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Global existence and large time behavior for a two-dimensional chemotaxis-Navier–Stokes system

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Abstract

This paper concerns the coupled chemotaxis-Navier–Stokes system in the two-dimensional setting. Such a system was proposed in [19] to describe the collective effects arising in bacterial suspensions in fluid drops. Under some basic assumptions on the parameter functions $\chi(\cdot)$, $k(\cdot)$ and the potential function ϕ, which are consistent with those used by the experimentalists but weaker than those appeared in the known mathematical works, we establish the global existence of weak solutions and classical solutions for both the Cauchy problem and the initial-boundary value problem supplemented with some initial data. For the initial-boundary value problem, we also assert that the solution converges in large time to the spatially homogeneous equilibrium $(\overline{n_0}, 0, 0)$ with $\overline{n_0} := \frac{1}{|\Omega|}\int_\Omega n_0(x)\,\mathrm{d}x$. Our results also show that the large diffusion of the cell density or the chemical concentration can rule out the finite-time blow-up even though the Navier–Stokes fluid is included.

MSC: 35K55; 35Q92; 35Q35; 92C17

Keywords: Chemotaxis; Navier–Stokes; Global existence; Convergence

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1. Introduction

Bacteria or microorganisms often live in fluid, in which the biology of aerotaxis is intimately related to the surrounding physics. Tuval et al. [19] proposed a coupled cell–fluid model to describe the dynamics of swimming bacteria, *Bacillus subtilis*, which not only consists of chemotaxis and diffusion, but also includes transport and viscous fluid dynamics. It is given as follows:

$$\begin{cases} n_t + \mathbf{u}\cdot\nabla n = \mathcal{D}_n \Delta n - \nabla\cdot\big(n\chi(c)\nabla c\big), & x\in\Omega,\ t>0,\\ c_t + \mathbf{u}\cdot\nabla c = \mathcal{D}_c \Delta c - k(c)n, & x\in\Omega,\ t>0,\\ \mathbf{u}_t + \mathbf{u}\cdot\nabla \mathbf{u} + \nabla P = \mathcal{D}_{\mathbf{u}}\Delta\mathbf{u} + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot\mathbf{u} = 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

where the unknowns $n(t,x)$, $c(t,x)$, $\mathbf{u}(t,x)$ and $P(t,x)$ denote the cell density, the oxygen concentration, the fluid velocity, and the corresponding scalar pressure, respectively. $\Omega\subset\mathbb{R}^2$ or $\mathbb{R}^3$ is a spatial domain where the cells and the fluid move and interact. The positive constants $\mathcal{D}_n$, $\mathcal{D}_c$ and $\mathcal{D}_{\mathbf{u}}$ are the corresponding diffusion coefficients for the cells, the oxygen and the fluid, respectively. The given functions $\chi(\cdot)$ and $k(\cdot)$ denote the chemotactic sensitivity and the oxygen consumption rate, respectively. The known function $\phi=\phi(x)$ is a time-independent one, and usually denotes potential function such as the gravitational force or centrifugal force. One example in the case of gravity is $\phi = ax_1$ for some constant $a\in\mathbb{R}$ depending on the ratio of the fluid mass density to the cell density and the gravity acceleration.

As usual, in order for the system (1.1) to be well-posed, it should be supplemented with some initial conditions

$$(n,c,\mathbf{u})|_{t=0} = (n_0(x), c_0(x), \mathbf{u}_0(x)), \qquad x\in\Omega \tag{1.2}$$

and some proper boundary conditions. Two typical cases for Ω are the whole space and the bounded domain. For the case of the whole space, the boundary condition is hidden in the decay of solutions at spatial infinity. For the case of the bounded domain, the most common boundary condition is that n and c satisfy the no-flux Neumann boundary value and u satisfies the no-slip boundary value, namely,

$$\frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0, \qquad \mathbf{u}=0 \qquad \text{on} \quad \partial\Omega, \tag{1.3}$$

where ν is the unit outward normal vector on $\partial\Omega$.

System (1.1) consists of two subsystems. One is the classical incompressible Navier–Stokes equations which are still lacking a complete existence and regularity theory, especially in the three-dimensional case (see [15]). The other is a variant of the classical Keller–Segel system

$$\begin{cases} n_t = \Delta n - \nabla\cdot\big(n\chi(c)\nabla c\big), & x\in\Omega,\ t>0,\\ c_t = \Delta c - c + n, & x\in\Omega,\ t>0. \end{cases} \tag{1.4}$$

It is well-known that the cross-diffusive term $-\nabla\cdot\big(n\chi(c)\nabla c\big)$ may destabilize the homogeneity of model (1.4) and even enforce blow-up of solutions (see [1] and references therein). Therefore, the mathematical analysis of the coupled chemotaxis-fluid model (1.1) faces large challenges.

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Up to now, there are only few analytic results which mainly focus on the local and global solvability of corresponding initial(-boundary)-value problems in either bounded or unbounded domains Ω, under various technical conditions on $\chi(\cdot)$ and $k(\cdot)$. In the case $\Omega = \mathbb{R}^2$, in [5], it was proved that there exists a global weak solutions for the chemotaxis-Stokes equations, *i.e*, the nonlinear convective term $\mathbf{u} \cdot \nabla \mathbf{u}$ is removed in the fluid equation of (1.1), by making use of quasi-energy functionals associated with (1.1), under the following conditions on $\chi(\cdot)$ and $k(\cdot)$:

$$\chi(c) > 0,\ \chi'(c) \geq 0,\ k(0) = 0,\ k'(c) > 0,\ \frac{\mathrm{d}^2}{\mathrm{d}c^2}\left(\frac{k(c)}{\chi(c)}\right) < 0, \tag{1.5}$$

and on ϕ and the initial data c_0:

$$\begin{gathered}\phi \geq 0,\ \nabla\phi \in L^\infty(\mathbb{R}^2),\\ \sup w|\nabla\phi| + \sup w^2|\nabla^2\phi| \ \text{ and }\ \|c_0\|_{L^4(\mathbb{R}^2)} \text{ are small;}\end{gathered} \tag{1.6}$$

or

$$k(0) = 0,\ k'(c) > 0, \tag{1.7}$$

and

$$\begin{gathered}\phi \geq 0,\ \nabla\phi \in L^\infty(\mathbb{R}^2),\\ \sup w|\nabla\phi| + \sup w^2|\nabla^2\phi| < \infty, \quad \|c_0\|_{L^\infty(\mathbb{R}^2)} \text{ is small,}\end{gathered} \tag{1.8}$$

where $w = w(x) = (1 + |x|)\big(1 + \ln(1 + |x|)\big)$. In [12], the global existence of weak solutions for the full chemotaxis-Navier–Stokes equations (1.1) are obtained, under the conditions on $\chi(\cdot)$ and $k(\cdot)$:

$$\begin{gathered}\chi(c),\ \chi'(c),\ k(c),\ k'(c) \geq 0,\\ \frac{\mathrm{d}^2}{\mathrm{d}c^2}\left(\frac{k(c)}{\chi(c)}\right) < 0, \quad \frac{\chi'(c)k(c) + \chi(c)k'(c)}{\chi(c)} > 0.\end{gathered} \tag{1.9}$$

The decay of the potential at infinity and the smallness of c_0 and ϕ in (1.6) and (1.8) are not required in [12,6]. Moreover, if the initial data are sufficiently smooth, it was proved in [2] that the global existence of smooth solutions could be established by assuming that $\phi(x)$, $\chi(c)$ and $k(c)$ satisfy

$$\phi(x),\ \ \chi(c),\ \ \chi'(c),\ \ k(c),\ \ k'(c) \geq 0,\ \ k(0) = 0 \tag{1.10}$$

and that there exists a constant μ such that

$$\sup_{c\geq 0} |\chi(c) - \mu k(c)| < \epsilon \quad \text{for a sufficiently small} \quad \epsilon > 0. \tag{1.11}$$

In [3], Chae et al. got rid of the condition (1.11) and obtained the global existence of smooth solution for some small initial data under only the assumptions (1.10). In the case $\Omega = \mathbb{R}^3$, the

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global classical solution near constant steady states and the global weak solutions in the special situation that $\chi(\cdot)$ precisely coincides with a fixed multiple of $k(\cdot)$ are constructed for the full chemotaxis-Navier–Stokes system (1.1) in [5] and [2], respectively.

In the case $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with smooth boundary, Winkler [23] proved the global existence of classical solution to the initial boundary value problem (1.1)–(1.3) under the assumption that the parameter functions satisfy that

$$\begin{cases} \chi \in C^2\big([0,\infty)\big), \ \ \chi(\cdot) > 0 \quad \text{in } [0,\infty), \\ k \in C^2([0,\infty)), \ \ k(0) = 0, \ \ k(\cdot) > 0 \qquad \text{in } (0,\infty), \\ \phi \in C^2(\overline{\Omega}), \end{cases} \tag{1.12}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{k(c)}{\chi(c)}\right) > 0, \quad \frac{\mathrm{d}^2}{\mathrm{d}c^2}\left(\frac{k(c)}{\chi(c)}\right) \le 0, \quad \Big(\chi(c)\cdot k(c)\Big)' \ge 0, \tag{1.13}$$

while the initial data satisfy that

$$\begin{cases} n_0 \in C^0(\bar{\Omega}), \ \ n_0 > 0 \quad \text{in } \overline{\Omega}, \\ c_0 \in W^{1,q}(\Omega) \quad \text{for some } q > 2, \ \ c_0 > 0 \quad \text{in } \overline{\Omega}, \\ \mathbf{u}_0 \in D(A^\alpha) \quad \text{for some} \quad \alpha \in (1/2,\ 1), \end{cases} \tag{1.14}$$

where A denotes the realization of the Stokes operator in the solenoidal subspace $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) | \nabla\cdot\varphi = 0\}$ of $L^2(\Omega)$. Then in [24], he further asserted that this solution stabilizes to the spatially uniform equilibrium $(\overline{n_0},\ 0,\ 0)$ with respect to the norm in $L^\infty(\Omega)$. More related works on the bounded domain case, we may refer to [16,20,21] for the two-dimensional case and [4,17,22,25,26] for the three dimensional case. In particular, the recent works [25,26] established the global existence of weak solutions as well as their eventual smoothness and stabilization to the 3D version of system (1.1), still under some strong structural assumptions on χ and k.

The main purpose of this paper is to establish the global existence for system (1.1) with initial(-boundary)-value condition under the weaker restrictions on the chemotactic sensitivity χ and the oxygen consumption rate k than those used in (1.5), (1.7), (1.9), (1.10) and (1.13). Indeed, Petroff and Libchaber [14] recently proposed a similar chemotaxis-fluid model to describe how the response of the sulfur-oxidizing bacterium *Thiovulum majus* to changing oxygen gradients causes cells to organize into large-scale fronts, where the chemotactic sensitivity $\chi(c)$ is preferred as $\chi'(c^*)(c-c^*)$ with $\chi'(c^*) < 0$. This assumption is based on the experimental observation in [7] and [18], that the sulfur-oxidizing bacterium *T. majus* shows a strong chemotactic response toward a specific concentration of oxygen, $c^* = 4\%$ air saturation. Moreover, the experimentalists in [19] used multiples of the Heaviside step function to model $\chi(\cdot)$ and $k(\cdot)$. Additionally, it is also reasonable to assume that $\chi(c) \to 0$ as $c \to \infty$, which indicates that at large oxygen concentrations chemotaxis is inhibited ([9,10]). As far as we know, there are few results involving parameter functions satisfying these conditions. Since different functional forms of χ and k are meaningful as well, then it is very interesting to investigate system (1.1) with the more general chemotactic sensitivity χ and the oxygen consumption rate k.

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Main results. We shall study both the Cauchy problem (1.1)–(1.2) and the initial-boundary value problem (1.1)–(1.3) in the two-dimensional setting. For the Cauchy problem (1.1)–(1.2), we first assume that:

(A). The chemotactic sensitivity $\chi(\cdot)$ and the oxygen consumption rate $k(\cdot)$ are locally bounded, $k(\cdot)$ is continuous at zero with $k(0)=0$, and $k(s)\geq 0$ for all $s\in\mathbb{R}$;

(A_1). $\nabla\phi\in L^\infty(\mathbb{R}^2)$;

(A_2). The initial data $(n_0, c_0, \mathbf{u}_0)$ satisfy that

$$n_0(x)\geq 0,\quad c_0(x)\geq 0,\quad \nabla\cdot\mathbf{u}_0(x)=0\quad \text{for all }\ x\in\mathbb{R}^2$$

and that

$$n_0(1+|x|+|\ln n_0|)\in L^1(\mathbb{R}^2),$$
$$c_0\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)\cap H^1(\mathbb{R}^2),\quad \mathbf{u}_0\in L^2(\mathbb{R}^2;\mathbb{R}^2).$$

Under these assumptions, we will first prove the global existence of weak solutions to the Cauchy problem (1.1)–(1.2). Here, the definition of global weak solutions is in the following sense:

Definition 1.1 *(Weak solution).* A triple $(n, c, \mathbf{u})$ is called a global weak solution to the Cauchy problem (1.1)–(1.2) if for any $T>0$,

(i) it holds that $n(t,x)\geq 0$, $c(t,x)\geq 0$ a.e. in $[0,T]\times\mathbb{R}^2$, and

$$\begin{cases} n(1+|x|+|\ln n|)\in L^\infty\big(0,T;L^1(\mathbb{R}^2)\big),\\ \nabla\sqrt{n}\in L^2\big(0,T;L^2(\mathbb{R}^2)\big),\\ c\in L^\infty\big(0,T;L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)\cap H^1(\mathbb{R}^2)\big),\\ \Delta c\in L^2\big(0,T;L^2(\mathbb{R}^2)\big)\\ \mathbf{u}\in L^\infty\big(0,T;L^2(\mathbb{R}^2,\mathbb{R}^2)\big)\cap L^2\big(0,T;H^1(\mathbb{R}^2,\mathbb{R}^2)\big); \end{cases} \tag{1.15}$$

(ii) it holds that

$$\int_0^T\int_{\mathbb{R}^2} n(\partial_t\psi+\mathcal{D}_n\Delta\psi+\nabla\psi\cdot\mathbf{u}+\chi(c)\nabla c\cdot\nabla\psi)\,\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}^2} n_0(x)\psi(0,x)\,\mathrm{d}x=0 \tag{1.16}$$

and

$$\int_0^T\int_{\mathbb{R}^2} c(\partial_t\psi+\mathcal{D}_c\Delta\psi+\nabla\psi\cdot\mathbf{u}-k(c)n\psi)\,\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}^2} c_0(x)\psi(0,x)\,\mathrm{d}x=0 \tag{1.17}$$

for any $\psi\in C^\infty([0,T)\times\mathbb{R}^2)$, and

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} \mathbf{u} \cdot \partial_t \varphi + \mathcal{D}_{\mathbf{u}} \mathbf{u} \cdot \Delta \varphi + (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \varphi - n \nabla \phi \cdot \varphi \, \mathrm{d}x \mathrm{d}t + \int_{\mathbb{R}^2} \mathbf{u}_0(x) \cdot \varphi(0, x) \, \mathrm{d}x \\
&\quad = 0
\end{aligned} \tag{1.18}
$$

for any $\varphi \in C^\infty([0, T) \times \mathbb{R}^2; \mathbb{R}^2)$ with $\nabla \cdot \varphi = 0$, where both ψ and φ have compact support in x, $\psi(T, \cdot) = 0$ and $\varphi(T, \cdot) = 0$.

