_x h \parallel^2_{\mathbb{H}^{s-1}}$, in the sense that there are of one order less than the estimation of $\int_0^t \parallel \partial_x h \parallel^2_{\mathbb{H}^{s-1}}$. Furthermore, if we add higher order viscous terms (order 4 or more) such as $-\mu\partial_x^2(h\partial_x(h\partial_x u))$, we are not able either to generalize the technique used in this work. In fact, in this latter case,

[6] in the sens that in all neighborhood of constant solutions, there exists an initial data for which a shock is created in a finite time.

the order of the first estimations are always less than the order of the estimates of $\int_0^t \| \partial_x h \|^2_{\mathbb{H}^{s-1}}$, with or without $-\kappa u + \mu u \partial_x(h\partial_x u)$. This means that the order 2 seems to be the only order of viscosity, our approach can be used for.

One of the other perspectives of this work is to study, in a general frame, the stability of equilibriums of locally-wellposed symmetrizable systems with a convenient friction or viscous term. In fact, the main difficulty of this generalization is to find the condition which leads to convenient estimates on the time integral of the spatial derivative of the solution. Let us note that in the case of hyperbolic systems, there are other equivalent formulations of the Kawashima–Shizuta condition [24,17] which may be more convenient for the generalization. One of these formulations for hyperbolic systems is the emptiness of the intersection of the eigenspaces of the symmetric positive definite matrix (the one equivalent to $\mathcal{A}_0$) and the symmetric matrix (the one equivalent to $\mathcal{A}_1$) with the kernel of the viscosity matrix at equilibriums. It is also interesting to mention that the Kawashima–Shizuta condition is not sharp for hyperbolic systems (see [22] or [4] for instance). A generalization of less sharp conditions may be another way to follow. The answer to this question may let us for instance, investigate the stability of equilibriums of 2D Green–Naghdi system. Let us recall that $\mathcal{A}_0(V)$ in 2-dimensional case is given by [18]

$$\mathcal{A}_0(V) = \begin{pmatrix} g - 3\alpha h(div(u,v))^2 & 0 & 0 \\ 0 & h - \alpha\partial_x(h^3\partial_x) & -\alpha\partial_x(h^3\partial_y) \\ 0 & -\alpha\partial_y(h^3\partial_x) & h - \alpha\partial_y(h^3\partial_y) \end{pmatrix}$$

where u (respectively v) represents the vertically averaged x-component (resp. y-component) of the speed. In this case, $\mathcal{A}_0(V)$ is quadratic near equilibriums, for the norm $\| \, . \, \|_{\mathbb{X}^0}$ defined by

$$\| f \|^2_{\mathbb{X}^0} = \| f \|^2_{\mathbb{L}^2} + \| div(f) \|^2_{\mathbb{L}^2} \, .$$

This is also the 0^{th} order norm of the local well-posedness space of the 2-dimensional system [1]. Indeed, the symmetric structure is coherent with the well-posedness space.

Acknowledgments

I would like to thank gratefully Nicolas Seguin. He has been a source of intuition for the creation of this work and played a very important role in its improvement.

Appendix A. Special symmetric structure

In this section, we consider a system of the form

$$\partial_t W + \partial_x F(W) = 0. \tag{55}$$

The unknown W is supposed to belong to $C([0,T);\mathcal{X})$ for some $T > 0$ where $\mathcal{X}$ is a Banach subspace of continuous functions of $\mathbb{L}^2(\mathbb{R}, \mathbb{R}^N)$ converging to 0 at infinity. We also assume that the derivative of all elements of $\mathcal{X}$ belongs to $\mathcal{X}$. Additionally, F is not anymore a function of $\mathbb{R}^N$ but a smooth application defined from $\mathcal{X}$ to $\mathcal{X}$. We also assume that (55) is a general Godunov system [10,18]. Therefore there exists a strictly convex functional $\mathcal{H}$ defined on a convex subset Ω of $\mathcal{X}$ such that $\delta^2\mathcal{H}(W)DF(W)$ is symmetric. Under theses assumptions, System (55) is symmetrizable under any change of unknown (see [18] for more details).

Proposition A.1. *Let us consider the decomposition $W=(U,V)$ of the unknown. Assume also that the application*

$$(U,V)\mapsto (U,\delta_V\mathcal{H}(U,.))$$

is a diffeomorphism. Then, (55) is written under the unknown $w=(U,\delta_V\mathcal{H}(W))$, as following

$$\mathcal{A}_0(w)\partial_t w+\mathcal{A}_1(w)\partial_x w=0. \tag{56}$$

Moreover, $\mathcal{A}_0(w)=D_wW^\star(w)\ \delta^2_W\mathcal{H}(W)\ D_wW(w)$ is a symmetric definite positive bloc diagonal operator and $\mathcal{A}_1(w)=D_wW^\star(w)\ \delta^2_W\mathcal{H}(W)\ D_WF(W)\ D_wW(w)$ is a symmetric one.

