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New characterizations of Morrey spaces and their preduals with applications to fractional Laplace equations ☆

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Abstract

In this article, the authors characterize the Morrey spaces as well as their preduals via quadratic functions related to the Taylor remainder of the kernel of the Riesz potential. As applications, the authors obtain some strong capacitary inequalities, which are then used to study the regularity of the duality/weak solution to the fractional Laplace equation with measure data.
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Contents

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1. Introduction and main results

Consider the n-dimensional Euclidean space $\mathbb{R}^n$ equipped with the Euclidean distance and the n-dimensional Lebesgue measure. It is known that the *Morrey space* $L^{p,\lambda}(\mathbb{R}^n)$ with $(p,\lambda) \in (0,\infty) \times (-\infty, n]$ was introduced by Morrey [21] and then used to study the regularity of solutions to some quasi-linear elliptic partial differential equations, where $L^{p,\lambda}(\mathbb{R}^n)$ comprises all Lebesgue measurable functions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} := \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \left[r^{\lambda-n} \int\limits_{B(x,r)} |f(z)|^p \, dz \right]^{1/p} < \infty,$$

where $B(x,r) := \{y \in \mathbb{R}^n : |y-x| < r\}$ is the Euclidean ball with center x and radius r. In particular, when $\lambda = n$, the space $L^{p,n}(\mathbb{R}^n)$ is just the *Lebesgue space* $L^p(\mathbb{R}^n)$, that is,

$$\|f\|_{L^p(\mathbb{R}^n)} := \left[\int\limits_{\mathbb{R}^n} |f(x)| \, dx \right]^{1/p}.$$

The predual of Morrey spaces was discussed in [3]. For any given $(p,\lambda) \in (1,\infty) \times (0,n)$, the *space* $H^{p,\lambda}(\mathbb{R}^n)$ consists of all Lebesgue measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{H^{p,\lambda}(\mathbb{R}^n)} := \inf_{\omega} \left\{ \int\limits_{\mathbb{R}^n} |f(x)|^p [\omega(x)]^{1-p} \, dx \right\}^{1/p} < \infty, \tag{1.1}$$

where the infimum is taken over all non-negative functions ω on $\mathbb{R}^n$ satisfying that

$$\|\omega\|_{L^1(\mathcal{H}^{(\infty)}_{n-\lambda})} := \int\limits_0^\infty \mathcal{H}^{(\infty)}_{n-\lambda} \left(\{x \in \mathbb{R}^n : |\omega(x)| > t\} \right) dt \le 1. \tag{1.2}$$

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Here and hereafter, for any given $\alpha \in (0, n)$, the *symbol* $\mathcal{H}_\alpha^{(\infty)}(E)$ denotes the α-th order Hausdorff capacity of a subset $E \subset \mathbb{R}^n$, which is defined by setting

$$\mathcal{H}_\alpha^{(\infty)}(E) := \inf \left\{ \sum_j r_j^\alpha : \ E \subset \bigcup_j B(x_j, r_j) \text{ with } x_j \in \mathbb{R}^n \text{ and } r_j \in (0, \infty) \right\}.$$

Based on [5, Theorem 7], the norm $\|\cdot\|_{H^{p,\lambda}(\mathbb{R}^n)}$ can be defined equivalently in the way that the infimum in (1.1) is taken over all non-negative functions $\omega \in A_1(\mathbb{R}^n)$ satisfying (1.2), where $A_1(\mathbb{R}^n)$ denotes the classical *Muckenhoupt weight class* consisting of all non-negative Lebesgue measurable functions on $\mathbb{R}^n$ such that

$$[\omega]_{A_1(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n,\, B \text{ is a ball}} \left[\frac{1}{|B|} \int_B \omega(x)\, dx \right] \left[\inf_{x \in B} \omega(x) \right]^{-1} < \infty.$$

According to [3], we have

$$\left(H^{p,\lambda}(\mathbb{R}^n)\right)^* = L^{p',\lambda}(\mathbb{R}^n), \tag{1.3}$$

where p' denotes the *conjugate index* of p, that is, $1/p + 1/p' = 1$.

For any $(p, \lambda) \in (1, \infty) \times (0, n)$, notice that $\|\cdot\|_{H^{p,\lambda}(\mathbb{R}^n)}$ is a norm, especially it satisfies the Minkowski inequality:

$$\|f + g\|_{H^{p,\lambda}(\mathbb{R}^n)} \le \|f\|_{H^{p,\lambda}(\mathbb{R}^n)} + \|g\|_{H^{p,\lambda}(\mathbb{R}^n)}, \qquad \forall\, f,\ g \in H^{p,\lambda}(\mathbb{R}^n). \tag{1.4}$$

Though this can not be obviously seen from the definition of $\|\cdot\|_{H^{p,\lambda}(\mathbb{R}^n)}$, but we can utilize [26, Theorem 4.3] and the fact

$$\|f\|_{L^{p',\lambda}(\mathbb{R}^n)} = \sup \left\{ \langle f, g\rangle : \ g \in H^{p,\lambda}(\mathbb{R}^n),\ \|g\|_{H^{p,\lambda}(\mathbb{R}^n)} \le 1 \right\} \tag{1.5}$$

in [3, Theorem 2.3] to derive that

$$\|f\|_{H^{p,\lambda}(\mathbb{R}^n)} = \sup \left\{ \langle f, g\rangle : \ g \in L^{p',\lambda}(\mathbb{R}^n),\ \|g\|_{L^{p',\lambda}(\mathbb{R}^n)} \le 1 \right\}, \tag{1.6}$$

while the latter easily implies (1.4).

Equivalent characterizations of $H^{p,\lambda}(\mathbb{R}^n)$ with any given $(p, \lambda) \in (1, \infty) \times (0, n)$ are established in [3, Theorem 3.3]. In particular, $H^{p,\lambda}(\mathbb{R}^n)$ coincides to the Zorko space $Z^{p,\lambda}(\mathbb{R}^n)$ introduced in [41, Proposition 5], as well as the Kalita space in [14, Theorem 1]. For any given $(p, \lambda) \in (1, \infty) \times (0, n)$, it is known that $C_c^\infty(\mathbb{R}^n)$ (that is, the space of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports) is not dense in $L^{p,\lambda}(\mathbb{R}^n)$ (see, for example, [11,36]), but it is dense in $H^{p,\lambda}(\mathbb{R}^n)$ (see [3]).

Nowadays, Morrey spaces and their preduals as well as their usual companion, namely, the Riesz potential operator, have been studied intensively in many literatures and found wide applications in analysis, geometry and partial differential equations; see, for instance, [1,4,6,17,19,20,33].

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Recall that, for any given $\alpha \in (0,n)$, the *Riesz potential operator* I_α on $\mathbb{R}^n$ is defined by the Fourier transform as follows:

$$\widehat{I_\alpha f}(\xi) := \gamma(\alpha)(2\pi|\xi|)^{-\alpha}\widehat{f}(\xi), \qquad \forall\, \xi \in \mathbb{R}^n,$$

where $\gamma(\alpha) := \frac{\pi^{n/2}2^\alpha\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}$ with $\Gamma(\cdot)$ being the usual gamma function and

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx, \qquad \forall\, \xi \in \mathbb{R}^n.$$

Based on [35,32], the operator I_α maps $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ onto $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, where $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ stands for the Schwartz distribution class $\mathcal{S}'(\mathbb{R}^n)$ modulo the polynomial space $\mathcal{P}(\mathbb{R}^n)$. It is known that $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ is topologically equivalent to $\mathcal{S}'_\infty(\mathbb{R}^n)$ (see, for example, [40, Proposition 8.1], [22, Theorem 6.28] and [27, Theorem 3.1]). For any sufficiently smooth function f which is small at infinity, one has (see Stein [29, p. 117])

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |y-x|^{\alpha-n} f(y)\,dy, \qquad \forall\, x \in \mathbb{R}^n.$$

Assume that $\alpha \in (0,n)$ is a non-integer and M the largest integer less than α. For any $r \in (0,\infty)$ and $x,\ y \in \mathbb{R}^n$, define

$$p_{r,y}(x) := |x+ry|^{\alpha-n} - \sum_{\beta\in\mathbb{Z}^n_+,\ |\beta|\le M} (ry)^\beta D^\beta |x|^{\alpha-n}(\beta!)^{-1}, \tag{1.7}$$

which is indeed the *Taylor remainder of the kernel of the Riesz potential*. Here and hereafter, for any multi-index $\beta := (\beta_1,\ldots,\beta_n) \in \mathbb{Z}^n_+ := (\mathbb{Z}_+)^n$ with $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$ and $\mathbb{N} := \{1,2,\ldots\}$, we use the following notation

$$\begin{cases} |\beta| := \sum_{j=1}^n \beta_j, \\ \beta! := \Pi_{j=1}^n \beta_j!, \\ D^\beta := \partial_{x_1}^{\beta_1}\cdots\partial_{x_n}^{\beta_n} = \left(\dfrac{\partial}{\partial x_1}\right)^{\beta_1}\cdots\left(\dfrac{\partial}{\partial x_n}\right)^{\beta_n}, \\ x^\beta := \Pi_{j=1}^n x_j^{\beta_j} \quad \text{if} \quad x=(x_1,\ldots,x_n)\in\mathbb{R}^n. \end{cases}$$

If f is sufficiently smooth and small at infinity, then it makes sense to define

$$T_\alpha f(x) := \left\{\int_0^\infty \left[\int_{B(\vec{0}_n,1)} |p_{r,y} * f(x)|\,dy\right]^2 \frac{dr}{r^{1+2\alpha}}\right\}^{1/2}, \qquad \forall\, x \in \mathbb{R}^n, \tag{1.8}$$

here and hereafter, $\vec{0}_n$ denotes the *origin* of $\mathbb{R}^n$. Such a quadratic operator T_α arises from the Riesz potential operator I_α and originates essentially from Stein [28]. Its current version was studied by Dahlberg [7] and then used to study the regularity of Riesz potentials on Lebesgue

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spaces. Another pioneer work regarding T_α for the special case $M = 0$ was due to Strichartz [31]. For any f in $L_c^\infty(\mathbb{R}^n)$ (that is, the set of all bounded functions with compact supports), it was proved in [7, Theorem 3] (see also [31, Theorem 2.3]) that $T_\alpha f$ is pointwisely well defined on $\mathbb{R}^n$ and

$$\|T_\alpha f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \qquad \forall\, f \in L_c^\infty(\mathbb{R}^n), \tag{1.9}$$

for some positive constant C independent of f, so that T_α can be extended to a bounded operator on $L^p(\mathbb{R}^n)$ via a standard density argument. Starting from this, Dahlberg [7, Theorem 3] characterized $L^p(\mathbb{R}^n)$ for any given $p \in (1, \infty)$ via T_α and then obtained

$$\|T_\alpha f\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)}, \qquad \forall\, f \in L^p(\mathbb{R}^n), \tag{1.10}$$

with the positive equivalence constants independent of f. It should be mentioned that the boundedness of T_α on mixed-norm Lebesgue spaces was obtained in [2] and then used to establish the capacitary inequalities for the Besov capacity. One of the main aims in this article is to extend (1.10) to the setting of Morrey spaces and their preduals; see Theorems 1.1 and 1.2 below, respectively.

Through this article, we *always restrict* the index $\alpha \in (0, n)$ to be a non-integer. Indeed, the fact $\lfloor\alpha\rfloor < \alpha < \lfloor\alpha\rfloor + 1$ is necessary for the below treatment of the kernel $p_{r,y}$. For the case when $\alpha \in (0, n)$ is an integer, the definition of T_α in (1.8) needs to be changed accordingly, perhaps with the kernel $p_{r,y}$ in (1.7) replaced by higher order differences. For example, when $\alpha = 1$, with $p_{r,y}(x)$ in (1.7) redefined by setting

$$p_{r,y}(x) := |x + ry|^{\alpha-n} + |x - ry|^{\alpha-n} - 2|x|^{\alpha-n}, \qquad \forall\, r \in (0, \infty),\ \forall\, x,\ y \in \mathbb{R}^n,$$

Strichartz [31, Theorem 2.3] obtained the boundedness of the corresponding operator T_α on Lebesgue spaces. In this article, we will not pursue the case when $\alpha \in (0, n)$ is an integer.

Let us emphasize more on the definition of $T_\alpha f$ in (1.8). Indeed, if f is a nice function, for example, f is in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, then one may easily observe that

$$p_{r,y} * f(x) = I_\alpha f(x + ry) - \sum_{\beta\in\mathbb{Z}_+^n,\ |\beta|\le M} (ry)^\beta D^\beta I_\alpha f(x)(\beta!)^{-1}, \quad \forall\, r \in (0, \infty),\ \forall\, x,\ y \in \mathbb{R}^n.$$

This identity fails for general functions $f \in \cup_{p\in(1,\infty)} L^p(\mathbb{R}^n)$, since $I_\alpha f$ might be infinite everywhere on $\mathbb{R}^n$. For instance, when $p \in (n/\alpha, \infty)$, if we take $\epsilon \in (0, \alpha - n/p]$ and consider the function

$$f(x) := (1 + |x|)^{-n/p-\epsilon}, \qquad \forall\, x \in \mathbb{R}^n,$$

then $f \in L^p(\mathbb{R}^n)$, but, for any $x \in \mathbb{R}^n$,

$$I_\alpha f(x) \ge \int_{|y|>1+|x|} |x - y|^{\alpha-n}(1 + |y|)^{-n/p-\epsilon}\, dy \ge \int_{|y|>1+|x|} (2|y|)^{\alpha-n-n/p-\epsilon}\, dy = \infty.$$

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However, for general $f \in \cup_{p\in(1,\infty)} L^p(\mathbb{R}^n)$, it is proved in Lemma 2.1 below that $p_{r,y} * f(x)$ is well defined for almost every $x \in \mathbb{R}^n$ and $\int_{B(\vec{0}_n,1)} |p_{r,y} * f(x)|\, dy$ has an upper bound irrelevant to $r \in [\epsilon, 1/\epsilon]$ (but, relevant to x and ϵ), where $\epsilon \in (0,1)$. So it makes sense to understand (1.8) as follows:

$$T_\alpha f = \lim_{\varepsilon\to 0} \left\{ \int_\varepsilon^{1/\varepsilon} \left[\int_{B(\vec{0}_n,1)} |p_{r,y} * f|\, dy \right]^2 \frac{dr}{r^{1+2\alpha}} \right\}^{1/2}, \quad \forall f \in \bigcup_{p\in(1,\infty)} L^p(\mathbb{R}^n). \quad (1.11)$$

Such a new understanding of T_α directly gives that T_α maps $L^p(\mathbb{R}^n)$ continuously into $L^p(\mathbb{R}^n)$; see Lemma 2.1 below.

Motivated by [7,31,28], in this article, we obtain the following characterization of Morrey spaces via the operator T_α.

Theorem 1.1. *Let $\lambda \in (0,n]$, $\alpha \in (0,n)$ be a non-integer and $p \in (1,\infty)$. Then there exists a positive constant C such that, for any $f \in L^{p,\lambda}(\mathbb{R}^n)$,*

$$C^{-1}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \le \|T_\alpha f\|_{L^{p,\lambda}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

The proof of Theorem 1.1 is given in Section 2 below. Notice that Theorem 1.1 for the case $\lambda = n$ goes back to the result of Dahlberg [7] (see also Strichartz [31] or Stein [28]). But, the method used in this article is totally different. Recall that the argument used in [7,31,28] strongly relies on the theory of Fourier transforms and the boundedness of the vector-valued Calderón–Zygmund operators, which certainly does not work for the setting of Morrey spaces. Due to the bad structure of the Morrey spaces, instead of the method used in [7,31,28], the proof for the boundedness of T_α on Morrey spaces relies on some quite delicate estimates of the kernel of T_α, while the converse counterpart needs the construction of a Calderón reproducing formula associated to the kernel of $\widetilde{T}_\alpha$ in (2.2), which is a variant of T_α but smaller than T_α. This operator $\widetilde{T}_\alpha$ in (2.2) looks like a classical Littlewood–Paley operator, but its kernel is obviously not as good as that of a classical Littlewood–Paley operator. In Section 2.1, we make a great effort to show that the kernel of $\widetilde{T}_\alpha$ is good enough for us to construct a Calderón reproducing formula needed in the proof of Theorem 1.1.

In analogy to Theorem 1.1, it is natural to ask the boundedness of T_α on the preduals of Morrey spaces. Indeed, we have the following conclusion.