The main results describing the global existence of weak solutions to the Cauchy problem are given as follows.

Theorem 1.1 *(Global existence of weak solution for $\Omega = \mathbb{R}^2$). Suppose that the assumptions (A), (A_1) and (A_2) hold. Let $M = \|c_0\|_{L^\infty(\mathbb{R}^2)}$ and C_{GN} be a positive constant resulted from the Gagliardo–Nirenberg inequality. If it holds that*

$$
\left(\frac{M^2 \sup\limits_{0 \le s \le M} \chi^4(s)}{4 \mathcal{D}_n^3 \mathcal{D}_c} + \frac{\sup\limits_{0 \le s \le M} k^2(s)}{\mathcal{D}_n \mathcal{D}_c} + \frac{4M^2}{\mathcal{D}_n \mathcal{D}_c \mathcal{D}_{\mathbf{u}}} \right) C_{GN} \|n_0\|_{L^1(\mathbb{R}^2)} < 1,
$$

then the Cauchy problem (1.1)–(1.2) admits at least a global-in-time weak solution $(n, c, \mathbf{u})$.

Whenever one enhances the regularity of the initial data $(n_0, c_0, \mathbf{u}_0)$ and of the parameter functions χ, k and ϕ, the corresponding solution $(n, c, \mathbf{u})$ can become more regular and thus the global classical solution may be obtained.

Theorem 1.2 *(Global existence of classical solution for $\Omega = \mathbb{R}^2$). Let $m \ge 3$. Under the assumptions of Theorem 1.1, if it additionally holds that the initial data $(n_0, c_0, \mathbf{u}_0) \in H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2; \mathbb{R}^2)$, $\chi, k \in C^m(\mathbb{R})$ and $\|\nabla^l \phi\|_{L^\infty(\mathbb{R}^2)} < \infty$ for $1 \le |l| \le m$, then system (1.1)–(1.2) admits a unique global-in-time classical solution $(n, c, \mathbf{u})$ satisfying for any $T > 0$*

$$
(n, c, \mathbf{u}) \in L^\infty\big(0, T;\; H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2; \mathbb{R}^2)\big)
$$

and

$$
(\nabla n, \nabla c, \nabla \mathbf{u}) \in L^2\big(0, T;\; H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2; \mathbb{R}^2)\big).
$$

For the initial boundary value problem (1.1)–(1.3), we will study both the global existence and the large-time behavior of classical solution under the basic regularity assumptions (1.12) and (1.14) but without the structural conditions (1.13).

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Suppose that the parameter functions χ, k and ϕ satisfy (1.12) and that the initial data $(n_0, c_0, \mathbf{u}_0)$ satisfies (1.14). Let $\widetilde{M} = \|c_0\|_{L^\infty(\Omega)}$ and C_{GN} be a positive constant resulted from the Gagliardo–Nirenberg inequality. If it holds that*

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$$\left(\frac{\widetilde{M}^2 \sup\limits_{0\le s\le \widetilde{M}} \chi^4(s)}{4\mathcal{D}_n^3\mathcal{D}_c}+\frac{\sup\limits_{0\le s\le \widetilde{M}} k'^2(s)}{\mathcal{D}_n\mathcal{D}_c}\right) C_{GN}\|n_0\|_{L^1(\Omega)}<\frac{4}{3},$$

then system (1.1)–(1.3) possesses a unique global in time classical solution with the regularity properties that, for all $T\in(0,\infty)$,

$$\begin{cases} n\in C^0\big([0,T);L^2(\Omega)\big)\cap L^\infty\big([0,T);C^0(\bar\Omega)\big)\cap C^{2,1}\big(\bar\Omega\times(0,T)\big),\\ c\in C^0\big([0,T);L^2(\Omega)\big)\cap L^\infty\big([0,T);W^{1,q}(\bar\Omega)\big)\cap C^{2,1}\big(\bar\Omega\times(0,T)\big),\\ \mathbf{u}\in C^0\big([0,T);L^2(\Omega)\big)\cap L^\infty\big([0,T);D(A^\alpha)\big)\cap C^{2,1}\big(\bar\Omega\times(0,T)\big),\\ P\in L^\infty\big((0,T);W^{1,2}(\Omega)\big). \end{cases} \tag{1.19}$$

Furthermore, the global solution converges in large time to the spatially uniform equilibrium $(\overline{n_0},0,0)$ with $\overline{n_0}:=\frac{1}{|\Omega|}\int_\Omega n_0(x)\,\mathrm{d}x$, in the sense that

$$n(\cdot,t)\to\overline{n_0},\quad c(\cdot,t)\to 0\quad \textit{and}\quad \mathbf{u}(\cdot,t)\to 0\quad \textit{as } t\to\infty$$

hold with respect to the norm in $L^\infty(\Omega)$.

Remark 1.1. Compared with the results in [5,12,23,24], we removed the monotonicity assumption and the structural conditions on χ and k.

Remark 1.2. Theorems 1.1, 1.2 and 1.3 also show that the large diffusion of the cell density or the chemical concentration can rule out the finite-time blow-up even though the Navier–Stokes fluid is included. Moreover, a similar conclusion as Theorem 1.1 still holds in the three dimensional case while it is not clear for the 3D analogies of Theorems 1.2 and 1.3 due to the outstanding open problem on the 3D Navier–Stokes system in fluid dynamics.

Remark 1.3. Smallness assumptions on $\|c_0\|_{L^\infty(\Omega)}$ are known to enforce global regularity also in other fluid-free chemotaxis systems involving signal consumption as in (1.1) but lacking convenient energy structures (see e.g. [11] for the chemotaxis system with rotational flux terms).

The rest of this paper is organized as follows. We first concern the Cauchy problem in Section 2 and then investigate the initial-boundary value problem in Section 3.

2. The Cauchy problem in $\Omega=\mathbb{R}^2$

In this section, we investigate the global existence of weak or classical solutions to the Cauchy problem of system (1.1)–(1.2). We begin with the nonnegativity of n and c, and the basic mass conservation of n as well as the boundedness of c.

Lemma 2.1. *Suppose that the assumptions (A), (A_1) and (A_2) hold. Then the global weak or strong solution $(n,c,\mathbf{u})$ to the Cauchy problem of system (1.1)–(1.2) satisfies*

$$n(t,x)\ge 0,\quad c(t,x)\ge 0\qquad \text{a.e. in}\quad [0,+\infty)\times\Omega, \tag{2.1}$$

and

$$\|n(t)\|_{L^1(\Omega)} \equiv \|n_0\|_{L^1(\Omega)} \qquad \text{for any} \quad t \geq 0, \tag{2.2}$$

as well as

$$\sup_{t\geq 0} \|c(t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)}. \tag{2.3}$$

Proof. We just show the case that $(n, c, \mathbf{u})$ is a strong solution to system (1.1)–(1.2). For the weak solution case, we can use the same argument to the regularized system (2.28)–(2.29) and then take an approximation procedure to obtain the desired result.

Firstly, it follows from the assumption (A) and the maximum principle that n and c preserve the nonnegativity of the initial data, which gives (2.1).

Next, integrating $(1.1)_1$ on x over $\mathbb{R}^2$ and using the nonnegativity of n, we see that (2.2) holds.

Finally, multiplying $(1.1)_2$ by pc^{p-1} with $p \geq 1$ and then integrating over $\mathbb{R}^2$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} c^p \,\mathrm{d}x + \frac{4(p-1)}{p}\int_{\mathbb{R}^2} |\nabla c^{p/2}|^2 \,\mathrm{d}x = -p\int_{\mathbb{R}^2} k(c)c^{p-1}n \,\mathrm{d}x \leq 0,$$

which implies that $\sup_{t\geq 0}\|c(t)\|_{L^p(\mathbb{R}^2)} \leq \|c_0\|_{L^p(\mathbb{R}^2)}$ for any $p \geq 1$. Then passing to the limit as $p \to \infty$ yields that $\sup_{t\geq 0}\|c(t)\|_{L^\infty(\mathbb{R}^2)} \leq \|c_0\|_{L^\infty(\mathbb{R}^2)}$. This completes the proof of (2.3) and hence Lemma 2.1. □

2.1. Global existence of weak solutions

The global existence of weak solution is based on deriving a key entropy functional inequality. To establish such an inequality for a weak solution $(n, c, \mathbf{u})$, we introduce

$$\mathcal{E}(t) := \int_{\mathbb{R}^2} \Big(n \ln n + 2n\sqrt{1+|x|^2} + |\nabla c|^2 + \frac{4C_{GN}M^2}{\mathcal{D}_{\mathbf{u}}\mathcal{D}_c}|\mathbf{u}|^2\Big)\mathrm{d}x,$$

and

$$\mathcal{F}(t) := \big(\mathcal{D}_n - K_1\big)\int_{\mathbb{R}^2} |\nabla\sqrt{n}|^2 \mathrm{d}x + \frac{\mathcal{D}_c}{2}\int_{\mathbb{R}^2} |\Delta c|^2 \mathrm{d}x + \frac{4C_{GN}M^2}{\mathcal{D}_c}\int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 \mathrm{d}x,$$

where $M = \|c_0\|_{L^\infty(\mathbb{R}^2)}$, K_1 is defined by (2.4) and C_{GN} is a uniform positive constant related to the Gagliardo–Nirenberg inequality, which may change from line to line.

We now state the entropy functional inequality for $\Omega = \mathbb{R}^2$.

Lemma 2.2. *Suppose that the assumptions* (A), (A_1) *and* (A_2) *hold. If it holds that*

$$K_1 := \left(\frac{C_{GN} M^2 \sup\limits_{0 \le c \le M} \chi^4(c)}{4\mathcal{D}_n^2 \mathcal{D}_c} + \frac{\sup\limits_{0 \le c \le M} k^2(c)}{\mathcal{D}_c} + \frac{4C_{GN} M^2}{\mathcal{D}_{\mathbf{u}} \mathcal{D}_c} \right) \times C_{GN} \|n_0\|_{L^1(\mathbb{R}^2)} < \mathcal{D}_n, \tag{2.4}$$

then the global weak solution $(n, c, \mathbf{u})$ *to the Cauchy problem of system* (1.1)–(1.2) *satisfies the entropy functional inequality*

$$\mathcal{E}(t) + \int_0^t \mathcal{F}(\tau)\,\mathrm{d}\tau \le \left(\mathcal{E}(0) + \frac{K_3}{K_2} \right) e^{K_2 t}, \qquad t > 0 \tag{2.5}$$

for $K_2 \neq 0$, *or*

$$\mathcal{E}(t) + \int_0^t \mathcal{F}(\tau)\,\mathrm{d}\tau \le \mathcal{E}(0) + K_3 t, \qquad t > 0 \tag{2.6}$$

for $K_2 = 0$, *where*

$$K_2 := \|\nabla \phi\|^2_{L^\infty(\mathbb{R}^2)} + \frac{\mathcal{D}_{\mathbf{u}} \mathcal{D}_c \|n_0\|_{L^1(\mathbb{R}^2)}}{2\mathcal{D}_n M^2} + \frac{2C_{GN} \|n_0\|_{L^1(\mathbb{R}^2)} \sup\limits_{0 \le c \le M} \chi^2(c)}{\mathcal{D}_n}$$

and

$$K_3 := 4\mathcal{D}_n \|n_0\|_{L^1(\mathbb{R}^2)} + 4K_2 e^{-1} \int_{\mathbb{R}^2} e^{-\frac{1}{2}\langle x \rangle}\,\mathrm{d}x.$$

Proof. To establish the entropy functional inequality (2.5) and (2.6), we divide the proof into several steps. Without loss of generality, we assume that $(n, c, \mathbf{u})$ is a strong solution to system (1.1)–(1.2). The general case can be dealt with by using the same argument to the regularized system (2.28)–(2.29) and then taking an approximation procedure.

Step 1. An evolution estimate for $n \ln n$.