Proof. Let us set $u=U$ and $v=\delta_V\mathcal{H}(W)$. Therefore $w=(u,v)$. It is easy to check that we obtain (56) by acting $D_wW^\star(w)\delta^2\mathcal{H}(w)$ on System (55). Let us now remark that

$$D_wW=\begin{pmatrix}1 & 0\\ D_uV & D_vV\end{pmatrix},$$

and

$$\delta^2_W\mathcal{H}(W)=\begin{pmatrix}\delta^2_U\mathcal{H}(W) & \delta^2_{VU}\mathcal{H}(W)\\ \delta^2_{UV}\mathcal{H}(W) & \delta^2_V\mathcal{H}(W)\end{pmatrix}.$$

Hence,

$$\mathcal{A}_0(w)=$$
$$\begin{pmatrix}\delta^2_U\mathcal{H}(W)+\delta^2_{VU}\mathcal{H}(W)\ D_uV+(D_uV)^T\delta^2_{UV}\mathcal{H}(W)+(D_uV)^T\delta^2_V\mathcal{H}(W)\ D_uV & \delta^2_{VU}\mathcal{H}(W)\ D_vV+(D_uV)^T\delta^2_V\mathcal{H}(W)D_vV\\ (D_vV)^T\ \delta^2_{UV}\mathcal{H}(W)+(D_vV)^T\delta^2_V\mathcal{H}(W)D_uV & (D_vV)^T\ \delta^2_V\mathcal{H}(W)\ D_vV\end{pmatrix}.$$

Then, $\mathcal{A}_0(w)$ is bloc diagonal considering the fact that

$$(D_vV)^T\ \delta^2_{UV}\mathcal{H}(W)+(D_vV)^T\delta^2_V\mathcal{H}(W)\ D_uV=0.$$

Indeed, $v=\delta_V\mathcal{H}(W)$ and $u=U$ give us

$$\begin{aligned}&(D_vV)^T\ \delta^2_{UV}\mathcal{H}(W)+(D_vV)^T\delta^2_V\mathcal{H}(W)\ D_uV=(D_vV)^T\ D_Uv+(D_vV)^T\ D_Vv\ D_uV\\&=(D_vV)^T\ D_Uv\ D_uU+(D_vV)^T\ D_Vv\ D_uV=(D_vV)^T\ (D_Uv\ D_uU+D_Vv\ D_uV)\\&=(D_vV)^T\ D_uv=0.\quad\square\end{aligned}$$

Let us now add a right hand side term of the following form to (55)

$$\begin{cases}\partial_tU+\partial_xF_1(U)=0,\\ \partial_tV+\partial_xF_2(V)=q(W),\end{cases} \tag{57}$$

where q is a smooth application of W and (U,V) is a decomposition of W satisfying the assumptions of Proposition A.1. Again, we act $D_wW^\star(w)\delta^2\mathcal{H}(w)$ on System (55) to find

$$\mathcal{A}_0(w)\partial_t w + \mathcal{A}_1(w)\partial_x w = G(w),$$

with

$$G(w) = (D_w W)^T \delta^2_W \mathcal{H}(W) Q(W).$$

We are now going to see that $Q(W) = (0, q(W))$ is an eigenvector for the eigenvalue 1 of $(D_w W)^T \delta^2_W \mathcal{H}(W)$. In fact, the following proposition holds true.

Proposition A.2. *The right hand side term $G(w)$ is equal to $Q(W)$.*

Proof. We have by assumptions

$$G(W) = (D_w W)^T \delta^2_W \mathcal{H}(W) Q(W) = \begin{pmatrix} \delta^2_{VU}\mathcal{H}\, q(W) + (D_u V)^T\ \delta^2_V \mathcal{H}\, q(W) \\ (D_v V)^T\ \delta^2_V \mathcal{H}\, q(W) \end{pmatrix}.$$

Considering the fact that the first components (associated to U) of $G(W)$ are the same as the up non-diagonal bloc of the operator $\mathcal{A}_0(w)$ considered in the proof of Proposition A.1 acting on $q(W)$, these components vanish. On the other hand,

$$(D_v V)^T\ \delta^2_V \mathcal{H}\, q(W) = (D_v V)^T\ (\delta^2_V \mathcal{H})^T\ q(W) = (D_v V)^T\ (D_V v)^T\ q(W)$$
$$= (D_V v\ D_v V)^T\ q(W) = (D_V v\ (D_V v)^{-1})^T\ q(W) = q(W). \quad \square$$