Theorem 1.2. *Let $\lambda \in (0,n)$, $\alpha \in (0,n)$ be a non-integer and $p \in (1,\infty)$. Then there exists a positive constant C such that, for any $f \in H^{p,\lambda}(\mathbb{R}^n)$,*

$$C^{-1}\|f\|_{H^{p,\lambda}(\mathbb{R}^n)} \le \|T_\alpha f\|_{H^{p,\lambda}(\mathbb{R}^n)} \le C\|f\|_{H^{p,\lambda}(\mathbb{R}^n)}.$$

The proof of Theorem 1.2 is presented in Section 3 below. The upper bound estimate can essentially be reduced to considering the boundedness of T_α on the weighted Lebesgue space $L^p(u)$ equipped with the norm

$$\|f\|_{L^p(u)} := \left[\int_{\mathbb{R}^n} |f(x)|^p u(x)\, dx \right]^{1/p}, \qquad \forall f \in L^p(u),$$

whenever $p \in (1, \infty)$. Here, u is in the *Muckenhoupt weight class* $A_p(\mathbb{R}^n)$, that is, u is a non-negative Lebesgue measurable function on $\mathbb{R}^n$ with its $A_p(\mathbb{R}^n)$-weight constant

$$[u]_{A_p(\mathbb{R}^n)} := \sup_{B \text{ is a ball of } \mathbb{R}^n} \left[\frac{1}{|B|}\int_B u(x)\,dx\right]\left[\frac{1}{|B|}\int_B u(x)^{-\frac{1}{p-1}}\,dx\right]^{p-1} < \infty.$$

Such weighted estimates usually follow from the sharp maximal function estimates of $T_\alpha f$ (see Lemma 3.1 below). The lower bound estimate relies on the Calderón reproducing formula as that used in the proof of Theorem 1.1.

For any $\alpha,\ \lambda \in (0, n)$ and $p \in (1, \infty)$, the *Riesz-type capacity* $\mathcal{R}_{\alpha,p,\lambda}(E)$ of an arbitrary set $E \subset \mathbb{R}^n$ is defined by setting

$$\mathcal{R}_{\alpha,p,\lambda}(E) := \inf\left\{\|f\|^p_{H^{p,\lambda}(\mathbb{R}^n)} : f \geq 0 \text{ and } I_\alpha f \geq \mathbf{1}_E\right\},$$

here and hereafter, we use $\mathbf{1}_E$ to denote the characteristic function of the set E. When $0 < \alpha < \lambda < n$ and $p > \lambda/\alpha$, it follows from Adams and Xiao [3, Theorem 7.4] that

$$\int_0^\infty \left[\mathcal{R}_{\alpha,p,\lambda}(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\})\right]^{1/p} dt \lesssim \|f\|_{H^{p,\lambda}(\mathbb{R}^n)}, \qquad \forall\, f \in H^{p,\lambda}(\mathbb{R}^n),$$

where the implicit positive constant is independent of f. As an application of Theorem 1.2, we establish the following regularity result for the Riesz potential $I_\alpha(H^{p,\lambda})$, which has an advantage over the aforementioned result of [3, Theorem 7.4] in the sense that p in Theorem 1.3 below can be as close as to 1.

Theorem 1.3. *Let $\lambda \in (0, n)$, $\alpha \in (0, n)$ be a non-integer and $p \in (1, \infty)$. Then there exists a positive constant C such that, for any $f \in C_c^\infty(\mathbb{R}^n)$,*

$$\int_0^\infty \mathcal{R}_{\alpha,p,\lambda}(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\})\,dt^p \leq C\|f\|^p_{H^{p,\lambda}(\mathbb{R}^n)}.$$

Indeed, the proof of Theorem 1.3 is almost trivial under the assumptions $\alpha + \lambda > n$ and $p \in [\frac{\lambda}{\alpha+\lambda-n}, \infty)$; see Remark 4.4 below. Thus, we only need to show Theorem 1.3 for any given

$$p \in \begin{cases} (1, \infty) & \text{if } \alpha + \lambda \leq n, \\ \left(1, \dfrac{\lambda}{\alpha + \lambda - n}\right) & \text{if } \alpha + \lambda > n. \end{cases} \tag{1.12}$$

For such a p, any $f \in H^{p,\lambda}(\mathbb{R}^n)$ ensures that the set $\{x \in \mathbb{R}^n : |I_\alpha f(x)| = \infty\}$ has both zero $\mathcal{R}_{\alpha,p,\lambda}$-capacity and Lebesgue measure; see Remark 4.3 below.

Applying Theorem 1.3, we immediately obtain the following restricting result for $I_\alpha(H^{p,\lambda})$, and then the regularity of the duality solution to the fractional Laplace equation.

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Corollary 1.4. *Let $\lambda \in (0, n)$, $\alpha \in (0, n)$ be a non-integer and $p \in (1, \infty)$. Suppose that μ is a non-negative Radon measure on $\mathbb{R}^n$.*

(i) *The following two assertions are equivalent:*
 (a) *There exists a positive constant C such that*

$$\mu(E) \le C\mathcal{R}_{\alpha,p,\lambda}(E), \qquad \forall\, E \subset \mathbb{R}^n.$$

 (b) *There exists a positive constant C such that, for any $f \in H^{p,\lambda}(\mathbb{R}^n)$,*

$$\|I_\alpha f\|_{L^p_\mu(\mathbb{R}^n)} := \left[\int_0^\infty \mu(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\})\, dt^p\right]^{1/p} \le C\|f\|_{H^{p,\lambda}(\mathbb{R}^n)}.$$

(ii) *Assume further that $0 < \alpha < 2 \le n$ and μ has compact support. Under the assumption of (a) or (b) in (i), if the fractional Laplace equation*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \mu, \\ \lim_{|x|\to\infty} u(x) = 0 \end{cases} \tag{1.13}$$

has a duality solution u in the sense of

$$\int_{\mathbb{R}^n} u(x)\phi(x)\, dx = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} I_\alpha \phi(x)\, d\mu(x), \qquad \forall\, \phi \in C_c^\infty(\mathbb{R}^n),$$

then

$$u \in \bigcup_{(p',\lambda)\in(\frac{n}{n-\alpha},\infty)\times(0,n)} L^{p',\lambda}(\mathbb{R}^n)$$

with

$$\|u\|_{L^{p',\lambda}(\mathbb{R}^n)} \le C|\mathrm{supp}\, \mu|^{1/p'},$$

where $\gamma(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$, C is a positive constant independent of μ and $1/p + 1/p' = 1$.

For Corollary 1.4(i), observe that (b) follows directly from Theorem 1.3 and (a). Conversely, for any $0 \le f \in H^{p,\lambda}(\mathbb{R}^n)$ satisfying $I_\alpha f \ge \mathbf{1}_E$, we use (b) to conclude that

$$\mu(E) = \|\mathbf{1}_E\|^p_{L^p_\mu(\mathbb{R}^n)} \le \|I_\alpha f\|^p_{L^p_\mu(\mathbb{R}^n)} \le C^p \|f\|^p_{H^{p,\lambda}(\mathbb{R}^n)},$$

which implies (a) via taking the infimum over all such f. Here, C is the same constant as in (b) of Corollary 1.4(i).

To obtain Corollary 1.4(ii), we apply Corollary 1.4(i) and the Hölder inequality to derive that, for any $\phi \in C_c^\infty(\mathbb{R}^n)$,

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$$
\begin{aligned}
\left|\int_{\mathbb{R}^n} u(x)\phi(x)\,dx\right| &= \frac{1}{\gamma(\alpha)}\left|\int_{\mathbb{R}^n} I_\alpha\phi(x)\,d\mu(x)\right| \\
&\le \frac{|\operatorname{supp}\mu|^{1/p'}}{\gamma(\alpha)}\|I_\alpha\phi\|_{L^p_\mu(\mathbb{R}^n)} \\
&\le \frac{C|\operatorname{supp}\mu|^{1/p'}}{\gamma(\alpha)}\|\phi\|_{H^{p,\lambda}(\mathbb{R}^n)},
\end{aligned}
$$

which, together with the density of $C_c^\infty(\mathbb{R}^n)$ in $H^{p,\lambda}(\mathbb{R}^n)$ as well as (1.5), implies $u \in L^{p',\lambda}(\mathbb{R}^n)$ with the desired norm estimate. Again, here C is the same constant as in (b) of Corollary 1.4(i).

Remark 1.5.

(i) When $\alpha \in (1,2)$ and μ is a non-negative Radon measure with compact support in $\mathbb{R}^n$, it was proved in [15, Theorem 1.1] that the duality solution u of the fractional Laplace equation exists uniquely and satisfies

$$
\int_B |u(x)|^q\,dx + \int_B\int_B \frac{|u(x)-u(y)|^q}{|x-y|^{n+q-2+\alpha}}\,dx\,dy < \infty, \qquad \forall \text{ ball } B \subset \mathbb{R}^n,
$$

whenever $1 < q < \frac{n+2-\alpha}{n+1-\alpha}$. Thus, Corollary 1.4(ii) partly improves [15, Theorem 1.1], because now the solution u proves locally $L^{p'}(\mathbb{R}^n)$ integrable with

$$
\frac{n+2-\alpha}{n+1-\alpha} < \frac{n}{n-\alpha} < p' < \infty,
$$

where $1/p + 1/p' = 1$.

(ii) Notice that u is a duality solution to (1.13) if and only if u is a weak solution to (1.13) in the sense of

$$
\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}} u(x)\varphi(x)\,dx = \int_{\mathbb{R}^n} \varphi(x)\,d\mu(x), \qquad \forall \varphi \in I_\alpha(C_c^\infty(\mathbb{R}^n)).
$$

This can be seen by using the idea in [23, Proposition 2.9]. To be precise, observe that u is a weak solution to (1.13) if and only if

$$
\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}} u(x) I_\alpha\phi(x)\,dx = \int_{\mathbb{R}^n} I_\alpha\phi(x)\,d\mu(x), \qquad \forall \phi \in C_c^\infty(\mathbb{R}^n).
$$

Using the Fubini theorem and the fact $I_\alpha(-\Delta)^{\frac{\alpha}{2}} u = \gamma(\alpha)u$ in $\mathcal{S}'(\mathbb{R}^n)$, we obtain

$$
\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}} u(x) I_\alpha\phi(x)\,dx = \int_{\mathbb{R}^n} I_\alpha(-\Delta)^{\frac{\alpha}{2}} u(x)\phi(x)\,dx = \gamma(\alpha)\int_{\mathbb{R}^n} u(x)\phi(x)\,dx,
$$

that is, u is a weak solution to (1.13) if and only if it is a duality solution to (1.13).

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The organization of this article is as follows.

Section 2 mainly deals with the proof of Theorem 1.1. The non-density of $C_c^\infty(\mathbb{R}^n)$ in Morrey spaces brings many difficulties when we consider the boundedness of operators on Morrey spaces. The reader may find various methods to overcome this deficit; see, for example, [24,25]. In this article, with a new understanding of T_α as in (1.11), we prove that $T_\alpha f$ is well defined almost everywhere on $\mathbb{R}^n$ whenever $f \in L^p(\mathbb{R}^n)$ with $p \in (1,\infty)$ (see Lemma 2.1 below), which leads to the well definedness of T_α on Morrey spaces. Combining this and a delicate estimate, we obtain the boundedness of T_α on Morrey spaces in Section 2.2, which gives the upper bound estimate of Theorem 1.1. In Section 2.3, we obtain the lower bound estimate of Theorem 1.1 via the Calderón reproducing formula and the boundedness of the Littlewood–Paley operator on $L^{p,\lambda}(\mathbb{R}^n)$ (see Theorem 2.4 below). For this part, we need some hard analysis on the kernel of $\widetilde{T}_\alpha$ (see Proposition 2.2 below), which is a variant of T_α and smaller than T_α, so that we can use the idea from [39, Lemma 2.1] to build a Calderón reproducing formula associated to the kernel of $\widetilde{T}_\alpha$ (see Proposition 2.3 below).

Section 3 is devoted to the proof of Theorem 1.2. In Section 3.1, we use the Hardy–Littlewood maximal function to dominate the sharp maximal function of $T_\alpha f$ when $f \in C_c^\infty(\mathbb{R}^n)$. This again needs some quite delicate estimates regarding $p_{r,y} * f$. In Section 3.2, we present the boundedness of T_α on $H^{p,\lambda}(\mathbb{R}^n)$, which can be attributed to the corresponding weighted estimates and then the aforementioned sharp maximal function estimates. In Section 3.3, we obtain the lower bound estimate of Theorem 1.2, via using both the Calderón reproducing formula built in Proposition 2.3 below and the boundedness of the Littlewood–Paley operator on $H^{p,\lambda}(\mathbb{R}^n)$ (see Theorem 2.4 below).

Section 4 focuses on the proof of Theorem 1.3. In Section 4.1, we first establish the monotonicity and a variant of the subadditivity as well as a capacitary weak-type inequality (see Lemmas 4.1 and 4.2 below) of the Riesz-type capacity $\mathcal{R}_{\alpha,p,\lambda}$, and then prove a differentiation theorem for $\mathcal{R}_{\alpha,p,\lambda}$ (see Lemma 4.5 below). With these and an idea from the proof of [7, Theorem 1], in Section 4.2, we establish the boundedness of I_α from $H^{p,\lambda}(\mathbb{R}^n)$ to the Lebesgue space L^p with respect to $\mathcal{R}_{\alpha,p,\lambda}$.

Throughout this article, we always adopt the following notation. Let

$$\mathbb{N} := \{1, 2, \dots\},\ \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\} \text{ and } \mathbb{Z}_+ := \mathbb{N} \cup \{0\}.$$

We always use C or c to denote a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $f \lesssim g$ (resp., $f \gtrsim g$) means $f \le Cg$ (resp., $f \ge Cg$) for a positive constant C, and $f \sim g$ amounts to $f \gtrsim g \gtrsim f$. We also use the following convention: If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$. For any $s \in \mathbb{R}$, denote by $\lfloor s \rfloor$ the largest integer not greater than s, and by $\lceil s \rceil$ the smallest integer greater than or equal to s. We always use $\vec{0}_n$ to denote the *origin* of $\mathbb{R}^n$. For any set $E \subset \mathbb{R}^n$, the symbol $⨍_E$ represents $\frac{1}{|E|}\int_E$, the symbol $\mathbf{1}_E$ its characteristic function and $E^\complement := \mathbb{R}^n \setminus E$. For any $p \in (1,\infty)$, let p' be the conjugate index of p, that is, $1/p + 1/p' = 1$.

2. Proof of Theorem 1.1

This section mainly deals with the proof of Theorem 1.1. To be precise, Section 2.1 concerns a detailed analysis for the kernel of T_α; Section 2.2 gives the upper bound estimate of Theorem 1.1 and Section 2.3 shows the lower bound estimate of Theorem 1.1.

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2.1. Analysis for the kernel of T_α

This section gives some preliminaries but important results regarding the operator T_α as well as its kernel.

Denote by $\mathcal{M}$ the *Hardy–Littlewood maximal operator* on $\mathbb{R}^n$, that is, for any locally integrable function u on $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$,

$$\mathcal{M}u(x) := \sup_{B \ni x,\ B \text{ is a ball of } \mathbb{R}^n} \fint_B |u(z)|\, dz.$$

Lemma 2.1. *For any $r \in (0, \infty)$ and $y \in B(\vec{0}_n, 1)$, define $p_{r,y}$ as in (1.7), with $\alpha \in (0, n)$ being a non-integer and $M := \lfloor \alpha \rfloor$. Let $f \in L^p(\mathbb{R}^n)$ with any given $p \in (1, \infty)$. Then*

(i) *there exists a positive constant C, independent of r, y and f, such that*

$$|p_{r,y} * f(x)| \le Cr^\alpha \left[\mathcal{M}f(x + ry) + \mathcal{M}f(x)\right], \qquad \forall\, x \in \mathbb{R}^n;$$

(ii) *$p_{r,y} * f$ is well defined almost everywhere on $\mathbb{R}^n$;*
(iii) *$T_\alpha f$ as in (1.11) is well defined and hence T_α bounded on $L^p(\mathbb{R}^n)$.*

Proof. From the boundedness of $\mathcal{M}$ on $L^p(\mathbb{R}^n)$ with any given $p \in (1, \infty)$, (ii) follows directly from (i).