Multiplying equation $(1.1)_1$ by $(1 + \ln n)$ and integrating the resulting equation in $\mathbb{R}^2$, we have

$$\frac{d}{\mathrm{d}t} \int_{\mathbb{R}^2} n \ln n\,\mathrm{d}x + \mathcal{D}_n \int_{\mathbb{R}^2} \frac{|\nabla n|^2}{n} \mathrm{d}x = \int_{\mathbb{R}^2} \nabla n \cdot \chi(c) \nabla c\,\mathrm{d}x := I_1, \tag{2.7}$$

by the integration by parts and the divergence free of $\mathbf{u}$. It follows from Young's inequality and (2.3) that

$$
\begin{aligned}
I_1 &\le \frac{\mathcal{D}_n}{2}\int_{\mathbb{R}^2}\frac{|\nabla n|^2}{n}\mathrm{d}x + \frac{1}{2\mathcal{D}_n}\int_{\mathbb{R}^2}\chi^2(c)n|\nabla c|^2\mathrm{d}x \\
&\le \frac{\mathcal{D}_n}{2}\int_{\mathbb{R}^2}\frac{|\nabla n|^2}{n}\mathrm{d}x + \frac{\sup\limits_{0\le c\le M}\chi^4(c)}{4\mathcal{D}_n^2\varepsilon}\int_{\mathbb{R}^2}n^2\mathrm{d}x + \frac{\varepsilon}{4}\int_{\mathbb{R}^2}|\nabla c|^4\mathrm{d}x
\end{aligned}
\tag{2.8}
$$

with $\varepsilon > 0$ to be specified later. Substituting (2.8) into (2.7), we obtain

$$
\frac{d}{\mathrm{d}t}\int_{\mathbb{R}^2} n\ln n\,\mathrm{d}x + \frac{\mathcal{D}_n}{2}\int_{\mathbb{R}^2}\frac{|\nabla n|^2}{n}\mathrm{d}x \le \frac{\sup\limits_{0\le c\le M}\chi^4(c)}{4\mathcal{D}_n^2\varepsilon}\int_{\mathbb{R}^2}n^2\mathrm{d}x + \frac{\varepsilon}{4}\int_{\mathbb{R}^2}|\nabla c|^4\mathrm{d}x. \tag{2.9}
$$

Step 2. An evolution estimate for ∇c.

To control the last term of (2.9), we multiply the equation $(1.1)_2$ by $-\Delta c$, integrate the resulting equation in $\mathbb{R}^2$ and use the integration by parts to obtain

$$
\frac{1}{2}\frac{d}{\mathrm{d}t}\int_{\mathbb{R}^2}|\nabla c|^2\mathrm{d}x + \mathcal{D}_c\int_{\mathbb{R}^2}|\Delta c|^2\mathrm{d}x = \int_{\mathbb{R}^2}k(c)n\Delta c\,\mathrm{d}x + \int_{\mathbb{R}^2}(\mathbf{u}\cdot\nabla c)\Delta c\,\mathrm{d}x := I_2 + I_3, \tag{2.10}
$$

due to $\nabla\cdot\mathbf{u} = 0$. We need to estimate I_2 and I_3. For I_2, Young's inequality and the boundedness (2.3) give that

$$
I_2 \le \frac{\mathcal{D}_c}{2}\int_{\mathbb{R}^2}|\Delta c|^2\mathrm{d}x + \frac{\sup\limits_{0\le c\le M}k^2(c)}{2\mathcal{D}_c}\int_{\mathbb{R}^2}n^2\mathrm{d}x. \tag{2.11}
$$

For I_3, invoking the divergence free of $\mathbf{u}$ and using the integration by parts and Young's inequality again, we have

$$
I_3 = -\int_{\mathbb{R}^2}\nabla\mathbf{u}\cdot\nabla c\cdot\nabla c\,\mathrm{d}x \le \frac{2}{\varepsilon}\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2\mathrm{d}x + \frac{\varepsilon}{8}\int_{\mathbb{R}^2}|\nabla c|^4\mathrm{d}x, \tag{2.12}
$$

with the same ε as that of (2.9). Thus substituting (2.11) and (2.12) into (2.10), we obtain

$$
\begin{aligned}
&\frac{d}{\mathrm{d}t}\int_{\mathbb{R}^2}|\nabla c|^2\mathrm{d}x + \mathcal{D}_c\int_{\mathbb{R}^2}|\Delta c|^2\mathrm{d}x \\
&\le \frac{\sup\limits_{0\le c\le M}k^2(c)}{\mathcal{D}_c}\int_{\mathbb{R}^2}n^2\mathrm{d}x + \frac{4}{\varepsilon}\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2\mathrm{d}x + \frac{\varepsilon}{4}\int_{\mathbb{R}^2}|\nabla c|^4\mathrm{d}x.
\end{aligned}
\tag{2.13}
$$

Step 3. A coupled evolution estimate for $n\ln n$ and ∇c.

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Collecting (2.9) to (2.13), we have

$$\frac{d}{dt}\Big(\int_{\mathbb{R}^2} n\ln n\,dx + \int_{\mathbb{R}^2} |\nabla c|^2 dx\Big) + \frac{\mathcal{D}_n}{2}\int_{\mathbb{R}^2} \frac{|\nabla n|^2}{n} dx + \mathcal{D}_c \int_{\mathbb{R}^2} |\Delta c|^2 dx$$
$$\le \left(\frac{\sup\limits_{0\le c\le M} \chi^4(c)}{4\mathcal{D}_n^2 \varepsilon} + \frac{\sup\limits_{0\le c\le M} k^2(c)}{\mathcal{D}_c}\right)\int_{\mathbb{R}^2} n^2 dx + \frac{4}{\varepsilon}\int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 dx + \frac{\varepsilon}{2}\int_{\mathbb{R}^2} |\nabla c|^4 dx.$$

Using the estimate

$$\|\nabla c\|^4_{L^4(\mathbb{R}^2)} \le C_{GN}\|c\|^2_{L^\infty(\mathbb{R}^2)}\|D^2 c\|^2_{L^2(\mathbb{R}^2)} \le C_{GN} M^2 \|\Delta c\|^2_{L^2(\mathbb{R}^2)}, \tag{2.14}$$

which follows from the Gagliardo–Nirenberg inequality and the boundedness (2.3), we have

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\mathbb{R}^2} n\ln n\,dx + \int_{\mathbb{R}^2} |\nabla c|^2 dx\Big) + \frac{\mathcal{D}_n}{2}\int_{\mathbb{R}^2} \frac{|\nabla n|^2}{n} dx + \mathcal{D}_c \int_{\mathbb{R}^2} |\Delta c|^2 dx \\ &\le \left(\frac{\sup\limits_{0\le c\le M} \chi^4(c)}{4\mathcal{D}_n^2 \varepsilon} + \frac{\sup\limits_{0\le c\le M} k^2(c)}{\mathcal{D}_c}\right)\int_{\mathbb{R}^2} n^2 dx + \frac{4}{\varepsilon}\int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 dx \\ &\quad + \frac{\varepsilon}{2} C_{GN} M^2 \|\Delta c\|^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{2.15}$$

To absorb the term involving the second order derivatives of c on the right-hand side of (2.15), we take $\varepsilon := \frac{\mathcal{D}_c}{C_{GN}M^2}$ and obtain

$$\begin{aligned}&\frac{d}{dt}\left(\int_{\mathbb{R}^2} n\ln n\,dx + \int_{\mathbb{R}^2} |\nabla c|^2 dx\right) + \frac{\mathcal{D}_n}{2}\int_{\mathbb{R}^2} \frac{|\nabla n|^2}{n} dx + \frac{\mathcal{D}_c}{2} \int_{\mathbb{R}^2} |\Delta c|^2 dx \\ &\le \left\{\frac{C_{GN}M^2 \sup\limits_{0\le c\le M} \chi^4(c)}{4\mathcal{D}_n^2 \mathcal{D}_c} + \frac{\sup\limits_{0\le c\le M} k^2(c)}{\mathcal{D}_c}\right\}\int_{\mathbb{R}^2} n^2 dx + \frac{4C_{GN}M^2}{\mathcal{D}_c}\int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 dx.\end{aligned} \tag{2.16}$$

Step 4. An evolution estimate for $\mathbf{u}$.

To absorb the last term of (2.16), we test the fluid equation $(1.1)_3$ against $2\mathbf{u}$ and obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} |\mathbf{u}|^2 dx + 2\mathcal{D}_{\mathbf{u}} \int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 dx = 2\int_{\mathbb{R}^2} n\nabla\phi\cdot\mathbf{u}\,dx, \tag{2.17}$$

where we have used $\nabla\cdot\mathbf{u} = 0$. Applying Young's inequality to the last term of (2.17), we have

$$\frac{d}{dt}\int_{\mathbb{R}^2} |\mathbf{u}|^2 dx + 2\mathcal{D}_{\mathbf{u}} \int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 dx \le \int_{\mathbb{R}^2} n^2 dx + \|\nabla\phi\|^2_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2} |\mathbf{u}|^2 dx. \tag{2.18}$$

Step 5. A coupled evolution estimate for $n\ln n$, ∇c and $\mathbf{u}$.

Multiplying (2.18) by $\frac{4C_{GN}M^2}{\mathcal{D}_\mathbf{u}\mathcal{D}_c}$ and adding the resulted equation to (2.16), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\Big(n\ln n+|\nabla c|^2+\frac{4C_{GN}M^2}{\mathcal{D}_\mathbf{u}\mathcal{D}_c}|\mathbf{u}|^2\Big)\mathrm{d}x\\
&\quad+\frac{\mathcal{D}_n}{2}\int_{\mathbb{R}^2}\frac{|\nabla n|^2}{n}\mathrm{d}x+\frac{\mathcal{D}_c}{2}\int_{\mathbb{R}^2}|\Delta c|^2\mathrm{d}x+\frac{4C_{GN}M^2}{\mathcal{D}_c}\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2\mathrm{d}x\\
&\le\left(\frac{C_{GN}M^2\sup\limits_{0\le c\le M}\chi^4(c)}{4\mathcal{D}_n^2\mathcal{D}_c}+\frac{\sup\limits_{0\le c\le M}k^2(c)}{\mathcal{D}_c}+\frac{4C_{GN}M^2}{\mathcal{D}_\mathbf{u}\mathcal{D}_c}\right)\int_{\mathbb{R}^2}n^2\mathrm{d}x\\
&\quad+\frac{4C_{GN}M^2\|\nabla\phi\|^2_{L^\infty(\mathbb{R}^2)}}{\mathcal{D}_\mathbf{u}\mathcal{D}_c}\int_{\mathbb{R}^2}|\mathbf{u}|^2\mathrm{d}x\\
&\le\left(\frac{C_{GN}M^2\sup\limits_{0\le c\le M}\chi^4(c)}{4\mathcal{D}_n^2\mathcal{D}_c}+\frac{\sup\limits_{0\le c\le M}k^2(c)}{\mathcal{D}_c}+\frac{4C_{GN}M^2}{\mathcal{D}_\mathbf{u}\mathcal{D}_c}\right)\\
&\quad\times C_{GN}\|n_0\|_{L^1(\mathbb{R}^2)}\int_{\mathbb{R}^2}|\nabla\sqrt{n}|^2\mathrm{d}x+\frac{4C_{GN}M^2\|\nabla\phi\|^2_{L^\infty(\mathbb{R}^2)}}{\mathcal{D}_\mathbf{u}\mathcal{D}_c}\int_{\mathbb{R}^2}|\mathbf{u}|^2\mathrm{d}x.
\end{aligned}\tag{2.19}$$

Here in the last inequality we used the estimate

$$\|n\|_{L^2(\mathbb{R}^2)}\le\sqrt{C_{GN}}\|n\|^{1/2}_{L^1(\mathbb{R}^2)}\|\nabla\sqrt{n}\|_{L^2(\mathbb{R}^2)}\le\sqrt{C_{GN}}\|n_0\|^{1/2}_{L^1(\mathbb{R}^2)}\|\nabla\sqrt{n}\|_{L^2(\mathbb{R}^2)},\tag{2.20}$$

which follows from the Gagliardo–Nirenberg inequality and the mass conservation equality (2.2).

Step 6. An evolution estimate of the first-order spatial moment of n.

For simplicity, we set $\langle x\rangle:=\sqrt{1+|x|^2}$. Multiplying the equation $(1.1)_1$ by $\langle x\rangle$ and integrating by parts, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2}n\langle x\rangle\,\mathrm{d}x&=\int_{\mathbb{R}^2}n\mathbf{u}\cdot\nabla\langle x\rangle\,\mathrm{d}x+\mathcal{D}_n\int_{\mathbb{R}^2}n\Delta\langle x\rangle\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^2}\chi(c)n\nabla c\cdot\nabla\langle x\rangle\,\mathrm{d}x:=I_4+I_5+I_6.
\end{aligned}\tag{2.21}$$

We now estimate I_4, I_5 and I_6 one by one. For I_4, by the Cauchy-Schwarz inequality, (2.20) and Young's inequality, we see that

$$
\begin{aligned}
I_4 &\leq \|n\|_{L^2(\mathbb{R}^2)}\|\mathbf{u}\|_{L^2(\mathbb{R}^2)}\|\nabla\langle x\rangle\|_{L^\infty(\mathbb{R}^2)} \\
&\leq \sqrt{C_{GN}}\|n_0\|_{L^1(\mathbb{R}^2)}^{1/2}\|\nabla\sqrt{n}\|_{L^2(\mathbb{R}^2)}\|\mathbf{u}\|_{L^2(\mathbb{R}^2)} \\
&\leq \frac{\mathcal{D}_n}{4}\|\nabla\sqrt{n}\|_{L^2(\mathbb{R}^2)}^2 + \frac{C_{GN}}{\mathcal{D}_n}\|n_0\|_{L^1(\mathbb{R}^2)}\|\mathbf{u}\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned}
\tag{2.22}
$$

For I_5, we have

$$
I_5 \leq \mathcal{D}_n\|\Delta\langle x\rangle\|_{L^\infty(\mathbb{R}^2)}\|n\|_{L^1(\mathbb{R}^2)} \leq 2\mathcal{D}_n\|n_0\|_{L^1(\mathbb{R}^2)}
\tag{2.23}
$$

by the mass conservation for n again. Finally, for I_6, we use a similar procedure as I_4 to obtain

$$
\begin{aligned}
I_6 &\leq \sup_{0\leq c\leq M}|\chi(c)|\,\|n\|_{L^2(\mathbb{R}^2)}\|\nabla c\|_{L^2(\mathbb{R}^2)}\|\nabla\langle x\rangle\|_{L^\infty(\mathbb{R}^2)} \\
&\leq \sup_{0\leq c\leq M}|\chi(c)|\sqrt{C_{GN}}\|n_0\|_{L^1(\mathbb{R}^2)}^{1/2}\|\nabla\sqrt{n}\|_{L^2(\mathbb{R}^2)}\|\nabla c\|_{L^2(\mathbb{R}^2)} \\
&\leq \frac{\mathcal{D}_n}{4}\|\nabla\sqrt{n}\|_{L^2(\mathbb{R}^2)}^2 + \frac{C_{GN}}{\mathcal{D}_n}\sup_{0\leq c\leq M}\chi^2(c)\,\|n_0\|_{L^1(\mathbb{R}^2)}\|\nabla c\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned}
\tag{2.24}
$$

Substituting (2.22), (2.23) and (2.24) into (2.21), we deduce that

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2} 2n\langle x\rangle\,\mathrm{d}x &\leq \mathcal{D}_n\int_{\mathbb{R}^2}|\nabla\sqrt{n}|^2\mathrm{d}x + \frac{2C_{GN}}{\mathcal{D}_n}\|n_0\|_{L^1(\mathbb{R}^2)}\int_{\mathbb{R}^2}|\mathbf{u}|^2\mathrm{d}x \\
&\quad + \frac{2C_{GN}}{\mathcal{D}_n}\sup_{0\leq c\leq M}\chi^2(c)\,\|n_0\|_{L^1(\mathbb{R}^2)}\int_{\mathbb{R}^2}|\nabla c|^2\mathrm{d}x \\
&\quad + 4\mathcal{D}_n\|n_0\|_{L^1(\mathbb{R}^2)}.
\end{aligned}
\tag{2.25}
$$

Step 7. Closing of the entropy estimates.