Appendix B. Local well-posedness

Let us first note that there exists $0 < \delta < h_e$ such that $\mathcal{A}_0(V)$ is invertible for all $V \in B_s(V_e, \delta)$. Then, consider the associated linear problem

$$\begin{cases} \partial_t V + \mathcal{A}_0^{-1}(\underline{V})\mathcal{A}_1(\underline{V})\partial_x V = \begin{pmatrix} 0 \\ \mu \mathcal{L}_{\underline{h}}^{-1}(\partial_x(\underline{h}\partial_x \underline{u})) \end{pmatrix} \\ V(0,x) = g_0(x) \end{cases} \tag{58}$$

where $\underline{V} \in C([0,T]; \mathbb{X}^s(\mathbb{R}))$ and $\partial_t \underline{V} \in \mathbb{X}^{s-1}(\mathbb{R})$ for some $s \geq 2$ and $g_0 \in \mathbb{X}^s(\mathbb{R})$. It is proved in [16] that the problem admits a unique solution V in $C([0,T]; \mathbb{X}^s(\mathbb{R}))$. We now consider the following iteration scheme

$$\begin{cases} \mathcal{A}_0(V^k)\partial_t V^{k+1} + \mathcal{A}_1(V^k)\partial_x V^{k+1} = \begin{pmatrix} 0 \\ \mu \partial_x (h^k \partial_x u^k) \end{pmatrix} \\ V^{k+1}(0,x) = g^{k+1}(x) \end{cases} \tag{59}$$

where $g^{k+1} = \epsilon^k V_0 \star \rho(\frac{\cdot}{\epsilon^k})$ for some mollifier ρ[7] with the positive real set $\epsilon^k = \frac{\beta}{2^k}$, with $\beta > 0$.

[7] $\rho : \mathbb{R} \to \mathbb{R}^+$ is infinity derivable compactly supported in the unit ball with $\int_{\mathbb{R}} \rho = 1$.

We initialize the iteration by $g^0 = V_0$. We know that (59) admits a unique solution for all positive integer k. Let us now assume that $V^l(t) \in B_s(V_e, \delta)$ for all $l \leq k$ and all $t \in [0, T]$. This implies by triangle inequality that

$$\| V^l - g^0 \|_{C([0,T];\mathbb{X}^s)} \leq 2\delta \tag{60}$$

for all $l \leq k$. We can show that there exists a suitable $T > 0$ such that the estimate (60) holds also true for $l = k + 1$. In fact, we consider the $\bar{s}^{\text{th}}$ derivative of (59), take the scalar product with $\partial_x^{\bar{s}+1}(V^{k+1} - g^0)$ and we sum over $\bar{s} \in \{0, \dots, s\}$. Then, using very similar logics as in 3.2, we find for all $0 \leq t \leq T$,

$$\| V^{k+1}(t) - g^0 \|_{\mathbb{X}^s}^2 \leq C_{\{\|g^0\|_{\mathbb{L}^\infty}\}}(\delta) \| g^{k+1} - g^0 \|_{\mathbb{X}^s}^2 + C_{\{\|g^0\|_{\mathbb{L}^\infty}, \mu\}}(\delta) \int_0^t \| V^{k+1}(t') - g^0 \|_{\mathbb{X}^s}^2 \, dt'$$
$$+ C_{\{\|g^0\|_{\mathbb{L}^\infty}, \mu\}}(\delta) t.$$

Then, Gronwall lemma leads us, for δ small enough, to

$$\| V^{k+1} - g^0 \|_{C([0,T];\mathbb{X}^s)}^2 \leq Ce^{\lambda T} \left(\| g^{k+1} - g^0 \|_{\mathbb{X}^s}^2 + T \right),$$

where C and λ are strictly positive reals independent of k. On the other hand, there exists by assumption, $\epsilon_0 > 0$ such that

$$\| g^{k+1} - g^0 \|_{\mathbb{X}^s} \leq \epsilon_0 \quad \text{for all} \quad k \in \mathbb{N}.$$

Then, choosing β small enough (therefore ϵ_0 small enough), there exists $T > 0$ such that the condition (60) is satisfied for all $l \in \mathbb{N}$. We assume from now that T and β are small enough to give us (60) for all positive integer. Then, we consider the $\bar{s}^{\text{th}}$ derivative of (59) for iterations k and $k - 1$, take the scalar product with $\partial_x^{\bar{s}+1}(V^{k+1} - V^k)$, subtract the two equations and sum over $\bar{s} \in \{0, \dots, s\}$. Likewise, we get

$$\| V^{k+1}(t) - V^k(t) \|_{\mathbb{X}^s}^2 \leq \gamma \| g^{k+1} - g^k \|_{\mathbb{X}^s}^2 + \theta \int_0^t \| V^k(t') - V^{k-1}(t') \|_{\mathbb{X}^s}^2 \, dt'$$
$$+ \theta \int_0^t \| V^{k+1}(t') - V^k(t') \|_{\mathbb{X}^s}^2 \, dt'$$

for some convenient positive γ, θ.

