



# A large deviations principle for stochastic flows of viscous fluids

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## Abstract

We study the well-posedness of a stochastic differential equation on the two dimensional torus  $\mathbb{T}^2$ , driven by an infinite dimensional Wiener process with drift in the Sobolev space  $L^2\left(0, T; H^1\left(\mathbb{T}^2\right)\right)$ . The solution corresponds to a stochastic Lagrangian flow in the sense of DiPerna Lions. By taking into account that the motion of a viscous incompressible fluid on the torus can be described through a suitable stochastic differential equation of the previous type, we study the inviscid limit. By establishing a large deviations principle, we show that, as the viscosity goes to zero, the Lagrangian stochastic Navier–Stokes flow approaches the Euler deterministic Lagrangian flow with an exponential rate function.

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## 1. Introduction

In the literature there are two major approaches to the study of fluid dynamics: the Eulerian approach, which consists in determining and studying the properties of certain physical quantities such as velocity, pressure, etc., at a certain fixed point  $x$  in space and time  $t$ ; and the so-called Lagrangian approximation, where the fluid is seen as a collection of particles that leave the initial

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position and as time progresses describe trajectories in the plane or space. In this perspective the fluid motion can be described by flow mappings more or less smooth (homeomorphisms, diffeomorphisms, etc.) on the region occupied by the fluid.

According to previous works [6], [7], [8], [9], [18] in certain incompressible viscous fluids, despite the deterministic nature of some physical Eulerian quantities, the evolution of particles is inherently stochastic. Thus, in its Lagrangian formulation, stochastic flows should be considered.

One of the classical problems in fluid mechanics is related with the turbulence, where the knowledge of the behaviour of the fluid under the inviscid limit transition is a key step to understand the phenomenon.

In the Eulerian context, the asymptotic study of solutions of the Navier–Stokes equations when the viscosity converges to zero is a difficult issue, still unsolved in three dimensions and in two dimensions in the case of Dirichlet boundary conditions. On a bi-dimensional domain with periodic or slip boundary conditions, it is known that the solutions of the Navier–Stokes equations converges to the solutions of the Euler equations, when the viscosity tends to zero (see [2], [3], [4] and references therein).

In this paper we address the inviscid limit problem for the incompressible viscous fluids from the stochastic Lagrangian point of view, more precisely, we consider the velocity field  $u^\epsilon$  given as solution of the Navier–Stokes equations in the 2-dimensional torus  $\mathbb{T}^2$

$$\begin{cases} \frac{\partial u^\epsilon}{\partial t} + (u^\epsilon \cdot \nabla) u^\epsilon = \epsilon \Delta u^\epsilon + \nabla p \\ \nabla \cdot u^\epsilon = 0 \\ u^\epsilon(x, 0) = u_0(x), \quad u_0 \in H^1(\mathbb{T}^2) \end{cases}$$

and according to [6], we define the stochastic flows as solution of the stochastic differential equation

$$dX_t^\epsilon(x) = u^\epsilon(X_t^\epsilon(x), t) dt + \sqrt{\epsilon} \sigma(X_t^\epsilon(x)) dW_t, \quad X_0^\epsilon = x, \quad x \in \mathbb{T}^2.$$

Then our main goal is to establish a Schilder's type theorem, in the sense of Freidlin and Wentzell, for the asymptotic behaviour of the Navier–Stokes flows  $X_t^\epsilon$ , when the viscosity  $\epsilon \rightarrow 0$ . This asymptotic result shows the exponential concentration of the viscous fluid of Navier–Stokes around the non-viscous fluid of Euler as the viscosity vanishes. Since this probabilist study allows to analyze the probability of rare events, we hope that our approach can give some new insight towards the understanding of turbulent fluids.

Our strategy to prove this asymptotic result is based on the equivalence between the Large deviations principle and the Laplace–Varadhan principle [24] conjugated with the weak convergence approach developed by P. Dupuis and R.S. Ellis [13].

Let us mention that in the study of Large deviations, the well-posedness of the involving dynamic equations is an essential requirement. Here, we deal with stochastic differential equations in which the drift, being a weak solution to the Navier–Stokes equations, does not satisfy the Lipschitz condition. On the other hand, the stochastic noise  $W_t$  is an infinite dimensional cylindrical Wiener process and the diffusion is a non-constant Hilbert–Schmidt operator. In this irregular context, the existence and uniqueness of solutions do not come from the classical methods. Then we will follow the theory of DiPerna Lions [12]. Such theory is based on the analysis of the corresponding transport equation and has been applied to stochastic transport equations with Hölder continuous drift and constant diffusion matrix in [17]. However, for non-constant diffusion operators and irregular drift, the study of the corresponding stochastic transport equations is not

an easy issue. In the deterministic framework, the authors in [10] obtained the same results as DiPerna Lions without using the transport equation, only by means of Lagrangian methods. This Lagrangian approach has been considered in [25] to study a stochastic differential equation with an  $\mathbb{R}^d$ -valued Brownian motion and turned out to be more appropriate in the case of non-constant diffusion operators.

In the present paper we follow those Lagrangian methods and prove the existence and the uniqueness of the flow, which can be considered as a random variable with values in the space  $C([0, T], L^2(\mathbb{T}^2))$ . This space will be endowed with its natural topology and for every set  $\Gamma$  in the Borel  $\sigma$ -field we study the asymptotic behaviour of  $\mathbb{P}(X_t^\epsilon \in \Gamma)$ , as  $\epsilon \rightarrow 0$ . One of the main difficulties is related with the tightness of the law of the flows  $X_t^\epsilon$  on that space. To overcome this difficulty we establish a suitable compactness criterion. For the large deviations analysis in the analogous framework we refer [23].

The paper is organized as follows: in Section 2, we introduce some fundamental definitions and lemmas that are required in the next sections to prove the main results. In Section 3, we prove a theorem concerning the existence and the uniqueness of solution for stochastic differential equations with drift in the Sobolev space  $L^2(0, T; H^1)$ . In Section 4, we establish the principle of large deviations.

## 2. Formulation of the problem and preliminaries results

This section is devoted to the formulation of the problem. We first introduce the general framework and give the appropriate definitions. Then we present some useful lemmas that will be required in the next sections in order to prove the main results.

Let  $W_t^{k_1}, W_t^{k_2}, k_1, k_2 \in \mathbb{Z}$  be  $\mathbb{R}$ -valued independent Brownian motions, defined on a probabilistic space  $(\Omega, \mathcal{F}, P)$ . We consider the stochastic process  $W_t$  defined by

$$W_t = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} W_t^{k_1} e_{k_1} + W_t^{k_2} e_{k_2} = (\dots, \underbrace{W_t^{k_1}, W_t^{k_2}}_{\text{position } k}, \dots)$$

where

$$\begin{aligned} e_{k_1} &= (\dots, 0, 0, \underbrace{1, 0}_{\text{k position}}, 0, 0, \dots) \\ e_{k_2} &= (\dots, 0, 0, \underbrace{0, 1}_{\text{k position}}, 0, 0, \dots) \end{aligned} \quad (2.1)$$

and  $k = (k_1, k_2)$ .

Let us notice that the set  $\{e_{k_1}, e_{k_2}, k \in \mathbb{Z}^2 \setminus \{(0,0)\}\}$  defines an orthonormal basis to the Hilbert space  $l^2$  of all sequences  $z = (z_k), k \in \mathbb{Z}^2 \setminus \{(0,0)\}$ , with norm

$$\|z\|_{l^2}^2 = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} |z_k|^2 < \infty \quad (2.2)$$

and inner product

$$\langle z, y \rangle_{l^2} = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} z_k y_k, \quad z = (z_k), \quad y = (y_k). \quad (2.3)$$

Thus, the stochastic process  $W_t$  can be understood as a cylindrical Wiener process over  $l^2$ . Its covariance operator  $Q$  corresponds to the identity operator in  $l^2$ . Since the identity operator is not a trace class operator, the stochastic process  $W_t$  can be considered as a  $Q$ -Wiener process in a bigger Hilbert space  $l^2_*$  in which  $l^2$  is embedded through a Hilbert–Schmidt operator (see [11]). We will consider on the probability space  $(\Omega, \mathcal{F}, P)$  the filtration  $\{\mathcal{F}_t\}$  induced by the process  $W_t$ .

Let us introduce the following stochastic differential equation,

$$dX_t(x) = u(X_t(x), t)dt + \sigma(X_t(x))dW_t, \quad X_0 = x, \quad x \in \mathbb{T}^2, \quad (2.4)$$

defined on the 2-dimensional torus  $\mathbb{T}^2$  that will be identified with the flat domain  $[0, 2\pi] \times [0, 2\pi]$  under periodic boundary conditions. The drift  $u$  belongs to the Sobolev space  $L^2(0, T; W^{1,p}(\mathbb{T}^2))$ ,  $p > 1$ , and the diffusion operator is defined by a Borel measurable map  $\sigma$  from  $\mathbb{T}^2$  with values in the space of the Hilbert–Schmidt operators from  $l^2$  to  $\mathbb{R}^2$  satisfying appropriate integrability conditions.

Given two Hilbert spaces  $U$  and  $V$ , we denote by  $HS(U, V)$  the space of Hilbert–Schmidt operators from  $U$  to  $V$  endowed with the Hilbert–Schmidt norm  $\|\cdot\|_{HS}$ .

The first goal of this work is to establish the existence and uniqueness of the solution to the stochastic differential equation (2.4) in the framework of the stochastic Lagrangian flows.

**Definition 2.1.** Let  $X_t(\omega, x)$  be a measurable stochastic field defined in  $\mathbb{R}_+ \times \Omega \times \mathbb{T}^2$ . We say that  $X_t$  is a a.e. stochastic incompressible flow solution of (2.4) if

- (a) for a.e.- $x \in \mathbb{T}^2$ ,  $X_t(x)$  is a continuous stochastic process adapted to the filtration  $\{\mathcal{F}_t\}$  such that for any  $T > 0$

$$\int_0^T |u(X_s(x), s)| ds + \int_0^T \|\sigma(X_s(x))\|_{HS}^2 ds < \infty, \quad \text{a.e.-}\omega \quad (2.5)$$

and  $X_t(x)$  solves

$$X_t(x) = x + \int_0^t u(X_s(x), s) ds + \int_0^t \sigma(X_s(x)) dW_s, \quad \forall t \in [0, T]; \quad (2.6)$$

- (b) for every  $t \geq 0$ , and a.e.- $\omega$ , the map  $X_t(\omega, \cdot) : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  preserves the Lebesgue measure, this means that for a.e.- $\omega$  and  $t \geq 0$  we have

$$\int_{\mathbb{T}^2} \varphi(X_t(x)) dx = \int_{\mathbb{T}^2} \varphi(x) dx \quad (2.7)$$

for every bounded Borel-measurable function  $\varphi : \mathbb{T}^2 \rightarrow \mathbb{R}$ .

We should emphasize that in the present work, we are especially interested in the particular case of equation (2.4), where the diffusion operator  $\sigma$  is formally represented by the infinite dimensional matrix

$$\sigma(x) = \begin{pmatrix} \dots, A_k^1(x), B_k^1(x), \dots \\ \dots, A_k^2(x), B_k^2(x), \dots \end{pmatrix} \quad (2.8)$$

where

$$\begin{aligned} A_k^1(x) &= \frac{k_2 \cos(k \cdot x)}{|k|^\beta}, & A_k^2(x) &= -\frac{k_1 \cos(k \cdot x)}{|k|^\beta} \\ B_k^1(x) &= \frac{k_2 \sin(k \cdot x)}{|k|^\beta}, & B_k^2(x) &= -\frac{k_1 \sin(k \cdot x)}{|k|^\beta} \end{aligned}$$

with  $k = (k_1, k_2) \in \mathbb{Z}^2 / \{(0, 0)\}$  and  $\beta > 3$ .

Let us recall from [6], the intrinsic relation between the solution of the stochastic differential equations and the Lagrangian description of a viscous incompressible flow. According to [6], the stochastic flow that is stationary for the energy functional is a weak solution to the following stochastic differential equation

$$dX_t^\epsilon(x) = u^\epsilon(X_t^\epsilon(x), t) dt + \sqrt{\epsilon} \sigma(X_t^\epsilon(x)) dW_t, \quad X_0^\epsilon = x, \quad x \in \mathbb{T}^2 \quad (2.9)$$

where  $\sigma$  is given by (2.8), and with drift  $u^\epsilon(x, t) : \mathbb{T}^2 \times [0, T] \rightarrow \mathbb{R}^2$  being solution of the Navier–Stokes equations

$$\begin{cases} \frac{\partial u^\epsilon}{\partial t} + (u^\epsilon \cdot \nabla) u^\epsilon = \epsilon \Delta u^\epsilon + \nabla p \\ \nabla \cdot u^\epsilon = 0 \\ u^\epsilon(x, 0) = u_0(x), \quad u_0 \in H^1(\mathbb{T}^2), \end{cases} \quad (2.10)$$

where  $\epsilon$  represents the fluid viscosity and  $p$  denotes the pressure.

Being motivated by the turbulence problem, our second goal is to study the inviscid limit of the stochastic Lagrangian Navier–Stokes flows by applying the large deviations techniques. The main role in this theory is played by the so-called rate function.

**Definition 2.2.** Let  $Y$  be a Banach space. A mapping  $I : Y \rightarrow [0, +\infty]$  is called a rate function if the level sets  $\{f \in Y : I(f) \leq M\}$ ,  $M > 0$ , are compact sets of  $Y$ .

The large deviations principle reads:

**Definition 2.3.** Given a Banach space  $Y$ , we say that a family of a  $Y$ -valued random variables  $\{X^\epsilon\}$  satisfies a large deviation principle on  $Y$  with rate function  $I$  if for every Borel set  $\Gamma$  of  $Y$  the following property holds

$$-\inf_{f \in \text{int}(\Gamma)} I(f) \leq \liminf_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P}\{X^\epsilon \in \text{int}(\Gamma)\} \leq \limsup_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P}\{X^\epsilon \in \overline{\Gamma}\} \leq -\inf_{f \in \overline{\Gamma}} I(f). \quad (2.11)$$

The Varadhan's result in [24] and the Bryc's converse lemma (see [13]) show that the large deviation principle (2.11) is equivalent to the following Laplace–Varadhan principle:

**Definition 2.4.** Given a Banach space  $Y$ , we say that a family of a  $Y$ -valued random variables  $\{X^\epsilon\}$  satisfies the Laplace–Varadhan principle on  $Y$  with rate function  $I$ , if the following asymptotic result holds

$$\lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E} \left\{ e^{-\frac{1}{\epsilon} g(X^\epsilon)} \right\} = - \inf_{f \in Y} \{g(f) + I(f)\} \quad (2.12)$$

for all bounded continuous function  $g : Y \rightarrow \mathbb{R}$ .