Since $\int_{\mathbb{R}^2}\frac{|\nabla n|^2}{n}\mathrm{d}x = 4\int_{\mathbb{R}^2}|\nabla\sqrt{n}|^2\mathrm{d}x$, we may combine (2.25) with (2.19) to obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\Big(n\ln n + 2n\langle x\rangle + |\nabla c|^2 + \frac{4C_{GN}M^2}{\mathcal{D}_{\mathbf{u}}\mathcal{D}_c}|\mathbf{u}|^2\Big)\mathrm{d}x \\
&\quad + \mathcal{D}_n\int_{\mathbb{R}^2}|\nabla\sqrt{n}|^2\mathrm{d}x + \frac{\mathcal{D}_c}{2}\int_{\mathbb{R}^2}|\Delta c|^2\mathrm{d}x + \frac{4C_{GN}M^2}{\mathcal{D}_c}\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2\mathrm{d}x \\
&\leq K_1\int_{\mathbb{R}^2}|\nabla\sqrt{n}|^2\mathrm{d}x + \left(\frac{4C_{GN}M^2\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}^2}{\mathcal{D}_{\mathbf{u}}\mathcal{D}_c} + \frac{2C_{GN}\|n_0\|_{L^1(\mathbb{R}^2)}}{\mathcal{D}_n}\right)\int_{\mathbb{R}^2}|\mathbf{u}|^2\mathrm{d}x \\
&\quad + \frac{2C_{GN}\|n_0\|_{L^1(\mathbb{R}^2)}\sup_{0\leq c\leq M}\chi^2(c)}{\mathcal{D}_n}\int_{\mathbb{R}^2}|\nabla c|^2\mathrm{d}x + 4\mathcal{D}_n\|n_0\|_{L^1(\mathbb{R}^2)}
\end{aligned}
\tag{2.26}
$$

$$\leq K_1 \int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \mathrm{d}x + K_2 \left(\int_{\mathbb{R}^2} |\nabla c|^2 \mathrm{d}x + \frac{4C_{GN}M^2}{\mathcal{D}_{\mathbf{u}}\mathcal{D}_c} \int_{\mathbb{R}^2} |\mathbf{u}|^2 \mathrm{d}x \right) + 4\mathcal{D}_n \|n_0\|_{L^1(\mathbb{R}^2)}.$$

To close the entropy estimates, we need to bound $\int_{\mathbb{R}^2} \big(n \ln n + 2\langle x\rangle n\big)\mathrm{d}x$ from below. Indeed, let $\mathbb{I}_X$ be the indicator function of a set X and then a direct calculation shows that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Big(n \ln n + 2n\langle x\rangle\Big)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^2} n|\ln n|\,\mathrm{d}x + 2\int_{\mathbb{R}^2} n \ln n\, \mathbb{I}_{\{x\in\mathbb{R}^2 |\, n(x)\leq 1\}} \mathrm{d}x + 2\int_{\mathbb{R}^2} n\langle x\rangle\,\mathrm{d}x \\
&= \int_{\mathbb{R}^2} n|\ln n|\,\mathrm{d}x + 2\int_{\mathbb{R}^2} n \ln n\, \mathbb{I}_{\{x\in\mathbb{R}^2 |\, 0\leq n(x)\leq e^{-\langle x\rangle}\}} \mathrm{d}x \\
&\quad + 2\int_{\mathbb{R}^2} n \ln n\, \mathbb{I}_{\{x\in\mathbb{R}^2 |\, e^{-\langle x\rangle}\leq n(x)\leq 1\}} \mathrm{d}x + 2\int_{\mathbb{R}^2} n\langle x\rangle\,\mathrm{d}x \\
&\geq \int_{\mathbb{R}^2} n|\ln n|\,\mathrm{d}x + 2\int_{\mathbb{R}^2} n \ln n\, \mathbb{I}_{\{x\in\mathbb{R}^2 |\, 0\leq n(x)\leq e^{-\langle x\rangle}\}} \mathrm{d}x.
\end{aligned}$$

By the basic fact that $\sqrt{x}\ln x \geq -2e^{-1}$ for any $x \geq 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Big(n \ln n + 2n\langle x\rangle\Big)\,\mathrm{d}x \\
&\geq \int_{\mathbb{R}^2} n|\ln n|\,\mathrm{d}x - 4e^{-1}\int_{\mathbb{R}^2} n^{\frac{1}{2}}\, \mathbb{I}_{\{x\in\mathbb{R}^2 |\, 0\leq n(x)\leq e^{-\langle x\rangle}\}} \mathrm{d}x \\
&\geq \int_{\mathbb{R}^2} n|\ln n|\,\mathrm{d}x - 4e^{-1}\int_{\mathbb{R}^2} e^{-\frac{1}{2}\langle x\rangle} \mathrm{d}x \\
&\geq -4e^{-1}\int_{\mathbb{R}^2} e^{-\frac{1}{2}\langle x\rangle} \mathrm{d}x.
\end{aligned} \tag{2.27}$$

Thus, substituting (2.27) into (2.26), we obtain

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\int_{\mathbb{R}^2} \Big(n \ln n + 2n\langle x\rangle + |\nabla c|^2 + \frac{4C_{GN}M^2}{\mathcal{D}_{\mathbf{u}}\mathcal{D}_c}|\mathbf{u}|^2\Big)\mathrm{d}x \\
&\quad + \big(\mathcal{D}_n - K_1\big)\int_{\mathbb{R}^2} |\nabla\sqrt{n}|^2\mathrm{d}x + \frac{\mathcal{D}_c}{2}\int_{\mathbb{R}^2} |\Delta c|^2\mathrm{d}x + \frac{4C_{GN}M^2}{\mathcal{D}_c}\int_{\mathbb{R}^2} |\nabla\mathbf{u}|^2\mathrm{d}x
\end{aligned}$$

$$\leq K_2 \int_{\mathbb{R}^2} \Big(n \ln n + 2n\langle x\rangle + |\nabla c|^2 + \frac{4C_{GN}M^2}{\mathcal{D}_{\mathbf{u}}\mathcal{D}_c}|\mathbf{u}|^2 \Big) \mathrm{d}x + 4\mathcal{D}_n \|n_0\|_{L^1(\mathbb{R}^2)}$$

$$+ 4K_2 e^{-1} \int_{\mathbb{R}^2} e^{-\frac{1}{2}\langle x\rangle} \mathrm{d}x,$$

that is,

$$\frac{d}{\mathrm{d}t}\mathcal{E}(t) + \mathcal{F}(t) \leq K_2\mathcal{E}(t) + K_3, \qquad t > 0.$$

Then Gronwall's inequality or a direct integration yields the desired entropy inequality (2.5) and (2.6). □

Remark 2.1. The inequality (2.27) gives the estimate

$$\int_{\mathbb{R}^2} n|\ln n| \,\mathrm{d}x \leq \int_{\mathbb{R}^2} \Big(n \ln n + 2n\langle x\rangle \Big) \mathrm{d}x + 4e^{-1} \int_{\mathbb{R}^2} e^{-\frac{1}{2}\langle x\rangle},$$

which together with (2.5) or (2.6) also yields the bounds of $\int_{\mathbb{R}^2} n|\ln n| \,\mathrm{d}x$.

We now prove the global existence of weak solutions to the Cauchy problem of system (1.1)–(1.2). Following the ideas of [5,12], our proof is based on the entropy functional estimate derived last section, which allows us to perform a compactness argument to the regularized system of (1.1)–(1.2).

Proof of Theorem 1.1. Firstly, we construct the following regularized system to (1.1)–(1.2) as [12]:

$$\begin{cases} n_t^\epsilon + \mathbf{u}^\epsilon \cdot \nabla n^\epsilon = \mathcal{D}_n \Delta n^\epsilon - \nabla \cdot \big(n^\epsilon [\chi(c^\epsilon)\nabla c^\epsilon] * \sigma^\epsilon \big), \\ c_t^\epsilon + \mathbf{u}^\epsilon \cdot \nabla c^\epsilon = \mathcal{D}_c \Delta c^\epsilon - [k(c^\epsilon)n^\epsilon] * \sigma^\epsilon, \\ \mathbf{u}_t^\epsilon + \mathbf{u}^\epsilon \cdot \nabla \mathbf{u}^\epsilon + \nabla P^\epsilon = \mathcal{D}_{\mathbf{u}} \Delta \mathbf{u}^\epsilon + (n^\epsilon \nabla\phi) * \sigma^\epsilon, \\ \nabla \cdot \mathbf{u}^\epsilon = 0 \end{cases} \tag{2.28}$$

in $\mathbb{R}^2 \times (0, \infty)$, with prescribed initial data

$$(n^\epsilon, c^\epsilon, \mathbf{u}^\epsilon)\big|_{t=0} = (n_0 * \sigma^\epsilon, c_0 * \sigma^\epsilon, \mathbf{u}_0 * \sigma^\epsilon) \tag{2.29}$$

in $\mathbb{R}^2$, where σ^ϵ is a mollifier. For each given ϵ, the local existence of strong solution $(n^\epsilon, c^\epsilon, \mathbf{u}^\epsilon)$ to the regularized system (2.28)–(2.29) can be obtained by applying Schauder fixed point theorem. Then the following uniform estimates also allows us to extend the local strong solution $(n^\epsilon, c^\epsilon, \mathbf{u}^\epsilon)$ to a global weak solution.

We now begin to establish the uniform estimates for $(n^\epsilon, c^\epsilon, \mathbf{u}^\epsilon)$ in ϵ. Following the same argument as Lemma 2.1, we can obtain the nonnegativity of n^ϵ and c^ϵ, and the basic mass conservation of n^ϵ as well as the boundedness of c^ϵ:

$\mathcal{B}_1$. $n^\epsilon \geq 0$ and $c^\epsilon \geq 0$ in $\Omega \times (0,\infty)$;

$\mathcal{B}_2$. $\|n^\epsilon\|_{L^1(\mathbb{R}^2)} = \|n_0^\epsilon\|_{L^1(\mathbb{R}^2)} \leq \|n_0\|_{L^1(\mathbb{R}^2)}$ and $\|c^\epsilon\|_{L^p(\mathbb{R}^2)} \leq \|c_0^\epsilon\|_{L^p(\mathbb{R}^2)} \leq \|c_0\|_{L^p(\mathbb{R}^2)}$ for any $1 \leq p \leq \infty$, where we used Young's inequality in the last two inequalities.

Then following the same argument as Lemma 2.2 and applying Young's inequality again to the terms involving the convolution, we can find that the same entropy functional inequality also works for this regular system. We first state the uniform boundedness of n^ϵ in ϵ:

$\mathcal{B}_3$. $\nabla\sqrt{n^\epsilon}$ is bounded in $L^2\big((0,T), L^2(\mathbb{R}^2)\big)$. This also yields that

$\mathcal{B}_4$. n^ϵ is bounded in $L^2\big((0,T)\times\mathbb{R}^2\big)$ by the estimate

$$\|n^\epsilon\|_{L^2(\mathbb{R}^2)} \leq C_{GN}\|n^\epsilon\|^{1/2}_{L^1(\mathbb{R}^2)}\|\nabla\sqrt{n^\epsilon}\|_{L^2(\mathbb{R}^2)} \leq C_{GN}\|n_0\|^{1/2}_{L^1(\mathbb{R}^2)}\|\nabla\sqrt{n^\epsilon}\|_{L^2(\mathbb{R}^2)},$$

where we used the Gagliardo–Nirenberg inequality and the mass conservation $\mathcal{B}_2$. The regularized versions of Remark 2.1 and Lemma 2.2 also give that

$\mathcal{B}_5$. $n^\epsilon|\ln n^\epsilon|$ is bounded in $L^\infty\big((0,T), L^1(\mathbb{R}^2)\big)$.

Next, for c^ϵ and $\mathbf{u}^\epsilon$, we have

$\mathcal{B}_6$. c^ϵ is bounded in $L^\infty\big((0,T), H^1(\mathbb{R}^2)\big) \cap L^2\big((0,T), H^2(\mathbb{R}^2)\big)$ for any $T > 0$, where we also used the boundedness $\mathcal{B}_2$ with $p = 2$;

$\mathcal{B}_7$. $\mathbf{u}^\epsilon$ is bounded in $L^\infty\big((0,T), L^2(\mathbb{R}^2;\mathbb{R}^2)\big) \cap L^2\big((0,T), H^1(\mathbb{R}^2;\mathbb{R}^2)\big)$.