Observe that (iii) is actually a consequence of (ii) and (1.9). Indeed, from (ii), it follows that $\{\int_\varepsilon^{1/\varepsilon}[\int_{B(\vec{0}_n,1)} |p_{r,y} * f(x)|\, dy]^2 \frac{dr}{r^{1+2\alpha}}\}^{1/2}$ is well defined for almost every $x \in \mathbb{R}^n$, as it is pointwise bounded by $\mathcal{M}f(x) + \mathcal{M}^2 f(x)$, up to a positive constant multiple. Because $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^n)$, we take a sequence $\{f_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $\lim_{j\to\infty} f_j = f$ both in $L^p(\mathbb{R}^n)$ and almost everywhere on $\mathbb{R}^n$. Then the Fatou lemma implies that, for any $\epsilon \in (0, 1)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\left\{\int_\varepsilon^{1/\varepsilon}\left[\int_{B(\vec{0}_n,1)} |p_{r,y} * f(x)|\, dy\right]^2 \frac{dr}{r^{1+2\alpha}}\right\}^{1/2} \\
&= \left\{\int_\varepsilon^{1/\varepsilon}\left[\int_{B(\vec{0}_n,1)} \left|p_{r,y} * (\lim_{j\to\infty} f_j)(x)\right| dy\right]^2 \frac{dr}{r^{1+2\alpha}}\right\}^{1/2} \\
&\le \liminf_{j\to\infty}\left\{\int_\varepsilon^{1/\varepsilon}\left[\int_{B(\vec{0}_n,1)} |p_{r,y} * f_j(x)|\, dy\right]^2 \frac{dr}{r^{1+2\alpha}}\right\}^{1/2} \\
&\le \liminf_{j\to\infty}\left\{\int_0^\infty\left[\int_{B(\vec{0}_n,1)} |p_{r,y} * f_j(x)|\, dy\right]^2 \frac{dr}{r^{1+2\alpha}}\right\}^{1/2}
\end{aligned}$$

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$$= \liminf_{j\to\infty} T_\alpha f_j(x). \tag{2.1}$$

Meanwhile, by (1.9) and the Fatou lemma, we have

$$\left\| \liminf_{j\to\infty} T_\alpha f_j \right\|_{L^p(\mathbb{R}^n)} \le \liminf_{j\to\infty} \|T_\alpha f_j\|_{L^p(\mathbb{R}^n)} \lesssim \liminf_{j\to\infty} \|f_j\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)}.$$

Combining the last two formulae implies that $T_\alpha f$ as in (1.11) is well defined in $L^p(\mathbb{R}^n)$ and hence almost everywhere on $\mathbb{R}^n$, with

$$\begin{aligned}
\|T_\alpha f\|_{L^p(\mathbb{R}^n)} &= \left\| \lim_{\varepsilon\to 0} \left\{ \int_\varepsilon^{1/\varepsilon} \left[\int_{B(\vec{0}_n,1)} |p_{r,y} * f|\,dy \right]^2 \frac{dr}{r^{1+2\alpha}} \right\}^{1/2} \right\|_{L^p(\mathbb{R}^n)} \\
&\le \liminf_{\varepsilon\to 0} \left\| \left\{ \int_\varepsilon^{1/\varepsilon} \left[\int_{B(\vec{0}_n,1)} |p_{r,y} * f|\,dy \right]^2 \frac{dr}{r^{1+2\alpha}} \right\}^{1/2} \right\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Thus, we conclude the proof of (iii).

It remains to show (i). Let $f \in L^p(\mathbb{R}^n)$ with any given $p \in (1,\infty)$. Fix $r \in (0,\infty)$ and $y \in B(\vec{0}_n, 1)$. For any $x \in \mathbb{R}^n$, we consider the ball $B := B(x, 3r)$ and define $f_1 := f\mathbf{1}_B$ and $f_2 := f\mathbf{1}_{B^\complement}$. It follows that

$$p_{r,y} * f = p_{r,y} * f_1 + p_{r,y} * f_2.$$

For the first part $p_{r,y} * f_1$, we write

$$\begin{aligned}
|p_{r,y} * f_1(x)| &\le \int_{|z-x|<3r} |p_{r,y}(x-z)||f(z)|\,dz \\
&\le \int_{|z-x|<3r} |x-z+ry|^{\alpha-n}|f(z)|\,dz \\
&\quad + \sum_{\beta\in\mathbb{Z}_+^n, |\beta|\le M} r^{|\beta|}(\beta!)^{-1} \int_{|z-x|<3r} |x-z|^{\alpha-n-|\beta|}|f(z)|\,dz.
\end{aligned}$$

Notice that

$$\int_{|z-x|<3r} |x-z+ry|^{\alpha-n}|f(z)|\,dz \le \int_{|x-z+ry|<4r} |x-z+ry|^{\alpha-n}|f(z)|\,dz$$

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$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \int_{2^{-j+1}r \le |x-z+ry| < 2^{-j+2}r} |x-z+ry|^{\alpha-n} |f(z)|\, dz \\
&\lesssim \sum_{j=0}^{\infty} (2^{-j}r)^{\alpha-n} \int_{|x-z+ry| < 2^{-j+2}r} |f(z)|\, dz \\
&\lesssim r^{\alpha} \mathcal{M} f(x+ry)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{|z-x|<3r} |x-z|^{\alpha-n-|\beta|} |f(z)|\, dz &\le \sum_{j=0}^{\infty} \int_{2^{-j+1}r \le |x-z| < 2^{-j+2}r} |x-z|^{\alpha-n-|\beta|} |f(z)|\, dz \\
&\lesssim \sum_{j=0}^{\infty} (2^{-j}r)^{\alpha-n-|\beta|} \int_{|x-z| < 2^{-j+2}r} |f(z)|\, dz \\
&\lesssim r^{\alpha-|\beta|} \mathcal{M} f(x).
\end{aligned}
$$

Consequently, we have

$$
|p_{r,y} * f_1(x)| \lesssim r^{\alpha} [\mathcal{M} f(x+ry) + \mathcal{M} f(x)],
$$

as desired.

Now we deal with the second part $p_{r,y} * f_2$. When $|x-z| \ge 3r$, by the Taylor theorem, we have

$$
|p_{r,y}(x-z)| \lesssim \sup_{\theta \in (0,1)} |ry|^{M+1} |x-z+\theta ry|^{\alpha-n-(M+1)} \lesssim r^{M+1} |x-z|^{\alpha-n-(M+1)},
$$

which further implies that

$$
\begin{aligned}
|p_{r,y} * f_2(x)| &\le \int_{|z-x|\ge 3r} |p_{r,y}(x-z)| |f(z)|\, dz \\
&\lesssim r^{M+1} \int_{|x-z|\ge 3r} |x-z|^{\alpha-n-(M+1)} |f(z)|\, dz \\
&\lesssim r^{M+1} \sum_{j=1}^{\infty} \int_{2^{j}r \le |x-z| < 2^{j+1}r} |x-z|^{\alpha-n-(M+1)} |f(z)|\, dz \\
&\lesssim r^{M+1} \sum_{j=1}^{\infty} (2^{j}r)^{\alpha-n-(M+1)} \int_{|x-z| < 2^{j+1}r} |f(z)|\, dz \\
&\lesssim r^{\alpha} \mathcal{M} f(x).
\end{aligned}
$$

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Combining the estimates for the two parts $p_{r,y} * f_1(x)$ and $p_{r,y} * f_2(x)$, we obtain (i) and hence complete the proof of Lemma 2.1. □

Fixing a non-integer α in $(0, n)$, instead of the operator T_α, we consider the following smaller one

$$\widetilde{T}_\alpha f(x) := \left[\int_0^\infty \left|\frac{1}{r^\alpha}\int_{B(\vec{0}_n,1)} p_{r,y} * f(x)\,dy\right|^2 \frac{dr}{r}\right]^{1/2}, \qquad \forall x \in \mathbb{R}^n. \tag{2.2}$$

Here and hereafter, we assume that f is a suitable function so that $\widetilde{T}_\alpha f$ makes sense. Of course, based on Lemma 2.1, one can take $f \in \cup_{p\in(1,\infty)} L^p(\mathbb{R}^n)$. For any $r \in (0, \infty)$ and $x \in \mathbb{R}^n$, let

$$K_r(x) := \frac{1}{r^\alpha}\int_{B(\vec{0}_n,1)} p_{r,y}(x)\,dy. \tag{2.3}$$

Then

$$\widetilde{T}_\alpha f(x) = \left[\int_0^\infty |K_r * f(x)|^2 \frac{dr}{r}\right]^{1/2}, \qquad \forall x \in \mathbb{R}^n.$$

For any $r \in (0, \infty)$, $y \in B(\vec{0}_n, 1)$ and $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, we use (1.7) and [29, p. 117] to derive that

$$\widehat{p_{r,y}}(\xi) = \gamma(\alpha)(2\pi)^{-\alpha}|\xi|^{-\alpha}\left[e^{2\pi i r y\cdot\xi} - \sum_{\beta\in\mathbb{Z}_+^n,\ |\beta|\le M} (ry)^\beta(2\pi i\xi)^\beta(\beta!)^{-1}\right],$$

which gives

$$\widehat{K_r}(\xi) = \frac{\gamma(\alpha)(2\pi)^{-\alpha}}{|r\xi|^\alpha}\int_{B(\vec{0}_n,1)}\left[e^{2\pi i r y\cdot\xi} - \sum_{\beta\in\mathbb{Z}_+^n,\ |\beta|\le M} y^\beta(2\pi i r\xi)^\beta(\beta!)^{-1}\right] dy. \tag{2.4}$$

More properties of the kernel K_r are presented in the following proposition.

Proposition 2.2. *For any $r \in (0, \infty)$, let K_r be as in (2.3) with its Fourier transform given in (2.4). When $r = 1$, write K_r simply by K. Then*

(i) *$\widehat{K_r}(\xi) = \widehat{K}(r\xi)$ for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$;*
(ii) *there exist small positive constants c, σ_1 and σ_2 such that $|\widehat{K}(\xi)| > c$ on the annulus $\{\xi \in \mathbb{R}^n : \sigma_1 < |\xi| < \sigma_2\}$;*

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(iii) *$\widehat{K}$ is infinitely differentiable on $\{\xi \in \mathbb{R}^n : \sigma_1 < |\xi| < \sigma_2\}$ and, for any multi-index $\beta \in \mathbb{Z}_+^n$,*

$$|D^\beta \widehat{K}(\xi)| \le C, \qquad \forall\, \xi \in \{\xi \in \mathbb{R}^n : \sigma_1 < |\xi| < \sigma_2\},$$

where C is a positive constant depending on σ_1, σ_2 and β.

Proof. Notice that (i) follows directly from the expression (2.4), and (iii) is a consequence of (ii) and the continuity of $D^\beta \widehat{K}$ outside of the origin.

It remains to prove (ii). By (2.4), it suffices to find small positive constants σ_1 and σ_2 such that the absolute value of the function

$$F(\xi) := \int_{B(\vec{0}_n,1)} \left[e^{2\pi i y\cdot\xi} - \sum_{\beta\in\mathbb{Z}_+^n,\ |\beta|\le M} y^\beta (2\pi i\xi)^\beta (\beta!)^{-1} \right] dy, \qquad \forall\, \xi \in \mathbb{R}^n,$$

has a positive lower bound on $\{\xi \in \mathbb{R}^n : \sigma_1 < |\xi| < \sigma_2\}$.

For any $\gamma \in \mathbb{Z}_+^n$ satisfying $|\gamma| \le M$, using the expression of F, we do simple calculation and obtain $D^\gamma F(\vec{0}_n) = 0$. For the case $\gamma := (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}_+^n$ such that $|\gamma| \ge M + 1$, by parity, one has

$$D^\gamma F(\vec{0}_n) = \int_{B(\vec{0}_n,1)} (2\pi i y)^\gamma \, dy = \begin{cases} 0 & \text{if some } \gamma_i \text{ is odd,} \\ \text{a non-zero real number} & \text{if each } \gamma_i \text{ is even.} \end{cases}$$

Take $L := M + 1$ when M is odd and $L := M + 2$ when M is even. Then L is always even. As $L - 1$ is odd and $L - 2 \le M$, one has

$$D^\gamma F(\vec{0}_n) = 0, \qquad \forall\, \gamma \in \mathbb{Z}_+^n \text{ with } |\gamma| \le L - 1.$$

For the moment, fix $\gamma := (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}_+^n$ with $|\gamma| = L$ and every γ_i being an even number. Then the sign of $D^\gamma F(\vec{0}_n)$ is determined only by the parity of $L/2$, because

$$D^\gamma F(\vec{0}_n) = (2\pi)^L (-1)^{L/2} \int_{B(\vec{0}_n,1)} y_1^{\gamma_1} \cdots y_n^{\gamma_n} \, dy_1 \cdots dy_n$$

and $\int_{B(\vec{0}_n,1)} y_1^{\gamma_1} \cdots y_n^{\gamma_n} \, dy_1 \cdots dy_n > 0$. For simplicity, below we consider only the case $L/2$ is even, in which case $D^\gamma F(\vec{0}_n) > 0$. By this and the continuity of $D^\gamma F$, we know that there exist constants c_0, $\sigma \in (0, \infty)$, uniformly in γ, such that

$$D^\gamma F(\xi) > c_0 > 0, \qquad \forall\, \xi \in \{\xi \in \mathbb{R}^n : |\xi| < \sigma\}. \tag{2.5}$$

Fix $\xi := (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ such that $|\xi| < \sigma$. By symmetry, we may as well assume that every $\xi_i \ge 0$ and $\xi_1 = \max_{1\le i\le n} \xi_i$ so that $\xi_1 \le |\xi| \le \sqrt{n}\xi_1$. Applying the Newton–Leibniz formula L-times, we obtain

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$$\begin{aligned} F(\xi_1, \dots, \xi_n) &= \int_0^{\xi_1} \int_0^{t_{L-1}} \cdots \int_0^{t_1} \partial_{x_1}^L F(t, \xi_2, \dots, \xi_n)\, dt\, dt_1 \cdots dt_{L-1} \\ &\quad + \sum_{k_1=0}^{L-1} \frac{1}{k_1!} \partial_{x_1}^{k_1} F(0, \xi_2, \dots, \xi_n) \xi_1^{k_1} \\ &=: \mathrm{Z}_1 + \mathrm{Z}_2. \end{aligned}$$

By (2.5) and the fact that L is even, we have

$$\mathrm{Z}_1 \ge c_0 \int_0^{\xi_1} \int_0^{t_{L-1}} \cdots \int_0^{t_1} dt\, dt_1 \cdots dt_{L-1} \ge c_0 \frac{\xi_1^L}{L!} \ge c_0 \frac{(|\xi|/\sqrt{n})^L}{L!}.$$

For the case $n = 1$, the term Z_2 is the linear combination of derivatives of F up to order $L - 1$ at the origin, which is zero.

Let us consider the case $n \ge 2$. For any $k_1 \in \{0, \dots, L-1\}$ such that k_1 is odd, we observe that

$$\partial_{x_1}^{k_1} F(0, \xi_2, \dots, \xi_n) = \int_{B(\vec{0}_n, 1)} (2\pi i y_1)^{k_1} e^{2\pi i \sum_{j=2}^n y_j \xi_j}\, dy_1 \cdots dy_n = 0$$

and hence

$$Z_2 = \sum_{\substack{k_1 \in \{0, \dots, L-1\} \\ k_1 \text{ even}}} \frac{1}{k_1!} \partial_{x_1}^{k_1} F(0, \xi_2, \dots, \xi_n) \xi_1^{k_1}.$$

Now, fixing $k_1 \in \{0, \dots, L-1\}$ such that k_1 is even, we then estimate $\partial_{x_1}^{k_1} F(0, \xi_2, \dots, \xi_n)$. Applying the Newton–Leibniz formula $L - k_1$ times, we have

$$\begin{aligned} \partial_{x_1}^{k_1} F(0, \xi_2, \dots, \xi_n) &= \int_0^{\xi_2} \int_0^{t_{L-k_1-1}} \cdots \int_0^{t_1} \partial_{x_2}^{L-k_1} \partial_{x_1}^{k_1} F(0, t, \xi_3, \dots, \xi_n)\, dt\, dt_1 \cdots dt_{L-k_1-1} \\ &\quad + \sum_{k_2=0}^{L-k_1-1} \frac{1}{k_2!} \partial_{x_2}^{k_2} \partial_{x_1}^{k_1} F(0, 0, \xi_3, \dots, \xi_n) \xi_2^{k_2} \\ &=: \mathrm{Z}_{2,1} + \mathrm{Z}_{2,2}. \end{aligned}$$

Since both k_1 and $L - k_1$ are even, we deduce from (2.5) that

$$\mathrm{Z}_{2,1} \ge c_0 \int_0^{\xi_2} \int_0^{t_{L-k_1-1}} \cdots \int_0^{t_1} dt\, dt_1 \cdots dt_{L-k_1-1} = c_0 \frac{\xi_2^{L-k_1}}{(L-k_1)!} \ge 0.$$

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As in the discussion for Z_2, the summation in $Z_{2,2}$ is non-zero only for those k_2 being even. Therefore, we have

$$\begin{aligned} Z_2 &= \text{a non-negative number} \\ &\quad + \sum_{\substack{k_1 \in \{0,\dots,L-1\} \\ k_1 \text{ even}}} \sum_{\substack{k_2 \in \{0,\dots,L-k_1-1\} \\ k_2 \text{ even}}} \frac{1}{k_2!k_1!} \partial_{x_2}^{k_2} \partial_{x_1}^{k_1} F(0,0,\xi_3,\dots,\xi_n) \xi_1^{k_1} \xi_2^{k_2}. \end{aligned}$$

In the case $n=2$, we find that Z_2 can be written as a non-negative number plus a linear combination of derivatives of F up to order $L-1$ at the origin, while the latter is equal to zero. This shows that $Z_2 \geq 0$ when $n=2$.