In Section 4, we will establish a large deviations principle on the Banach space  $C([0, T], L^2(\mathbb{T}^2))$ , showing that, as  $\epsilon \rightarrow 0$ , the Navier–Stokes flows, which can be interpreted as solutions of the stochastic differential equations (2.9), will converge with an exponential rate function to the deterministic Euler flow  $X_t$ , defined as the solution in the DiPerna Lions sense of the differential equation

$$\frac{dX_t(x)}{dt} = u(X_t(x), t), \quad X_0 = x, \quad x \in \mathbb{T}^2, \quad (2.13)$$

with velocity field  $u(x, t)$  being solution of the Euler equations

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla) u = \nabla p \\ \nabla \cdot u = 0 \\ u(x, 0) = u_0(x), \quad u_0 \in H^1(\mathbb{T}^2). \end{cases} \quad (2.14)$$

Let us notice that in the study of the large deviations, the drift usually is independent of the parameter  $\epsilon$ , such parameter only defines the small random perturbations of the deterministic system, in our case the parameter  $\epsilon$  being the viscosity of the fluid, the drift depends on  $\epsilon$  and the large deviations principle has a physical meaning corresponding to inviscid limit transition.

Before we introduce some lemmas that will be crucial to establish the well-posedness of equations (2.4) and (2.9), we will set some useful notations. Given  $h \in l^2$  we represent  $\langle \sigma(x), h \rangle$  as the vector field in  $\mathbb{R}^2$  corresponding to the image of  $h$  by  $\sigma(x)$ . Since  $\sigma(x)$  can be formally identified with the matrix (2.8), we will take  $\sigma_i \cdot (x)$  and  $\sigma \cdot_j (x)$  as the  $i$ th line and  $j$ th column of  $\sigma$ , respectively. Therefore we can write

$$\langle \sigma(x), h \rangle = (\langle \sigma_1 \cdot (x), h \rangle_{l^2}, \langle \sigma_2 \cdot (x), h \rangle_{l^2})$$

where  $\langle \cdot, \cdot \rangle_{l^2}$  is the inner product in  $l^2$ . In the case where  $u, z$  are vectors in  $\mathbb{R}^2$ ,  $\langle u, z \rangle$  is the regular inner product in  $\mathbb{R}^2$ .

We denote by  $\sigma_n$  the finite dimensional approximation of the operator  $\sigma$ . For each  $x \in \mathbb{T}^2$ ,  $\sigma_n(x)$  is defined as the  $2 \times 2d$ -matrix, where  $d = \#\{k \in \mathbb{Z}^2 / \{(0, 0)\} : |k| \leq n\}$ , which corresponds to the matrix constructed from (2.8) by excluding the entries with  $|k| > n$ . When convenient, we also identify  $\sigma_n(x)$  with its natural extension as an Hilbert–Schmidt operator from  $l^2$  with values in  $\mathbb{R}^2$ , having null entries for  $|k| > n$ .

Throughout the paper, we will denote by  $C$  a generic constant with a not specified value that can change from one inequality to another.

The remaining part of this section is devoted to the presentation of some lemmas that are fundamental to prove the existence and uniqueness results in Section 3.

We start by establishing three lemmas dealing with the regularity properties of the diffusion operator  $\sigma$  defined by (2.8).

**Lemma 2.5.** *Let  $\sigma(x)$  be defined as in (2.8). Then the following propositions hold:*

- (i)  $\sigma : \mathbb{T}^2 \rightarrow HS(l^2, \mathbb{R}^2)$  and  $\|\sigma(x)\|_{HS}$  is bounded,
- (ii)  $\operatorname{div} \sigma(x)e_{k_i} = 0, \quad \forall k \in \mathbb{Z}^2/\{(0, 0)\}, \quad i = 1, 2.$

**Proof.** We first prove the statement (i). For any  $x \in \mathbb{T}^2$ , the Hilbert–Schmidt norm of  $\sigma(x)$  is given by

$$\begin{aligned} \|\sigma(x)\|_{HS}^2 &= \sum_{k \in \mathbb{Z}^2/\{(0,0)\}} |A_k^1(x)e_{k_1}|^2 + |A_k^2(x)e_{k_1}|^2 + |B_k^1(x)e_{k_2}|^2 + |B_k^2(x)e_{k_2}|^2 \\ &= \sum_{k \in \mathbb{Z}^2/\{(0,0)\}} \frac{1}{|k|^{2\beta}} ((k_2 \cos(k.x))^2 + (k_2 \sin(k.x))^2) \\ &\quad + \frac{1}{|k|^{2\beta}} ((k_1 \cos(k.x))^2 + (k_1 \sin(k.x))^2) = \sum_{k \in \mathbb{Z}^2/\{(0,0)\}} \frac{1}{|k|^{2\beta-2}} \end{aligned}$$

and this series converges for  $\beta > 2$ .

The proof of (ii) follows directly from the fact that for any  $k \in \mathbb{Z}^2/\{(0, 0)\}$

$$\partial_1 A_k^1(x) + \partial_2 A_k^2(x) = \partial_1 \left( \frac{k_2 \cos(k.x)}{|k|^\beta} \right) - \partial_2 \left( \frac{k_1 \cos(k.x)}{|k|^\beta} \right) = 0$$

and

$$\partial_1 B_k^1(x) + \partial_2 B_k^2(x) = \partial_1 \left( \frac{k_2 \sin(k.x)}{|k|^\beta} \right) - \partial_2 \left( \frac{k_1 \sin(k.x)}{|k|^\beta} \right) = 0. \quad \square$$

In order to define the regularity of the diffusion operator with respect to the space variable, we set

$$\|\nabla \sigma(x)\| := \sum_{k \in \mathbb{Z}^2/\{(0,0)\}} |\nabla(\sigma(x)e_{k_1})|^2 + |\nabla(\sigma(x)e_{k_2})|^2$$

and introduce the spaces

$$\begin{aligned} \mathbb{H}^1 \left( \mathbb{T}^2, HS(l^2, \mathbb{R}^2) \right) &:= \left\{ \sigma : \mathbb{T}^2 \rightarrow HS(l^2, \mathbb{R}^2) \text{ is a measurable function} \right. \\ &\quad \left. \text{with } \int_{\mathbb{T}^2} \left( \|\sigma(x)\|_{HS}^2 + \|\nabla \sigma(x)\|^2 \right) dx < \infty \right\} \end{aligned} \quad (2.15)$$

and

$$\mathbb{H}_V^1 \left( \mathbb{T}^2, HS(l^2, \mathbb{R}^2) \right) \quad (2.16)$$

as the space of all elements in  $\mathbb{H}^1 \left( \mathbb{T}^2, HS(l^2, \mathbb{R}^2) \right)$  with  $\operatorname{div} \sigma(x) e_{k_i} = 0$ ,  $i = 1, 2, k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ .

**Lemma 2.6.** *Let  $\sigma$  be defined in (2.8) with  $\beta > 3$ . Then  $\forall x \in \mathbb{T}^2$ , we have*

$$\|\nabla \sigma(x)\| \leq C.$$

**Proof.** We have

$$\begin{aligned} |\nabla(\sigma(x)e_{k_1})|^2 &= \left| \begin{pmatrix} \partial_1 A_k^1(x) & \partial_2 A_k^1(x) \\ \partial_1 A_k^2(x) & \partial_2 A_k^2(x) \end{pmatrix} \right|^2 \\ &= |\partial_1 A_k^1(x)|^2 + |\partial_2 A_k^1(x)|^2 + |\partial_1 A_k^2(x)|^2 + |\partial_2 A_k^2(x)|^2 \end{aligned}$$

and similarly

$$|\nabla(\sigma(x)e_{k_2})|^2 = |\partial_1 B_k^1(x)|^2 + |\partial_2 B_k^1(x)|^2 + |\partial_1 B_k^2(x)|^2 + |\partial_2 B_k^2(x)|^2.$$

Therefore

$$\begin{aligned} \|\nabla \sigma(x)\| &= \sum_{k \in \mathbb{Z}^2 / \{(0,0)\}} |\nabla(\sigma(x)e_{k_1})|^2 + |\nabla(\sigma(x)e_{k_2})|^2 \\ &= \sum_{k \in \mathbb{Z}^2 / \{(0,0)\}} |\partial_1 A_k^1(x)|^2 + |\partial_2 A_k^1(x)|^2 + |\partial_1 A_k^2(x)|^2 + |\partial_2 A_k^2(x)|^2 \\ &\quad + |\partial_1 B_k^1(x)|^2 + |\partial_2 B_k^1(x)|^2 + |\partial_1 B_k^2(x)|^2 + |\partial_2 B_k^2(x)|^2. \end{aligned}$$

So, having

$$|\partial_1 A_k^1(x)|^2 + |\partial_2 A_k^1(x)|^2 = \frac{1}{|k|^{2\beta}} ((k_1^2 + k_2^2) k_2^2 \sin^2(kx)) \leq \frac{k_2^2}{|k|^{2\beta-2}}$$

and analogous estimates for the other terms, we deduce

$$\|\nabla \sigma(x)\| \leq \sum_{k \in \mathbb{Z}^2 / \{(0,0)\}} \frac{1}{|k|^{2\beta-2}} (2k_1^2 + 2k_2) = 2 \sum_{k \in \mathbb{Z}^2 / \{(0,0)\}} \frac{1}{|k|^{2\beta-4}}.$$

Since this series converges, we obtain the claimed result.  $\square$

**Lemma 2.7.** *Let us consider the SDE (2.4) with  $\sigma$  defined in (2.8). Then the Itô contraction term is zero, i.e. for  $\sigma$  defined in (2.8) the Stratonovich and Itô integrals are equal.*



The proof of the previous lemma can be found in [6].

Now, we are able to introduce two lemmas that are the basis to prove the existence and uniqueness of solution to the equation (2.4).

**Lemma 2.8.** Assume that  $X_t$  and  $\hat{X}_t$  are two a.e. incompressible stochastic flows which are solutions to the stochastic differential equation (2.4) with coefficients  $(u, \sigma)$  and  $(\hat{u}, \hat{\sigma})$ , respectively, with  $u, \hat{u} \in L^2(0, T; H^1(\mathbb{T}^2))$  and  $\sigma, \hat{\sigma} \in \mathbb{H}^1(\mathbb{T}^2, HS(l^2, \mathbb{R}^2))$ . Then for all  $T > 0$  and  $\delta > 0$ , we have

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{T}^2} \log \left( \frac{\sup_{t \in [0, T]} |X_t(x) - \hat{X}_t(x)|^2}{\delta^2} + 1 \right) dx \\ & \leq \frac{C_1}{\delta} \left( \int_0^T \int_{\mathbb{T}^2} |u(x, t) - \hat{u}(x, t)| dx dt + \left\{ \int_0^T \int_{\mathbb{T}^2} \|\sigma(x) - \hat{\sigma}(x)\|_{HS}^2 dx dt \right\}^{\frac{1}{2}} \right. \\ & \quad \left. + \frac{1}{\delta} \int_0^T \int_{\mathbb{T}^2} \|\sigma(x) - \hat{\sigma}(x)\|_{HS}^2 dx dt \right) + C_2, \end{aligned} \quad (2.17)$$

with

$$C_2 = C \left( 1 + \int_0^T \int_{\mathbb{T}^2} |\nabla \hat{u}(x, t)|^2 dx dt + \left\{ \int_0^T \int_{\mathbb{T}^2} \|\nabla \sigma(x)\|^2 dx dt \right\}^{\frac{1}{2}} + \int_{\mathbb{T}^2} \|\nabla \hat{\sigma}(x)\|^2 dx \right), \quad (2.18)$$

where the constants  $C_1$  and  $C$  are independent of  $u, \hat{u}, \sigma, \hat{\sigma}$ .

**Proof.** Defining  $Z_t(x) = X_t(x) - \hat{X}_t(x)$ , the Itô's formula yields

$$\begin{aligned} \log \left( \frac{|Z_t|^2}{\delta^2} + 1 \right) &= 2 \int_0^t \frac{\langle Z_s, u(X_s, s) - \hat{u}(\hat{X}_s, s) \rangle}{|Z_s|^2 + \delta^2} ds + 2 \int_0^t \frac{\langle Z_s, (\sigma(X_s) - \hat{\sigma}(\hat{X}_s)) dW_s \rangle}{|Z_s|^2 + \delta^2} \\ &\quad + \int_0^t \frac{\|\sigma(X_s) - \hat{\sigma}(\hat{X}_s)\|_{HS}^2}{|Z_s|^2 + \delta^2} ds - 2 \int_0^t \frac{|(\sigma(X_s) - \hat{\sigma}(\hat{X}_s))^\top Z_s|^2}{|Z_s|^2 + \delta^2} ds \\ &= I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned}$$

For the first term  $I_1(t)$ , we have

$$I_1(t) = 2 \int_0^t \frac{\langle Z_s, u(X_s, s) - \hat{u}(\hat{X}_s, s) \rangle}{|Z_s|^2 + \delta^2} ds =$$

$$\begin{aligned}
&= 2 \int_0^t \frac{\langle Z_s, u(X_s, s) - \hat{u}(X_s, s) \rangle}{|Z_s|^2 + \delta^2} ds + 2 \int_0^t \frac{\langle Z_s, \hat{u}(X_s, s) - \hat{u}(\hat{X}_s, s) \rangle}{|Z_s|^2 + \delta^2} ds \\
&\leq 2 \int_0^t \frac{|Z_s| |u(X_s, s) - \hat{u}(X_s, s)|}{|Z_s|^2 + \delta^2} ds + 2 \int_0^t \frac{|Z_s| |\hat{u}(X_s, s) - \hat{u}(\hat{X}_s, s)|}{|Z_s|^2 + \delta^2} ds,
\end{aligned}$$

which gives

$$I_1(t) \leq \frac{1}{\delta} \int_0^t |u(X_s, s) - \hat{u}(X_s, s)| ds + 2 \int_0^t \frac{|\hat{u}(X_s, s) - \hat{u}(\hat{X}_s, s)|}{\sqrt{|Z_s|^2 + \delta^2}} ds = I_{11}(t) + I_{12}(t).$$

Taking into account the incompressibility property of the flow  $X_t$ , we derive

$$\begin{aligned}
\mathbb{E} \int_{\mathbb{T}^2} \sup_{t \in [0, T]} |I_{11}(t)| dx &\leq \frac{1}{\delta} \mathbb{E} \int_0^T \int_{\mathbb{T}^2} |u(X_t, t) - \hat{u}(X_t, t)| dx dt \\
&= \frac{1}{\delta} \int_0^T \int_{\mathbb{T}^2} |u(x, t) - \hat{u}(x, t)| dx dt.
\end{aligned}$$

Let us introduce the constant  $R := \sup_{x, y \in \mathbb{T}^2} |x - y|$ . Using the inequalities *D1* and *D2* in the Appendix, and the incompressibility property of the flows  $X_t$  and  $\hat{X}_t$ , we obtain

$$\begin{aligned}
\mathbb{E} \int_{\mathbb{T}^2} \sup_{t \in [0, T]} |I_{12}(t)| &\leq 2 \mathbb{E} \int_{\mathbb{T}^2} \int_0^T \frac{|\hat{u}(X_t, t) - \hat{u}(\hat{X}_t, t)|}{\sqrt{|Z_t|^2 + \delta^2}} dt dx \\
&\leq C \mathbb{E} \int_0^T \int_{\mathbb{T}^2} ([M_R |\nabla \hat{u}|](X_t, t) + [M_R |\nabla \hat{u}|](\hat{X}_t, t)) dx dt \\
&\leq C \int_0^T \mathbb{E} \int_{\mathbb{T}^2} [M_R |\nabla \hat{u}|](X_t, t) + \int_{\mathbb{T}^2} [M_R |\nabla \hat{u}|](\hat{X}_t, t) dx dt \\
&\leq C \left( \int_0^T \int_{\mathbb{T}^2} [M_R |\nabla \hat{u}|]^2(x, t) dx dt \right)^{\frac{1}{2}} \\
&\leq C \left( \int_0^T \int_{\mathbb{T}^2} |\nabla \hat{u}(x, t)|^2 dx dt \right)^{\frac{1}{2}}.
\end{aligned}$$