Finally, to apply the Aubin–Lions lemma, we need to show the boundedness of n_t^ϵ, c_t^ϵ and $\mathbf{u}_t^\epsilon$. Indeed, we have

$\mathcal{B}_8$. n_t^ϵ is bounded in $L^2\big((0,T), H^{-3}(\mathbb{R}^2)\big)$. This can be verified as follows: it follows from the integration by parts, Hölder's inequality, Young's inequality and Sobolev's embedding that

$$\begin{aligned}
\big|\langle n_t^\epsilon, \varphi\rangle\big| &= \big|\mathcal{D}_n\langle n^\epsilon, \Delta\varphi\rangle + \langle n^\epsilon(\chi(c^\epsilon)\nabla c^\epsilon * \sigma^\epsilon) + \mathbf{u}^\epsilon n^\epsilon, \nabla\varphi\rangle\big| \\
&\leq \mathcal{D}_n\|n^\epsilon\|_{L^2(\mathbb{R}^2)}\|\Delta\varphi\|_{L^2(\mathbb{R}^2)} + C(n^\epsilon, c^\epsilon, \mathbf{u}^\epsilon)\|\nabla\varphi\|_{L^\infty(\mathbb{R}^2)} \\
&\leq \Big(\mathcal{D}_n\|n^\epsilon\|_{L^2(\mathbb{R}^2)} + C(n^\epsilon, c^\epsilon, \mathbf{u}^\epsilon)\Big)\|\varphi\|_{H^3(\mathbb{R}^2)}
\end{aligned}$$

for any $\varphi \in H^3(\mathbb{R}^2)$, with

$$C(n^\epsilon, c^\epsilon, \mathbf{u}^\epsilon) := \sup_{0\leq s\leq M}\chi(s)\|n^\epsilon\|_{L^2(\mathbb{R}^2)}\|\nabla c^\epsilon\|_{L^2(\mathbb{R}^2)} + \|\mathbf{u}^\epsilon\|_{L^2(\mathbb{R}^2)}\|n^\epsilon\|_{L^2(\mathbb{R}^2)}.$$

This together with the uniform boundedness $\mathcal{B}_4$, $\mathcal{B}_6$ and $\mathcal{B}_7$ implies the desired estimate.

$\mathcal{B}_9$. c_t^ϵ is bounded in $L^2\big((0,T)\times\mathbb{R}^2\big)$. This can be seen from the fact

$$\begin{aligned}
\|c_t^\epsilon\|_{L^2(\mathbb{R}^2)} &\leq \|\mathbf{u}^\epsilon\|_{L^4(\mathbb{R}^2)}\|\nabla c^\epsilon\|_{L^4(\mathbb{R}^2)} + \mathcal{D}_c\|\Delta c^\epsilon\|_{L^2(\mathbb{R}^2)} \\
&\quad + \sup_{0\leq s\leq M}|k(s)|\|n^\epsilon\|_{L^2(\mathbb{R}^2)} \\
&\leq \Big(\|\mathbf{u}^\epsilon\|_{H^1(\mathbb{R}^2)} + \mathcal{D}_c\Big)\|c^\epsilon\|_{H^2(\mathbb{R}^2)} + \sup_{0\leq s\leq M}|k(s)|\|n^\epsilon\|_{L^2(\mathbb{R}^2)}
\end{aligned}$$

and the uniform boundedness $\mathcal{B}_4$, $\mathcal{B}_6$ and $\mathcal{B}_7$.

$\mathcal{B}_{10}$. $\mathbf{u}_t^\epsilon$ is bounded in $L^2\big((0,T), H^{-1}(\mathbb{R}^2)\big)$. To see this, we apply the project operator $\mathcal{P}$ to $(2.28)_3$ and use the L^2–L^2 boundedness of Riesz operator to deduce that

$$
\begin{aligned}
\big|\langle \mathbf{u}_t^\epsilon, \varphi\rangle\big| &= \big|\mathcal{D}_{\mathbf{u}}\langle \nabla \mathbf{u}^\epsilon, \nabla\mathcal{P}\varphi\rangle + \langle (n^\epsilon\nabla\phi) * \sigma^\epsilon, \mathcal{P}\varphi\rangle + \langle \mathbf{u}^\epsilon\otimes\mathbf{u}^\epsilon, \nabla\mathcal{P}\varphi\rangle\big| \\
&\le \mathcal{D}_{\mathbf{u}}\|\nabla\mathbf{u}^\epsilon\|_{L^2(\mathbb{R}^2)}\|\nabla\mathcal{P}\varphi\|_{L^2(\mathbb{R}^2)} + \|n^\epsilon\|_{L^2(\mathbb{R}^2)}\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}\|\mathcal{P}\varphi\|_{L^2(\mathbb{R}^2)} \\
&\quad + \|\mathbf{u}^\epsilon\|^2_{L^4(\mathbb{R}^2)}\|\nabla\mathcal{P}\varphi\|_{L^2(\mathbb{R}^2)} \\
&\le \Big(\mathcal{D}_{\mathbf{u}}\|\mathbf{u}^\epsilon\|_{H^1(\mathbb{R}^2)} + \|n^\epsilon\|_{L^2(\mathbb{R}^2)}\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)} + \|\mathbf{u}^\epsilon\|_{L^2(\mathbb{R}^2)}\|\mathbf{u}^\epsilon\|_{H^1(\mathbb{R}^2)}\Big)\|\varphi\|_{H^1(\mathbb{R}^2)}
\end{aligned}
$$

for any $\varphi \in H^1(\mathbb{R}^2;\mathbb{R}^2)$, which together with the uniform boundedness $\mathcal{B}_4$ and $\mathcal{B}_7$ yields the desired estimate.

With the above uniform estimates at hand, we turn to passing the limit of $(n^\epsilon, c^\epsilon, \mathbf{u}^\epsilon)$. Firstly, we see that n^ϵ will strongly converge to some n in $L^2\big([0,T]; L^2_{loc}(\mathbb{R}^2)\big)$ by using the Aubin–Lions lemma together with $\mathcal{B}_4$ and $\mathcal{B}_8$, and $\nabla\sqrt{n^\epsilon}$ weakly converges to $\nabla\sqrt{n}$ in $L^2\big((0,T), L^2_{loc}(\mathbb{R}^2)\big)$ by $\mathcal{B}_3$.

Secondly, c^ϵ converges strongly to some c in $L^2\big([0,T]; H^1_{loc}(\mathbb{R}^2)\big)$ by applying the Aubin–Lions lemma to $\mathcal{B}_6$ and $\mathcal{B}_9$.

Thirdly, by applying the Aubin–Lions lemma to $\mathcal{B}_7$ and $\mathcal{B}_{10}$, $\mathbf{u}^\epsilon$ converges strongly to some $\mathbf{u}$ in $L^2\big([0,T]; L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)\big)$, and further converges weakly to $\mathbf{u}$ in $L^2\big([0,T]; H^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)\big)$ by $\mathcal{B}_7$ again.

The above convergences also imply that $\mathbf{u}^\epsilon n^\epsilon$ and $\mathbf{u}^\epsilon c^\epsilon$ will strongly converge to $\mathbf{u}n$ and $\mathbf{u}c$ in $L^1\big([0,T]; L^1_{loc}(\mathbb{R}^2)\big)$, respectively, while $\mathbf{u}^\epsilon\cdot\nabla\mathbf{u}^\epsilon$ will weakly converge to $\mathbf{u}\cdot\nabla\mathbf{u}$ in $L^1\big([0,T]; L^1_{loc}(\mathbb{R}^2)\big)$.

Summarily, we have shown that $(n,c,\mathbf{u})$ satisfies the weak form (1.16)–(1.18) and the corresponding entropy functional inequalities (2.5) and (2.6) hold, which yield the regularity (1.15). Thus $(n,c,\mathbf{u})$ is a weak solution to system (1.1)–(1.2). This completes the proof of Theorem 1.1. □

2.2. Global existence of classical solutions

In this subsection, we prove that the weak solution obtained afore is regular under some additional assumptions on $\chi(\cdot)$, $k(\cdot)$ and $\phi(\cdot)$, and on the initial data. To achieve this goal, we first recall the existence and uniqueness of smooth local solution as well as the corresponding extensibility criterion.

Lemma 2.3 *(See Theorem 1 and Theorem 2 in [2]). Let $m \ge 3$. Assume that $\chi(\cdot), k(\cdot) \in C^m(\mathbb{R}_+)$ with $k(0) = 0$, and that $\|\nabla^l\phi\|_{L^\infty(\mathbb{R}^2)} < \infty$ for $1 \le |l| \le m$. Then there exists $T^* > 0$, the maximal time of existence, such that, if the initial data $(n_0, c_0, \mathbf{u}_0) \in H^{m-1}(\mathbb{R}^2)\times H^m(\mathbb{R}^2)\times H^m(\mathbb{R}^2;\mathbb{R}^2)$, then there exists a unique classical solution $(n,c,\mathbf{u})$ to system (1.1)–(1.2) satisfying for any $T < T^*$*

$$(n,c,\mathbf{u}) \in L^\infty\big(0,T;\ H^{m-1}(\mathbb{R}^2)\times H^m(\mathbb{R}^2)\times H^m(\mathbb{R}^2;\mathbb{R}^2)\big)$$

and

$$(\nabla n, \nabla c, \nabla\mathbf{u}) \in L^2\big(0,T;\ H^{m-1}(\mathbb{R}^2)\times H^m(\mathbb{R}^2)\times H^m(\mathbb{R}^2;\mathbb{R}^2)\big).$$

Moreover, if the maximal time of existence $T^* < \infty$, then

$$\int_0^{T^*} \|\nabla c(\tau)\|^2_{L^\infty(\mathbb{R}^2)} \mathrm{d}\tau = \infty. \tag{2.30}$$

The global existence will be proved by showing that the local classical solutions can be extended at any time $T > 0$ due to the extensibility criterion.

Proof of Theorem 1.2. We will prove the global existence of classical solutions by contradictory arguments. Assuming that the maximal time T^* is finite, we will show that

$$\int_0^{T^*} \|\nabla c(\tau)\|^2_{L^\infty(\mathbb{R}^2)} \mathrm{d}\tau < \infty,$$

which leads to a contradiction to the extensibility criterion (2.30). Indeed, since

$$\|\nabla c\|^2_{L^\infty(\mathbb{R}^2)} \le C_{GN} \|\nabla c\|_{L^2(\mathbb{R}^2)} \|\nabla^3 c\|_{L^2(\mathbb{R}^2)} \le \frac{C_{GN}}{2} \left(\|\nabla c\|^2_{L^2(\mathbb{R}^2)} + \|\nabla^3 c\|^2_{L^2(\mathbb{R}^2)} \right),$$

we only need to verify that

$$\int_0^{T^*} \int_{\mathbb{R}^2} |\nabla c(x,\tau)|^2 \mathrm{d}x \mathrm{d}\tau < \infty, \tag{2.31}$$

and that

$$\int_0^{T^*} \int_{\mathbb{R}^2} |\nabla^3 c(x,\tau)|^2 \mathrm{d}x \mathrm{d}\tau < \infty. \tag{2.32}$$

To see (2.31), we first substitute (2.27) into (2.5) or (2.6) and then obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} |\nabla c(x,t)|^2 \mathrm{d}x + \int_{\mathbb{R}^2} |\mathbf{u}(x,t)|^2 \mathrm{d}x + \int_0^{T^*} \int_{\mathbb{R}^2} |\nabla \sqrt{n}(x,\tau)|^2 \mathrm{d}x \mathrm{d}\tau \\ &+ \int_0^{T^*} \int_{\mathbb{R}^2} |\Delta c(x,\tau)|^2 \mathrm{d}x \mathrm{d}\tau + \int_0^{T^*} \int_{\mathbb{R}^2} |\nabla \mathbf{u}(x,\tau)|^2 \mathrm{d}x \mathrm{d}\tau \le C_1, \end{aligned} \tag{2.33}$$

for all $t \in (0, T^*)$, where C_1 is a positive constant depending only on the initial data and the maximal time T^*. Thus a direct integration from 0 to T^* yields (2.31).