In general, we repeat the above argument to treat the terms like

$$\partial_{x_2}^{k_2} \partial_{x_1}^{k_1} F(0,0,\xi_3,\dots,\xi_n)$$

with both k_2 and k_1 being even, then we finally obtain

$$\begin{aligned} Z_2 &= \text{a non-negative number} \\ &\quad + \sum_{\substack{k_1 \in \{0,\dots,L-1\} \\ k_1 \text{ even}}} \cdots \sum_{\substack{k_n \in \{0,\dots,L-k_{n-1}-1\} \\ k_n \text{ even}}} \frac{1}{k_1!\cdots k_n!} \partial_{x_n}^{k_n} \cdots \partial_{x_1}^{k_1} F(\vec{0}_n) \xi_1^{k_1} \cdots \xi_n^{k_n}. \end{aligned}$$

This implies that Z_2 is a non-negative number. Invoking the estimate of Z_1, we conclude that, for any $\xi \in \mathbb{R}^n$ satisfying $|\xi| < \sigma$,

$$|F(\xi)| \geq c_0 \frac{(|\xi|/\sqrt{n})^L}{L!}.$$

Let $\sigma_2 := \sigma$ and σ_1 be an arbitrary number in $(0,\sigma)$. Then, when $\sigma_1 < |\xi| < \sigma_2$, we have $|F(\xi)| \geq c_0 \frac{(\sigma_1/\sqrt{n})^L}{L!}$, which is as desired. This concludes the proof of (ii) and hence of Proposition 2.2. □

Combining Proposition 2.2 and the argument from the proof of [39, Lemma 2.1], we obtain a Calderón reproducing formula invoking the kernel K_r.

Proposition 2.3. *Let σ_1, σ_2 and K be as in Proposition 2.2. Then there exists ϕ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ having the following properties:*

(i) *there exist $\sigma_3,\ \sigma_4 \in (0,\infty)$ such that $\sigma_1 < \sigma_3 < \sigma_4 < \sigma_2$ and*

$$\operatorname{supp} \widehat{\phi} \subset \{\xi \in \mathbb{R}^n : \sigma_3 < |\xi| < \sigma_4\};$$

(ii) *there exist $c,\ \sigma_5,\ \sigma_6 \in (0,\infty)$ such that $\sigma_3 < \sigma_5 < \sigma_6 < \sigma_4$ and*

$$|\widehat{\phi}(\xi)| > c \ \text{ whenever } \sigma_5 \leq |\xi| \leq \sigma_6;$$

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(iii) *for any* $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$,

$$\int_0^\infty \widehat{K}(r\xi)\widehat{\phi}(r\xi)\,\frac{dr}{r} = 1;$$

(iv) *for any* f *in* $\mathcal{S}(\mathbb{R}^n)$ *(resp., in* $\mathcal{S}'(\mathbb{R}^n)$*, or in* $L^p(\mathbb{R}^n)$ *with any given* $p \in [1, \infty)$*),*

$$\int_0^\infty K_r * \phi_r * f\,\frac{dr}{r} = f$$

in $\mathcal{S}(\mathbb{R}^n)$ *(resp., in* $\mathcal{S}'(\mathbb{R}^n)$*, or in* $L^p(\mathbb{R}^n)$ *with any given* $p \in [1, \infty)$*), where* $\phi_r(\cdot) := r^{-n}\phi(r^{-1}\cdot)$ *for any* $r \in (0, \infty)$.

Proof. Choose $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{g} \geq 0$, $\operatorname{supp}\widehat{g} \subset \{\xi \in \mathbb{R}^n : \sigma_3 < |\xi| < \sigma_4\}$ and $\widehat{g}(\xi) > c_1$ whenever $\sigma_5 < |\xi| < \sigma_6$, where c_1 is a positive constant and $\sigma_3, \sigma_4, \sigma_5, \sigma_6$ are as in the statement of the proposition.

For any $\xi \in \mathbb{R}^n$, define $F(\xi) := \int_0^\infty \widehat{g}(r\xi)\frac{dr}{r}$. Then F is a bounded function, F has a positive lower bound on $\mathbb{R}^n \setminus \{\vec{0}_n\}$, and $F(t\xi) = F(\xi)$ for any $t \in (0, \infty)$ and $\xi \in \mathbb{R}^n$. Define

$$h(\xi) := \frac{\widehat{g}(\xi)}{F(\xi)}, \qquad \forall\, \xi \in \mathbb{R}^n.$$

Clearly, $\operatorname{supp} h \subset \{\xi \in \mathbb{R}^n : \sigma_3 < |\xi| < \sigma_4\}$ and h has a positive lower bound on $\{\xi \in \mathbb{R}^n : \sigma_5 < |\xi| < \sigma_6\}$. Moreover, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, one has

$$\int_0^\infty h(r\xi)\frac{dr}{r} = \int_0^\infty \frac{\widehat{g}(r\xi)}{F(r\xi)}\frac{dr}{r} = \frac{1}{F(\xi)}\int_0^\infty \widehat{g}(r\xi)\frac{dr}{r} = 1.$$

Let ϕ be a function such that $\widehat{\phi} := h/\widehat{K}$. Since $\operatorname{supp} h \subset \{\xi \in \mathbb{R}^n : \sigma_3 < |\xi| < \sigma_4\}$ and $\widehat{K}$ has a positive lower bound on a bigger annulus $\{\xi \in \mathbb{R}^n : \sigma_1 < |\xi| < \sigma_2\}$, it follows that $\widehat{\phi}$ is well defined on $\mathbb{R}^n$. By Proposition 2.2(iii), it is easy to see that ϕ enjoys properties (i), (ii) and (iii).

From Proposition 2.2(iii) again, we deduce that $\widehat{\phi}$ is infinitely differentiable on $\{\xi \in \mathbb{R}^n : \sigma_3 < |\xi| < \sigma_4\}$, which contains $\operatorname{supp}\widehat{\phi}$. This implies that $\widehat{\phi} \in C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$. Consequently, $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Finally, (iv) follows from a standard argument (see, for example, [9, Appendix] or [38, Lemma 2.1]) and the already proved properties of ϕ. This finishes the proof of Proposition 2.3. □

Using the Schwartz function ϕ constructed in Proposition 2.3, one can introduce the *Littlewood–Paley g-function* $\mathcal{G}$ as follows:

$$\mathcal{G}(f)(x) := \left[\int_0^\infty |\phi_r * f(x)|^2\,\frac{dr}{r}\right]^{1/2}, \qquad \forall\, x \in \mathbb{R}^n,$$

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which is well defined for any sufficiently smooth function f. In particular, it is well known that $\mathcal{G}$ is bounded on $L^p(\mathbb{R}^n)$ for any given $p \in (1, \infty)$ (see, for example, [34, (3.8)]). The next proposition concerns the boundedness of the operator $\mathcal{G}$ on Morrey spaces and their preduals.

Theorem 2.4. *Let $(p, \lambda) \in (1, \infty) \times (0, n)$. Assume that $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (i) and (ii) of Proposition 2.3. Then the operator $\mathcal{G}$ is bounded on both $L^{p,\lambda}(\mathbb{R}^n)$ and $H^{p,\lambda}(\mathbb{R}^n)$.*

Proof. The boundedness of $\mathcal{G}$ on $L^{p,\lambda}(\mathbb{R}^n)$ follows immediately from [37, Corollary 1.5]. It remains to consider the boundedness of $\mathcal{G}$ on $H^{p,\lambda}(\mathbb{R}^n)$.

Let $f \in H^{p,\lambda}(\mathbb{R}^n)$. Without loss of generality, we may as well assume that

$$\|f\|_{H^{p,\lambda}(\mathbb{R}^n)} = 1.$$

By the definition of $\|\cdot\|_{H^{p,\lambda}(\mathbb{R}^n)}$, there exists a non-negative function ω satisfying (1.2) and

$$\|f\|_{L^p(\omega^{1-p})} \leq 2\|f\|_{H^{p,\lambda}(\mathbb{R}^n)} = 2.$$

Fix $\theta \in (\lambda/n, 1)$ and let $\omega_\theta := (\mathcal{M}\omega^{1/\theta})^\theta$. Based on the argument in [5, p. 211], we know that there exists a positive constant c_0, depending only on θ, n and λ, such that

$$\int_{\mathbb{R}^n} \omega_\theta \, d\Lambda_{n-\lambda}^{(\infty)} \leq c_0.$$

Let $\widetilde{\omega}_\theta := c_0^{-1}\omega_\theta$. Then $\widetilde{\omega}_\theta$ satisfies (1.2) and hence

$$\|\mathcal{G}(f)\|_{H^{p,\lambda}(\mathbb{R}^n)} \leq \left\{ \int_{\mathbb{R}^n} [\mathcal{G}(f)(x)]^p \, [\widetilde{\omega}_\theta(x)]^{1-p} \, dx \right\}^{1/p}.$$

Next, we consider the function $u := \widetilde{\omega}_\theta^{1-p}$, that is, $u = c_0^{p-1}\omega_\theta^{1-p}$. According to the argument in [8, pp. 140-141], the function ω_θ belongs to $A_1(\mathbb{R}^n)$, with $[\omega_\theta]_{A_1(\mathbb{R}^n)}$ depending only on n and θ. Further, applying [8, p. 136, Proposition 7.2(3)], we find that

$$u = c_0^{p-1}\omega_\theta^{1-p} \in A_p(\mathbb{R}^n),$$

with $[u]_{A_p(\mathbb{R}^n)}$ depending only on n, θ and c_0. Moreover, it follows from [18, Theorem 1.1] that the operator norm of $\mathcal{G}$ on $L^p(u)$ with $u \in A_p(\mathbb{R}^n)$ depends only on ϕ, n, p and $[u]_{A_p(\mathbb{R}^n)}$. Noticing that $\omega \leq \omega_\theta$ almost everywhere on $\mathbb{R}^n$, we hence have $u \lesssim \omega^{1-p}$ almost everywhere on $\mathbb{R}^n$. Therefore,

$$\|\mathcal{G}(f)\|_{H^{p,\lambda}(\mathbb{R}^n)} \leq \|\mathcal{G}(f)\|_{L^p(u)} \lesssim \|f\|_{L^p(u)} \lesssim \|f\|_{L^p(\omega^{1-p})} \lesssim 1.$$

This proves that $\mathcal{G}$ is bounded on $H^{p,\lambda}(\mathbb{R}^n)$, and hence finishes the proof of Theorem 2.4. $\square$

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2.2. The upper bound estimate of Theorem 1.1

Proof of the upper bound estimate of Theorem 1.1. We are about to show that

$$\|T_\alpha f\|_{L^{p,\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}, \qquad \forall f \in L^{p,\lambda}(\mathbb{R}^n).$$

By homogeneity, we may assume that $f \in L^{p,\lambda}(\mathbb{R}^n)$ with $\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = 1$. Then it suffices to prove that

$$r_0^{\lambda-n} \int_{B(x_0,r_0)} |T_\alpha f(x)|^p \, dx \lesssim 1, \tag{2.6}$$

where the implicit positive constant is independent of $x_0 \in \mathbb{R}^n$ and $r_0 \in (0,\infty)$.

Split $f = f_1 + f_2$, where $f_1 := f\mathbf{1}_{B(x_0,3r_0)}$ and $f_2 := f\mathbf{1}_{B(x_0,3r_0)^\complement}$. Notice that Lemma 2.1(iii) guarantees that $T_\alpha f_1$ is well defined in $L^p(\mathbb{R}^n)$ and hence almost everywhere on $\mathbb{R}^n$. Moreover, the boundedness of T_α on $L^p(\mathbb{R}^n)$ implies that

$$r_0^{\lambda-n} \int_{B(x_0,r_0)} |T_\alpha f_1(x)|^p \, dx \le r_0^{\lambda-n} \|T_\alpha f_1\|^p_{L^p(\mathbb{R}^n)} \lesssim r_0^{\lambda-n} \|f_1\|^p_{L^p(\mathbb{R}^n)} \lesssim \|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)} \lesssim 1.$$

Thus, the proof of (2.6) reduces to the estimate

$$r_0^{\lambda-n} \int_{B(x_0,r_0)} |T_\alpha f_2(x)|^p \, dx \lesssim 1,$$

which follows directly from the claim that

$$|T_\alpha f_2(x)| \lesssim r_0^{-\lambda/p}, \qquad \forall x \in B(x_0,r_0). \tag{2.7}$$

To prove (2.7), let M be the largest integer less than α and $L := M+1$. For any $x \in B(x_0,r_0)$, we write

$$\begin{aligned}
|T_\alpha f_2(x)|^2 &\le \int_0^\infty \left[\int_{B(\vec{0}_n,1)} \int_{|z-x|\ge 2r_0} |p_{r,y}(x-z) f(z)| \, dz \, dy \right]^2 \frac{dr}{r^{1+2\alpha}} \\
&\le \int_0^\infty \left[\int_{B(\vec{0}_n,1)} \int_{|z-x|\ge 2\max\{r,r_0\}} |p_{r,y}(x-z) f(z)| \, dz \, dy \right]^2 \frac{dr}{r^{1+2\alpha}} \\
&\quad + \int_{r_0}^\infty \left[\int_{B(\vec{0}_n,1)} \int_{2r>|z-x|\ge 2r_0} \cdots \, dz \, dy \right]^2 \frac{dr}{r^{1+2\alpha}} \\
&=: \mathrm{Y}_1 + \mathrm{Y}_2.
\end{aligned}$$

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For any $y \in B(\vec{0}_n, 1)$ and $|z-x| \geq 2\max\{r, r_0\}$, we have $|ry| \leq r < |x-z|/2$, which, together with the Taylor theorem, implies that

$$|p_{r,y}(x-z)| \lesssim \sup_{\theta \in (0,1)} |ry|^L |x-z+\theta ry|^{\alpha-n-L} \lesssim r^L |x-z|^{\alpha-n-L}$$

and hence, by the Hölder inequality, we further have

$$\begin{aligned}
&\int_{|z-x|\geq 2\max\{r,r_0\}} |p_{r,y}(x-z)||f(z)|\,dz \\
&\lesssim r^L \int_{|z-x|\geq 2\max\{r,r_0\}} |x-z|^{\alpha-n-L}|f(z)|\,dz \\
&\lesssim r^L \sum_{j=1}^{\infty} \int_{2^j \max\{r,r_0\} \leq |z-x| < 2^{j+1}\max\{r,r_0\}} |x-z|^{\alpha-n-L}|f(z)|\,dz \\
&\lesssim r^L \sum_{j=1}^{\infty} (2^j \max\{r, r_0\})^{\alpha-L-\lambda/p} \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \\
&\lesssim r^L (\max\{r, r_0\})^{\alpha-L-\lambda/p}.
\end{aligned}$$

Then an easy calculation leads to that

$$\mathrm{Y}_1 \lesssim \int_0^{\infty} \left[r^L (\max\{r, r_0\})^{\alpha-L-\lambda/p}\right]^2 \frac{dr}{r^{1+2\alpha}} \lesssim r_0^{-2\lambda/p}.$$

To conclude the proof of (2.7), we still need to estimate Y_2. For any $x \in B(x_0, r_0)$, by the expression of $p_{r,y}(x-z)$, we have

$$\begin{aligned}
\mathrm{Y}_2 \lesssim &\int_{r_0}^{\infty} \left[\int_{B(\vec{0}_n,1)} \int_{2r>|z-x|\geq 2r_0} |x-z+ry|^{\alpha-n}|f(z)|\,dz\,dy\right]^2 \frac{dr}{r^{1+2\alpha}} \\
&+ \sum_{\beta\in\mathbb{Z}_+^n,\,|\beta|\leq M} \int_{r_0}^{\infty} \left[r^{|\beta|} \int_{2r>|z-x|\geq 2r_0} |x-z|^{\alpha-n-|\beta|}|f(z)|\,dz\right]^2 \frac{dr}{r^{1+2\alpha}} \\
=: &\ \mathrm{Y}_{2,0} + \sum_{\beta\in\mathbb{Z}_+^n,\,|\beta|\leq M} \mathrm{Y}_{2,\beta}.
\end{aligned}$$

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Let us first estimate $Y_{2,0}$. Since $x \in B(x_0, r_0)$ and $r > r_0$, it follows that

$$\int_{2r>|z-x|\geq 2r_0} |x-z+ry|^{\alpha-n}|f(z)|\,dz \leq \int_{|z-x_0|<3r} |x-z+ry|^{\alpha-n}|f(z)|\,dz \lesssim I_\alpha(|f|\mathbf{1}_{B(x_0,3r)})(x+ry)$$

and hence

$$\begin{aligned}\int_{B(\vec{0}_n,1)}\int_{2r>|z-x|\geq 2r_0} |x-z+ry|^{\alpha-n}|f(z)|\,dz\,dy &\lesssim \int_{B(\vec{0}_n,1)} I_\alpha(|f|\mathbf{1}_{B(x_0,3r)})(x+ry)\,dy\\ &\sim r^{-n}\int_{B(x,r)} I_\alpha(|f|\mathbf{1}_{B(x_0,3r)})(w)\,dw\\ &\lesssim r^{-n}\int_{B(x_0,3r)} I_\alpha(|f|\mathbf{1}_{B(x_0,3r)})(w)\,dw.\end{aligned}$$