The stochastic term  $I_2(t)$  can be estimated with the help of the Burkholder–Davis–Gundy inequality

$$\begin{aligned} \mathbb{E} \int_{\mathbb{T}^2} \sup_{t \in [0, T]} |I_2(t)| &= 2 \mathbb{E} \int_{\mathbb{T}^2} \sup_{t \in [0, T]} \left| \int_0^t \frac{\langle Z_s, (\sigma(X_s) - \hat{\sigma}(\hat{X}_s)) dW_s \rangle}{|Z_s|^2 + \delta^2} \right| \\ &\leq C \left\{ \int_{\mathbb{T}^2} \mathbb{E} \int_0^T \frac{|Z_t|^2 \|\sigma(X_t) - \hat{\sigma}(\hat{X}_t)\|_{HS}^2}{(|Z_t|^2 + \delta^2)^2} dt \right\}^{\frac{1}{2}}. \end{aligned}$$

Reasoning as in the estimate of  $I_1(t)$  by splitting the right hand side in two terms, we deduce

$$\begin{aligned} \mathbb{E} \int_{\mathbb{T}^2} \sup_{t \in [0, T]} |I_2(t)| dx &\leq \frac{C_1}{\delta} \left\{ \int_{\mathbb{T}^2} \mathbb{E} \int_0^T \|\sigma(X_t) - \hat{\sigma}(X_t)\|_{HS}^2 dt dx \right\}^{\frac{1}{2}} \\ &\quad + C \left\{ \int_{\mathbb{T}^2} \mathbb{E} \int_0^T \frac{\|\hat{\sigma}(X_t) - \hat{\sigma}(\hat{X}_t)\|_{HS}^2}{|Z_t|^2 + \delta^2} dt dx \right\}^{\frac{1}{2}} \\ &= I_{21} + I_{22}, \end{aligned}$$

where the constant  $C_1$  only depends of the domain  $\mathbb{T}^2$ . Therefore, as in the previous cases, the incompressibility of the flow  $X_t$  gives

$$I_{21} \leq \frac{C_1}{\delta} \left\{ \int_0^T \int_{\mathbb{T}^2} \|\sigma(x) - \hat{\sigma}(x)\|_{HS}^2 dx dt \right\}^{\frac{1}{2}}.$$

Again, using the inequalities D1 and D2 in the Appendix and the incompressibility of the flow  $X_t$  and  $\hat{X}_t$ , we derive

$$\begin{aligned} I_{22} &\leq C \left\{ \mathbb{E} \int_0^T \int_{\mathbb{T}^2} ([M_R \llbracket \nabla \hat{\sigma} \rrbracket](X_t) + [M_R \llbracket \nabla \hat{\sigma} \rrbracket](\hat{X}_t)) dx dt \right\}^{\frac{1}{2}} \\ &\leq C \left\{ \int_0^T \int_{\mathbb{T}^2} [M_R \llbracket \nabla \hat{\sigma} \rrbracket]^2(x) dx dt \right\}^{\frac{1}{4}} \leq C \left\{ \int_0^T \int_{\mathbb{T}^2} \llbracket \nabla \hat{\sigma}(x) \rrbracket^2 dx dt \right\}^{\frac{1}{4}}. \end{aligned}$$

Similar arguments show that

$$\mathbb{E} \int_{\mathbb{T}^2} \sup_{t \in [0, T]} |I_3(t)| dx = \mathbb{E} \int_0^T \int_{\mathbb{T}^2} \frac{\|\hat{\sigma}(X_t) - \hat{\sigma}(\hat{X}_t)\|_{HS}^2}{|Z_t|^2 + \delta} dx dt$$

$$\leq \frac{1}{\delta^2} \mathbb{E} \int_0^T \int_{\mathbb{T}^2} \|\hat{\sigma}(x) - \hat{\sigma}(x)\|_{HS}^2 dx dt$$

$$+ C \int_0^T \left( \int_{\mathbb{T}^2} \|\nabla \hat{\sigma}(x)\|^2 dx dt \right)^{\frac{1}{2}}.$$

Finally,  $I_4(t)$  is negative and can be abandoned. Therefore, collecting all deduced estimates, we obtain the claimed relation (2.17).  $\square$

**Lemma 2.9.** Let  $\phi(\omega, x) = \sup_{t \in [0, T]} |X_t(\omega, x) - \hat{X}_t(\omega, x)|^2$ . If it exists  $M > 0$  such that

$$\int_{\mathbb{T}^2} \log \left( \frac{\phi(\omega, x)}{\delta^2} + 1 \right) dx \leq M.$$

Then, there exists  $R > 0$  such that

$$\int_{\mathbb{T}^2} \phi(\omega, x) dx \leq \frac{4R^2}{M} + \delta^2(e^{M^2} - 1)|\mathbb{T}^2|.$$

**Proof.** Let  $f(\omega, x) = \log \left( \frac{\phi(\omega, x)}{\delta^2} + 1 \right)$  and  $A(\omega) = \{x \in \mathbb{T}^2 : f(\omega, x) \geq M^2\}$ . We know that

$$\int_{\mathbb{T}^2} \phi(\omega, x) dx = \int_{\mathbb{T}^2} \phi(\omega, x) \mathbf{1}_{A(\omega)} dx + \int_{\mathbb{T}^2} \phi(\omega, x) \mathbf{1}_{A(\omega)^c} dx = I_1(\omega) + I_2(\omega).$$

If  $f(\omega, x) \leq M^2$  then  $\phi(\omega, x) \leq \delta^2(e^{M^2} - 1)$ , thus

$$I_2(\omega) \leq \int_{\mathbb{T}^2} \delta^2(e^{M^2} - 1) \mathbf{1}_{A(\omega)^c} dx \leq \delta^2(e^{M^2} - 1)|\mathbb{T}^2|.$$

On the other hand, since  $X_t(x)$  belongs to the bounded set  $\mathbb{T}^2$ , there exists  $R > 0$  such that  $|X_t| \leq R, \forall t \in [0, T]$ . Therefore

$$I_1(\omega) \leq 4R^2 \int_{\mathbb{T}^2} \mathbf{1}_{A(\omega)} dx.$$

Using the Chebyshev inequality, we deduce

$$I_1(\omega) \leq \frac{4R^2}{M^2} \int_{A(\omega)} f(\omega, x) dx \leq \frac{4R^2}{M^2} \int_{\mathbb{T}^2} \log \left( \frac{\phi(\omega, x)}{\delta^2} + 1 \right) dx \leq \frac{4R^2}{M}. \quad \square$$

The next lemma deals with appropriate smooth regular approximations for the vector field  $u$  and finite dimensional smooth approximations for the diffusion operator  $\sigma$ .

**Lemma 2.10.** *Let  $u \in L^2(0, T; H^1(\mathbb{T}^2))$  such that  $\operatorname{div} u = 0$ , and consider the diffusion operator  $\sigma$  defined in (2.8). Then there exists  $u_n \in L^2(0, T; C^\infty(\mathbb{T}^2))$  such that*

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\mathbb{T}^2} |u_n(x, t) - u(x, t)| dx dt = 0, \quad \sup_n \left( \int_0^T \int_{\mathbb{T}^2} |\nabla u_n(x, t)|^2 dx dt \right) < \infty, \quad (2.19)$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}^2} \|\sigma_n(x) - \sigma(x)\|_{HS}^2 dx = 0, \quad \sup_n \left( \int_{\mathbb{T}^2} \|\nabla \sigma_n(x)\|^2 dx \right) < \infty, \quad (2.20)$$

where  $\sigma_n \in C^\infty(\mathbb{T}^2, HS(l^2, \mathbb{R}^2))$  is the finite dimensional approximation of the operator  $\sigma$ . Moreover, we have  $\operatorname{div} u_n = 0$  and  $\operatorname{div} \sigma_n e_{k_i} = 0$ ,  $\forall k \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ ,  $i = 1, 2$ .

**Proof.** Defining  $u_n(x, t) = u(x, t) * \varrho_n(x)$ , where  $\varrho_n(x) = n^d \varrho(nx)$  with  $\varrho \in C_c^\infty(\mathbb{T}^2)$ ,  $\varrho \leq 1$  and  $\int_{\mathbb{T}^2} \varrho(x) dx = 1$ , it is clear that (2.20) holds. Taking into account the results of Lemmas 2.5, 2.6 and the definition of  $\sigma_n$ , we verify (2.20).  $\square$

### 3. Existence and uniqueness

In this section, we will establish a theorem concerning the existence and uniqueness of the solution for the stochastic differential equation (2.4), which is defined through an infinite dimensional Brownian motion, having a Hilbert–Schmidt diffusion operator, and drift in the Sobolev space  $H^1$  with zero divergence. Our motivation to study this equation follows from the paper [6], where this equations is interpreted as the law of motion for viscous incompressible fluids, with velocity given by the solution of the Navier–Stokes equations. The solution of equation (2.4) will correspond to a stochastic flow map with  $L^2$ -regularity, that preserves the Lebesgue measure, in the sense of the Definition 2.1. In spite of our restriction to divergence free vector fields, the mathematical analysis turns out that, with simple modifications, the existence and the uniqueness result is also true for bounded divergence vector fields.

Let us also mention that in the last section we will apply this well-posedness result to solve specific stochastic differential equations of the form (2.4) that appears in the analysis of the large deviation principle.

**Theorem 3.1.** *Let  $u \in L^2(0, T; H^1(\mathbb{T}^2))$  such that  $\operatorname{div} u = 0$ , and  $\sigma$  in the space  $\mathbb{H}_V^1(\mathbb{T}^2, HS(l^2, \mathbb{R}^2))$ . Then, there exists a unique stochastic incompressible a.e. flow  $X_t$  defined almost surely in the variables  $\omega$  and  $x$  that solves*

$$dX_t = u(X_t, t)dt + \sigma(X_t)dW_t, \quad X_0 = x \quad (3.1)$$

in the sense of the Definition 2.1. Moreover for a.e.- $\omega$  the path  $X(\omega, \cdot)$  belongs to  $C([0, T], L^2(\mathbb{T}^2))$ .

**Proof.** The proof is split into four parts.

*Part 1. Existence.* Let us consider the smooth approximations  $u_n$  and  $\sigma_n$  of  $u$  and  $\sigma$ , respectively, introduced in Lemma 2.10. The classical theory of stochastic differential equation guarantees the existence and uniqueness of a smooth solution  $X_n$  for the equation

$$dX_{n,t} = u_n(X_{n,t}, t)dt + \sigma_n(X_{n,t})dW_t, \quad X_{n,0} = x. \quad (3.2)$$

Throughout the proof we will assume that for a.e.- $\omega$ ,  $X_n$  preserves the Lebesgue measure, i.e.

$$\int_{\mathbb{T}^2} \varphi(X_{n,t}(x))dx = \int_{\mathbb{T}^2} \varphi(x)dx, \quad \forall t \in [0, T], \quad (3.3)$$

for every measurable and bounded function  $\varphi$ . This fact will be verified a posteriori.

We start by proving that for every  $T > 0$ ,  $X_n$  is a Cauchy sequence in  $L^2$ , i.e.

$$\lim_{n,m \rightarrow \infty} \mathbb{E} \int_{\mathbb{T}^2} \sup_{t \in [0, T]} |X_{n,t} - X_{m,t}|^2 dx = 0. \quad (3.4)$$

Since for all  $t \in [0, T]$ ,  $X_{n,t} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  and  $\mathbb{T}^2$  is bounded, there exists  $R > 0$  such that  $\mathbb{E}|X_{n,T}|^2 < R$ . Therefore, it is enough to verify the following convergence in probability,  $\forall \eta > 0$

$$\lim_{n,m \rightarrow \infty} P \left\{ \omega : \int_{\mathbb{T}^2} \Phi_{n,m}(\omega, x) dx \geq \eta \right\} = 0, \quad (3.5)$$

where  $\Phi_{n,m}(\omega, x) = \sup_{t \in [0, T]} |X_{n,t}(\omega, x) - X_{m,t}(\omega, x)|^2$ . Let us define

$$\xi_{n,m}(\omega) = \int_{\mathbb{T}^2} \log \left( \frac{\Phi_{n,m}(\omega, x)}{\delta^2} + 1 \right) dx.$$

Applying the result of the Lemma 2.8 by taking

$$\delta = \delta_{n,m} = \int_0^T \int_{\mathbb{T}^2} |u_n(x, t) - u_m(x, t)| dx dt + \left\{ \int_0^T \int_{\mathbb{T}^2} |\sigma_n(x) - \sigma_m(x)|^2 dx dt \right\}^{\frac{1}{2}},$$

we obtain

$$\begin{aligned} \mathbb{E}(\xi_{n,m}) &\leq \frac{C_1}{\delta_{n,m}} \left( \int_0^T \int_{\mathbb{T}^2} |u_n(x, t) - u_m(x, t)| dx dt \right. \\ &\quad \left. + \left\{ \int_0^T \int_{\mathbb{T}^2} |\sigma_n(x) - \sigma_m(x)|^2 dx dt \right\}^{\frac{1}{2}} \right) \end{aligned}$$

$$+ \frac{C_1}{\delta_{n,m}^2} \int_0^T \int_{\mathbb{T}^2} |\sigma_n(x) - \sigma_m(x)|^2 dx dt + C_2 \leq C,$$

where  $C$  is a constant independent of  $n, m$ . Therefore, by the Chebyshev's inequality, there exists  $M_1 > 0$  such that  $\forall M > M_1$  we have

$$P(\xi_{n,m} > M) \leq \epsilon$$

for every  $n, m \in \mathbb{N}$ .