To see (2.32), we apply Δ to equation (1.1)$_2$, multiply Δc with the resulted equation, and integrate over $\mathbb{R}^2$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{\mathrm{d}t}\int_{\mathbb{R}^2}|\Delta c|^2\mathrm{d}x+\mathcal{D}_c\int_{\mathbb{R}^2}|\nabla\Delta c|^2\mathrm{d}x\\
&=\int_{\mathbb{R}^2}\big(\nabla(\mathbf{u}\cdot\nabla c)+k'(c)n\nabla c+k(c)\nabla n\big)\cdot\nabla\Delta c\mathrm{d}x\\
&\le\frac{\mathcal{D}_c}{2}\int_{\mathbb{R}^2}|\nabla\Delta c|^2\mathrm{d}x+\frac{1}{\mathcal{D}_c}\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2|\nabla c|^2\mathrm{d}x+\frac{1}{\mathcal{D}_c}\int_{\mathbb{R}^2}|\mathbf{u}|^2|D^2c|^2\mathrm{d}x\\
&\quad+\frac{1}{\mathcal{D}_c}\sup_{0\le s\le M}k'^2(s)\int_{\mathbb{R}^2}n^2|\nabla c|^2\mathrm{d}x+\frac{1}{\mathcal{D}_c}\sup_{0\le s\le M}k^2(s)\int_{\mathbb{R}^2}|\nabla n|^2\mathrm{d}x.
\end{aligned}$$

Absorbing the first term on the right-hand side, we have

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\int_{\mathbb{R}^2}|\Delta c|^2\mathrm{d}x+\mathcal{D}_c\int_{\mathbb{R}^2}|\nabla\Delta c|^2\mathrm{d}x\\
&\le\frac{2}{\mathcal{D}_c}\|\nabla\mathbf{u}\|^2_{L^3(\mathbb{R}^2)}\|\nabla c\|^2_{L^6(\mathbb{R}^2)}+\frac{2}{\mathcal{D}_c}\|\mathbf{u}\|^2_{L^\infty(\mathbb{R}^2)}\|D^2c\|^2_{L^2(\mathbb{R}^2)}\\
&\quad+\frac{2}{\mathcal{D}_c}\sup_{0\le s\le M}k'^2(s)\|\nabla c\|^2_{L^6(\mathbb{R}^2)}\|n\|^2_{L^3(\mathbb{R}^2)}+\frac{2}{\mathcal{D}_c}\sup_{0\le s\le M}k^2(s)\int_{\mathbb{R}^2}|\nabla n|^2\mathrm{d}x\\
&\le C_2\Big(\|\mathbf{u}\|^2_{H^2(\mathbb{R}^2)}+\|n\|^2_{H^1(\mathbb{R}^2)}\Big)\|\nabla c\|^2_{H^1(\mathbb{R}^2)}+C_2\int_{\mathbb{R}^2}|\nabla n|^2\mathrm{d}x,
\end{aligned}\tag{2.34}$$

where C_2 is a positive constant related to the Sobolev's embedding and the initial data. Combining (2.13) with (2.34) and using (2.14), we deduce that

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\|\nabla c(t)\|^2_{H^1(\mathbb{R}^2)}+\mathcal{D}_c\|\nabla^2c(t)\|^2_{H^1(\mathbb{R}^2)}\\
&\le C_3\Big(1+\|n\|^2_{H^1(\mathbb{R}^2)}+\|\mathbf{u}\|^2_{H^2(\mathbb{R}^2)}\Big)\|\nabla c\|^2_{H^1(\mathbb{R}^2)}\\
&\quad+C_3\Big(\|n\|^2_{H^1(\mathbb{R}^2)}+\|\mathbf{u}\|^2_{H^1(\mathbb{R}^2)}\Big)
\end{aligned}$$

for some positive constant C_3. It then follows from Gronwall's inequality that

$$\begin{aligned}
&\|\nabla c(t)\|^2_{H^1(\mathbb{R}^2)}+\mathcal{D}_c\int_0^{T^*}\|\nabla^2c(\tau)\|^2_{H^1(\mathbb{R}^2)}\mathrm{d}\tau\\
&\le\|c_0\|^2_{H^2(\mathbb{R}^2)}e^{C_3\int_0^{T^*}(1+\|n(\tau)\|^2_{H^1(\mathbb{R}^2)}+\|\mathbf{u}(\tau)\|^2_{H^2(\mathbb{R}^2)})\mathrm{d}\tau}
\end{aligned}$$

$$+ C_3 \int_0^{T^*} \left(\|n(\tau)\|^2_{H^1(\mathbb{R}^2)} + \|\mathbf{u}(\tau)\|^2_{H^1(\mathbb{R}^2)} \right) d\tau$$

for all $t \in (0, T^*)$. Thus we have verified (2.32) provided that

$$\int_0^{T^*} \left(\|n(\tau)\|^2_{H^1(\mathbb{R}^2)} + \|\mathbf{u}(\tau)\|^2_{H^2(\mathbb{R}^2)} \right) d\tau < \infty. \tag{2.35}$$

We now turn to prove (2.35). Firstly, we may employ the L^2 scalar product for equation $(1.1)_1$ and integrate by parts to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} n^2 dx + \mathcal{D}_n \int_{\mathbb{R}^2} |\nabla n|^2 dx &= \int_{\mathbb{R}^2} n\chi(c)\nabla c \cdot \nabla n \, dx \\ &\le \frac{\mathcal{D}_n}{4}\int_{\mathbb{R}^2} |\nabla n|^2 dx + \frac{\sup\limits_{0\le c\le M} \chi^2(c)}{\mathcal{D}_n} \int_{\mathbb{R}^2} n^2 |\nabla c|^2 dx. \end{aligned}$$

The integral of the second term on the right can be further bounded as

$$\begin{aligned} \int_{\mathbb{R}^2} n^2|\nabla c|^2 dx &\le \|n\|^2_{L^4(\mathbb{R}^2)} \|\nabla c\|^2_{L^4(\mathbb{R}^2)} \\ &\le C_{GN} \|n\|_{L^2(\mathbb{R}^2)} \|\nabla n\|_{L^2(\mathbb{R}^2)} \|c\|_{L^\infty(\mathbb{R}^2)} \|\Delta c\|_{L^2(\mathbb{R}^2)} \\ &\le \frac{\mathcal{D}_n}{4} \int_{\mathbb{R}^2} |\nabla n|^2 dx + C_3 \int_{\mathbb{R}^2} |\Delta c|^2 dx \int_{\mathbb{R}^2} n^2 dx, \end{aligned}$$

with $C_3 = \mathcal{D}_n^{-3} C_{GN}^2 M^2 \sup_{0\le c\le M} \chi^4(c)$, where we used Sobolev's embedding, Young's inequality and the boundedness of c. Then, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} n^2 dx + \mathcal{D}_n \int_{\mathbb{R}^2} |\nabla n|^2 dx \le \frac{\mathcal{D}_n}{2} \int_{\mathbb{R}^2} |\nabla n|^2 dx + C_3 \int_{\mathbb{R}^2} |\Delta c|^2 dx \int_{\mathbb{R}^2} n^2 dx.$$

Absorbing the first term on the right hand side, we have

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^2 dx + \mathcal{D}_n \int_{\mathbb{R}^2} |\nabla n|^2 dx \le 2C_3 \int_{\mathbb{R}^2} |\Delta c|^2 dx \int_{\mathbb{R}^2} n^2 dx.$$

It then follows from Gronwall's inequality that

$$\int_{\mathbb{R}^2} n^2(x,t)\,\mathrm{d}x + \mathcal{D}_n \int_0^{T^*}\int_{\mathbb{R}^2} |\nabla n(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau \le C_4$$

for all $t \in (0, T^*)$ and some positive constant C_4 depending on the initial data and the maximal time T^*, where we also used the fact

$$\int_0^{T^*}\int_{\mathbb{R}^2} |\Delta c(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau < \infty$$

due to (2.33). Thus a direct integration on $[0, T^*]$ yields that

$$\int_0^{T^*} \|n(\tau)\|^2_{H^1(\mathbb{R}^2)} \mathrm{d}\tau < \infty. \tag{2.36}$$

On the other hand, by using (2.33) again, we also have

$$\int_0^{T^*} \|\mathbf{u}(\tau)\|^2_{H^1(\mathbb{R}^2)} \mathrm{d}\tau < \infty. \tag{2.37}$$

It remains to investigate the integrability of the second derivative of $\mathbf{u}$. We first let $\omega := \nabla^{\perp}\mathbf{u}$ be the vorticity of $\mathbf{u}$ and then consider the vorticity equation

$$\omega_t + \mathbf{u}\cdot\nabla\omega = \mathcal{D}_{\mathbf{u}}\Delta\omega + \nabla^{\perp}(n\nabla\phi).$$

A direct energy method gives that

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\int_{\mathbb{R}^2} \omega^2 \mathrm{d}x + \mathcal{D}_{\mathbf{u}}\int_{\mathbb{R}^2} |\nabla\omega|^2 \mathrm{d}x &= -\int_{\mathbb{R}^2} \nabla^{\perp}\omega\cdot(n\nabla\phi)\mathrm{d}x \\ &\le \frac{\mathcal{D}_{\mathbf{u}}}{2}\int_{\mathbb{R}^2} |\nabla\omega|^2 \mathrm{d}x + \frac{1}{2\mathcal{D}_{\mathbf{u}}}\int_{\mathbb{R}^2} n^2|\nabla\phi|^2 \mathrm{d}x,\end{aligned}$$

which leads to the vorticity estimate

$$\frac{d}{\mathrm{d}t}\int_{\mathbb{R}^2} \omega^2 \mathrm{d}x + \mathcal{D}_{\mathbf{u}}\int_{\mathbb{R}^2} |\nabla\omega|^2 \mathrm{d}x \le C_5 \int_{\mathbb{R}^2} n^2 \mathrm{d}x,$$

where C_5 is a positive constant depending on $\nabla\phi$. Then Gronwall's inequality together with (2.36) implies that

$$\int_{\mathbb{R}^2} \omega^2(x,t)\mathrm{d}x + \mathcal{D}_{\mathbf{u}} \int_0^{T^*} \int_{\mathbb{R}^2} |\nabla \omega(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau \le C_6 \tag{2.38}$$

for all $t \in (0, T^*)$ with $C_6 := \int_{\mathbb{R}^2} \omega^2(x,0)\mathrm{d}x + C_4 C_5 T^*$. Thus we can use (2.38) and the Biot–Savart law to obtain

$$\int_0^{T^*} \|\nabla^2 \mathbf{u}(\tau)\|^2_{L^2(\mathbb{R}^2)} \mathrm{d}\tau < \infty. \tag{2.39}$$

Summarily, by (2.36), (2.37) and (2.39), we have verified the validness of (2.35). This complete the Proof of Theorem 1.2. □

3. The initial-boundary value problem

In this section, we will focus on the global existence and large time behavior of classical solutions to the initial-boundary value problem of system (1.1)–(1.3). We first recall the local existence, uniqueness and the extensibility criterion of classical solutions to system (1.1)–(1.3).

Lemma 3.1 *(See Lemma 2.1 in [23]). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Suppose that (1.12) and (1.14) hold. Then there exist $T^* \in (0, \infty]$ and a classical solution $(n, c, \mathbf{u}, P)$ to the initial-boundary value problem of system (1.1)–(1.3) in $\Omega \times (0, T^*)$. For any $T \in (0, T^*)$, this solution is unique, up to addition of constants to P, among all functions satisfying (1.19). Moreover, if $T^* < \infty$, then*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\alpha \mathbf{u}(\cdot,t)\|_{L^2(\Omega)} \to \infty \quad \text{as} \quad t \nearrow T^*. \tag{3.1}$$

To establish an entropy functional inequality, we define

$$\widetilde{\mathcal{E}}(t) := \int_\Omega \Big(n \ln n + |\nabla c|^2 + \frac{9 C_{GN} \widetilde{M}^2}{\mathcal{D}_c \mathcal{D}_{\mathbf{u}}} |\mathbf{u}|^2 \Big) \mathrm{d}x,$$

and

$$\widetilde{\mathcal{F}}(t) := \Big(\frac{4\mathcal{D}_n}{3} - \widetilde{K}_1 \Big) \int_\Omega |\nabla \sqrt{n}|^2 \mathrm{d}x + \frac{\mathcal{D}_c}{2} \int_\Omega |D^2 c|^2 \mathrm{d}x + \frac{C_{GN} \widetilde{M}^2}{\mathcal{D}_c} \int_\Omega |\nabla \mathbf{u}|^2 \mathrm{d}x,$$

where $\widetilde{M} := \|c_0\|_{L^\infty(\Omega)}$, $\widetilde{K}_1$ is defined by (3.2) and C_{GN} is a positive constant related to the Gagliardo–Nirenberg inequality, which may change from line to line. Here compared with the case of $\mathbb{R}^2$, we removed the first-order spatial moment $\int_\Omega n\langle x\rangle \mathrm{d}x$ of n in the definition of $\widetilde{\mathcal{E}}$ due to the boundedness of Ω and the fact $x \ln x \ge -e^{-1}$ for any $x \ge 0$.

We now state the entropy functional inequality for bounded domain Ω.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Suppose that (1.12) and (1.14) hold. If it holds that*

$$\widetilde{K}_1 := \left(\frac{C_{GN} \widetilde{M}^2 \sup\limits_{0 \le s \le \widetilde{M}} \chi^4(s)}{4\mathcal{D}_n^2 \mathcal{D}_c} + \frac{\sup\limits_{0 \le s \le \widetilde{M}} k'^2(s)}{\mathcal{D}_c} \right) C_{GN} \|n_0\|_{L^1(\Omega)} < \frac{4\mathcal{D}_n}{3}, \tag{3.2}$$

then the solution $(n, c, \mathbf{u})$ to the initial-boundary value problem of system (1.1)–(1.3) satisfies

$$\widetilde{\mathcal{E}}(t) + \int_0^t \widetilde{\mathcal{F}}(\tau)\,\mathrm{d}\tau \le Ct \qquad \text{for all} \quad t \in (0, T^*) \tag{3.3}$$

with some positive constant C depending on Ω and the initial data.

Proof. The proof is similar to that of Lemma 2.2. Here we only give the key steps.

We first note that the nonnegativity of n and c, and the basic mass conservation of n as well as the boundedness of c established in Lemma 2.1 still hold for the initial boundary value problem (1.1)–(1.3).