According to [6, Theorem 3.1(i)], the last integral can be estimated as follows:

$$\int_{B(x_0,3r)} I_\alpha(|f|\mathbf{1}_{B(x_0,3r)})(w)\,dw \lesssim r^{n+\alpha-\lambda/p}\|f\mathbf{1}_{B(x_0,3r)}\|_{L^{p,\lambda}(\mathbb{R}^n)} \lesssim r^{n+\alpha-\lambda/p}.$$

This in turn gives

$$\int_{B(\vec{0}_n,1)}\int_{2r>|z-x|\geq 2r_0} |x-z+ry|^{\alpha-n}|f(z)|\,dz\,dy \lesssim r^{\alpha-\lambda/p}.$$

Consequently, we have

$$Y_{2,0} \lesssim \int_{r_0}^{\infty} r^{2(\alpha-\lambda/p)}\frac{dr}{r^{1+2\alpha}} \lesssim r_0^{-2\lambda/p}.$$

Now we estimate $Y_{2,\beta}$ with $\beta \in \mathbb{Z}_+^n$ and $|\beta| \leq M$. Observe that

$$\begin{aligned}\int_{2r>|z-x|\geq 2r_0} |x-z|^{\alpha-n-|\beta|}|f(z)|\,dz &\leq \sum_{\{j\in\mathbb{N}:\,2^j r_0<2r\}}\int_{2^{j+1}r_0>|z-x|\geq 2^j r_0} |x-z|^{\alpha-n-|\beta|}|f(z)|\,dz\\ &\lesssim \sum_{\{j\in\mathbb{N}:\,2^j r_0<2r\}} (2^j r_0)^{\alpha-|\beta|-\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}\end{aligned}$$

$$\lesssim \begin{cases} r_0^{\alpha-|\beta|-\lambda/p} & \text{if } \alpha-|\beta|-\lambda/p<0, \\ \ln\dfrac{2r}{r_0} & \text{if } \alpha-|\beta|-\lambda/p=0, \\ r^{\alpha-|\beta|-\lambda/p} & \text{if } \alpha-|\beta|-\lambda/p>0. \end{cases}$$

Notice that $|\beta|\le M<\alpha$. When $\alpha-|\beta|-\lambda/p<0$, we have

$$\mathrm{Y}_{2,\beta} \lesssim r_0^{2(\alpha-|\beta|-\lambda/p)} \int_{r_0}^{\infty} r^{2|\beta|} \frac{dr}{r^{1+2\alpha}} \lesssim r_0^{-2\lambda/p}.$$

When $\alpha-|\beta|-\lambda/p=0$, we obtain

$$\begin{aligned} \mathrm{Y}_{2,\beta} &\lesssim \int_{r_0}^{\infty} r^{2|\beta|}\left(\ln\frac{2r}{r_0}\right)^2 \frac{dr}{r^{1+2\alpha}} \\ &\sim \sum_{j=1}^{\infty} \int_{2^{j-1}r_0}^{2^j r_0} r^{2|\beta|}\left(\ln\frac{2r}{r_0}\right)^2 \frac{dr}{r^{1+2\alpha}} \\ &\lesssim \sum_{j=1}^{\infty} j^2 (2^j r_0)^{2|\beta|-2\alpha} \\ &\lesssim r_0^{-2\lambda/p}. \end{aligned}$$

When $\alpha-|\beta|-\lambda/p>0$, we also have

$$\mathrm{Y}_{2,\beta} \lesssim \int_{r_0}^{\infty} r^{-2\lambda/p} \frac{dr}{r} \lesssim r_0^{-2\lambda/p}.$$

Combining all these three cases, we obtain the desired estimate of $\mathrm{Y}_{2,\beta}$.

Altogether, we obtain $\mathrm{Y}_2 \lesssim r_0^{-2\lambda/p}$, which finishes the proof of (2.7) and hence of the upper bound estimate of Theorem 1.1. □

2.3. *The lower bound estimate of Theorem 1.1*

In this section, we establish the lower bound estimate of Theorem 1.1, via using both a duality argument and Theorem 2.4.

Proof of the lower bound estimate of Theorem 1.1. Due to [7, Theorem 3], we only need to consider the case $\lambda \in (0,n)$. For any non-integer α in $(0,n)$, instead of T_α, we consider the smaller operator $\widetilde{T}_\alpha$ defined in Section 2.1. Then it suffices to prove that

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \lesssim \|\widetilde{T}_\alpha f\|_{L^{p,\lambda}(\mathbb{R}^n)}, \qquad \forall f \in L^{p,\lambda}(\mathbb{R}^n). \tag{2.8}$$

Fix $f \in L^{p,\lambda}(\mathbb{R}^n)$. One may easily verify that $\int_{\mathbb{R}^n}(1+|x|)^{-n-1}|f(x)|\,dx < \infty$, which implies $f \in \mathcal{S}'(\mathbb{R}^n)$ in terms of [30, p. 21, Example (3)]. With all the notations same as in Proposition 2.3, we have

$$f = \int_0^\infty K_r * \phi_r * f \,\frac{dr}{r} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n).$$

From this, it follows that, for any $g \in C_c^\infty(\mathbb{R}^n)$ satisfying $\|g\|_{H^{p',\lambda}(\mathbb{R}^n)} \le 1$,

$$\langle f, g\rangle = \int_0^\infty \langle K_r * \phi_r * f,\ g\rangle \,\frac{dr}{r} = \int_0^\infty \langle K_r * f,\ \widetilde{\phi}_r * g\rangle \,\frac{dr}{r},$$

where $\widetilde{\phi}(\cdot) := \phi(-\cdot)$. Further, by the Fubini theorem, the Hölder inequality, Theorem 2.4 and (1.3), we obtain

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} |K_r * f(x)||\widetilde{\phi}_r * g(x)|\,dx\,\frac{dr}{r} \\
&\quad\le \int_{\mathbb{R}^n} \left[\int_0^\infty |K_r * f(x)|^2 \,\frac{dr}{r}\right]^{1/2} \left[\int_0^\infty |\widetilde{\phi}_r * g(x)|^2 \,\frac{dr}{r}\right]^{1/2} dx \\
&\quad\le \left\| \left[\int_0^\infty |K_r * f|^2 \,\frac{dr}{r}\right]^{1/2} \right\|_{L^{p,\lambda}(\mathbb{R}^n)} \left\| \left[\int_0^\infty |\widetilde{\phi}_r * g|^2 \,\frac{dr}{r}\right]^{1/2} \right\|_{H^{p',\lambda}(\mathbb{R}^n)} \\
&\quad\lesssim \left\| \left[\int_0^\infty |K_r * f|^2 \,\frac{dr}{r}\right]^{1/2} \right\|_{L^{p,\lambda}(\mathbb{R}^n)} \|g\|_{H^{p',\lambda}(\mathbb{R}^n)} \\
&\quad\lesssim \|\widetilde{T}_\alpha f\|_{L^{p,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

Taking supremum over all $g \in C_c^\infty(\mathbb{R}^n)$ with $\|g\|_{H^{p',\lambda}(\mathbb{R}^n)} \le 1$, we find that

$$\sup\left\{\langle f, g\rangle : g \in C_c^\infty(\mathbb{R}^n),\ \|g\|_{H^{p',\lambda}(\mathbb{R}^n)} \le 1\right\} \lesssim \|\widetilde{T}_\alpha f\|_{L^{p,\lambda}(\mathbb{R}^n)}. \tag{2.9}$$

Consequently, applying (1.5) and the density of $C_c^\infty(\mathbb{R}^n)$ in $H^{p',\lambda}(\mathbb{R}^n)$, we conclude that

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup\left\{\langle f, g\rangle : g \in C_c^\infty(\mathbb{R}^n),\ \|g\|_{H^{p',\lambda}(\mathbb{R}^n)} \le 1\right\} \lesssim \|\widetilde{T}_\alpha f\|_{L^{p,\lambda}(\mathbb{R}^n)},$$

which leads to (2.8). This finishes the proof of the lower bound estimate of Theorem 1.1. □

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Remark 2.5. Notice that the lower bound estimate of Theorem 1.2 remains true if we assume only $f \in \mathcal{S}'(\mathbb{R}^n)$ and $T_\alpha f \in L^{p,\lambda}(\mathbb{R}^n)$. Indeed, in this case, the above argument remains valid and we still have (2.9), which indicates that f induces a bounded linear functional on $H^{p',\lambda}(\mathbb{R}^n)$. In other words, f coincides to an element in $(H^{p',\lambda}(\mathbb{R}^n))^* = L^{p,\lambda}(\mathbb{R}^n)$. In this sense, we say that $f \in L^{p,\lambda}(\mathbb{R}^n)$ and $\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \le C\|T_\alpha f\|_{L^{p,\lambda}(\mathbb{R}^n)}$ with the positive constant C independent of f.

3. Proof of Theorem 1.2

In this section, we show Theorem 1.2. To be precise, in Section 3.1, we use the Hardy–Littlewood maximal function to dominate the sharp maximal function of $T_\alpha f$ when $f \in C_c^\infty(\mathbb{R}^n)$. In Sections 3.2 and 3.3, we prove the upper and the lower bound estimates of Theorem 1.2, respectively.

3.1. The sharp maximal function of $T_\alpha f$

For any locally integrable function f on $\mathbb{R}^n$, its *sharp maximal function* $\mathcal{M}^\sharp f$ is defined by setting

$$\mathcal{M}^\sharp f(x) := \sup_{B \ni x} \fint_B |f(y) - f_B|\, dy, \qquad \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ containing x and $f_B := \fint_B f(y)\, dy$.

The main aim of this section is to show the following estimate regarding the sharp maximal function $\mathcal{M}^\sharp(T_\alpha f)$, which is needed in the proof of the upper bound estimate of Theorem 1.2.

Lemma 3.1. *Let $\alpha \in (0, n)$ be a non-integer and $\sigma \in (1, \infty)$. Then there exists a positive constant C such that, for any $f \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$\mathcal{M}^\sharp(T_\alpha f)(x) \le C\left\{\mathcal{M}f(x) + \mathcal{M}^2 f(x) + [\mathcal{M}(|f|^\sigma)(x)]^{1/\sigma}\right\}.$$

Proof. Let $f \in C_c^\infty(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. To obtain the desired estimate for $\mathcal{M}^\sharp(T_\alpha f)(x_0)$, we only need to prove that, for any given ball $B \subset \mathbb{R}^n$ containing x_0, there exists a constant $C_B \in \mathbb{R}$ such that

$$\fint_B |T_\alpha f(x) - C_B|\, dx \lesssim \mathcal{M}f(x_0) + \mathcal{M}^2 f(x_0) + [\mathcal{M}(|f|^\sigma)(x_0)]^{1/\sigma}. \tag{3.1}$$

Now we prove (3.1). Define $f_1 := f\mathbf{1}_{8B}$ and $f_2 := f - f_1$. By Lemma 2.1(iii), we know that $T_\alpha f_2$ is well defined almost everywhere on $\mathbb{R}^n$, so there exists some point $x_B \in B$ such that $T_\alpha f_2(x_B)$ is finite. Choose

$$C_B := T_\alpha f_2(x_B).$$

We claim that

$$|T_\alpha f(x) - T_\alpha f_2(x_B)| \le |T_\alpha f_1(x)| + |T_\alpha f_2(x) - T_\alpha f_2(x_B)|, \qquad \forall x \in B. \tag{3.2}$$

Indeed, if $T_\alpha f(x) > T_\alpha f_2(x_B)$, then the Minkowski inequality implies $T_\alpha f(x) \le T_\alpha f_1(x) + T_\alpha f_2(x)$ and hence

$$T_\alpha f(x) - T_\alpha f_2(x_B) \le T_\alpha f_1(x) + T_\alpha f_2(x) - T_\alpha f_2(x_B) \le |T_\alpha f_1(x)| + |T_\alpha f_2(x) - T_\alpha f_2(x_B)|\,;$$

if $T_\alpha f(x) \le T_\alpha f_2(x_B)$, then the Minkowski inequality also implies $T_\alpha f_2(x) \le T_\alpha f(x) + T_\alpha f_1(x)$ and hence

$$\begin{aligned} T_\alpha f_2(x_B) - T_\alpha f(x) &\le T_\alpha f_2(x_B) - [T_\alpha f_2(x) - T_\alpha f_1(x)] \\ &\le |T_\alpha f_1(x)| + |T_\alpha f_2(x) - T_\alpha f_2(x_B)|\,. \end{aligned}$$

Thus, we obtain (3.2).

By (3.2) and $C_B = T_\alpha f_2(x_B)$, we control the left-hand side of (3.1) by

$$\fint_B |T_\alpha f_1(x)|\,dx + \fint_B |T_\alpha f_2(x) - T_\alpha f_2(x_B)|\,dx.$$

Then the boundedness of T_α on $L^\sigma(\mathbb{R}^n)$ with $\sigma \in (1,\infty)$ gives

$$\fint_B |T_\alpha f_1(x)|\,dx \le \left[\fint_B |T_\alpha f_1(x)|^\sigma\,dx\right]^{1/\sigma} \lesssim \left[\frac{1}{|B|}\int_{\mathbb{R}^n} |f_1(x)|^\sigma\,dx\right]^{1/\sigma} \lesssim [\mathcal{M}(|f|^\sigma)(x_0)]^{1/\sigma}.$$

So the proof of (3.1) falls into estimating

$$\fint_B |T_\alpha f_2(x) - T_\alpha f_2(x_B)|\,dx \lesssim \mathcal{M} f(x_0) + \mathcal{M}^2 f(x_0). \tag{3.3}$$

To show (3.3), we may assume that the radius of B is r_0. Notice that, for any $x \in B$ and $z \in \operatorname{supp} f_2 \subset (8B)^{\complement}$, we have $|z - x_B| \ge 7r_0$ and $\frac{5}{7}|x_B - z| < |x - z| < \frac{9}{7}|x_B - z|$, which implies that

$$\text{either } \min\{|z - x_B|,\, |z - x|\} \ge 5\max\{r, r_0\} \quad \text{or} \quad 5r_0 \le |z - x| < 7r,$$

by considering the two cases $|z - x| \ge 7r$ and $|z - x| < 7r$, respectively. Therefore,

$$\begin{aligned} &|T_\alpha f_2(x) - T_\alpha f_2(x_B)| \\ &\le \left\{\int_0^\infty \left[\int_{B(\vec{0}_n,1)} \int_{\mathbb{R}^n} |p_{r,y}(x - z) - p_{r,y}(x_B - z)||f_2(z)|\,dz\,dy\right]^2 \frac{dr}{r^{1+2\alpha}}\right\}^{1/2} \\ &\le \left\{\int_0^\infty \left[\int_{B(\vec{0}_n,1)} \int_{\min\left\{|z-x_B|,\,|z-x|\right\}\ge 5\max\{r,r_0\}} |p_{r,y}(x - z) - p_{r,y}(x_B - z)|\right.\right. \end{aligned}$$

$$\times |f_2(z)|\,dz\,dy\Bigg]^2 \frac{dr}{r^{1+2\alpha}}\Bigg\}^{1/2}$$

$$+\left\{\int_0^\infty\Bigg[\int_{B(\vec{0}_n,1)}\int_{5r_0\le|z-x|<7r}\cdots\,dz\,dy\Bigg]^2 \frac{dr}{r^{1+2\alpha}}\right\}^{1/2}$$

$$=: \mathrm{Z}_1(x)+\mathrm{Z}_2(x).$$

In the expression of $\mathrm{Z}_2(x)$, the restriction $5r_0 \le |z-x| < 7r$ forces the integral interval of r to be $r > \frac{5}{7}r_0$. Then, to obtain (3.3), we only need to show

$$\mathrm{Z}_1(x) \lesssim \mathcal{M}f(x_0) \quad \text{and} \quad \mathrm{Z}_2(x) \lesssim \mathcal{M}f(x_0)+\mathcal{M}^2 f(x_0), \qquad \forall\, x\in B.$$

This is done by the following two parts.