Let us consider  $R$  as the constant in the [Lemma 2.9](#). We can choose  $M \geq M_1 \vee \frac{8R^2}{\eta}$  and  $n, m$  large enough obtaining

$$\delta_{n,m} < \left( \frac{\eta}{2(e^{M^2} - 1)|\mathbb{T}^2|} \right)^{\frac{1}{2}}.$$

Therefore, by the [Lemma 2.9](#) we get that

$$\Omega_{n,m}^M = \left\{ \omega : \int_{\mathbb{T}^2} \Phi_{n,m} dx \geq \eta; \xi_{n,m} \leq M \right\} = \emptyset$$

and thus

$$P \left\{ \omega : \int_{\mathbb{T}^2} \Phi_{n,m}(\omega, x) dx \geq \eta \right\} \leq P(\Omega_{n,m}^M) + P(\xi_{n,m} > M) \leq \epsilon,$$

which proves [\(3.5\)](#). Denoting by  $X_t(\omega, x) \in L^2(\Omega \times \mathbb{T}^2, C[0, T])$  the limit process, there exists a subsequence  $X_{n_k}$  such that almost every  $(\omega, x) \in \Omega \times \mathbb{T}^2$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |X_{n_k, t}(\omega, x) - X_t(\omega, x)| = 0.$$

*Part 2. Solution of the equation.* We will now verify that  $X_t(\omega, x)$  satisfies the equation [\(3.1\)](#). In order to simplify the notation we will ignore the index  $k$  in the subsequence  $X_{n_k}$ . We have

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{T}^2} \int_0^t |u(X_s, s) - u_n(X_{n,s}, s)| dx ds \\ & \leq \mathbb{E} \int_{\mathbb{T}^2} \int_0^T |u(X_s, s) - u(X_{n,s}, s)| dx ds + \mathbb{E} \int_{\mathbb{T}^2} \int_0^T |u(X_{n,s}, s) - u_n(X_{n,s}, s)| dx ds \\ & = I_1 + I_2. \end{aligned}$$

On the other hand, the Burkholder–Davis–Gundy inequality yields

$$\begin{aligned}
 \mathbb{E} \int_{\mathbb{T}^2} \left| \int_0^t (\sigma(X_s) - \sigma_n(X_{n,s})) dW_s \right| dx &\leq \int_{\mathbb{T}^2} \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t (\sigma(X_s) - \sigma_n(X_{n,s})) dW_s \right| dx \\
 &\leq C \left\{ \int_{\mathbb{T}^2} \mathbb{E} \int_0^T \|\sigma(X_t) - \sigma_n(X_{n,t})\|_{HS}^2 dx dt \right\}^{\frac{1}{2}} \\
 &\leq C \left\{ \int_{\mathbb{T}^2} \mathbb{E} \int_0^T \|\sigma(X_t) - \sigma(X_{n,t})\|_{HS}^2 dx dt \right. \\
 &\quad \left. + \int_{\mathbb{T}^2} \mathbb{E} \int_0^T \|\sigma(X_{n,t}) - \sigma_n(X_{n,t})\|_{HS}^2 dx dt \right\}^{\frac{1}{2}} \\
 &= C \left\{ I_3 + I_4 \right\}^{\frac{1}{2}}.
 \end{aligned}$$

To deal with  $I_1$ , we look at the function  $u(x, t)$ . By the Lusin's Theorem, given  $\epsilon > 0$  there exists a compact set  $K_\epsilon \subset \mathbb{T}^2 \times [0, T]$  with  $m(K_\epsilon) > 1 - \frac{\epsilon^2}{9}$ , where  $m$  denotes the product measure on  $\mathbb{T}^2 \times [0, T]$ , such that the restriction of  $u$  to  $K_\epsilon$  is uniformly continuous. Therefore we have that

$$\begin{aligned}
 I_1 = \mathbb{E} \int_{\mathbb{T}^2} \int_0^T &|u(X_t, t) - u(X_{n,t}, t)| \{ 1_{K_\epsilon}(X_t, t) 1_{K_\epsilon}(X_{n,t}, t) \\
 &+ 1_{K_\epsilon}(X_t, t) 1_{K_\epsilon^c}(X_{n,t}, t) + 1_{K_\epsilon^c}(X_t, t) 1_{K_\epsilon}(X_{n,t}, t) \\
 &+ 1_{K_\epsilon^c}(X_t, t) 1_{K_\epsilon^c}(X_{n,t}, t) \} dt dx = I_{11} + I_{12} + I_{13} + I_{14}.
 \end{aligned}$$

Taking into account the uniform continuity, we obtain the following convergence

$$I_{11} = \mathbb{E} \int_{\mathbb{T}^2} \int_0^T |u(X_t, t) - u(X_{n,t}, t)| 1_{K_\epsilon}(X_t, t) 1_{K_\epsilon}(X_{n,t}, t) dt dx \rightarrow 0, \quad n \rightarrow \infty.$$

By applying the Cauchy–Schwarz inequality to the second integral  $I_{12}$ , we deduce

$$\begin{aligned}
 I_{12} &= \mathbb{E} \int_{\mathbb{T}^2} \int_0^T |u(X_t, t) - u(X_{n,t}, t)| 1_{K_\epsilon}(X_t, t) 1_{K_\epsilon^c}(X_{n,t}, t) dt dx \\
 &\leq \left\{ \mathbb{E} \int_{\mathbb{T}^2} \int_0^T |u(X_t, t) - u(X_{n,t}, t)|^2 dt dx \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \int_{\mathbb{T}^2} \int_0^T 1_{K_\epsilon}(X_t, t) 1_{K_\epsilon^c}(X_{n,t}, t) dt dx \right\}^{\frac{1}{2}}
 \end{aligned}$$



$$\leq \left\{ \mathbb{E} \int_{\mathbb{T}^2} \int_0^T |u(X_t, t) - u(X_{n,t}, t)|^2 dt dx \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \int_{\mathbb{T}^2} \int_0^T 1_{K_\epsilon^c}(X_{n,t}, t) dt dx \right\}^{\frac{1}{2}} \leq C \frac{\epsilon}{3}.$$

Analogous arguments allow to verify that  $I_{13} \leq C \frac{\epsilon}{3}$  and  $I_{14} \leq C \frac{\epsilon}{3}$ . Therefore the arbitrariness of  $\epsilon$  gives

$$\mathbb{E} \int_{\mathbb{T}^2} I_1(t) dx \rightarrow 0, \quad n \rightarrow \infty.$$

A similar reasoning can be used to show that  $I_3 \rightarrow 0$ , as  $n \rightarrow \infty$ .

Now, from the invariance of the measure, we derive

$$\begin{aligned} I_2 &= \int_0^T \mathbb{E} \int_{\mathbb{T}^2} |u(X_{n,s}, s) - u_n(X_{n,s}, s)| dx ds \\ &= \int_0^T \int_{\mathbb{T}^2} |u(x, s) - u_n(x, s)| dx ds \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

and

$$\begin{aligned} I_4 &= \mathbb{E} \int_{\mathbb{T}^2} \int_0^T \|\sigma(X_{n,t}) - \sigma_n(X_{n,t})\|_{HS}^2 dt dx \\ &= \mathbb{E} \int_{\mathbb{T}^2} \int_0^T \sum_{k \in \mathbb{Z}^2: |k| > n} |A_k^1 e_{k_1}|^2 + |A_k^2 e_{k_1}|^2 + |B_k^1 e_{k_2}|^2 + |B_k^2 e_{k_2}|^2 dx dt. \end{aligned}$$

By the [Lemma 2.5](#), we know that  $\|\sigma(x)\|_{HS}^2 < C$ , and by the definition of the finite dimensional approximation  $\sigma_n$ , we have  $\|\sigma_n(x)\|_{HS}^2 < \|\sigma(x)\|_{HS}^2$ . Therefore

$$\mathbb{E} \int_{\mathbb{T}^2} \int_0^T \sum_{k \in \mathbb{Z}^2: |k| > n} |A_k^1(x) e_{k_1}|^2 + |A_k^2(x) e_{k_1}|^2 + |B_k^1(x) e_{k_2}|^2 + |B_k^2(x) e_{k_2}|^2 dx dt \rightarrow 0,$$

as  $n \rightarrow \infty$ . Taking into account these convergence results, we are able to pass to the limit the equation (3.2) and showing that the limit process  $X_t$  is a solution of the stochastic differential equation (3.1).

*Part 3. Invariance of the Lebesgue measure.* To verify that the flow map  $X_t(\omega, \cdot)$  verifies the incompressibility condition (2.7) we take the limit of the expression (3.3), as  $n \rightarrow \infty$ . Taking into account that  $\varphi$  is bounded, we apply the Lusin's Theorem to deduce that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}^2} \varphi(X_{n,t}(x)) dx = \int_{\mathbb{T}^2} \varphi(X_t(x)) dx. \quad (3.6)$$

*Part 4. Uniqueness.* Let us assume that  $X_t(x)$  and  $\hat{X}_t(x)$  are two solutions of (3.1). Reasoning as for the existence, we show that

$$\mathbb{E} \int_{\mathbb{T}^2} \log \left( \frac{|X_t - \hat{X}_t|^2}{\delta^2} + 1 \right) dx \leq C.$$

Since  $C$  is independent of  $\delta$ , we can take the limit as  $\delta \rightarrow \infty$  deducing

$$|X_t - \hat{X}_t|^2 = 0,$$

which gives the uniqueness.  $\square$

The next lemma establishes the propriety (3.3) that we assumed as valid along the proof of Theorem 3.1.

**Lemma 3.2.** *Let  $\varphi$  be a bounded Borel-measurable function and  $X_{n,t}(x)$  the solution of the equation (3.2). Then we have*

$$\int_{\mathbb{T}^2} \varphi(X_{n,t}(x)) dx = \int_{\mathbb{T}^2} \varphi(x) dx, \quad \forall n \in \mathbb{N}, \forall t \in [0, T]. \quad (3.7)$$

**Proof.** Let us denote by  $W_t^n$  the finite dimensional approximation of  $W_t$ . By Lemma 2.7, the Itô integral in the equation (3.2) can be written as a Stratonovich integral, so the equation (3.2) reads

$$dX_{n,t} = u_n(X_{n,t}, t)dt + \sigma_n(X_{n,t}) \circ dW_t^n, \quad X_0 = x.$$

Let  $W_t^l$  be the linearized approximation of  $W_t^n$ , then we consider the deterministic equation

$$dX_t^l = u_n(X_t^l, t)dt + \sigma_n(X_t^l) \partial_t W_t^l dt.$$

Classical results give

$$\det(\nabla X_t^l(x)) = \exp \left\{ \int_0^t \operatorname{div} u_n(X_s^l(x), s) ds + \int_0^t \operatorname{div} \sigma_n(X_s^l(x)) \partial_s W_s^l ds \right\}.$$

From the results on regular stochastic flows in [19], we can evaluate this expression at the limit, as  $l \rightarrow \infty$ , deducing

$$\det(\nabla X_{n,t}(x)) = \exp \left\{ \int_0^t \operatorname{div} u_n(X_{n,s}(x), s) ds + \int_0^t \operatorname{div} \sigma_n(X_{n,s}(x)) \circ dW_s \right\}. \quad (3.8)$$

Let us now define

$$Y_{s,\tau}^t(x) = x - \int_s^\tau u(Y_{r,\tau}^t(x), r) dr + \int_s^\tau \sigma(Y_{r,\tau}^t(x)) dW_r^t, \quad (3.9)$$

where  $W_r^t := W_{t-r} - W_t$ . In order to introduce the inverse flow, we take, for a moment, the initial time in the notations by setting  $X_{n,0,t} := X_{n,t}$  and  $X_{n,s,t}$ ,  $s \leq t$ . From [16] we know that  $Y_{s,t}^t := X_{n,s,t}^{-1}$ . Since  $Y_{s,\tau}^t$  verifies the equation (3.9), the reasoning that we used to verify (3.8) also gives

$$\det(\nabla X_{n,0,t}^{-1}) = \exp \left\{ \int_0^t -\operatorname{div} u_n(X_{n,r,t}^{-1}, r) dr + \int_0^t \operatorname{div} \sigma_n(X_{n,r,t}^{-1}) dW_r^n \right\}.$$

We recall from Lemmas 2.10 and 2.5 that  $\operatorname{div} u_n = 0$  and each column  $\sigma_{\cdot j}$  of the matrix  $\sigma$  is equal zero, therefore we have

$$\det(\nabla X_{n,t}^{-1}(x)) = 1, \quad t \in [0, T], \text{ and a.e. } -\omega, \quad (3.10)$$

which implies the claimed result.  $\square$

#### 4. Principle of large deviations

In this section, we will establish a large deviations principle for the family of stochastic flows  $\{X^\epsilon, \epsilon > 0\}$  defined as solutions of the stochastic differential equations (2.9) where the drift  $u^\epsilon(x, t) : \mathbb{T}^2 \times [0, T] \rightarrow \mathbb{R}^2$  is solution of the Navier–Stokes equations (2.10). Our result will imply that as  $\epsilon \rightarrow 0$ , the stochastic Navier–Stokes flow will converges with an exponential velocity to the Euler flow  $X_t$ , defined by the solution of deterministic equation (2.13), where the velocity field  $u(x, t)$  is a solution to the Euler equations (2.14).

Let us introduce two well known results concerning the asymptotic behaviour of  $u^\epsilon(x, t)$  when  $\epsilon$  tends to zero. We refer [14] and [20] for a developed study on the inviscid limit in domains without boundary.

**Lemma 4.1.** *Let  $u^\epsilon(x, t)$  be defined as to solution to the Navier–Stokes equations (2.10). Then  $u^\epsilon(x, t) \in L^2(0, T; H^1(\mathbb{T}^2)) \cap C([0, T], L^2(\mathbb{T}^2))$  and*

$$\|u^\epsilon\|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \leq C, \quad \|\partial_t u^\epsilon\|_{L^\infty(0,T;H^{-1}(\mathbb{T}^2))} \leq C, \quad (4.1)$$

where  $C$  does not depend on  $\epsilon$ .

**Proof.** The transport equation for the vorticity  $\omega^\epsilon = \operatorname{curl} u^\epsilon$  is given by

$$\partial_t \omega^\epsilon + u^\epsilon \nabla \omega^\epsilon = \epsilon \Delta \omega^\epsilon.$$

Multiplying this equation by  $\omega^\epsilon$  and integrating over  $\mathbb{T}^2$ , we obtain  $\|\omega^\epsilon(\cdot, t)\|_{L^2(\mathbb{T}^2)}^2 \leq C$ . On the other hand, since  $\operatorname{div} u^\epsilon = 0$ , there exists a stream function  $h$  such that  $u^\epsilon = (-\partial_2 h^\epsilon, \partial_1 h^\epsilon)$  and  $\Delta h^\epsilon = \omega^\epsilon$ . Then we have

$$\|u^\epsilon\|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \leq \|\omega^\epsilon\|_{L^\infty((0,T)\times\mathbb{T}^2)} \leq C \quad (4.2)$$

and moreover  $\partial_t h^\epsilon = \partial_t \omega^\epsilon$ . Since

$$\|\partial_t h^\epsilon(\cdot, t)\|_{L_2(\mathbb{T}^2)} \leq C \|G^\epsilon(\cdot, t)\|_{H^{-2}(\mathbb{T}^2)} \quad \text{a.e. } t \in (0, T) \quad (4.3)$$

with  $G^\epsilon := \operatorname{div}[-u^\epsilon \omega^\epsilon + \epsilon \nabla \omega^\epsilon]$ . The inequality (4.2) yields  $\|G^\epsilon(\cdot, t)\|_{H^{-2}(\mathbb{T}^2)} \leq C$  which implies the second estimate of (4.1).  $\square$

**Lemma 4.2.** *Let  $(u^\epsilon)$ ,  $\epsilon > 0$ , be defined as solutions to (2.10) so, there exists a subsequence  $(u^\epsilon)$  such that*

$$u^\epsilon \rightharpoonup u \quad \text{weakly-}^* \text{ in } L^\infty(0, T; H^1(\mathbb{T}^2)), \quad (4.4)$$

$$u^\epsilon \rightarrow u \quad \text{strongly in } L^2(\mathbb{T}^2 \times (0, T)) \quad (4.5)$$

and the limit function  $u$  is the solution to (2.14).

**Proof.** We observe that (4.4) is a direct consequence of (4.1). The convergence of (4.5) follows from (4.1) and the compactness result in the Corollary 4 of [22]. Using the convergence results (4.4) and (4.5) we can obtain the Euler equations by passing to the limit in the sense of distributions the Navier–Stokes equations.  $\square$

We have now fulfilled the necessary conditions to start working towards the main result of this section.