Then similar to Step 1–Step 3 in the proof of Lemma 2.2, we can deduce that

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\Big(\int_\Omega n \ln n \,\mathrm{d}x + \int_\Omega |\nabla c|^2 \mathrm{d}x \Big) + 2\mathcal{D}_n \int_\Omega |\nabla \sqrt{n}|^2 \mathrm{d}x + \mathcal{D}_c \int_\Omega |\Delta c|^2 \mathrm{d}x \\
&\le \left(\frac{\sup\limits_{0 \le c \le \widetilde{M}} \chi^4(c)}{4\mathcal{D}_n^2 \varepsilon} + \frac{\sup\limits_{0 \le c \le \widetilde{M}} k^2(c)}{\mathcal{D}_c} \right) \int_\Omega n^2 \mathrm{d}x + \frac{4}{\varepsilon} \int_\Omega |\nabla \mathbf{u}|^2 \mathrm{d}x + \frac{\varepsilon}{2} \int_\Omega |\nabla c|^4 \mathrm{d}x.
\end{aligned}$$

To control the last term on the right-hand side, we use the Gagliardo–Nirenberg inequality to obtain

$$\begin{aligned}
\|\nabla c\|_{L^4(\Omega)}^4 &\le C_{GN} \|c\|_{L^\infty(\Omega)}^2 \|D^2 c\|_{L^2(\Omega)}^2 + C_{GN} \|c\|_{L^\infty(\Omega)}^4 \\
&\le C_{GN} \|c_0\|_{L^\infty(\Omega)}^2 \|D^2 c\|_{L^2(\Omega)}^2 + C_{GN} \|c_0\|_{L^\infty(\Omega)}^4
\end{aligned} \tag{3.4}$$

and thus

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\Big(\int_\Omega n \ln n \,\mathrm{d}x + \int_\Omega |\nabla c|^2 \mathrm{d}x \Big) + 2\mathcal{D}_n \int_\Omega |\nabla \sqrt{n}|^2 \mathrm{d}x + \mathcal{D}_c \int_\Omega |\Delta c|^2 \mathrm{d}x \\
&\le \left(\frac{\sup\limits_{0 \le c \le \widetilde{M}} \chi^4(c)}{4\mathcal{D}_n^2 \varepsilon} + \frac{\sup\limits_{0 \le c \le \widetilde{M}} k^2(c)}{\mathcal{D}_c} \right) \int_\Omega n^2 \mathrm{d}x + \frac{4}{\varepsilon} \int_\Omega |\nabla \mathbf{u}|^2 \mathrm{d}x \\
&\quad + \frac{\varepsilon}{2} C_{GN} \widetilde{M}^2 \|D^2 c\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2} C_{GN} \widetilde{M}^4.
\end{aligned} \tag{3.5}$$

To cancel $\|D^2c\|^2_{L^2(\Omega)}$, we invoke the pointwise identity $|\Delta c|^2 = \nabla\cdot(\Delta c\nabla c) - \nabla c\cdot\nabla\Delta c$, and $\Delta|\nabla c|^2 = 2\nabla c\cdot\nabla\Delta c + 2|D^2c|^2$, as well as the no-flux Neumann boundary condition for c to rewrite $\int_\Omega |\Delta c|^2 \mathrm{d}x$ as

$$\begin{aligned}\int_\Omega |\Delta c|^2\mathrm{d}x &= -\int_\Omega \nabla c\cdot\nabla\Delta c\,\mathrm{d}x + \int_{\partial\Omega} \Delta c\frac{\partial c}{\partial\nu}dS\\ &= \int_\Omega |D^2c|^2\mathrm{d}x - \frac{1}{2}\int_\Omega \Delta|\nabla c|^2\mathrm{d}x\\ &= \int_\Omega |D^2c|^2\mathrm{d}x - \frac{1}{2}\int_{\partial\Omega} \frac{\partial|\nabla c|^2}{\partial\nu}dS.\end{aligned}\tag{3.6}$$

For the rightmost item, one can first invoke Lemma 4.2 in [13] to obtain

$$\frac{1}{2}\int_{\partial\Omega}\frac{\partial|\nabla c|^2}{\partial\nu}dS \le \kappa(\Omega)\int_{\partial\Omega}|\nabla c|^2dS,\tag{3.7}$$

where $\kappa(\Omega) > 0$ is an upper bound for the curvatures of $\partial\Omega$. Moreover, by the trace theorem (see Proposition 4.22(ii) and Theorem 4.24(i) in [8]), it holds that

$$\int_{\partial\Omega}|\nabla c|^2dS \le C(\Omega,s)\|c\|^2_{H^{\frac{3+s}{2}}(\Omega)} \qquad \text{for any} \quad s\in\big(0,1\big),$$

where $C(\Omega,s) > 0$ depends only on Ω and s, which can be fixed, for instance, $s=\frac{1}{2}$. On the other hand, by the interpolation inequality and Young's inequality, we have

$$\begin{aligned}\kappa(\Omega)C(\Omega,s)\|c\|^2_{H^{\frac{3+s}{2}}(\Omega)} &\le C_1\Big(\|D^2c\|^{\frac{3+s}{2}}_{L^2(\Omega)}\|c\|^{\frac{1-s}{2}}_{L^2(\Omega)} + \|c\|^2_{L^2(\Omega)}\Big)\\ &\le \frac{1}{4}\int_\Omega |D^2c|^2\mathrm{d}x + C_2\widetilde{M}^2\end{aligned}\tag{3.8}$$

for some positive constants C_1 and C_2 depending on Ω and s, where we also used the boundedness of c in the last inequality. Thus, substituting (3.7)–(3.8) into (3.6), we obtain

$$\int_\Omega |\Delta c|^2\mathrm{d}x \ge \frac{3}{4}\int_\Omega |D^2c|^2\mathrm{d}x - C_2\widetilde{M}^2.\tag{3.9}$$

Then taking $\varepsilon = \frac{\mathcal{D}_c}{2C_{GN}\widetilde{M}^2}$ in (3.5) and using (3.9), we have

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega n\ln n\,\mathrm{d}x+\int_\Omega|\nabla c|^2\mathrm{d}x\Big)+2\mathcal{D}_n\int_\Omega|\nabla\sqrt{n}|^2\mathrm{d}x+\frac{\mathcal{D}_c}{2}\int_\Omega|D^2c|^2\mathrm{d}x\\
&\le\left\{\frac{C_{GN}\widetilde{M}^2\sup\limits_{0\le c\le\widetilde{M}}\chi^4(c)}{4\mathcal{D}_n^2\mathcal{D}_c}+\frac{\sup\limits_{0\le c\le\widetilde{M}}k^2(c)}{\mathcal{D}_c}\right\}\int_\Omega n^2\mathrm{d}x \qquad (3.10)\\
&\quad+\frac{8C_{GN}\widetilde{M}^2}{\mathcal{D}_c}\int_\Omega|\nabla\mathbf{u}|^2\mathrm{d}x+C_3
\end{aligned}
$$

for some positive C_3 depending on Ω and the initial data.

In order to absorb the term involving $\int_\Omega|\nabla\mathbf{u}|^2\mathrm{d}x$, similar to Step 4 in the proof of Lemma 2.2, we first invoke the identity

$$
\frac{d}{dt}\int_\Omega|\mathbf{u}|^2\mathrm{d}x+2\mathcal{D}_{\mathbf{u}}\int_\Omega|\nabla\mathbf{u}|^2\mathrm{d}x=2\int_\Omega n\nabla\phi\cdot\mathbf{u}\,\mathrm{d}x. \qquad (3.11)
$$

By the Hölder inequality along with the assumed boundedness of ϕ, we have

$$
2\int_\Omega n\nabla\phi\cdot\mathbf{u}\,\mathrm{d}x\le 2\|\nabla\phi\|_{L^\infty(\Omega)}\|n\|_{L^{\frac{3}{2}}(\Omega)}\|\mathbf{u}\|_{L^3(\Omega)}. \qquad (3.12)
$$

Since $\Omega\subset\mathbb{R}^2$ is bounded and $\mathbf{u}=0$ on $\partial\Omega$, the continuous embedding $W_0^{1,2}(\Omega)\hookrightarrow L^3(\Omega)$ and the Poincaré inequality in $W_0^{1,2}(\Omega)$ ensure that there exists a positive constant C_p such that

$$
\|\mathbf{u}\|_{L^3(\Omega)}\le C_p\|\nabla\mathbf{u}\|_{L^2(\Omega)}.
$$

Thus by Young's inequality and (3.12), we have

$$
2\int_\Omega n\nabla\phi\cdot\mathbf{u}\,\mathrm{d}x\le\mathcal{D}_{\mathbf{u}}\|\nabla\mathbf{u}\|^2_{L^2(\Omega)}+\frac{C_p^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}{\mathcal{D}_{\mathbf{u}}}\|n\|^2_{L^{\frac{3}{2}}(\Omega)},
$$

which together with (3.11) yields

$$
\frac{d}{dt}\int_\Omega|\mathbf{u}|^2\mathrm{d}x+\mathcal{D}_{\mathbf{u}}\int_\Omega|\nabla\mathbf{u}|^2\mathrm{d}x\le\frac{C_p^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}{\mathcal{D}_{\mathbf{u}}}\|n\|^2_{L^{\frac{3}{2}}(\Omega)}. \qquad (3.13)
$$

Combining (3.10) with (3.13), we deduce that

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega n\ln n\,\mathrm{d}x+\int_\Omega|\nabla c|^2\mathrm{d}x+\frac{9C_{GN}\widetilde{M}^2}{\mathcal{D}_c\mathcal{D}_{\mathbf{u}}}\int_\Omega|\mathbf{u}|^2\mathrm{d}x\right)\\
&\quad+\left(2\mathcal{D}_n\int_\Omega|\nabla\sqrt{n}|^2\mathrm{d}x+\frac{\mathcal{D}_c}{2}\int_\Omega|D^2c|^2\mathrm{d}x+\frac{C_{GN}\widetilde{M}^2}{\mathcal{D}_c}\int_\Omega|\nabla\mathbf{u}|^2\mathrm{d}x\right)\\
&\le\left(\frac{C_{GN}\widetilde{M}^2\sup\limits_{0\le c\le\widetilde{M}}\chi^4(c)}{4\mathcal{D}_n^2\mathcal{D}_c}+\frac{\sup\limits_{0\le c\le\widetilde{M}}k'^2(c)}{\mathcal{D}_c}\right)\int_\Omega n^2\mathrm{d}x+C_4\|n\|^2_{L^{\frac{3}{2}}(\Omega)}+C_3,
\end{aligned}
\tag{3.14}
$$

with

$$
C_4=\frac{9C_p^2C_{GN}\widetilde{M}^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}{\mathcal{D}_c\mathcal{D}_{\mathbf{u}}^2}.
$$

Here we used $k(0)=0$ and $k(\cdot)\in C^1([0,\infty))$. In view of the Gagliardo–Nirenberg inequality and the mass identity (2.3), we have

$$
\begin{aligned}
\|n\|^2_{L^{\frac{3}{2}}(\Omega)}&\le C_{GN}\|n\|^{\frac{4}{3}}_{L^1(\Omega)}\|\nabla\sqrt{n}\|^{\frac{4}{3}}_{L^2(\Omega)}+C_{GN}\|n\|^2_{L^1(\Omega)}\\
&\le C_{GN}\|n_0\|^{\frac{4}{3}}_{L^1(\Omega)}\|\nabla\sqrt{n}\|^{\frac{4}{3}}_{L^2(\Omega)}+C_{GN}\|n_0\|^2_{L^1(\Omega)},
\end{aligned}
$$

which implies that

$$
C_4\|n\|^2_{L^{\frac{3}{2}}(\Omega)}\le\frac{2\mathcal{D}_n}{3}\|\nabla\sqrt{n}\|^2_{L^2(\Omega)}+C_5 \tag{3.15}
$$

with $C_5=\frac{C_4^3C_{GN}^3\|n_0\|^4_{L^1(\Omega)}}{3\mathcal{D}_n^2}+C_4C_{GN}\|n_0\|^2_{L^1(\Omega)}$ by Young's inequality. Similarly, we have

$$
\begin{aligned}
\int_\Omega n^2\mathrm{d}x&\le C_{GN}\|n\|_{L^1(\Omega)}\|\nabla\sqrt{n}\|^2_{L^2(\Omega)}+C_{GN}\|n\|^2_{L^1(\Omega)}\\
&\le C_{GN}\|n_0\|_{L^1(\Omega)}\|\nabla\sqrt{n}\|^2_{L^2(\Omega)}+C_{GN}\|n_0\|^2_{L^1(\Omega)}.
\end{aligned}
\tag{3.16}
$$

Substituting (3.15) and (3.16) into (3.14), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega n\ln n\,\mathrm{d}x+\int_\Omega|\nabla c|^2\mathrm{d}x+\frac{9C_{GN}\widetilde{M}^2}{\mathcal{D}_c\mathcal{D}_{\mathbf{u}}}\int_\Omega|\mathbf{u}|^2\mathrm{d}x\right)\\
&\quad+\frac{4\mathcal{D}_n}{3}\int_\Omega|\nabla\sqrt{n}|^2\mathrm{d}x+\frac{\mathcal{D}_c}{2}\int_\Omega|D^2c|^2\mathrm{d}x+\frac{C_{GN}\widetilde{M}^2}{\mathcal{D}_c}\int_\Omega|\nabla\mathbf{u}|^2\mathrm{d}x\\
&\le\widetilde{K}_1\|\nabla\sqrt{n}\|^2_{L^2(\Omega)}+C_6,
\end{aligned}
$$

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where $\widetilde{K}_1$ is given in (3.2), and

$$C_6 = \widetilde{K}_1 \|n_0\|_{L^1(\Omega)} + C_3 + C_5.$$

Absorbing the first term on the right-hand side, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega n \ln n \,\mathrm{d}x + \int_\Omega |\nabla c|^2 \mathrm{d}x + \frac{9C_{GN}\widetilde{M}^2}{\mathcal{D}_c\mathcal{D}_{\mathbf{u}}}\int_\Omega |\mathbf{u}|^2 \mathrm{d}x\right)\\
&+\left(\frac{4\mathcal{D}_n}{3} - \widetilde{K}_1\right)\int_\Omega |\nabla\sqrt{n}|^2 \mathrm{d}x + \frac{\mathcal{D}_c}{2}\int_\Omega |D^2 c|^2 \mathrm{d}x + \frac{C_{GN}\widetilde{M}^2}{\mathcal{D}_c}\int_\Omega |\nabla \mathbf{u}|^2 \mathrm{d}x\\
&\le C_6.
\end{aligned}$$

That is,

$$\frac{d}{dt}\widetilde{\mathcal{E}}(t) + \widetilde{\mathcal{F}}(t) \le C_6 \qquad \text{for all} \quad t \in (0, T^*). \tag{3.17}$$

Thus integrating (3.17) from 0 to t, we may obtain the desired estimate (3.3). This completes the proof of Lemma 3.2. □

Recalling the fact that $x \ln x \ge -e^{-1}$ for any $x \ge 0$ and the estimate (3.4) holds true, we can deduce the following estimates from the entropy functional (3.3).