Part 1) Estimation of $\mathrm{Z}_1(x)$. For any $x \in B$, we have

$$\mathrm{Z}_1(x) \le \left\{\int_0^{r_0}\Bigg[\int_{B(\vec{0}_n,1)}\int_{|z-x|\ge 5r_0} |p_{r,y}(x-z)||f(z)|\,dz\,dy\Bigg]^2 \frac{dr}{r^{1+2\alpha}}\right\}^{1/2}$$

$$+\left\{\int_0^{r_0}\Bigg[\int_{B(\vec{0}_n,1)}\int_{|z-x_B|\ge 5r_0} |p_{r,y}(x_B-z)||f(z)|\,dz\,dy\Bigg]^2 \frac{dr}{r^{1+2\alpha}}\right\}^{1/2}$$

$$+\Bigg\{\int_{r_0}^\infty\Bigg[\int_{B(\vec{0}_n,1)}\int_{\min\{|z-x|,|z-x_B|\}\ge 5r} |p_{r,y}(x-z)-p_{r,y}(x_B-z)|$$

$$\times |f(z)|\,dz\,dy\Bigg]^2 \frac{dr}{r^{1+2\alpha}}\Bigg\}^{1/2}$$

$$=: \mathrm{Z}_{1,1}(x)+\mathrm{Z}_{1,2}(x)+\mathrm{Z}_{1,3}(x).$$

Let M be the largest integer less than α and $L := M+1$. To estimate the term $\mathrm{Z}_{1,1}$, if $|x-z| \ge 5r_0 \ge 5r$, then the Taylor theorem gives

$$|p_{r,y}(x-z)| \lesssim \sup_{\theta\in(0,1)} |ry|^L|x-z+\theta ry|^{\alpha-n-L} \lesssim r^L|x-z|^{\alpha-n-L}$$

and hence

$$\int_{|z-x|\ge 5r_0} |p_{r,y}(x-z)||f(z)|\,dz \lesssim r^L \int_{|z-x|\ge 5r_0} |x-z|^{\alpha-n-L}|f(z)|\,dz$$

$$\lesssim r^L \sum_{j=2}^\infty \int_{2^j r_0\le|z-x|<2^{j+1}r_0} |x-z|^{\alpha-n-L}|f(z)|\,dz$$

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$$\lesssim r^L \sum_{j\in\mathbb{N}} (2^j r_0)^{\alpha-n-L} \int_{|z-x|<2^{j+1}r_0} |f(z)|\,dz$$

$$\lesssim r^L \sum_{j\in\mathbb{N}} (2^j r_0)^{\alpha-n-L} \int_{|z-x_0|<2^{j+2}r_0} |f(z)|\,dz$$

$$\lesssim r^L r_0^{\alpha-L} \mathcal{M}f(x_0).$$

This further implies that

$$\mathrm{Z}_{1,1}(x) \lesssim r_0^{\alpha-L} \mathcal{M}f(x_0) \left[\int_0^{r_0} r^{2L} \frac{dr}{r^{1+2\alpha}}\right]^{1/2} \lesssim \mathcal{M}f(x_0).$$

By an argument similar to that used in the estimation of $\mathrm{Z}_{1,1}$, we also have

$$\mathrm{Z}_{1,2}(x) \lesssim \mathcal{M}f(x_0).$$

For the term $\mathrm{Z}_{1,3}$, if $x,\ x_B \in B$, $z \in (8B)^{\complement}$ and $\min\{|z-x|, |z-x_B|\} \geq 5r \geq 5r_0$, then the Taylor theorem implies that

$$\begin{aligned}
&|p_{r,y}(x-z) - p_{r,y}(x_B - z)| \\
&\quad \leq |x - x_B| \sup_{\theta\in(0,1)} \left|\nabla p_{r,y}(x-z+\theta(x-x_B))\right| \\
&\quad < 2r_0 \sup_{\theta\in(0,1)} \sup_{1\leq j\leq n} \left| (\alpha-n) \sum_{|\beta|=L} \frac{1}{\beta!} (ry)^\beta D^\beta \left(\frac{\xi_j}{|\xi|}|\xi|^{\alpha-n-1}\right)\Big|_{\xi = x-z+\theta(x-x_B)+\tilde{\theta}ry} \right| \\
&\quad \lesssim r_0 r^L \sup_{\theta\in(0,1)} |x-z+\theta(x-x_B)+\widetilde{\theta}ry|^{\alpha-n-1-L} \\
&\quad \lesssim r_0 r^L |x-z|^{\alpha-n-1-L},
\end{aligned}$$

where $\widetilde{\theta} \in (0, 1)$ and the last step is due to

$$|x-z+\theta(x-x_B)+\widetilde{\theta}ry| \geq |x-z| - 2r_0 - r \geq |x-z| - 3r \sim |x-z|.$$

Thus, we have

$$\begin{aligned}
&\int_{\min\{|z-x|,|z-x_B|\}\geq 5r} |p_{r,y}(x-z) - p_{r,y}(x_B-z)||f(z)|\,dz \\
&\quad \lesssim r_0 r^L \int_{|z-x|\geq 5r} |x-z|^{\alpha-n-1-L} |f(z)|\,dz \\
&\quad \lesssim r_0 r^L \sum_{j=2}^{\infty} (2^j r)^{\alpha-n-1-L} \int_{2^j r\leq |z-x|<2^{j+1}r} |f(z)|\,dz
\end{aligned}$$

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$$\lesssim r_0 r^L \sum_{j=2}^{\infty} (2^j r)^{\alpha-n-1-L} \int\limits_{|z-x_0|<2^{j+2}r} |f(z)|\,dz$$
$$\lesssim r_0 r^{\alpha-1} \mathcal{M} f(x_0),$$

which further implies that

$$\mathrm{Z}_{1,3}(x) \lesssim r_0 \mathcal{M} f(x_0) \left[\int_{r_0}^{\infty} r^{2(\alpha-1)} \frac{dr}{r^{1+2\alpha}} \right]^{1/2} \lesssim \mathcal{M} f(x_0).$$

Combining the estimates of $\mathrm{Z}_{1,1}$ through $\mathrm{Z}_{1,3}$ gives $\mathrm{Z}_1(x) \lesssim \mathcal{M} f(x_0)$.

Part 2) Estimation of Z_2. For $5r_0 \le |z-x| < 7r$, applying the mean value theorem, we obtain

$$\begin{aligned}
&|p_{r,y}(x-z) - p_{r,y}(x_B - z)| \\
&\quad \le \left| |x-z+ry|^{\alpha-n} - |x_B - z + ry|^{\alpha-n} \right| + \sum_{|\beta|\le M} \frac{1}{\beta!} r^{|\beta|} \left| |x-z|^{\alpha-n-|\beta|} - |x_B - z|^{\alpha-n-|\beta|} \right| \\
&\quad \le \left| |x-z+ry|^{\alpha-n} - |x_B - z + ry|^{\alpha-n} \right| \\
&\qquad + \sum_{|\beta|\le M} \frac{1}{\beta!} r^{|\beta|} |x - x_B| \sup_{\theta\in(0,1)} |x - z + \theta(x - x_B)|^{\alpha-n-|\beta|-1} \\
&\quad \lesssim \left| |x-z+ry|^{\alpha-n} - |x_B - z + ry|^{\alpha-n} \right| + \sum_{|\beta|\le M} r_0 r^{|\beta|} |x-z|^{\alpha-n-|\beta|-1}
\end{aligned}$$

and hence

$$\begin{aligned}
\mathrm{Z}_2(x) &\lesssim \left\{ \int_{\frac{5}{7}r_0}^{\infty} \left[\int\limits_{B(\vec{0}_n,1)} \int\limits_{\substack{5r_0\le|z-x|<7r \\ |x-z+ry|<4r_0}} \left| |x-z+ry|^{\alpha-n} - |x_B - z + ry|^{\alpha-n} \right| \right.\right. \\
&\qquad\qquad \left.\left. \times |f(z)|\,dz\,dy \right]^2 \frac{dr}{r^{1+2\alpha}} \right\}^{1/2} \\
&\quad + \left\{ \int_{\frac{5}{7}r_0}^{\infty} \left[\int\limits_{B(\vec{0}_n,1)} \int\limits_{\substack{5r_0\le|z-x|<7r \\ |x-z+ry|\ge 4r_0}} \cdots\,dz\,dy \right]^2 \frac{dr}{r^{1+2\alpha}} \right\}^{1/2} \\
&\quad + \left\{ \int_{\frac{5}{7}r_0}^{\infty} \left[\int\limits_{B(\vec{0}_n,1)} \int\limits_{5r_0\le|z-x|<7r} \sum_{|\beta|\le M} r_0 r^{|\beta|} |x-z|^{\alpha-n-|\beta|-1} |f(z)|\,dz\,dy \right]^2 \frac{dr}{r^{1+2\alpha}} \right\}^{1/2} \\
&=: \mathrm{Z}_{2,1}(x) + \mathrm{Z}_{2,2}(x) + \mathrm{Z}_{2,3}(x).
\end{aligned}$$

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To estimate $\mathrm{Z}_{2,1}(x)$, we observe that

$$\begin{aligned}
&\int\limits_{|x-z+ry|<4r_0} |x-z+ry|^{\alpha-n}|f(z)|\,dz \\
&\le \sum_{j=0}^{\infty} \int\limits_{2^{-j+1}r_0\le|x-z+ry|<2^{-j+2}r_0} |x-z+ry|^{\alpha-n}|f(z)|\,dz \\
&\lesssim \sum_{j=0}^{\infty}(2^{-j}r_0)^{\alpha-n} \int\limits_{|x-z+ry|<2^{-j+2}r_0} |f(z)|\,dz \\
&\lesssim r_0^{\alpha}\mathcal{M}f(x+ry).
\end{aligned}$$

Similarly, if $|x-z+ry|<4r_0$, then $|x_B-z+ry|\le|x-z+ry|+|x_B-x|<4r_0+2r_0=6r_0$ and

$$\begin{aligned}
\int\limits_{|x-z+ry|<4r_0} |x_B-z+ry|^{\alpha-n}|f(z)|\,dz &\le \int\limits_{|x_B-z+ry|<6r_0} |x_B-z+ry|^{\alpha-n}|f(z)|\,dz \\
&\lesssim r_0^{\alpha}\mathcal{M}f(x_B+ry).
\end{aligned}$$

Since $r>\frac{5}{7}r_0$ implies that $B(x,r)\cup B(x_B,r)\subset B(x_0,5r)$, we then combine the last two estimates to obtain

$$\begin{aligned}
\mathrm{Z}_{2,1}(x) &\lesssim r_0^{\alpha}\left\{\int_{\frac{5}{7}r_0}^{\infty}\left[\int\limits_{B(\vec{0}_n,1)}[\mathcal{M}f(x+ry)+\mathcal{M}f(x_B+ry)]\,dy\right]^2\frac{dr}{r^{1+2\alpha}}\right\}^{1/2} \\
&\lesssim r_0^{\alpha}\left\{\int_{\frac{5}{7}r_0}^{\infty}\left[r^{-n}\int\limits_{B(x,r)}\mathcal{M}f(u)\,du+r^{-n}\int\limits_{B(x_B,r)}\mathcal{M}f(u)\,du\right]^2\frac{dr}{r^{1+2\alpha}}\right\}^{1/2} \\
&\lesssim r_0^{\alpha}\left\{\int_{\frac{5}{7}r_0}^{\infty}\left[r^{-n}\int\limits_{B(x_0,5r)}\mathcal{M}f(u)\,du\right]^2\frac{dr}{r^{1+2\alpha}}\right\}^{1/2} \\
&\lesssim \mathcal{M}^2f(x_0). \qquad (3.4)
\end{aligned}$$

Next we estimate $\mathrm{Z}_{2,2}(x)$. When $|x-z+ry|\ge4r_0$, we use the fact $|x-x_B|<2r_0$ and the mean value theorem to derive that

$$\begin{aligned}
\left||x-z+ry|^{\alpha-n}-|x_B-z+ry|^{\alpha-n}\right| &\lesssim r_0\sup_{\theta\in(0,1)}|x-z+ry+\theta(x_B-x)|^{\alpha-n-1} \\
&\sim r_0|x-z+ry|^{\alpha-n-1}.
\end{aligned}$$

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Since $\alpha \notin \mathbb{N}$, we consider the cases $\alpha \in (0,1)$ and $\alpha \in (1,n)$, respectively. If $\alpha \in (0,1)$, then

$$\begin{aligned}
&\int\limits_{\substack{5r_0 \le |z-x| < 7r \\ |x-z+ry| \ge 4r_0}} \left| |x-z+ry|^{\alpha-n} - |x_B - z + ry|^{\alpha-n} \right| |f(z)|\,dz \\
&\lesssim r_0 \int\limits_{|x-z+ry| \ge 4r_0} |x-z+ry|^{\alpha-n-1} |f(z)|\,dz \\
&\lesssim r_0 \sum_{j=2}^{\infty} \int\limits_{2^j r_0 \le |x-z+ry| < 2^{j+1} r_0} |x-z+ry|^{\alpha-n-1} |f(z)|\,dz \\
&\lesssim r_0 \sum_{j=2}^{\infty} (2^j r_0)^{\alpha-n-1} \int\limits_{|x-z+ry| < 2^{j+1} r_0} |f(z)|\,dz \\
&\lesssim r_0^{\alpha} \mathcal{M} f(x+ry),
\end{aligned}$$

which, together with the argument used in (3.4), implies that

$$\mathrm{Z}_{2,2}(x) \lesssim \mathcal{M}^2 f(x_0).$$

If $\alpha \in (1,n)$, then

$$\begin{aligned}
&\int\limits_{\substack{5r_0 \le |z-x| < 7r \\ |x-z+ry| \ge 4r_0}} \left| |x-z+ry|^{\alpha-n} - |x_B - z + ry|^{\alpha-n} \right| |f(z)|\,dz \\
&\lesssim r_0 \int\limits_{|x-z+ry| < 8r} |x-z+ry|^{\alpha-n-1} |f(z)|\,dz \\
&\lesssim r_0 \sum_{j=0}^{\infty} \int\limits_{2^{-j+2} r \le |x-z+ry| < 2^{-j+3} r} |x-z+ry|^{\alpha-n-1} |f(z)|\,dz \\
&\lesssim r_0 \sum_{j=0}^{\infty} (2^{-j} r)^{\alpha-n-1} \int\limits_{|x-z+ry| < 2^{-j+3} r} |f(z)|\,dz \\
&\lesssim r_0 r^{\alpha-1} \mathcal{M} f(x+ry)
\end{aligned}$$

and hence

$$\mathrm{Z}_{2,2}(x) \lesssim r_0 \left\{ \int\limits_{\frac{5}{7}r_0}^{\infty} r^{2(\alpha-1)} \left[\int\limits_{B(\vec{0}_n,1)} \mathcal{M} f(x+ry)\,dy \right]^2 \frac{dr}{r^{1+2\alpha}} \right\}^{1/2}$$

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$$\lesssim r_0 \left\{ \int_{\frac{5}{7}r_0}^{\infty} r^{2(\alpha-1)} \left[r^{-n} \int_{B(x_0,5r)} \mathcal{M} f(u)\, du \right]^2 \frac{dr}{r^{1+2\alpha}} \right\}^{1/2}$$
$$\lesssim \mathcal{M}^2 f(x_0).$$

Finally, we consider $Z_{2,3}(x)$. From $x,\ x_0 \in B$ and $5r_0 \le |z-x| < 7r$, we easily deduce that $|z-x| \sim |z-x_0|$ and $2r_0 \le |z-x_0| < 10r$. With this, for any multi-index β satisfying $|\beta| \le M-1 < \alpha-1$, we have

$$\begin{aligned}\int_{5r_0\le|z-x|<7r} |x-z|^{\alpha-n-|\beta|-1}|f(z)|\,dz &\lesssim \int_{2r_0\le|z-x_0|<10r} |z-x_0|^{\alpha-n-|\beta|-1}|f(z)|\,dz\\ &\lesssim \sum_{j=0}^{\infty} \int_{2^{-j+3}r\le|z-x_0|<2^{-j+4}r} |z-x_0|^{\alpha-n-|\beta|-1}|f(z)|\,dz\\ &\lesssim \sum_{j=0}^{\infty} (2^{-j}r)^{\alpha-n-|\beta|-1} \int_{|z-x_0|<2^{-j+4}r} |f(z)|\,dz\\ &\lesssim r^{\alpha-|\beta|-1}\mathcal{M} f(x_0),\end{aligned}$$

which further implies that

$$\begin{aligned}&\left\{ \int_{\frac{5}{7}r_0}^{\infty} \left[\int_{B(\vec{0}_n,1)} \int_{5r_0\le|z-x|<7r} \sum_{|\beta|\le M-1} r_0 r^{|\beta|}|x-z|^{\alpha-n-|\beta|-1}|f(z)|\,dz\,dy \right]^2 \frac{dr}{r^{1+2\alpha}} \right\}^{1/2}\\ &\lesssim r_0 \mathcal{M} f(x_0) \left\{ \int_{\frac{5}{7}r_0}^{\infty} r^{2(\alpha-1)} \frac{dr}{r^{1+2\alpha}} \right\}^{1/2}\\ &\lesssim \mathcal{M} f(x_0). \end{aligned} \tag{3.5}$$

Likewise, for any multi-index β satisfying $|\beta| = M > \alpha-1$, we have

$$\begin{aligned}&\int_{5r_0\le|z-x|<7r} |x-z|^{\alpha-n-|\beta|-1}|f(z)|\,dz\\ &\lesssim \int_{|z-x_0|\ge 2r_0} |z-x_0|^{\alpha-n-|\beta|-1}|f(z)|\,dz\\ &\lesssim \sum_{j=1}^{\infty} \int_{2^j r_0\le|z-x_0|<2^{j+1}r_0} |z-x_0|^{\alpha-n-|\beta|-1}|f(z)|\,dz\end{aligned}$$

$$\lesssim \sum_{j=1}^{\infty} (2^j r_0)^{\alpha-n-|\beta|-1} \int\limits_{|z-x_0|<2^{j+1}r_0} |f(z)|\,dz$$

$$\lesssim r_0^{\alpha-|\beta|-1} \mathcal{M} f(x_0)$$

and hence

$$\left\{ \int\limits_{\frac{5}{7}r_0}^{\infty} \left[\int\limits_{B(\vec{0}_n,1)} \int\limits_{5r_0 \le |z-x|<7r} \sum_{|\beta|=M} r_0 r^{|\beta|} |x-z|^{\alpha-n-|\beta|-1} |f(z)|\,dz\,dy \right]^2 \frac{dr}{r^{1+2\alpha}} \right\}^{1/2}$$

$$\lesssim r_0^{\alpha-M} \mathcal{M} f(x_0) \left\{ \int\limits_{\frac{5}{7}r_0}^{\infty} r^{2M} \frac{dr}{r^{1+2\alpha}} \right\}^{1/2}$$

$$\lesssim \mathcal{M} f(x_0). \tag{3.6}$$

Combining (3.5) and (3.6), we obtain $\mathrm{Z}_{2,3}(x) \lesssim \mathcal{M} f(x_0)$.