#### 4.1. Rate function

Let us consider the Hilbert space  $l^2$  endowed with the norm and inner product defined in (2.2)–(2.3). We define  $\mathbb{H}$  as the space of the  $l^2$ -valued absolutely continuous functions  $h$  defined on  $[0, T]$  with square integrable time derivative  $\dot{h}$ , i.e.

$$\mathbb{H} = \{h : [0, T] \rightarrow l^2 : \|h\|_{\mathbb{H}}^2 = \int_0^T \|\dot{h}(s)\|_{l^2}^2 ds < \infty\}.$$

For any  $N \in \mathbb{N}$ , we denote by  $B_N$  the ball in  $\mathbb{H}$  with center 0 and radius  $N$ , namely

$$B_N = \{h \in \mathbb{H} : \|h\|_{\mathbb{H}} \leq N\}.$$

We know that  $B_N$  is compact with respect to the weak topology of  $\mathbb{H}$ . Let  $A$  be the class of all  $\mathbb{H}$ -valued processes  $h(t)$ ,  $\mathcal{F}_t$ -predictable, satisfying

$$\mathbb{E} \left( \int_0^T \|\dot{h}(s)\|_{l^2}^2 ds \right) < \infty.$$

Given  $h \in \mathbb{H}$ , we will consider the following deterministic control equation

$$dX_t^h(x) = x + \int_0^t \langle \sigma(X_s^h(x)), \dot{h}(s) \rangle_{\ell^2} ds + \int_0^t u(X_s^h(x), s) ds. \quad (4.6)$$

We define the functional  $S$  by

$$\begin{aligned} S : \mathbb{H} &\rightarrow C([0, T], L^2(\mathbb{T}^2)) \\ h &\rightarrow S(h) := X_t^h(x) \end{aligned} \quad (4.7)$$

$X_t^h$  is the solution to (4.6). The next lemma guaranties that  $S$  is well defined.

**Lemma 4.3.** Assume that  $u \in L^2(0, T; H^1(\mathbb{T}^2))$  and  $\operatorname{div} u = 0$ . Then for any  $h \in \mathbb{H}$ , the equation (4.6) has a unique solution  $X_t^h$ .

**Proof.** Let us notice that if the vector field  $u^h(x, t) = \langle \sigma(x), \dot{h}(t) \rangle + u(x, t)$  has divergence equal to zero and belongs to  $L^2(0, T; H^1(\mathbb{T}^2))$ , the existence and the uniqueness of the solution to equation (4.6) follows from the deterministic results in the papers [5], [10], [12]. We also mention that the result of Theorem 3.1 still holds if we take the diffusion operator equal to the null operator yielding the existence and the uniqueness of solution to the deterministic equation (4.6). Knowing that each column of the operator  $\sigma$  has divergence equal to zero and  $h(t)$  is independent of the variable  $x$ , we get  $\operatorname{div} \langle \sigma(x), \dot{h}(s) \rangle = 0$ . On the other hand

$$\int_0^T \int_{\mathbb{T}^2} |\langle \sigma(x), \dot{h}(t) \rangle|^2 dx dt \leq \int_0^T \|\dot{h}(t)\|_{\ell^2}^2 dt \int_{\mathbb{T}^2} \|\sigma(x)\|_{HS}^2 dx < \infty$$

and by Lemma 2.6 we deduce that

$$\int_0^T \int_{\mathbb{T}^2} |\nabla \langle \sigma(x), \dot{h}(t) \rangle|^2 dx dt \leq \int_0^T \|\dot{h}(t)\|_{\ell^2}^2 dt \int_{\mathbb{T}^2} \|\nabla \sigma(x)\| dx < \infty.$$

Therefore we have  $\operatorname{div} u^h = 0$  and  $u^h \in L^2(0, T; H^1(\mathbb{T}^2))$ , which give the claimed result.  $\square$

We define the mapping  $I : C([0, T], L^2(\mathbb{T}^2)) \mapsto \mathbb{R}$  as

$$I(f) := \frac{1}{2} \inf_{\{h \in \mathbb{H} : S(h)=f\}} \|h\|_{\mathbb{H}}^2, \quad (4.8)$$

and our next aim is to show that  $I$  is a rate function. We begin by proving the following two lemmas.

**Lemma 4.4.**  $\forall N > 0$  fixed, the set  $\{S(h) : \|h\|_{\mathbb{H}} \leq N\}$  is precompact in  $C([0, T], L^2(\mathbb{T}^2))$ .

**Proof.** Let  $\{S_n\}_{n \geq 1}$  be a sequence of elements of the set  $\{S(h) : \|h\|_{\mathbb{H}} \leq N\}$ . Then, there exists a sequence  $\{h_n\}_{n \geq 1}$  of elements of the set  $\{h : \|h\|_{\mathbb{H}} \leq N\}$ , such that  $S_n = S(h_n)$ . Since  $\mathbb{H}$  is locally compact for the weak topology, we can extract a subsequence  $\{h_{n_k}\}_{n_k \geq 1}$  that converges weakly to  $h \in \{h : \|h\|_{\mathbb{H}} \leq N\}$ . In order to simplify the notation we will omit the index  $k$  in the notation of subsequences. Now, we prove that the sequence  $\{S(h_n)\}_{n \geq 1}$ , where  $h_n \rightharpoonup h$ , has a Cauchy subsequence, i.e.

$$\lim_{n,m \rightarrow +\infty} \int_{\mathbb{T}^2} \sup_{t \in [0,T]} |X_t^{h_n} - X_t^{h_m}|^2 dx = 0 \quad (4.9)$$

where  $X_t^{h_n}$  and  $X_t^{h_m}$  are solutions to the equation (4.6) with velocity fields  $u^{h_n}(x, t) = \langle \sigma(x), \dot{h}_n(t) \rangle + u(x, t)$  and  $u^{h_m}(x, t) = \langle \sigma(x), \dot{h}_m(t) \rangle + u(x, t)$ , respectively. To prove (4.9) we follow the same ideas as in the first part of the proof of the Theorem 3.1 to verify the convergence (3.4). Therefore it is enough to show that  $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n, m > N$  we have

$$\int_{\mathbb{T}^2} \Phi_{h_n, h_m} dx \leq \epsilon$$

with  $\Phi_{h_n, h_m} = \sup_{t \in [0, T]} |X_t^{h_n} - X_t^{h_m}|^2$ . Let us define

$$\xi_{h_n, h_m} = \int_{\mathbb{T}^2} \log \left( \frac{\Phi_{h_n, h_m}}{\delta^2} + 1 \right) dx.$$

Considering in Lemma 2.8  $u = u^{h_n}$ ,  $\hat{u} = u^{h_m}$  and  $\sigma = \hat{\sigma} = 0$ , we obtain

$$\xi_{h_n, h_m} \leq \frac{C_1}{\delta} \left( \int_0^T \int_{\mathbb{T}^2} |u^{h_n}(x, t) - u^{h_m}(x, t)| dx dt \right) + C_2$$

with

$$C_2 = C \left( 1 + \int_0^T \int_{\mathbb{T}^2} |\nabla u^{h_m}(x, t)|^2 dx dt \right),$$

where the constants  $C_1$  and  $C$  are independent of  $u^{h_n}$  and  $u^{h_m}$ . Choosing

$$\delta = \delta_{h_n, h_m} := \int_0^T \int_{\mathbb{T}^2} |u^{h_n}(x, t) - u^{h_m}(x, t)| dx dt$$

we derive

$$\begin{aligned}\xi_{h_n, h_m} &\leq C \left( 1 + \int_{\mathbb{T}^2} \int_0^T |\nabla \langle \sigma(x), \dot{h}_m(s) \rangle|^2 + |\nabla u(x, t)|^2 dt dx \right) \\ &\leq C \left( 1 + \int_0^T \|\dot{h}_m(t)\|_{l^2}^2 dt \int_{\mathbb{T}^2} \|\nabla \sigma(x)\| dx + \int_{\mathbb{T}^2} \int_0^T |\nabla u(x, t)|^2 dt dx \right) \\ &\leq C \left( 1 + N^2 \int_{\mathbb{T}^2} \|\nabla \sigma(x)\| dx + \int_{\mathbb{T}^2} \int_0^T |\nabla u(x, t)|^2 dt dx \right) = M_1\end{aligned}$$

with  $M_1$  independent of  $h_n$  and  $h_m$ . Thus, since there is  $R > 0$  such that  $|X_t^{h_n}| \leq R$  we can use the [Lemma 2.9](#) obtaining

$$\int_{\mathbb{T}^2} \Phi_{h_n, h_m} dx \leq \frac{4R^2}{M} + \delta^2(e^{M^2} - 1)|\mathbb{T}^2|,$$

$\forall M \geq M_1$ . Now, from the weak convergence of  $\{h_n\}$  in  $l^2$  and the dominated convergence theorem we get

$$\begin{aligned}\lim_{n, m \rightarrow 0} \int_0^T \int_{\mathbb{T}^2} |u^{h_n}(x, t) - u^{h_m}(x, t)| dx dt &= \lim_{n, m \rightarrow 0} \int_0^T \int_{\mathbb{T}^2} |\langle \sigma(x), h_n - h_m \rangle| dx dt \\ &= \lim_{n, m \rightarrow 0} \int_0^T \int_{\mathbb{T}^2} |(\langle \sigma_1, \cdot(x) \rangle, h_n - h_m)_{l^2}, \langle \sigma_2, \cdot(x) \rangle, h_n - h_m)_{l^2}| dx dt \\ &= 0.\end{aligned}$$

Therefore we can choose  $M \geq M_1 \vee \frac{8R^2}{\epsilon}$  and  $n, m$  big enough such that

$$\xi_{h_n, h_m} < \left( \frac{\epsilon}{2(e^{M^2} - 1)|\mathbb{T}^2|} \right)^{\frac{1}{2}}$$

deducing

$$\int_{\mathbb{T}^2} \Phi_{h_n, h_m} dx \leq \epsilon. \quad \square$$

**Lemma 4.5.** *The mapping  $h \mapsto S(h)$  is continuous for the weak topology.*

**Proof.** The proof of this lemma is analogous to the proof of [Lemma 4.4](#) and for this reason will be omitted.  $\square$

With the help of [Lemmas 4.4 and 4.5](#) we are able to show that  $I$  is a rate function.

**Lemma 4.6.** *The following propositions hold:*

- (i) *For every  $f \in C([0, T], L^2(\mathbb{T}^2))$  such that  $I(f) < \infty$ , there exists  $h_0 \in \mathbb{H}$  that verifies  $2I(f) = \|h_0\|_{\mathbb{H}}^2$ .*
- (ii) *The functional  $I$  is a rate function in  $C([0, T], L^2(\mathbb{T}^2))$ .*

**Proof.** Let us verify (i). By definition of the functional  $I$ , we can choose a sequence  $\{h_n\}_{n \geq 1}$  of elements of  $\mathbb{H}$  such that  $\|h_n\|_{\mathbb{H}}^2 \searrow 2I(f)$  and  $S(h_n) = f$ . Let us define  $N := \sup_n \|h_n\|_{\mathbb{H}}$ , then we can choose a subsequence  $\{h_{n_k}\}$  and  $h_0$  such that  $h_{n_k} \rightharpoonup h_0$  in  $B_N$ . So  $\|h_0\|_{\mathbb{H}}^2 \leq \liminf_{k \rightarrow \infty} \|h_{n_k}\|_{\mathbb{H}}^2 = 2I(f)$ . Applying [Lemma 4.5](#) we obtain  $S(h_0) = f$  and consequently  $2I(f) = \|h_0\|_{\mathbb{H}}^2$ .

Now, we consider (ii). For every  $a \leq \infty$ , we define

$$E := \{f : I(f) \leq a\} \subset \{S(h); \|h\|_{\mathbb{H}}^2 \leq 2a\}.$$

So, according to [Lemma 4.4](#), we only need to see that  $E$  is closed in  $C([0, T], L^2(\mathbb{T}^2))$ . Let  $\{f_n\}$  be a sequence of elements of  $E$  such that  $f_n \rightarrow f$  in  $C([0, T], L^2(\mathbb{T}^2))$ . By (i) we can choose  $h_n \in B_{2a}$  such that  $S(h_n) = f_n$ . From the compactness of  $B_{2a}$  we can choose a subsequence  $\{h_{n_k}\}$  and  $h \in B_{2a}$  such that  $h_{n_k} \rightharpoonup h$  in  $B_{2a}$ . So, using the [Lemma 4.5](#) we get that  $f_{n_k} = S(h_{n_k}) \rightarrow S(h)$ , therefore  $f = S(h)$  and  $E$  is closed.  $\square$

#### 4.2. Variational representation

To establish the principle of large deviations, we need a variational representation of certain functionals of the process  $X_t^\epsilon$  (solution to [\(2.9\)](#)).

Recalling that  $A$  was defined as the class of all  $\mathbb{H}$ -valued stochastic processes  $h(t)$ ,  $\mathcal{F}_t$ -predictable, satisfying

$$\mathbb{E} \left( \int_0^T \|\dot{h}(s)\|_{\mathbb{H}}^2 ds \right) < \infty,$$

let us introduce the following variational representation for the cylindrical Wiener process obtained in [\[1\]](#).

**Lemma 4.7.** *Assume the same hypothesis and notations as in Theorem 3.6 in [\[1\]](#), namely  $W$  is a Brownian motion with values in  $H$ , and  $H_0$  is the Cameron–Martin space (endowed with the norm  $\|\cdot\|_0$ ). For every bounded measurable function  $F : C([0, T], H) \rightarrow \mathbb{R}$ , we have the following representation*

$$-\log \mathbb{E}(e^{-F(W)}) = \inf_{v \in \mathcal{A}} \mathbb{E} \left( F(W + \int_0^\cdot v(s) ds) + \frac{1}{2} \int_0^T \|v(s)\|_0^2 ds \right).$$



Now, for each  $h \in A$ , we consider the so-called stochastic control equation

$$dX_t^{\epsilon, h} = u^\epsilon(X_t^{\epsilon, h}, t)dt + \langle \sigma(X_t^{\epsilon, h}), \dot{h}_t \rangle_{l^2} dt + \sqrt{\epsilon} \sigma(X_t^{\epsilon, h}) dW_t, \quad X_0^h = x \in \mathbb{T}^2. \quad (4.10)$$

Using the same reasoning as in the proof of the [Lemma 4.3](#), we are able to apply the result of the [Theorem 3.1](#) to deduce that the equation (4.10) has a unique strong solution  $X_t^{\epsilon, h}$  and, in addition,  $X_t^{\epsilon, h}$  belongs to  $C([0, T], L^2(\mathbb{T}^2))$  for a.e.- $\omega$ .