Corollary 3.1. *Let $T \in (0, T^*)$. Under the assumption of Lemma 3.2, there exists a positive constant C depending on T, Ω and the initial data such that*

$$\int_\Omega n \ln n(x,t)\,\mathrm{d}x \le C \qquad \text{for all} \quad t \in (0, T),$$

$$\int_\Omega |\nabla c(x,t)|^2 \mathrm{d}x \le C \qquad \text{for all} \quad t \in (0, T), \tag{3.18}$$

$$\int_\Omega |\mathbf{u}(x,t)|^2 \mathrm{d}x \le C \qquad \text{for all} \quad t \in (0, T), \tag{3.19}$$

$$\int_0^T \int_\Omega |\nabla\sqrt{n}(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau \le C, \tag{3.20}$$

$$\int_0^T \int_\Omega |\nabla \mathbf{u}(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau \le C, \tag{3.21}$$

$$\int_0^T \int_\Omega |\nabla c(x,\tau)|^4 \mathrm{d}x\mathrm{d}\tau \le C. \tag{3.22}$$

The following integrability result is an improvement of the above estimates (3.20)–(3.22) and will be used to investigate the large-time behavior.

Corollary 3.2. *Under the assumption of Lemma 3.2, there exists a positive constant C depending only on the initial data such that for all $t > 0$,*

$$\int_0^t \int_\Omega e^{-\kappa_0(t-\tau)} |\nabla \sqrt{n}(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau \le C \qquad \text{for all} \quad t \in (0, T^*), \tag{3.23}$$

$$\int_0^t \int_\Omega e^{-\kappa_0(t-\tau)} |\nabla \mathbf{u}(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau \le C \qquad \text{for all} \quad t \in (0, T^*), \tag{3.24}$$

$$\int_0^t \int_\Omega e^{-\kappa_0(t-\tau)} |\nabla c(x,\tau)|^4 \mathrm{d}x\mathrm{d}\tau \le C \qquad \text{for all} \quad t \in (0, T^*), \tag{3.25}$$

where

$$\kappa_0 := \min\left\{ \frac{1}{C_{GN}\|n_0\|_{L^1(\Omega)}}\Big(\frac{4\mathcal{D}_n}{3} - \widetilde{K}_1\Big), \frac{\mathcal{D}_c}{C_{GN}\widetilde{M}^2}, \frac{\mathcal{D}_\mathbf{u}}{18C_\Omega} \right\} > 0.$$

Proof. By $x\ln x \le \frac{1}{2}x^2$ for any $x \ge 0$ and (3.16), we obtain

$$\int_\Omega n\ln n\,\mathrm{d}x \le \frac{1}{2}\int_\Omega n^2\mathrm{d}x \le \frac{1}{2}C_{GN}\|n_0\|_{L^1(\Omega)}\int_\Omega |\nabla\sqrt{n}|^2\mathrm{d}x + \frac{1}{2}C_{GN}\|n_0\|^2_{L^1(\Omega)},$$

which implies that

$$\begin{aligned} &\frac{1}{2}\Big(\frac{4\mathcal{D}_n}{3} - \widetilde{K}_1\Big)\int_\Omega |\nabla\sqrt{n}|^2\mathrm{d}x \\ &\ge \frac{1}{C_{GN}\|n_0\|_{L^1(\Omega)}}\Big(\frac{4\mathcal{D}_n}{3} - \widetilde{K}_1\Big)\int_\Omega n\ln n\,\mathrm{d}x - \frac{1}{2}\Big(\frac{4\mathcal{D}_n}{3} - \widetilde{K}_1\Big)\|n_0\|_{L^1(\Omega)} \\ &\ge \kappa_0\int_\Omega n\ln n\mathrm{d}x + \kappa_0 e^{-1} \\ &\quad - \frac{e^{-1}}{C_{GN}\|n_0\|_{L^1(\Omega)}}\Big(\frac{4\mathcal{D}_n}{3} - \widetilde{K}_1\Big) - \frac{1}{2}\Big(\frac{4\mathcal{D}_n}{3} - \widetilde{K}_1\Big)\|n_0\|_{L^1(\Omega)}, \end{aligned} \tag{3.26}$$

where we used $x\ln x \ge -e^{-1}$ for any $x \ge 0$ in the last inequality. By Young's inequality and (3.4), we also have

$$\int_\Omega |\nabla c|^2 \mathrm{d}x \le \frac{1}{4}\int_\Omega |\nabla c|^4 \mathrm{d}x + |\Omega| \le \frac{1}{4} C_{GN}\widetilde{M}^2 \int_\Omega |D^2 c|^2 \mathrm{d}x + \frac{1}{4} C_{GN}\widetilde{M}^4 + |\Omega|,$$

and thus

$$\begin{aligned}\frac{\mathcal{D}_c}{4}\int_\Omega |D^2 c|^2 \mathrm{d}x &\ge \frac{\mathcal{D}_c}{C_{GN}\widetilde{M}^2}\int_\Omega |\nabla c|^2 \mathrm{d}x - \frac{1}{4}\mathcal{D}_c\widetilde{M}^2 - \frac{\mathcal{D}_c|\Omega|}{C_{GN}\widetilde{M}^2} \\ &\ge \kappa_0 \int_\Omega |\nabla c|^2 \mathrm{d}x - \frac{1}{4}\mathcal{D}_c\widetilde{M}^2 - \frac{\mathcal{D}_c|\Omega|}{C_{GN}\widetilde{M}^2}.\end{aligned} \tag{3.27}$$

Finally, Poincare's inequality $\int_\Omega |\mathbf{u}|^2 \mathrm{d}x \le C_\Omega \int_\Omega |\nabla \mathbf{u}|^2 \mathrm{d}x$ implies that

$$\frac{C_{GN}\widetilde{M}^2}{2\mathcal{D}_c}\int_\Omega |\nabla \mathbf{u}|^2 \mathrm{d}x \ge \frac{C_{GN}\widetilde{M}^2}{2\mathcal{D}_c C_\Omega}\int_\Omega |\mathbf{u}|^2 \mathrm{d}x \ge \kappa_0 \frac{9 C_{GN}\widetilde{M}^2}{\mathcal{D}_c\mathcal{D}_{\mathbf{u}}}\int_\Omega |\mathbf{u}|^2 \mathrm{d}x. \tag{3.28}$$

Combining (3.26), (3.27) and (3.28), we see that

$$\frac{1}{2}\widetilde{\mathcal{F}}(t) \ge \kappa_0 \widetilde{\mathcal{E}}(t) - C_7,$$

for some positive constant C_7 depending only on Ω and the initial data, which together with (3.17) yields

$$\frac{d}{\mathrm{d}t}\widetilde{\mathcal{E}} + \kappa_0\widetilde{\mathcal{E}} + \frac{1}{2}\widetilde{\mathcal{F}} \le C_6 + C_7.$$

Then a direct calculation gives that

$$\widetilde{\mathcal{E}}(t) + \frac{1}{2}\int_0^t e^{-\kappa_0(t-\tau)}\widetilde{\mathcal{F}}(\tau)\mathrm{d}\tau \le \widetilde{\mathcal{E}}(0) + \kappa_0^{-1}\big(C_6 + C_7\big),$$

for all $t \in (0, T^*)$. Thus, the inequalities (3.23) and (3.24) are acquired. Moreover, we also have

$$\int_0^t \int_\Omega e^{-\kappa_0(t-\tau)} |D^2 c|^2 \mathrm{d}x\mathrm{d}\tau \le C_8$$

for some $C_8 > 0$, which together with (3.4) leads to

$$\begin{aligned}&\int_0^t\int_\Omega e^{-\kappa_0(t-\tau)}|\nabla c|^4\mathrm{d}x\mathrm{d}\tau\\&\le C_{GN}\|c_0\|^2_{L^\infty(\Omega)}\int_0^t\int_\Omega e^{-\kappa_0(t-\tau)}|D^2c|^2\mathrm{d}x\mathrm{d}\tau+C_{GN}\|c_0\|^4_{L^\infty(\Omega)}\int_0^t e^{-\kappa_0(t-\tau)}\mathrm{d}\tau\\&\le C_{GN}C_8\|c_0\|^2_{L^\infty(\Omega)}+C_{GN}\kappa_0^{-1}\|c_0\|^4_{L^\infty(\Omega)}.\end{aligned}$$

We then obtain the inequality (3.25). This completes the proof of Corollary 3.2. □

Proof of Theorem 1.3. We divide the proof into two parts. For the global existence, we will prove it by the contradiction argument. Assuming that $T^*<\infty$, we will show that

$$\sup_{t\in(0,T^*)}\left(\|n(\cdot,t)\|_{L^\infty(\Omega)}+\|c(\cdot,t)\|_{W^{1,q}(\Omega)}+\|A^\alpha\mathbf{u}(\cdot,t)\|_{L^2(\Omega)}\right)<\infty, \tag{3.29}$$

which contradicts to the extensibility criterion (3.1).

We now verify (3.29). Due to $T^*<\infty$, the bound (3.22) can be read as

$$\int_0^{T^*}\int_\Omega|\nabla c(x,\tau)|^4\mathrm{d}x\mathrm{d}\tau<\infty.$$

This is sufficient to guarantee that for any $p>1$, there exists a positive constant C_1 such that

$$\int_\Omega n^p(x,t)\,\mathrm{d}x\le C_1\qquad\text{for all}\quad t\in(0,\ T^*) \tag{3.30}$$

by a standard regularity argument which relies on testing the first equation in (1.1) by n^{p-1} (see Lemma 4.5 in [23] for details). On the other hand, inequalities (3.19) and (3.21) together with $T^*<\infty$ show that

$$\|\mathbf{u}(\cdot,t)\|_{L^2(\Omega)}\le C_2\qquad\text{for all}\quad t\in(0,T^*)$$

and

$$\int_0^{T^*}\int_\Omega|\nabla\mathbf{u}(x,\tau)|^2\mathrm{d}x\mathrm{d}\tau\le C_2$$

for some $C_2>0$. Then by the same procedure as that in [23], we can use the variation-of constants formula for $\mathbf{u}$ and the contractivity of the Stokes semigroup in L^2 along with (3.30) to obtain

$$\sup_{t\in(0,\ T^*)}\|A^\alpha\mathbf{u}(\cdot,t)\|_{L^2(\Omega)}<\infty \tag{3.31}$$

with α taken from the hypothesis of the theorem (see (4.16)–(4.22) in [23] for details).

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Next, (3.31) together with Sobolev's embedding implies $|\mathbf{u}| \le C_3$ in $\Omega \times (0, T^*)$ for some $C_3 > 0$. In view of $T^* < \infty$ and (3.18), we have

$$\int_\Omega |\nabla c(x,t)|^2 \mathrm{d}x \le C_4 \qquad \text{for all} \quad t \in (0, T^*)$$

for some $C_4 > 0$. Thus the variation-of constants formula of c and the well-known smoothing estimates for the Neumann heat semigroup then yield $\|\nabla c(\cdot,t)\|_{L^q(\Omega)} \le C_4$ for all $t \in (0, T^*)$, which together with (2.3) leads to

$$\sup_{t\in(0,T^*)} \|c(\cdot,t)\|_{W^{1,q}(\Omega)} < \infty. \tag{3.32}$$

Finally, we apply B^β with $\beta \in (\frac{1}{q}, \frac{1}{2})$ to the variation-of-constants of n, where B denotes the realization of $-\Delta + 1$ with homogeneous Neumann boundary conditions in $L^r(\Omega)$ with $r \in (\frac{1}{\beta}, q)$, and using the embedding $D(B^\beta) \hookrightarrow L^\infty(\Omega)$ and the maximum principle to obtain

$$\sup_{t\in(0,T^*)} \|n(\cdot,t)\|_{L^\infty(\Omega)} < \infty \tag{3.33}$$

(see (4.26) in [23] for details).

By (3.31), (3.32) and (3.33), we have verified (3.29) and thus infer that $T^* = \infty$. This completes the proof of global existence.

We now turn to the proof of the convergence. Our proof is very similar to the corresponding one in [24] and we only give a sketch for completeness. It follows from (3.23) that

$$\int_t^{t+1}\int_\Omega |\nabla\sqrt{n}(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau \le e^{\kappa_0} \int_t^{t+1}\int_\Omega e^{-\kappa_0(t+1-\tau)}|\nabla\sqrt{n}(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau \le C_5 e^{\kappa_0},$$

for all $t > 0$ and for some $C_5 > 0$, which combining with the Gagliardo–Nirenberg inequality

$$\int_\Omega n^2 \mathrm{d}x \le C_{GN}\|n_0\|_{L^1(\Omega)} \int_\Omega |\nabla\sqrt{n}|^2 \mathrm{d}x + C_{GN}\|n_0\|^2_{L^1(\Omega)},$$

yields that for all $t > 0$,

$$\int_t^{t+1}\int_\Omega n^2 \mathrm{d}x ds \le C_5 C_{GN} e^{\kappa_0}\|n_0\|_{L^1(\Omega)} + C_{GN}\|n_0\|^2_{L^1(\Omega)}. \tag{3.34}$$

Similarly by (3.25), we have that for all $t > 0$,

$$\int_t^{t+1}\int_\Omega |\nabla c(x,\tau)|^4 \mathrm{d}x\mathrm{d}\tau \le C_6 e^{\kappa_0}, \tag{3.35}$$

for some $C_6 > 0$. With (3.34) and (3.35) at hand, we can follow the proof of Corollary 4.4 in [24] to see that

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \tag{3.36}$$

Applying the integrability of n to the energy identity for $\mathbf{u}$, we have $\|\mathbf{u}(t)\|_{L^2(\Omega)} \to 0$ as $t \to \infty$. Then by the variation-of-constants formula for $\mathbf{u}$ and the interpolation, we can obtain

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \tag{3.37}$$

See the proof of Lemma 6.3 in [24].

By the variation-of-constants formula for n and a compactness argument, we can show that

$$\|n(\cdot,t) - \overline{n_0}\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty \tag{3.38}$$

(see the proof of Lemma 8.2 in [24] for details). From (3.36), (3.37) and (3.38), we complete the proof of Theorem 1.3. □

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