Applying the Gronwall lemma, we have for all $k \in \mathbb{N}$

$$\| V^{k+1} - V^k \|_{C([0,T];\mathbb{X}^s)}^2 \leq e^{\lambda T} \left(\| g^{k+1} - g^k \|_{\mathbb{X}^s}^2 + \theta \int_0^T \| V^k(t') - V^{k-1}(t') \|_{\mathbb{X}^s}^2 \, dt' \right). \tag{61}$$

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Now, we sum (61) on $k \in \mathbb{N}$. This leads us to

$$(1 - \theta T e^{\lambda T}) \sum_{k\in\mathbb{N}} \| V^{k+1} - V^k \|^2_{C([0,T];\mathbb{X}^s)} \le e^{\lambda T} \sum_{k\in\mathbb{N}} \| g^{k+1} - g^k \|^2_{\mathbb{X}^s} .$$

Then, considering the fact the T is small and the fact that the sum $\sum_{k\in\mathbb{N}} \| g^{k+1} - g^k \|^2_{\mathbb{X}^s}$ is convergent, we conclude that the set V^k is convergent in $C([0,T];\mathbb{X}^s(\mathbb{R}))$. The uniqueness can be proved by the same way. In fact, we obtain a very similar approximation to (61) for $\| V^1 - V^2 \|_{\mathbb{X}^s}$ considering two solutions $V^1(x,t)$ and $V^2(x,t)$ for the initial conditions $V_1(x)$ and $V_2(x)$. Hence, the local well-posedness is proved.

Appendix C. Linear stability of equilibriums of the Green–Naghdi equation

In this part we are going to see another use of the symmetric structure of the Green–Naghdi equation. In fact, this structure enables us to prove the linear stability of an equilibrium $V_e = (h_e, u_e)$ with $h_e > 0$, for the system without any dissipative right hand side term. To see this, let us consider the solution $V \in C([0,T);\mathbb{X}^s(\mathbb{R}))$ of the linearized system

$$\mathcal{A}_0(V_e)\partial_t V + \mathcal{A}_1(V_e)\partial_x V = 0, \tag{62}$$

act ∂_x^ℓ on (62) for $0 \le \ell \le s$, and take the scalar product by $\partial_x^\ell(V - V_e)$:

$$\int_0^t \int_{\mathbb{R}} \mathcal{A}_0(V_e)\partial_t\partial_x^\ell V \cdot \partial_x^\ell(V - V_e) + \int_0^T \int_{\mathbb{R}} \mathcal{A}_1(V_e)\partial_x^{\ell+1} V \cdot \partial_x^\ell(V - V_e) = 0. \tag{63}$$

Now, considering the facts that

$$\int_{\mathbb{R}} \mathcal{A}_0(V_e)\partial_t\partial_x^\ell V \cdot \partial_x^\ell(V - V_e) = \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}} A_0(V_e)\partial_x^\ell(V - V_e) \cdot \partial_x^\ell(V - V_e),$$

and

$$\int_0^t \int_{\mathbb{R}} \mathcal{A}_1(V_e)\partial_x^{\ell+1} V \cdot \partial_x^\ell(V - V_e) = 0,$$

together with the $\mathbb{X}^0$-quadraticity of $\mathcal{A}_0(V_e)$, we get the following estimate,

$$\| \partial_x^\ell(V(t) - V_e) \|^2_{\mathbb{X}^0} \le C \| \partial_x^\ell(V(0) - V_e) \|^2_{\mathbb{X}^0}, \tag{64}$$

where C is a strictly positive constant depending only on h_e, α and g. Hence, we have the following proposition,

Proposition C.1. *Let $s \geq 2$ be an integer and consider the initial data $V_0 \in \mathbb{X}^s(\mathbb{R})$. Then, there exists $C > 0$ such that the solution V of (62) satisfies for all time,*

$$\| V(t) - V_e \|_{\mathbb{X}^s}^2 \leq C \| V_0 - V_e \|_{\mathbb{X}^s}^2 . \tag{65}$$

This gives us the linear stability of the equilibrium of (1).

Theorem C.2. *Let $s \geq 2$ be an integer and consider the Green–Naghdi system,*

$$\begin{cases} \partial_t h + \partial_x hu = 0, \\ \partial_t hu + \partial_x (hu^2) + \partial_x (gh^2/2 + \alpha h^2 \ddot{h}) = 0. \end{cases} \tag{66}$$

Then, the equilibrium solutions $V_e = (h_e, u_e)$, with $h_e > 0$, are linearly stable for the $\mathbb{X}^s$ norm.

Let us note that this theorem can be generalized to all locally well-posed symmetrizable system of the form (62) such that $\mathcal{A}_0(V_e)$ is quadratic.

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