**Lemma 4.8.** *Let  $X_t^\epsilon$ ,  $\epsilon > 0$ , be the strong solution to equation (2.9). Then for any bounded function  $f : C([0, T], L^2(\mathbb{T}^2)) \rightarrow \mathbb{R}$ , the following variational representation hold*

$$-\epsilon \log \mathbb{E}(e^{-\frac{1}{\epsilon} f(X_t^\epsilon)}) = \inf_{h \in A} \mathbb{E} \left( f(X_t^{\epsilon, h}) + \frac{1}{2} \|h\|_{\mathbb{H}}^2 \right). \quad (4.11)$$

**Proof.** Since  $X^\epsilon$  is the strong solution of (2.9), it can be written as a functional of the cylindrical Wiener process  $W_t$ , more precisely there exists a measurable function  $\Phi^\epsilon : C([0, T], l_\ast^2) \mapsto C([0, T], L^2(\mathbb{T}^2))$  such that

$$X_t^\epsilon(x) = \Phi^\epsilon(W)(x, t), \quad \text{a.e.-(x, t) in } \mathbb{T}^2 \times [0, T].$$

On the other hand, we have

$$\begin{aligned} \Phi^\epsilon \left( W + \frac{h}{\sqrt{\epsilon}} \right) (x, t) &= x + \sqrt{\epsilon} \int_0^t \sigma \left( \Phi^\epsilon \left( W + \frac{h}{\sqrt{\epsilon}} \right) (x, s) \right) d \left( W + \frac{h}{\sqrt{\epsilon}} \right) (s) \\ &\quad + \int_0^t u \left( \Phi^\epsilon \left( W + \frac{h}{\sqrt{\epsilon}} \right) (x, s), s \right) ds \\ &= x + \sqrt{\epsilon} \int_0^t \sigma \left( \Phi^\epsilon \left( W + \frac{h}{\sqrt{\epsilon}} \right) (x, s) \right) dW_s \\ &\quad + \int_0^t \left\langle \sigma \left( \Phi^\epsilon \left( W + \frac{h}{\sqrt{\epsilon}} \right) (x, s) \right), \dot{h}_s \right\rangle_{l^2} ds \\ &\quad + \int_0^t u \left( \Phi^\epsilon \left( W + \frac{h}{\sqrt{\epsilon}} \right) (s, x), s \right) ds, \end{aligned}$$

then  $\Phi^\epsilon \left( W + \frac{h}{\sqrt{\epsilon}} \right)$  is a solution to the equation (4.10). Therefore the uniqueness gives

$$X^{\epsilon, h} = \Phi^\epsilon \left( W + \frac{h}{\sqrt{\epsilon}} \right).$$

Finally, taking into account that  $W_t$  can be considered as a Brownian motion with values in  $l_*^2$ , Lemma 4.7 yields

$$\begin{aligned}
 -\epsilon \log \mathbb{E}(e^{-\frac{1}{\epsilon} f(X^\epsilon)}) &= -\epsilon \log \mathbb{E}(e^{-\frac{1}{\epsilon} f \circ \Phi^\epsilon(W)}) \\
 &= \epsilon \inf_{h \in A} \mathbb{E} \left( \frac{1}{\epsilon} f \circ \Phi^\epsilon(W + h) + \frac{1}{2} \|h\|_{\mathbb{H}}^2 \right) \\
 &= \inf_{h \in A} \mathbb{E} \left( f \circ \Phi^\epsilon(W + h) + \frac{1}{2} \|\sqrt{\epsilon} h\|_{\mathbb{H}}^2 \right) \\
 &= \inf_{h \in A} \mathbb{E} \left( f \circ \Phi^\epsilon \left( W + \frac{h}{\sqrt{\epsilon}} \right) + \frac{1}{2} \|h\|_{\mathbb{H}}^2 \right) \\
 &= \inf_{h \in A} \mathbb{E} \left( f(X_t^{\epsilon, h}) + \frac{1}{2} \|h\|_{\mathbb{H}}^2 \right). \quad \square
 \end{aligned}$$

### 4.3. Tightness

For any fixed  $N \in \mathbb{N}$ , let us introduce the subspace of  $A$ , of all predictable processes with a.e. paths in  $B_N$ , namely

$$A_N := \{h \in A : \|h\|_{\mathbb{H}} \leq N \text{ for a.e. } \omega\}.$$

Given a family  $\{h^\epsilon, \epsilon > 0\}$  in  $A_N$ , we define  $X_t^{\epsilon, h^\epsilon}$  as the solution to (4.10) with  $h = h^\epsilon$ , i.e.

$$\begin{cases} dX_t^{\epsilon, h^\epsilon} = \sqrt{\epsilon} \sigma(X_t^{\epsilon, h^\epsilon}) dW_t + \langle \sigma(X_t^{\epsilon, h^\epsilon}), \dot{h}_t^\epsilon \rangle_{l^2} dt + u^\epsilon(X_t^{\epsilon, h^\epsilon}) dt, \\ X_0^{\epsilon, h^\epsilon} = x, \quad x \in \mathbb{T}^2. \end{cases} \quad (4.12)$$

In order to prove the tightness of the laws of  $(h^\epsilon, X_t^{h^\epsilon})$  in the space  $B_N \times C([0, T], L^2(\mathbb{T}^2))$  with  $\epsilon \in [0, \epsilon_0]$ ,  $\epsilon_0 > 0$ , we introduce the following auxiliary lemmas.

**Lemma 4.9.** *Let  $X_t^{\epsilon, h^\epsilon}$ ,  $\epsilon > 0$ , be the solutions to (4.12). Then there exists a continuous function  $g$ , independent on  $\epsilon$ , such that*

$$\lim_{r \rightarrow 0} g(r) = 0$$

and

$$\mathbb{E} \left\{ \frac{1}{|B_r(0)|} \int_{B_r(0)} \int_{\mathbb{T}^2} \sup_{t \in [0, T]} |X_t^{\epsilon, h^\epsilon}(x) - X_t^{\epsilon, h^\epsilon}(x + z)|^2 dx dz \right\} \leq g(r)$$

for every  $z \in \mathbb{T}^2$ .

**Proof.** Let us define  $Z_t^\epsilon(x) := |X_t^{\epsilon, h^\epsilon}(x) - X_t^{\epsilon, h^\epsilon}(x + z)|$ ,  $\sigma^\epsilon(x) := \sqrt{\epsilon} \sigma(x)$  and  $u^{h^\epsilon}(x, t) := \langle \sigma(x), \dot{h}_t^\epsilon \rangle_{l^2} + u^\epsilon(x, t)$ . By Itô's formula, we have

$$\begin{aligned} \log \left( \frac{|Z_t^\epsilon|^2}{\delta^2} + 1 \right) &= \log \left( \frac{|z|^2}{\delta^2} + 1 \right) \\ &+ 2 \int_0^t \frac{\langle Z_s^\epsilon, u^{h^\epsilon}(X_s^{\epsilon, h^\epsilon}(x), s) - u^{h^\epsilon}(X_s^{\epsilon, h^\epsilon}(x + z), s) \rangle}{|Z_s^\epsilon|^2 + \delta^2} ds \\ &+ 2 \int_0^t \frac{\langle Z_s^\epsilon, (\sigma^\epsilon(X_s^{\epsilon, h^\epsilon}(x)) - \sigma^\epsilon(X_s^{\epsilon, h^\epsilon}(x + z))) dW_s \rangle}{|Z_s^\epsilon|^2 + \delta^2} \\ &+ \int_0^t \frac{\|\sigma^\epsilon(X_s^{\epsilon, h^\epsilon}(x)) - \sigma^\epsilon(X_s^{\epsilon, h^\epsilon}(x + z))\|_{HS}^2}{|Z_s^\epsilon|^2 + \delta^2} ds \\ &- 2 \int_0^t \frac{|\sigma^\epsilon(X_s^{\epsilon, h^\epsilon}(x)) - \sigma^\epsilon(X_s^{\epsilon, h^\epsilon}(x + z))^\top Z_s^\epsilon|^2}{|Z_s^\epsilon|^2 + \delta^2} ds \\ &= \log \left( \frac{|z|^2}{\delta^2} + 1 \right) + I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned}$$

Using the same arguments as in the proof of the [Lemma 2.8](#), we deduce estimates for the integrals of the terms  $I_1(t)$ ,  $I_2(t)$  and  $I_3(t)$ . The term  $I_4(t)$  being negative can be abandoned.

The stochastic integral  $I_2(t)$  is analogous to the stochastic integral that appeared in the proof of the [Lemma 2.8](#), then, as before, with the help of the Burkholder–Davis–Gundy inequality, the incompressibility of the flow and the inequalities D1 and D2 in the Appendix, we deduce

$$\begin{aligned} &E \left\{ \frac{1}{|B_r(0)|} \int_{B_r(0)} \int_{\mathbb{T}^2} \sup_{t \in [0, T]} |I_2(t)| dx dz \right\} \\ &\leq \frac{C}{|B_r(0)|} \mathbb{E} \left\{ \int_0^T \int_{B_r(0)} \int_{\mathbb{T}^2} M_R \|\nabla \sigma^\epsilon(X_t^{\epsilon, h^\epsilon}(x))\| + M_R \|\nabla \sigma^\epsilon(X_t^{\epsilon, h^\epsilon}(x + z))\| dx dz dt \right\}^{\frac{1}{2}} \\ &\leq C \left\{ \int_0^T \int_{\mathbb{T}^2} M_R \|\nabla \sigma^\epsilon(x)\|^2 dx dt \right\}^{\frac{1}{4}} \leq C \left\{ \int_0^T \int_{\mathbb{T}^2} \|\nabla \sigma^\epsilon(x)\|^2 dx dt \right\}^{\frac{1}{4}} \\ &\leq C \left\{ \epsilon_0 \int_0^T \int_{\mathbb{T}^2} \|\nabla \sigma(x)\|^2 dx dt \right\}^{\frac{1}{4}} = C_2. \end{aligned}$$

The same reasoning gives

$$\begin{aligned}
 & E \left\{ \frac{1}{|B_r(0)|} \int_{B_r(0)} \int_{\mathbb{T}^2} \sup_{t \in [0, T]} |I_3(t)| dx dz \right\} \\
 & \leq \frac{C}{|B_r(0)|} \int_0^T \mathbb{E} \int_{B_r(0)} \int_{\mathbb{T}^2} M_R \|\nabla \sigma^\epsilon(X_t^{\epsilon, h^\epsilon}(x))\| + M_R \|\nabla \sigma^\epsilon(X_t^{\epsilon, h^\epsilon}(x+z))\| dx dz dt \\
 & \leq C \epsilon_0 \left( \int_0^T \int_{\mathbb{T}^2} M_R \|\nabla \sigma(x)\|^2 dx dt \right)^{\frac{1}{2}} \leq C \left( \int_0^T \int_{\mathbb{T}^2} \|\nabla \sigma(x)\|^2 dx dt \right)^{\frac{1}{2}} = C3.
 \end{aligned}$$

Finally for  $I_1(t)$ , we have

$$\begin{aligned}
 & \frac{1}{|B_r(0)|} E \int_{B_r(0)} \int_{\mathbb{T}^2} \sup_{t \in [0, T]} |I_1(t)| dx dz \\
 & \leq \frac{C}{|B_r(0)|} \int_0^T \mathbb{E} \int_{B_r(0)} \int_{\mathbb{T}^2} M_R |\nabla u^{h^\epsilon}(X_t^{h^\epsilon}(x), t)| + M_R |\nabla u^{h^\epsilon}(X_t^{h^\epsilon}(x+z), t)| dx dz dt \\
 & \leq C \mathbb{E} \int_0^T \left( \int_{\mathbb{T}^2} |\nabla u^{h^\epsilon}(X_t^{h^\epsilon}(x), t)|^2 dx \right)^{\frac{1}{2}} dt \\
 & \quad + \frac{C}{|B_r(0)|} \mathbb{E} \int_{B_r(0)} \int_0^T \left( \int_{\mathbb{T}^2} |\nabla u^{h^\epsilon}(X_t^{h^\epsilon}(x+z), t)|^2 dx \right)^{\frac{1}{2}} dt dz \\
 & = J_1 + J_2.
 \end{aligned}$$

By the definition of  $u^{h^\epsilon}$ , we write

$$\begin{aligned}
 & \left( \int_{\mathbb{T}^2} |\nabla u^{h^\epsilon}(X_t^{h^\epsilon}(x+z), t)|^2 dx \right)^{\frac{1}{2}} \\
 & \leq \sqrt{2} \left( \int_{\mathbb{T}^2} |\nabla u^\epsilon(X_t^\epsilon(x+z), t)|^2 dx \right)^{\frac{1}{2}} + \sqrt{2} \left( \int_{\mathbb{T}^2} |\nabla \langle \sigma^\epsilon(X_t^{h^\epsilon}(x+z)), \dot{h}_t^\epsilon \rangle_{l^2}|^2 dx \right)^{\frac{1}{2}}
 \end{aligned}$$

$$\leq \sqrt{2} \left( \int_{\mathbb{T}^2} |\nabla u^\epsilon(X_t^{h^\epsilon}(x+z), t)|^2 dx \right)^{\frac{1}{2}} + \sqrt{2}\epsilon_0 \|\dot{h}_t^\epsilon\|_{l^2} \left( \int_{\mathbb{T}^2} \|\nabla \sigma(X_t^{h^\epsilon}(x+z))\|^2 dx \right)^{\frac{1}{2}}.$$

Recalling that  $h^\epsilon \in A_N$  and applying the [Lemmas 2.6, 4.1](#) as well as the incompressibility of the flow, we deduce

$$\begin{aligned} J_2 &\leq \frac{C}{|B_r(0)|} \mathbb{E} \int_{B_r(0)} \int_0^T \left( \int_{\mathbb{T}^2} |\nabla u^\epsilon(X_t^{h^\epsilon}(x+z), t)|^2 dx \right)^{\frac{1}{2}} dt dz \\ &\quad + \frac{C}{|B_r(0)|} \mathbb{E} \int_{B_r(0)} \int_0^T \|\dot{h}_t^\epsilon\|_{l^2} \left( \int_{\mathbb{T}^2} \|\nabla \sigma(X_t^{h^\epsilon}(x+z))\|^2 dx \right)^{\frac{1}{2}} dt dz \\ &\leq C \int_0^T \left( \int_{\mathbb{T}^2} |\nabla u^\epsilon(x, t)|^2 dx \right)^{\frac{1}{2}} dt \\ &\quad + \frac{C}{|B_r(0)|} \mathbb{E} \left( \int_0^T \|\dot{h}_t^\epsilon\|_{l^2}^2 dt \right)^{\frac{1}{2}} \left( \mathbb{E} \int_{B_r(0)} \int_0^T \int_{\mathbb{T}^2} \|\nabla \sigma(X_t^{h^\epsilon}(x+z))\|^2 dx dt dz \right)^{\frac{1}{2}} \\ &\leq C \left( \int_0^T \int_{\mathbb{T}^2} |\nabla u(x, t)|^2 dx dt \right)^{\frac{1}{2}} + CN \left( \int_0^T \int_{\mathbb{T}^2} \|\nabla \sigma(x)\|^2 dx dt \right)^{\frac{1}{2}} \leq C. \end{aligned}$$

Likewise, we derive  $J_1 \leq C$ . As a consequence of all estimates, we obtain

$$\begin{aligned} &\mathbb{E} \left\{ \frac{1}{|B_r(0)|} \int_{B_r(0)} \int_{\mathbb{T}^2} \log \left( \frac{\sup_{t \in [0, T]} |X_t^{\epsilon, h^\epsilon}(x) - X_t^{\epsilon, h^\epsilon}(x+z)|^2}{\delta^2} + 1 \right) dx dz \right\} \\ &\leq C \log \left( \frac{|r|^2}{\delta^2} + 1 \right) + C \end{aligned} \quad (4.13)$$

where the constant  $C$  is independent of  $h^\epsilon$  and  $\epsilon$ .