Summarizing the estimates of $\mathrm{Z}_{2,1}(x)$ through $\mathrm{Z}_{2,3}(x)$, we conclude that

$$\mathrm{Z}_2(x) \lesssim \mathcal{M} f(x_0) + \mathcal{M}^2 f(x_0),$$

as desired.

Altogether, we complete the proof of Lemma 3.1. □

3.2. *The upper bound estimate of Theorem 1.2*

Denote by $\mathcal{M}_d$ the *dyadic Hardy–Littlewood maximal operator*, that is, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}_d f(x) := \sup_{Q \text{ dyadic},\ Q \ni x} \fint_Q |f(y)|\,dy,$$

where the supremum is taken over all dyadic cubes of $\mathbb{R}^n$ containing x. We need the following well-known result (see, for example, [8, Lemma 7.10]) : if $1 \le p_0 \le p < \infty$ and $\omega \in A_p(\mathbb{R}^n)$, then there exists a positive constant C such that, for any function f satisfying $\mathcal{M}_d f \in L^{p_0}(\omega) \cap L^p(\omega)$,

$$\|\mathcal{M}_d f\|_{L^p(\omega)} \le C \|\mathcal{M}^\sharp f\|_{L^p(\omega)}.$$

Proof of the upper bound estimate of Theorem 1.2. Let $f \in H^{p,\lambda}(\mathbb{R}^n)$. By the definition of $\|\cdot\|_{H^{p,\lambda}(\mathbb{R}^n)}$ in (1.1), we know that there exists a non-negative function ω satisfying (1.2) and

$$\|f\|_{L^p(\omega^{1-p})} \le 2\|f\|_{H^{p,\lambda}(\mathbb{R}^n)}.$$

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By the argument used in the proof of Theorem 2.4, if we let $\widetilde{\omega}_\theta := c_0^{-1}(\mathcal{M}\omega^{1/\theta})^\theta$, where c_0 is some positive constant depending only on n, λ and $\theta \in (\lambda/n, 1)$, then $\widetilde{\omega}_\theta$ satisfies (1.2). Moreover, one has $\nu := \widetilde{\omega}_\theta^{1-p} \in A_p(\mathbb{R}^n)$ and $[\nu]_{A_p(\mathbb{R}^n)}$ depends only on c_0, n, λ and θ. Since $C_c^\infty(\mathbb{R}^n)$ is dense in the weighted Lebesgue space $L^p(\nu)$, it follows that there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that

$$\lim_{j\to\infty} f_j = f$$

holds true both in $L^p(\nu)$ and almost everywhere on $\mathbb{R}^n$.

Applying the reverse Hölder inequality in [12, Theorem 2.3] as well as the self-improvement property of the $A_p(\mathbb{R}^n)$-weight class (see, for example, [8, Corollary 7.6]), we find that there exists $\sigma_0 \in (1, p)$ sufficiently close to 1 and depending only on $[\nu]_{A_p(\mathbb{R}^n)}$ such that $\nu \in A_{p/\sigma}(\mathbb{R}^n)$ whenever $\sigma \in (1, \sigma_0)$, with $[\nu]_{A_{p/\sigma}(\mathbb{R}^n)}$ depending only on c_0, n, λ and θ. Consequently, from [10, Theorem 7.1.9(b)], we deduce the boundedness of $\mathcal{M}$ on the weighted spaces $L^p(\nu)$ and $L^{p/\sigma}(\nu)$, with operator norms depending only on c_0, n, λ and θ.

Fix such a σ. For any $j \in \mathbb{N}$, applying Lemma 3.1 and [8, Lemma 7.10], we conclude that

$$\begin{aligned}
\|T_\alpha f_j\|_{L^p(\nu)} &\le \|\mathcal{M}_d(T_\alpha f_j)\|_{L^p(\nu)} \\
&\lesssim \|\mathcal{M}^\sharp(T_\alpha f_j)\|_{L^p(\nu)} \\
&\lesssim \|\mathcal{M} f_j + \mathcal{M}^2 f_j + [\mathcal{M}(|f_j|^\sigma)]^{1/\sigma}\|_{L^p(\nu)} \\
&\lesssim \|f_j\|_{L^p(\nu)}.
\end{aligned}$$

Notice that $\nu \lesssim \omega^{1-p}$ almost everywhere on $\mathbb{R}^n$. Therefore,

$$\sup_{j\in\mathbb{N}} \|f_j\|_{L^p(\nu)} \lesssim \|f\|_{L^p(\nu)} \lesssim \|f\|_{L^p(\omega^{1-p})} \lesssim \|f\|_{H^{p,\lambda}(\mathbb{R}^n)}.$$

By (1.11), (2.1) and the Fatou lemma, we obtain

$$\|T_\alpha f\|_{L^p(\nu)} \le \left\| \liminf_{j\to\infty} T_\alpha f_j \right\|_{L^p(\nu)} \le \liminf_{j\to\infty} \|T_\alpha f_j\|_{L^p(\nu)} \lesssim \|f\|_{H^{p,\lambda}(\mathbb{R}^n)},$$

which further implies that

$$\|T_\alpha f\|_{H^{p,\lambda}(\mathbb{R}^n)} \le \left\{ \int_{\mathbb{R}^n} [T_\alpha f(x)]^p [\widetilde{\omega}_\theta(x)]^{1-p}\, dx \right\}^{1/p} = \|T_\alpha f\|_{L^p(\nu)} \lesssim \|f\|_{H^{p,\lambda}(\mathbb{R}^n)}.$$

This finishes the proof of the upper bound estimate of Theorem 1.2. $\square$

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3.3. The lower bound estimate of Theorem 1.2

Let $(p, \lambda) \in (1, \infty) \times (0, n)$. According to [5, Section 3.2] (see also [24,25]), the predual of the Zorko space $Z^{p,\lambda}(\mathbb{R}^n)$ is the space

$$L_0^{p',\lambda}(\mathbb{R}^n) := \left\{\text{closure of } C_c^\infty(\mathbb{R}^n) \text{ in } L^{p',\lambda}(\mathbb{R}^n)\right\}.$$

Recall that $H^{p,\lambda}(\mathbb{R}^n) = Z^{p,\lambda}(\mathbb{R}^n)$ with equivalent norms (see [3, Theorem 3.3]). Thus, any $f \in H^{p,\lambda}(\mathbb{R}^n)$ satisfies

$$\|f\|_{H^{p,\lambda}(\mathbb{R}^n)} \sim \sup\left\{|\langle f, g\rangle| : g \in C_c^\infty(\mathbb{R}^n),\ \|g\|_{L^{p',\lambda}(\mathbb{R}^n)} \le 1\right\}, \tag{3.7}$$

with the equivalence constants being positive and independent of f.

Proof of the lower bound estimate of Theorem 1.2. Let $f \in H^{p,\lambda}(\mathbb{R}^n)$. Notice that

$$H^{p,\lambda}(\mathbb{R}^n) \subset \bigcup_{\nu \in A_p(\mathbb{R}^n)} L^p(\nu)$$

implies $H^{p,\lambda}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. With all the notation as in Proposition 2.3, we have

$$f = \int_0^\infty K_r * \phi_r * f \,\frac{dr}{r} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Let $\widetilde{\phi}(\cdot) := \phi(-\cdot)$. For any $g \in C_c^\infty(\mathbb{R}^n)$ satisfying $\|g\|_{L^{p',\lambda}(\mathbb{R}^n)} \le 1$, we obtain

$$\langle f, g\rangle = \int_0^\infty \langle K_r * \phi_r * f,\ g\rangle \frac{dr}{r} = \int_0^\infty \langle K_r * f,\ \widetilde{\phi}_r * g\rangle \frac{dr}{r},$$

which, together with the Fubini theorem and the Hölder inequality as well as Theorem 2.4 and (1.3), implies that

$$\begin{aligned}
\left|\int_0^\infty \langle K_r * f,\ \widetilde{\phi}_r * g\rangle \frac{dr}{r}\right| &\le \int_0^\infty \int_{\mathbb{R}^n} |K_r * f(x)||\widetilde{\phi}_r * g(x)|\,dx\,\frac{dr}{r} \\
&\le \int_{\mathbb{R}^n} \left[\int_0^\infty |K_r * f(x)|^2 \frac{dr}{r}\right]^{\frac12} \left[\int_0^\infty |\widetilde{\phi}_r * g(x)|^2 \frac{dr}{r}\right]^{\frac12} dx \\
&\le \left\|\left[\int_0^\infty |K_r * f|^2 \frac{dr}{r}\right]^{\frac12}\right\|_{H^{p,\lambda}(\mathbb{R}^n)} \left\|\left[\int_0^\infty |\widetilde{\phi}_r * g|^2 \frac{dr}{r}\right]^{\frac12}\right\|_{L^{p',\lambda}(\mathbb{R}^n)} \\
&\lesssim \|T_\alpha f\|_{H^{p,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

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Consequently,

$$\sup\left\{|\langle f, g\rangle| : g \in C_c^\infty(\mathbb{R}^n),\ \|g\|_{L^{p',\lambda}(\mathbb{R}^n)} \le 1\right\} \lesssim \|T_\alpha f\|_{H^{p,\lambda}(\mathbb{R}^n)}. \tag{3.8}$$

Combining this and (3.7) gives

$$\|f\|_{H^{p,\lambda}(\mathbb{R}^n)} \lesssim \|T_\alpha f\|_{H^{p,\lambda}(\mathbb{R}^n)}.$$

This finishes the proof of the lower bound estimate of Theorem 1.2. □

Remark 3.2. The condition $f \in H^{p,\lambda}(\mathbb{R}^n)$ in the lower bound estimate of Theorem 1.2 can be relaxed. For instance, given any $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfying $T_\alpha f \in H^{p,\lambda}(\mathbb{R}^n)$, one can show that $f \in H^{p,\lambda}(\mathbb{R}^n)$ and $\|f\|_{H^{p,\lambda}(\mathbb{R}^n)} \le C\|T_\alpha f\|_{H^{p,\lambda}(\mathbb{R}^n)}$ with the positive constant C independent of f.

Indeed, since the assumption $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ also implies that $f \in \mathcal{S}'(\mathbb{R}^n)$, we still have (3.8) in this case, so that f induces a bounded linear functional on $L_0^{p',\lambda}(\mathbb{R}^n)$. Using the assumption $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ as well as the fact that the dual space of $L_0^{p',\lambda}(\mathbb{R}^n)$ is $H^{p,\lambda}(\mathbb{R}^n)$, we know that f coincides to an $H^{p,\lambda}(\mathbb{R}^n)$-function almost everywhere on $\mathbb{R}^n$, that is, $f \in H^{p,\lambda}(\mathbb{R}^n)$.

4. Proof of Theorem 1.3

In this section, we show Theorem 1.3. In particular, we first establish some basic properties of the Riesz-type capacity $\mathcal{R}_{\alpha,p,\lambda}$ in Section 4.1, and then give the proof of Theorem 1.3 in Section 4.2.

4.1. Properties of the Riesz-type capacity $\mathcal{R}_{\alpha,p,\lambda}$

We begin with the following properties.

Lemma 4.1. *Let $\alpha,\ \lambda \in (0,n)$ and $p \in (1,\infty)$. Then $\mathcal{R}_{\alpha,p,\lambda}$ has the following properties:*

(i) *for any subsets $E_1,\ E_2 \subset \mathbb{R}^n$ satisfying $E_1 \subset E_2$, $\mathcal{R}_{\alpha,p,\lambda}(E_1) \le \mathcal{R}_{\alpha,p,\lambda}(E_2)$;*
(ii) *for any sequence $\{E_j\}_{j\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$,*

$$\left[\mathcal{R}_{\alpha,p,\lambda}\left(\bigcup_{j\in\mathbb{N}} E_j\right)\right]^{1/p} \le \sum_{j\in\mathbb{N}} [\mathcal{R}_{\alpha,p,\lambda}(E_j)]^{1/p}.$$

Proof. Notice that (i) follows directly from the definition of $\mathcal{R}_{\alpha,p,\lambda}$.

To show (ii), we may assume that $\sum_{j\in\mathbb{N}}[\mathcal{R}_{\alpha,p,\lambda}(E_j)]^{1/p} < \infty$; otherwise, (ii) holds true trivially. Then, for any $\varepsilon \in (0,\infty)$ and $j \in \mathbb{N}$, there exists $f_j \ge 0$ such that $I_\alpha f_j \ge \mathbf{1}_{E_j}$ and

$$\|f_j\|^p_{H^{p,\lambda}(\mathbb{R}^n)} \le \mathcal{R}_{\alpha,p,\lambda}(E_j) + 2^{-j}\varepsilon.$$

Let $f := \sup_{j\in\mathbb{N}} f_j$. Clearly, $f \ge 0$ and $I_\alpha f \ge \mathbf{1}_{\cup_{j\in\mathbb{N}} E_j}$. Applying (1.4), we obtain

$$\|f\|_{H^{p,\lambda}(\mathbb{R}^n)} \le \sum_{j\in\mathbb{N}} \|f_j\|_{H^{p,\lambda}(\mathbb{R}^n)} \le \sum_{j\in\mathbb{N}} [\mathcal{R}_{\alpha,p,\lambda}(E_j) + 2^{-j}\varepsilon]^{1/p},$$

which further gives

$$\mathcal{R}_{\alpha,p,\lambda}\left(\bigcup_{j\in\mathbb{N}} E_j\right) \le \|f\|^p_{H^{p,\lambda}(\mathbb{R}^n)} \le \left[\sum_{j\in\mathbb{N}} [\mathcal{R}_{\alpha,p,\lambda}(E_j)]^{1/p} + (2^{\frac{1}{p}} - 1)^{-1}\varepsilon^{1/p}\right]^p.$$

Letting $\varepsilon \to 0$, we obtain (ii). This finishes the proof of Lemma 4.1. $\square$

Next, we show the weak-type capacitary inequality for the Riesz-type capacity $\mathcal{R}_{\alpha,p,\lambda}$.

Lemma 4.2. *Let $\alpha,\ \lambda \in (0,n)$ and $p \in (1,\infty)$. Then, for any $t \in (0,\infty)$ and $f \in H^{p,\lambda}(\mathbb{R}^n)$,*

$$\mathcal{R}_{\alpha,p,\lambda}(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}) \le t^{-p}\|f\|^p_{H^{p,\lambda}(\mathbb{R}^n)}.$$

Proof. For any $t \in \mathbb{R}$, let $E_t := \{x \in \mathbb{R}^n : I_\alpha(|f|)(x) > t\}$ and $f_t := t^{-1}|f|$. Then

$$I_\alpha(f_t)(x) = t^{-1} I_\alpha(|f|)(x) > 1, \qquad \forall x \in E_t.$$

From this, Lemma 4.1(i) and the definition of $\mathcal{R}_{\alpha,p,\lambda}$, we deduce that

$$\mathcal{R}_{\alpha,p,\lambda}(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}) \le \mathcal{R}_{\alpha,p,\lambda}(E_t) \le \|f_t\|^p_{H^{p,\lambda}(\mathbb{R}^n)} = t^{-p}\|f\|^p_{H^{p,\lambda}(\mathbb{R}^n)},$$

which completes the proof of Lemma 4.2. $\square$

Remark 4.3. Let $\alpha,\ \lambda \in (0,n)$ and $p \in (1,\infty)$.

(i) For any $f \in H^{p,\lambda}(\mathbb{R}^n)$, we use Lemma 4.2 to deduce that $\mathcal{R}_{\alpha,p,\lambda}(\{x \in \mathbb{R}^n : |I_\alpha f(x)| = \infty\}) = 0$. That is, $I_\alpha f$ is finite outside a set of vanishing $\mathcal{R}_{\alpha,p,\lambda}$-capacity on $\mathbb{R}^n$.