Taking into account the concavity of the logarithmic function and knowing that  $X_t^{\epsilon, h^\epsilon}$  is bounded, i.e. there exists  $R' > 0$  such that  $|X_t^{\epsilon, h^\epsilon}| \leq \frac{\sqrt{R'}}{2}$ , we derive

$$\log \left( \frac{\sup_{t \in [0, T]} |Z_t^\epsilon|^2}{\delta^2} + 1 \right) R' \geq \log \left( \frac{R'}{\delta^2} + 1 \right) \sup_{t \in [0, T]} |Z_t^\epsilon|^2. \quad (4.14)$$

Thus setting  $\delta^2 = |r|^2$ , the inequalities (4.13) and (4.14) yield

$$\begin{aligned} \mathbb{E} \left\{ \frac{1}{|B_r(0)|} \int_{B_r(0)} \int_{\mathbb{T}^2} \sup_{t \in [0, T]} |X_t^{\epsilon, h^\epsilon}(x) - X_t^{\epsilon, h^\epsilon}(x+z)|^2 dx dz \right\} \\ \leq \frac{R'}{\log \left( \frac{R'}{|r|^2} + 1 \right)} C(\log 2 + 1). \end{aligned} \quad (4.15)$$

Therefore, if we define

$$g(r) := \frac{R'}{\log \left( \frac{R'}{|r|^2} + 1 \right)} C(\log 2 + 1)$$

the claimed results follow.  $\square$

**Lemma 4.10.** *Let  $X_t^{\epsilon, h^\epsilon}$ ,  $\epsilon > 0$ , be the solutions to (4.12). Then there exists a continuous function  $f$  independent of  $\epsilon$  such that*

$$\lim_{\tau \rightarrow 0} f(\tau) = 0$$

and for any  $r, t > 0$  satisfying  $0 \leq t \leq t+r \leq T$ , we have

$$\mathbb{E} \int_{\mathbb{T}^2} \sup_{t \in [0, T-r]} |X_{t+r}^{\epsilon, h^\epsilon}(x) - X_t^{\epsilon, h^\epsilon}(x)|^2 dx \leq f(|r|).$$

**Proof.** Let us define  $Z_t(x) := X_{t+r}^{\epsilon, h^\epsilon}(x) - X_t^{\epsilon, h^\epsilon}(x)$ . As in the proof of the previous lemma we have

$$\begin{aligned} \mathbb{E} \int_{\mathbb{T}^2} |X_{t+r}^{\epsilon, h^\epsilon}(x) - X_t^{\epsilon, h^\epsilon}(x)|^2 \\ = \mathbb{E} \int_{\mathbb{T}^2} \int_t^{t+r} \langle Z_s, u^{h^\epsilon}(X_s^{\epsilon, h^\epsilon}, s) \rangle ds dx + \mathbb{E} \int_{\mathbb{T}^2} \int_t^{t+r} \|\sigma(X_s^{\epsilon, h^\epsilon})\|_{HS}^2 dx dt \\ + \mathbb{E} \int_{\mathbb{T}^2} \int_t^{t+r} \langle Z_s, \sigma(X_s^{\epsilon, h^\epsilon}) dW_s \rangle dx. \end{aligned}$$

Now, we estimate each term in the right hand side. Since the flow lives in the torus, there exists a constant  $R'$ , such that  $|Z_t(x)| \leq R'$  for all  $t \in [0, T]$  and a.e.- $\omega$ ,  $x$ . Recalling that  $h^\epsilon \in A_N$ , the incompressibility of the flow and Lemmas 2.5, 4.1 yield

$$\begin{aligned}
 & \mathbb{E} \int_{\mathbb{T}^2} \sup_{t \in [0, T-r]} \int_t^{t+r} \langle Z_s, u^{h^\epsilon}(X_s^{h^\epsilon}, s) \rangle ds dx \leq R' \mathbb{E} \int_{\mathbb{T}^2} \int_{t_1}^{t_1+r} |u^{h^\epsilon}(X_s^{h^\epsilon}, s)| ds dx \\
 & \leq R' \mathbb{E} \int_{\mathbb{T}^2} \int_{t_1}^{t_1+r} |u^\epsilon(X_s^{h^\epsilon}, s)| ds dx + R' \mathbb{E} \int_{\mathbb{T}^2} \int_{t_1}^{t_1+r} |\langle \sigma^\epsilon(X_s^{h^\epsilon}), \dot{h}_t^\epsilon \rangle_{l^2}| ds dx \\
 & = R' \int_{t_1}^{t_1+r} \int_{\mathbb{T}^2} |u^\epsilon(x, s)| dx ds + \epsilon_0 R' C \left( N \int_{t_1}^{t_1+r} \int_{\mathbb{T}^2} \|\sigma(x)\|_{HS}^2 dx ds \right)^{\frac{1}{2}} \\
 & \leq C |r|^{\frac{1}{2}} \left( \int_{\mathbb{T}^2} \int_{t_1}^{t_1+r} |u(x, s)|^2 ds dx \right)^{\frac{1}{2}} + C |r|^{\frac{1}{2}} \leq C |r|^{\frac{1}{2}}.
 \end{aligned} \tag{4.16}$$

For  $I_2(t)$  we get

$$\begin{aligned}
 & \mathbb{E} \int_{\mathbb{T}^2} \sup_{t \in [0, T-r]} \int_t^{t+r} \|\sigma(X_s^{h^\epsilon})\|_{HS}^2 ds dx \leq \mathbb{E} \int_{\mathbb{T}^2} \int_{t_2}^{t_2+r} \|\sigma(X_s^{h^\epsilon})\|_{HS}^2 ds dx \\
 & = \int_{\mathbb{T}^2} \int_{t_2}^{t_2+r} \|\sigma(x)\|_{HS}^2 ds dx \leq |r| \int_{\mathbb{T}^2} \|\sigma(x)\|_{HS}^2 dx \leq C |r|.
 \end{aligned} \tag{4.17}$$

Finally, using the Burkholder–Davis–Gundy inequality, we deduce the following estimate for the stochastic integral

$$\begin{aligned}
 & \mathbb{E} \int_{\mathbb{T}^2} \sup_{t \in [0, T-r]} \int_t^{t+r} \langle Z_s, \sigma(X_s^{h^\epsilon}) dW_s \rangle dx \leq \int_{\mathbb{T}^2} E \sup_{t \in [0, T-r]} \left| \int_t^{t+r} \langle Z_s, \sigma(X_s^{h^\epsilon}) dW_s \rangle \right| dx \\
 & \leq C \left\{ \int_{\mathbb{T}^2} E \sup_{t \in [0, T-r]} \int_t^{t+r} |Z_s|^2 \|\sigma(X_s^{h^\epsilon})\|_{HS}^2 ds dx \right\}^{\frac{1}{2}} \\
 & = C \left\{ \int_{\mathbb{T}^2} \mathbb{E} \int_{t_3}^{t_3+r} |Z_s|^2 \|\sigma(X_s^{h^\epsilon})\|_{HS}^2 ds dx \right\}^{\frac{1}{2}} \\
 & \leq C R' \left\{ \int_{\mathbb{T}^2} \mathbb{E} \int_{t_3}^{t_3+r} \|\sigma(X_s^{h^\epsilon})\|_{HS}^2 ds dx \right\}^{\frac{1}{2}} = C |r|^{\frac{1}{2}}.
 \end{aligned} \tag{4.18}$$

Joining the three estimates (4.16), (4.17) and (4.18) with  $|r| \leq 1$ , we obtain

$$\mathbb{E} \int_{\mathbb{T}^2} \sup_{t \in [0, T-r]} |X_{t+r}^{h^\epsilon}(x) - X_t^{h^\epsilon}(x)|^2 dx \leq C|r|^{\frac{1}{2}},$$

where the constant  $C$  is independent on  $h^\epsilon$  and  $\epsilon$ . Thus if we define  $f(\tau) := C|\tau|^{\frac{1}{2}}$ , the claimed result follows.  $\square$

Here we are interested in the identification of the compact sets of the space  $C([0, T], L^2(\mathbb{T}^2))$  endowed with the topology defined by the norm

$$\|f\|_{C([0, T], L^2(\mathbb{T}^2))} = \sup_{t \in [0, T]} \left( \int_{\mathbb{T}^2} |f(t, x)|^2 dx \right)^{\frac{1}{2}}.$$

We will state a lemma that is an adaptation of the Arzelà–Ascoli’s Theorem, for the space of continuous functions defined on the interval  $[0, T]$  with values in  $L^p$ ,  $p > 1$ . The proof of this result can be found in [22].

**Lemma 4.11.** *Let us consider the space  $C([0, T], L^p(S))$ ,  $p \geq 1$ , where  $S$  is a bounded subset of  $\mathbb{R}^d$ . A family of functions  $\{f^\epsilon(t, \cdot)\} \subset C([0, T], L^p(S))$  is a relatively compact set in  $C([0, T], L^p(S))$ , if the following three conditions are satisfied:*

- (i) *for every  $t \in [0, T]$ , we have  $\int_S |f^\epsilon(t, x+z) - f^\epsilon(t, x)|^p dx \rightarrow 0$  uniformly in  $\epsilon$ , as  $|z| \rightarrow 0$ ;*
- (ii) *for every  $t \in [0, T]$ , the set  $\{f^\epsilon(t, \cdot)\}$  in a bounded subset of  $L^p(S)$ ;*
- (iii) *the family  $\{f^\epsilon(t, \cdot), \epsilon > 0\}$  is equicontinuous.*

Under the assumptions of Lemma 4.11, we establish a criterion to analyze the tightness of a family of probability measures  $\{P^\epsilon, \epsilon > 0\}$  on the space  $C([0, T], L^p(S))$ ,  $p \geq 1$ . Our motivation arises from Section 4 of the paper [10].

**Proposition 4.12.** *Assume that  $S$  is a bounded open subset of  $\mathbb{R}^d$ . A family of probability measures  $\{P^\epsilon, \epsilon > 0\}$  on the space  $C([0, T], L^p(S))$ ,  $p \geq 1$ , is tight if for all  $\rho > 0$  the following properties hold*

- (1)  $\lim_{r \searrow 0} \sup_{\epsilon > 0} P^\epsilon \left\{ f \in C([0, T], L^p(S)) : \sup_{0 \leq t \leq T} \frac{1}{|B_r(0)|} \int_{B_r(0)} \int_S |f(t, x) - f(t, x+z)|^p dx dz \geq \rho \right\} = 0,$
- (2)  $\lim_{M \nearrow \infty} \sup_{\epsilon > 0} P^\epsilon \left\{ f \in C([0, T], L^p(S)) : \sup_{0 \leq t \leq T} \|f(t, \cdot)\|_{L^p(S)}^p \geq M \right\} = 0,$
- (3)  $\lim_{\delta \searrow 0} \sup_{\epsilon > 0} P^\epsilon \left\{ f \in C([0, T], L^p(S)) : \sup_{\substack{|s-t| \leq \delta \\ 0 \leq t \leq T}} \|f(s, \cdot) - f(t, \cdot)\|_{L^p(S)}^p \geq \rho \right\} = 0.$



**Proof.** Assuming that (1), (2) and (3) hold. Then for every  $\eta > 0$ , there exists  $\delta_0 > 0$ ,  $r_0 > 0$  and  $M_0 > 0$  such that  $P^\epsilon(D) < \frac{\eta}{3}$ ,  $P^\epsilon(E) < \frac{\eta}{3}$  and  $P^\epsilon(F) < \frac{\eta}{3}$ , for every  $\epsilon > 0$ , where

$$D := \left\{ f \in C([0, T], L^p(S)) : \sup_{0 \leq t \leq T} \frac{1}{|B_{r_0}(0)|} \int_{B_{r_0}(0)} \int_S |f(t, x) - f(t, x+z)|^p dx dz \geq \rho \right\},$$

$$E := \left\{ f \in C([0, T], L^p(S)) : \sup_{0 \leq t \leq T} \|f(t, \cdot)\|_{L^p(S)}^p \geq M_0 \right\},$$

$$F := \left\{ f \in C([0, T], L^p(S)) : \sup_{\substack{|s-t| \leq \delta \leq \delta_0 \\ 0 \leq t \leq T}} \|f(s, \cdot) - f(t, \cdot)\|_{L^p(S)}^p \geq \rho \right\}.$$

Let us notice that there exists a dimensional constant  $\alpha_d$  such that  $D^c \subset \tilde{D}$  with

$$\tilde{D} = \left\{ f \in C([0, T], L^p(S)) : \sup_{\substack{|z| \leq \frac{r_0}{2} \\ 0 \leq t \leq T}} \int_S |f(t, x) - f(t, x+z)|^p dx \leq \frac{2\rho}{\alpha_d} \right\}$$

$$= \left\{ f \in C([0, T], L^p(S)) : \forall t \in [0, T] \lim_{z \rightarrow 0} \int_S |f(t, x) - f(t, x+z)|^p dx = 0 \right\}$$

(see Section 4 of [10]). For every  $t \in [0, T]$  we have

$$\lim_{z \rightarrow 0} \int_S |f(t, x) - f(t, x+z)|^p dx = 0, \quad \text{uniformly for } f \in \tilde{D}.$$

Thus if we set  $K := \tilde{D} \cap E^c \cap F^c \supset (D \cup E \cup F)^c$ , the result of Lemma 4.11 gives the compactness of  $K$  in the space  $C([0, T], L^p(S))$ . Moreover we have  $P^\epsilon(K) \geq P^\epsilon(D \cup E \cup F)^c = 1 - P^\epsilon(D \cup E \cup F) \geq 1 - \eta$ .  $\square$

Now, we will apply the compactness criterion in Proposition 4.12 to show that the laws of  $X_t^{\epsilon, h^\epsilon}$  defined over the Borel  $\sigma$ -field of  $C([0, T], L^2(\mathbb{T}^2))$  are tight.

**Lemma 4.13.** *The family  $\{P^\epsilon, \epsilon > 0\}$  of the laws of the processes  $X_t^{\epsilon, h^\epsilon}$  on the space  $C([0, T], L^2(\mathbb{T}^2))$  is tight.*

**Proof.** Applying the Chebyshev's inequality and Lemma 4.9, we deduce

$$\limsup_{r \searrow 0} \sup_{\epsilon > 0} P^\epsilon \left\{ f \in C([0, T], L^2(\mathbb{T}^2)) : \sup_{0 \leq t \leq T} \frac{1}{|B_r(0)|} \int_{B_r(0)} \int_{\mathbb{T}^2} |f(t, x) - f(t, x+z)|^p dx dz \geq \rho \right\}$$

$$\begin{aligned}
&= \limsup_{r \searrow 0} \sup_{\epsilon > 0} P \left\{ \sup_{0 \leq t \leq T} \frac{1}{|B_r(0)|} \int_{B_r(0)} \int_{\mathbb{T}^2} \left| X_t^{\epsilon, h^\epsilon}(x) - X_t^{\epsilon, h^\epsilon}(x+z) \right|^2 dx dz \geq \rho \right\} \\
&\leq \limsup_{r \searrow 0} \sup_{\epsilon > 0} \frac{1}{\rho} \mathbb{E} \left( \sup_{0 \leq t \leq T} \frac{1}{|B_r(0)|} \int_{B_r(0)} \int_{\mathbb{T}^2} \left| X_t^{\epsilon, h^\epsilon}(x) - X_t^{\epsilon, h^\epsilon}(x+z) \right|^2 dx dz \right) \\
&\leq \lim_{r \searrow 0} \frac{1}{\rho} g(r) = 0.
\end{aligned}$$

Again, the Chebyshev's inequality and the incompressibility of the flow yield

$$\begin{aligned}
&\lim_{M \nearrow \infty} \sup_{\epsilon > 0} P^\epsilon \left\{ f \in C([0, T], L^2(\mathbb{T}^2)) : \sup_{t \in [0, T]} \|f(t, \cdot)\|_{L^2(S)}^2 \geq M \right\} \\
&\lim_{M \nearrow \infty} \sup_{\epsilon > 0} P \left\{ \sup_{t \in [0, T]} \int_{\mathbb{T}^2} \left| X_t^{\epsilon, h^\epsilon}(x) \right|^2 dx \geq M \right\} \leq \lim_{M \nearrow \infty} \frac{1}{M} \mathbb{E} \left( \sup_{t \in [0, T]} \int_{\mathbb{T}^2} |x|^2 dx \right) = 0.
\end{aligned}$$

Using the same reasoning and [Lemma 4.10](#), we derive

$$\begin{aligned}
&\limsup_{\delta \searrow 0} \sup_{\epsilon > 0} P^\epsilon \left\{ f \in C([0, T], L^2(S)) : \sup_{|s-t| \leq \delta} \|f(s, \cdot) - f(t, \cdot)\|_{L^2(S)}^2 \geq \rho \right\} \\
&\frac{1}{\rho} \limsup_{\delta \searrow 0} \sup_{\epsilon > 0} \mathbb{E} \left( \int_{\mathbb{T}^2} \left| X_s^{\epsilon, h^\epsilon}(x) - X_t^{\epsilon, h^\epsilon}(x) \right|^2 dx \right) \leq \frac{1}{\rho} \lim_{\delta \searrow 0} f(|\delta|) = 0. \quad \square
\end{aligned}$$

In the next lemma, we consider  $B_N$ ,  $N \in \mathbb{N}$ , endowed with the weak topology of  $\mathbb{H}$ . From [\[21\]](#),  $B_N$  is a compact Polish space.

**Lemma 4.14.** *Let us fix  $N \in \mathbb{N}$ . Assume that  $\{h^\epsilon, \epsilon > 0\} \subset A_N$  and consider the corresponding solutions  $X_t^{\epsilon, h^\epsilon}$ ,  $\epsilon > 0$ , of the equation (4.12). Then the following propositions hold:*

- (i) *The laws of  $\{(h^\epsilon, X_t^{\epsilon, h^\epsilon}, W)\}$  are tight in  $B_N \times C([0, T], L^2(\mathbb{T}^2)) \times C([0, T], l_*^2)$ .*
- (ii) *There exist a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ , a sequence of stochastic processes  $\{(\tilde{h}^\epsilon, \tilde{X}_t^{\epsilon, \tilde{h}^\epsilon}, \tilde{W}^\epsilon)\}$  and a stochastic process  $\{(h, X_t^h, \tilde{W})\}$  defined in this probability space taking values in  $B_N \times C([0, T], L^2(\mathbb{T}^2)) \times C([0, T], l_*^2)$  such that:*
  - (a)  *$\{(\tilde{h}^\epsilon, \tilde{X}_t^{\epsilon, \tilde{h}^\epsilon}, \tilde{W}^\epsilon)\}$  has the same law as  $\{(h^\epsilon, X_t^{\epsilon, h^\epsilon}, W)\}$  for every  $\epsilon$ ;*
  - (b)  *$\{(\tilde{h}^\epsilon, \tilde{X}_t^{\epsilon, \tilde{h}^\epsilon}, \tilde{W}^\epsilon)\} \rightarrow \{(h, X_t^h, \tilde{W})\}$  in  $B_N \times C([0, T], L^2(\mathbb{T}^2)) \times C([0, T], l_*^2)$  a.e.- $\tilde{P}$ , when  $\epsilon \rightarrow 0$ ;*
  - (c) *for a.e.- $\tilde{P}$ ,  $X^h$  is the solution of the differential equation (4.6).*

**Proof.** To show the property (i), we notice that by hypothesis, for each  $\epsilon > 0$ ,  $h^\epsilon$  is a  $B_N$ -valued random variable, then its law has support in the compact  $B_N$ , and as a consequence the family of the laws of  $h^\epsilon$ ,  $\epsilon > 0$ , is tight. On the other hand, the tightness of the laws of  $X_t^{\epsilon, h^\epsilon}$  follows from [Lemma 4.13](#). Let us verify (ii). The existence of  $\{(\tilde{h}^\epsilon, \tilde{X}_t^{\epsilon, \tilde{h}^\epsilon}, \tilde{W}^\epsilon)\}$  and  $\{(h, X_t^h, \tilde{W})\}$  as well as propositions (a) and (b) are a direct consequence of (i) and of the Skorohod's representation

**Theorem.** To see that (c) holds, we first notice that for each  $\epsilon$ , the stochastic process  $\tilde{W}^\epsilon$  has the same law as  $W$ , therefore  $\tilde{W}^\epsilon$  is a Brownian motion defined on the probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ . Let  $\Phi^\epsilon$  be the functional defined in the proof of [Lemma 4.8](#). The property (a) and the fact that

$$X^{\epsilon, h^\epsilon} = \Phi^\epsilon(W + h^\epsilon) \quad \text{a.e.-}P$$

yield

$$\tilde{X}^{\epsilon, \tilde{h}^\epsilon} = \Phi^\epsilon(\tilde{W}^\epsilon + \tilde{h}^\epsilon) \quad \text{a.e.-}\tilde{P}.$$

Therefore by the uniqueness argument, we conclude that  $\tilde{X}^{\epsilon, \tilde{h}^\epsilon}$  satisfies the following differential equation

$$\begin{aligned} \tilde{X}_t^{\epsilon, \tilde{h}^\epsilon}(x) = x + \sqrt{\epsilon} \int_0^t \sigma(\tilde{X}_s^{\epsilon, \tilde{h}^\epsilon}(x)) d\tilde{W}_s^\epsilon + \int_0^t \langle \sigma(\tilde{X}_s^{\epsilon, \tilde{h}^\epsilon}(x)), \dot{\tilde{h}}_s^\epsilon \rangle_{l^2} ds \\ + \int_0^t u^\epsilon(\tilde{X}_s^{\epsilon, \tilde{h}^\epsilon}(x), s) ds. \end{aligned}$$

By taking into account the result of [Lemma 4.2](#), the proof that for a.e.- $\tilde{P}$ ,  $X^h$  satisfies the equation (4.6) for  $h$  is similar to the second part of the proof of the [Theorem 3.1](#) then, by this reason, will be omitted.  $\square$

#### 4.4. Schilder's Theorem

Now, we have the necessary tools to establish the main result of this section. We follow the methods developed in [\[1\]](#) in order to establish the Laplace–Varadhan Principle, which is equivalent to the large deviation principle, according to [\[13\]](#).

**Theorem 4.15. Laplace–Varadhan Principle:** Let  $\{X^\epsilon\}$  be the solutions of the equation (2.9). Then  $\{X^\epsilon\}$  satisfies the Laplace's Principle in  $C([0, T], L^2(\mathbb{T}^2))$  with the rate function  $I(f)$  defined in (4.8).

**Proof.** Let  $g$  be a bounded continuous function defined on  $C([0, T], L^2(\mathbb{T}^2))$ .

*Lower bound of the Laplace–Varadhan Principle:* By the [Lemma 4.8](#), we have that

$$-\epsilon \log \mathbb{E} \left( \exp \left[ -\frac{g(X^\epsilon)}{\epsilon} \right] \right) = \inf_{h \in A} \mathbb{E} \left( g(X^{\epsilon, h}) + \frac{1}{2} \|h\|_{\mathbb{H}}^2 \right). \quad (4.19)$$

Let us fix  $\delta > 0$ . For every  $\epsilon > 0$ , there exists  $h^\epsilon \in A$  such that

$$\inf_{h \in A} \mathbb{E} \left( g(X^{\epsilon, h}) + \frac{1}{2} \|h\|_{\mathbb{H}}^2 \right) \geq \mathbb{E} \left( g(X^{\epsilon, h^\epsilon}) + \frac{1}{2} \|h^\epsilon\|_{\mathbb{H}}^2 \right) - \delta.$$

Since  $g$  is bounded, we have

$$\frac{1}{2} \sup_{\epsilon > 0} \mathbb{E}(\|h^\epsilon\|_{\mathbb{H}}^2) \leq 2\|g\|_\infty + \delta.$$

Let us now define

$$\tau_N^\epsilon := \left\{ t \in [0, T] : \int_0^t \|\dot{h}^\epsilon(s)\|_{l^2}^2 ds \geq N \right\}$$

and

$$h_N^\epsilon(t) := h^\epsilon(t \wedge \tau_N^\epsilon).$$

We have that  $h_N^\epsilon(t) \in A_N$  and

$$P(\omega : \|h_N^\epsilon(\omega) - h^\epsilon(\omega)\|_{\mathbb{H}} \neq 0) = P(\omega : \tau_N^\epsilon(\omega) < T) \leq \frac{2(2\|g\|_\infty + \delta)}{N}.$$

Thus

$$\mathbb{E}\left(g(X^{\epsilon, h^\epsilon}) + \frac{1}{2}\|h^\epsilon\|_{\mathbb{H}}^2\right) - \delta \geq \mathbb{E}\left(g(X^{\epsilon, h_N^\epsilon}) + \frac{1}{2}\|h_N^\epsilon\|_{\mathbb{H}}^2\right) - \frac{2\|g\|_\infty(2\|g\|_\infty + \delta)}{N} - \delta.$$

By the [Lemma 4.14](#) we obtain

$$\begin{aligned} & \liminf_{\epsilon \rightarrow 0} \mathbb{E}\left(g(X^{\epsilon, h_N^\epsilon}) + \frac{1}{2}\|h_N^\epsilon\|_{\mathbb{H}}^2\right) \\ &= \liminf_{\epsilon \rightarrow 0} \mathbb{E}^{\tilde{P}}\left(g(\tilde{X}^{\epsilon, \tilde{h}_N^\epsilon}) + \frac{1}{2}\|\tilde{h}_N^\epsilon\|_{\mathbb{H}}^2\right) \\ &\geq \mathbb{E}^{\tilde{P}}\left(g(S(h)) + \frac{1}{2}\|h\|_{\mathbb{H}}^2\right) \\ &\geq \inf_{\{(f, h) \in C([0, T], L^2(\mathbb{T}^2)) \times \mathbb{H} : f = S(h)\}} \left(g(f) + \frac{1}{2}\|h\|_{\mathbb{H}}^2\right) \\ &\geq \inf_{f \in C([0, T], L^2(\mathbb{T}^2))} \{g(f) + I(f)\}. \end{aligned}$$

In particular we used the fact that for a.e.  $\omega \in \tilde{\Omega}$  fixed,  $X^h(\omega) = S(h(\omega))$ . Therefore we have

$$\begin{aligned} & \liminf_{\epsilon \rightarrow 0} -\epsilon \log \mathbb{E}\left(\exp\left[-\frac{g(X^\epsilon)}{\epsilon}\right]\right) \\ &\geq \inf_{f \in C([0, T], L^2(\mathbb{T}^2))} \{g(f) + I(f)\} - \frac{2\|g\|_\infty(2\|g\|_\infty + \delta)}{N} - \delta \end{aligned}$$

Taking  $N \rightarrow \infty$  and  $\delta \rightarrow 0$ , we obtain the lower bound of the Laplace's Principle.

*Upper bound of the Laplace–Varadhan Principle:* Let us set  $\delta > 0$ . Due to the boundedness of  $g$ , we can find  $f_0 \in C([0, T], L^2(\mathbb{T}^2))$  such that

$$g(f_0) + I(f_0) \leq \inf_{f \in C([0, T], L^2(\mathbb{T}^2))} \{g(f) + I(f)\} + \delta.$$

Thus, we pick  $h_0 \in \mathbb{H}$  such that  $\frac{1}{2}\|h_0\|_{\mathbb{H}}^2 = I(f_0)$  and  $f_0 = S(h_0)$ . By (4.19) and Lemma 4.14 we have that

$$\begin{aligned} \limsup_{\epsilon \rightarrow 0} -\epsilon \log \mathbb{E} \left( \exp \left[ -\frac{g(X^\epsilon)}{\epsilon} \right] \right) \\ &= \limsup_{\epsilon \rightarrow 0} \inf_{h \in A} \mathbb{E} \left( g(X^{\epsilon, h}) + \frac{1}{2}\|h\|_{\mathbb{H}}^2 \right) \\ &\leq \limsup_{\epsilon \rightarrow 0} \mathbb{E} \left( g(X^{\epsilon, h_0}) + \frac{1}{2}\|h_0\|_{\mathbb{H}}^2 \right) \\ &= g(S(h_0)) + \frac{1}{2}\|h_0\|_{\mathbb{H}}^2 \leq g(f_0) + I(f) \\ &\leq \inf_{f \in C([0, T], L^2(\mathbb{T}^2))} \{g(f) + I(f)\} + \delta. \end{aligned}$$

Taking  $\delta \rightarrow 0$ , we obtain the upper bound of the Laplace Principle.  $\square$

**Theorem 4.16. Principle of Large Deviations:** Let  $\{X^\epsilon\}$  be the solutions of the equation (2.9). Then  $\{X^\epsilon\}$  satisfies the principle of large deviations in  $C([0, T], L^2(\mathbb{T}^2))$  with the rate function  $I(f)$ .

**Proof.** The proof follows from the equivalence between the Laplace–Varadhan principle and the principle of large deviations.  $\square$

## 5. Appendix

The aim of this section is to recall well known results on maximal functions that are of main importance to prove Lemma 2.8. These results can be found for instance in [10], [15] and [25].

Let  $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ , its local maximal function is defined by

$$M_R f(x) = \sup_{0 < r < R} \frac{1}{|B_r|} \int_{B_r} f(x+y) dy, \quad \forall R > 0.$$

(D1) Let  $f \in L^1_{\text{loc}}(\mathbb{R}^d)$  with  $\nabla f \in L^1_{\text{loc}}(\mathbb{R}^d)$ . Then there exists  $C_d > 0$  and a negligible measurable set  $A$  such that, for any  $x, y \in \mathbb{R}^d \setminus A$  with  $|x - y| \leq R$ , the following inequality holds

$$|f(x) - f(y)| \leq C_d |x - y| (M_R |\nabla f|(x) + M_R |\nabla f|(y))$$

(D2) Let  $f \in L^p_{\text{loc}}(\mathbb{R}^d)$  with  $p > 1$ . Then there exists a constant  $C_{d,p} > 0$  such that for any  $N, R > 0$ , we have

$$\left( \int_{B_N} (M_R |f|)^p \right)^{\frac{1}{p}} \leq C_{d,p} \left( \int_{B_{N+R}} |f|^p \right)^{\frac{1}{p}}.$$

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