(ii) Assume further that p satisfies (1.12), which implies $(n-\alpha)p' > \lambda$. For such p', we can choose $\bar{p} = \frac{n}{\lambda/p'+\alpha} \in (1, n/\alpha)$, that is $p' = \frac{\lambda\bar{p}}{n-\alpha\bar{p}}$. Then [19, Theorem 1.1] implies that I_α maps $L^{\bar{p}}(\mathbb{R}^n)$ continuously into $L^{p',\lambda}(\mathbb{R}^n)$. Thus, for any $f \in H^{p,\lambda}(\mathbb{R}^n)$ and $g \in L^{\bar{p}}(\mathbb{R}^n)$, we have

$$|\langle I_\alpha f, g\rangle| = |\langle f, I_\alpha g\rangle| \le \|f\|_{H^{p,\lambda}(\mathbb{R}^n)} \|I_\alpha g\|_{L^{p',\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p,\lambda}(\mathbb{R}^n)} \|g\|_{L^{\bar{p}}(\mathbb{R}^n)},$$

where the implicit positive constant is independent of f and g, which further implies that $I_\alpha f \in L^{(\bar{p})'}(\mathbb{R}^n)$ and hence $I_\alpha f$ is finite almost everywhere on $\mathbb{R}^n$. Therefore, the set $\{x \in \mathbb{R}^n : |I_\alpha f(x)| = \infty\}$ has both zero $\mathcal{R}_{\alpha,p,\lambda}$-capacity and zero Lebesgue measure.

Remark 4.4. Let $\alpha \in (0,n)$ be a non-integer, $\lambda \in (0,n)$ and $p \in (1,\infty)$. Below we only show Theorem 1.3 for the case p satisfying (1.12), because the proof of Theorem 1.3 is almost trivial under the assumptions $\alpha + \lambda > n$ and $p \in [\frac{\lambda}{\alpha+\lambda-n}, \infty)$. Let us be more precise.

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(i) Consider first the case $p = \frac{\lambda}{\alpha+\lambda-n}$. Applying the Hölder inequality (see also [3, Remark 6.3]), we obtain

$$|I_\alpha f(x)| \le \|f\|_{H^{p,\lambda}(\mathbb{R}^n)} \left\| |x-\cdot|^{\alpha-n} \right\|_{L^{p',\lambda}(\mathbb{R}^n)}, \qquad \forall\, x \in \mathbb{R}^n.$$

The translation invariant property of the norm $\|\cdot\|_{L^{p',\lambda}(\mathbb{R}^n)}$ leads to that

$$\left\| |x-\cdot|^{\alpha-n} \right\|^{p'}_{L^{p',\lambda}(\mathbb{R}^n)} = \sup_{(y,r)\in\mathbb{R}^n\times(0,\infty)} r^{\lambda-n} \int_{B(y,r)} |z|^{-\lambda}\, dz,$$

which is a finite number independent of x, because

$$\begin{cases} \text{if } |y|<2r, \text{ then } r^{\lambda-n} \displaystyle\int_{B(y,r)} |z|^{-\lambda}\, dz \le r^{\lambda-n} \int_{B(0,3r)} |z|^{-\lambda}\, dz \lesssim 1; \\ \text{if } |y|\ge 2r, \text{ then } r^{\lambda-n} \displaystyle\int_{B(y,r)} |z|^{-\lambda}\, dz \sim r^{\lambda-n} \int_{B(y,r)} |y|^{-\lambda}\, dz \sim (r/|y|)^{\lambda} \lesssim 1. \end{cases}$$

We therefore obtain that $|I_\alpha f|$ is pointwisely bounded by a positive constant multiple of $\|f\|_{H^{p,\lambda}(\mathbb{R}^n)}$. Meanwhile, we know from [3, Theorem 6.4(i), Corollary 6.2] that $\mathcal{R}_{\alpha,p,\lambda}(\mathbb{R}^n) < \infty$. Combining the above two facts, we easily obtain the desired estimate of Theorem 1.3.

(ii) To treat the case $p > \frac{\lambda}{\alpha+\lambda-n}$, we choose $\delta \in (n-\lambda/p', \alpha)$ and consider the function $f(x) := (1+|x|)^{-\delta}$ for any $x \in \mathbb{R}^n$. Using (1.6), one can easily show that

$$\begin{aligned} \|f\|_{H^{p,\lambda}(\mathbb{R}^n)} &= \sup_{\|g\|_{L^{p',\lambda}(\mathbb{R}^n)}\le 1} \left| \int_{\mathbb{R}^n} (1+|x|)^{-\delta} g(x)\, dx \right| \\ &\le \sup_{\|g\|_{L^{p',\lambda}(\mathbb{R}^n)}\le 1} \left[\int_{|x|<1} g(x)\, dx + \sum_{j=1}^{\infty} \int_{2^{j-1}\le|x|<2^j} (2^{j-1})^{-\delta} g(x)\, dx \right] \\ &\lesssim 1 + \sum_{j=1}^{\infty} 2^{-j(\delta+\lambda/p'-n)} \\ &\lesssim 1. \end{aligned}$$

Meanwhile, by an easy calculation, we obtain

$$I_\alpha f(x) \ge \int_{|y|>1+|x|} |x-y|^{\alpha-n}(1+|y|)^{-\delta}\, dy \ge \int_{|y|>1+|x|} (2|y|)^{\alpha-n-\delta}\, dy = \infty, \quad \forall\, x \in \mathbb{R}^n.$$

Combining the last two facts and Lemma 4.2, we conclude that $\mathcal{R}_{\alpha,p,\lambda}(\mathbb{R}^n) = 0$, which further indicates that Theorem 1.3 holds true automatically.

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Finally, we end this section with a differentiation theorem for the Riesz-type capacity $\mathcal{R}_{\alpha,p,\lambda}$, where $\alpha,\ \lambda \in (0,n)$ and $p \in (1,\infty)$.

Lemma 4.5. *Let $\alpha,\ \lambda \in (0,n)$ and $p \in (1,\infty)$. For any given $f \in H^{p,\lambda}(\mathbb{R}^n)$, it holds*

$$\lim_{\varepsilon\to 0} \fint_{B(\vec{0}_n,1)} |I_\alpha f(x+\varepsilon y) - I_\alpha f(x)|\, dy = 0$$

outside a set of vanishing $\mathcal{R}_{\alpha,p,\lambda}$-capacity on $\mathbb{R}^n$.

Proof. Let $f \in H^{p,\lambda}(\mathbb{R}^n)$. Due to Remark 4.3, we may as well assume that $I_\alpha f$ is pointwisely finite on $\mathbb{R}^n$. Notice that the desired conclusion of Lemma 4.5 follows directly from

$$\mathcal{R}_{\alpha,p,\lambda}\left(\left\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0} \fint_{B(\vec{0}_n,1)} |I_\alpha f(x+\varepsilon y) - I_\alpha f(x)|\, dy > 0\right\}\right) = 0.$$

To obtain the above identity, it suffices to show that, for any $\delta \in (0,\infty)$,

$$\mathcal{R}_{\alpha,p,\lambda}\left(\left\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0} \fint_{B(\vec{0}_n,1)} |I_\alpha f(x+\varepsilon y) - I_\alpha f(x)|\, dy > \delta\right\}\right) = 0. \tag{4.1}$$

Indeed, once we have proved (4.1), then Lemma 4.1(ii) implies that

$$\begin{aligned}
&\mathcal{R}_{\alpha,p,\lambda}\left(\left\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0} \fint_{B(\vec{0}_n,1)} |I_\alpha f(x+\varepsilon y) - I_\alpha f(x)|\, dy > 0\right\}\right)\\
&\quad = \mathcal{R}_{\alpha,p,\lambda}\left(\bigcup_{j\in\mathbb{N}}\left\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0} \fint_{B(\vec{0}_n,1)} |I_\alpha f(x+\varepsilon y) - I_\alpha f(x)|\, dy > 2^{-j}\right\}\right)\\
&\quad \lesssim \left\{\sum_{j\in\mathbb{N}}\left[\mathcal{R}_{\alpha,p,\lambda}\left(\left\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0} \fint_{B(\vec{0}_n,1)} |I_\alpha f(x+\varepsilon y) - I_\alpha f(x)|\, dy > 2^{-j}\right\}\right)\right]^{1/p}\right\}^p\\
&\quad = 0,
\end{aligned}$$

as desired.

It remains to show (4.1). For any given $\eta \in (0,\infty)$, by the density of $C_c^\infty(\mathbb{R}^n)$ in $H^{p,\lambda}(\mathbb{R}^n)$, we know that there exists $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|f - \varphi\|_{H^{p,\lambda}(\mathbb{R}^n)} < \eta.$$

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Then we write

$$\fint_{B(\vec{0}_n,1)} |I_\alpha f(x+\varepsilon y) - I_\alpha f(x)|\,dy \le \fint_{B(\vec{0}_n,1)} I_\alpha(|f-\varphi|)(x+\varepsilon y)\,dy + \fint_{B(\vec{0}_n,1)} |I_\alpha \varphi(x+\varepsilon y) - I_\alpha \varphi(x)|\,dy + I_\alpha(|\varphi - f|)(x).$$

To deal with the first term in the right-hand side of the above formula, we let

$$q_\epsilon := |B(\vec{0}_n,\epsilon)|^{-1}\mathbf{1}_{B(\vec{0}_n,\epsilon)} \tag{4.2}$$

and utilize the fact (see [7, (2)]) that there exists a positive constant $C_{(\alpha,n)}$, depending on α and n, such that

$$I_\alpha q_\varepsilon(x) \le C_{(\alpha,n)}|x|^{\alpha-n}, \qquad \forall\, x\in\mathbb{R}^n,\ \forall\,\varepsilon\in(0,\infty),$$

thereby obtaining

$$\fint_{B(\vec{0}_n,1)} I_\alpha(|f-\varphi|)(x+\varepsilon y)\,dy = q_\varepsilon * [I_\alpha(|f-\varphi|)](x) \le C_{(\alpha,n)} I_\alpha(|f-\varphi|)(x).$$

Meanwhile, from $\varphi\in C_c^\infty(\mathbb{R}^n)$, it follows that $I_\alpha\varphi$ is a continuous locally integrable function on $\mathbb{R}^n$, so the Lebesgue differentiation implies that

$$\lim_{\varepsilon\to 0} \fint_{B(\vec{0}_n,1)} |I_\alpha\varphi(x+\varepsilon y) - I_\alpha\varphi(x)|\,dy = 0, \qquad \forall\, x\in\mathbb{R}^n,$$

because any $x\in\mathbb{R}^n$ is a Lebesgue point of $I_\alpha\varphi$. Therefore,

$$\limsup_{\varepsilon\to 0} \fint_{B(\vec{0}_n,1)} |I_\alpha f(x+\varepsilon y) - I_\alpha f(x)|\,dy \le [C_{(\alpha,n)}+1]\, I_\alpha(|f-\varphi|)(x).$$

With this and Lemma 4.2, we conclude that

$$\begin{aligned}
&\mathcal{R}_{\alpha,p,\lambda}\left(\left\{x\in\mathbb{R}^n : \limsup_{\varepsilon\to 0} \fint_{B(\vec{0}_n,1)} |I_\alpha f(x+\varepsilon y) - I_\alpha f(x)|\,dy > \delta\right\}\right)\\
&\quad\le \mathcal{R}_{\alpha,p,\lambda}\left(\left\{x\in\mathbb{R}^n :\ I_\alpha(|f-\varphi|)(x) > \frac{\delta}{C_{(\alpha,n)}+1}\right\}\right)\\
&\quad\lesssim \delta^{-p}\|f-\varphi\|^p_{H^{p,\lambda}(\mathbb{R}^n)}\\
&\quad\lesssim \delta^{-p}\eta^p.
\end{aligned}$$

By this and the arbitrariness of η, we obtain (4.1), which completes the proof of Lemma 4.5. $\square$

4.2. Proof of Theorem 1.3

In this section, we show Theorem 1.3 by first establishing the following three auxiliary lemmas. The first lemma stems from [7, Lemma 1].

Lemma 4.6. *Let $\{f_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $H^{p,\lambda}(\mathbb{R}^n)$, where $\alpha,\ \lambda\in(0,n)$ and p satisfies (1.12). Assume that there exists a measurable function F on $\mathbb{R}^n$ such that*

$$\lim_{j\to\infty} I_\alpha f_j(x) = F(x) \quad \textit{for almost every } x\in\mathbb{R}^n.$$

Then there exists a function $f\in H^{p,\lambda}(\mathbb{R}^n)$ such that

$$\|f\|_{H^{p,\lambda}(\mathbb{R}^n)} \le C\sup_{j\in\mathbb{N}} \|f_j\|_{H^{p,\lambda}(\mathbb{R}^n)}$$

and

$$I_\alpha f(x) = F(x) \quad \textit{for almost every } x\in\mathbb{R}^n,$$

where C is a positive constant independent of f.

Proof. According to [5, Section 3.2] (see also Section 3.3), we know that the predual space of $H^{p,\lambda}(\mathbb{R}^n)$ is $L_0^{p',\lambda}(\mathbb{R}^n)$, where $1/p+1/p'=1$. By the Banach–Alaoglu theorem (see [26, Theorem 3.15]), we know that there exist $f\in H^{p,\lambda}(\mathbb{R}^n)$ and a subsequence of $\{f_j\}_{j\in\mathbb{N}}$, still denoted by $\{f_j\}_{j\in\mathbb{N}}$, such that $\lim_{j\to\infty} f_j = f$ in the weak-$*$ topology, that is,

$$\lim_{j\to\infty}\int_{\mathbb{R}^n} f_j(x)\varphi(x)\,dx = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx, \qquad \forall\,\varphi\in L_0^{p',\lambda}(\mathbb{R}^n).$$

This further gives

$$\|f\|_{H^{p,\lambda}(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{N}} \|f_j\|_{H^{p,\lambda}(\mathbb{R}^n)}.$$

It remains to show that $I_\alpha f(x)=F(x)$ for almost every $x\in\mathbb{R}^n$. Without loss of generality, we may assume that $f=0$ almost everywhere on $\mathbb{R}^n$ and aim to show that $F=0$ almost everywhere on $\mathbb{R}^n$. Define

$$g(x) := |x|^{\alpha-n}\mathbf{1}_{B(\vec{0}_n,1)}(x) \qquad \text{and} \qquad h(x) := |x|^{\alpha-n} - g(x), \qquad \forall\, x\in\mathbb{R}^n.$$

For any $\varepsilon\in(0,\infty)$, let $g_\varepsilon(\cdot) := \varepsilon^{-n} g(\varepsilon^{-1}\cdot)$ and h_ε be defined in a similar way. Observe that

$$|x|^{\alpha-n} = \varepsilon^\alpha g_\varepsilon(x) + \varepsilon^\alpha h_\varepsilon(x), \qquad \forall\, x\in\mathbb{R}^n.$$

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Let us first prove that $h_\varepsilon(x-\cdot) \in L_0^{p',\lambda}(\mathbb{R}^n)$ whenever $x \in \mathbb{R}^n$ and $\varepsilon \in (0,\infty)$. By translation and dilation invariance, this is equivalent to validating that $h \in L_0^{p',\lambda}(\mathbb{R}^n)$. To this end, it suffices to find a sequence $\{h_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that

$$\|h_j - h\|_{L^{p',\lambda}(\mathbb{R}^n)} \to 0 \text{ as } j \to \infty. \tag{4.3}$$

Indeed, by the Hölder inequality, one easily finds that

$$\|\cdot\|_{L^{p',\lambda}(\mathbb{R}^n)} \le \|\cdot\|_{L^{p'n/\lambda}(\mathbb{R}^n)}.$$

Meanwhile, noticing that the assumption (1.12) implies that $\frac{(\alpha-n)p'n}{\lambda} + n < 0$, we then have

$$\|h\|_{L^{p'n/\lambda}(\mathbb{R}^n)} = \left[\int_{|x|\ge 1} |x|^{\frac{(\alpha-n)p'n}{\lambda}}\, dx\right]^{\frac{\lambda}{p'n}} \sim \left[\int_1^\infty \rho^{\frac{(\alpha-n)p'n}{\lambda}+n-1}\, d\rho\right]^{\frac{\lambda}{p'n}} < \infty.$$

This gives (4.3), because any function in the Lebesgue space $L^{p'n/\lambda}(\mathbb{R}^n)$ can be approximated by functions in $C_c^\infty(\mathbb{R}^n)$.

For any $x \in \mathbb{R}^n$ and $\varepsilon \in (0,\infty)$, by $h_\varepsilon(x-\cdot) \in L_0^{p',\lambda}(\mathbb{R}^n)$ and the convergence of $\{f_j\}_{j\in\mathbb{N}} \subset H^{p,\lambda}(\mathbb{R}^n)$ to $f=0$ in the weak-$*$ topology, we obtain

$$\lim_{j\to\infty} h_\varepsilon * f_j(x) = \lim_{j\to\infty} \int_{\mathbb{R}^n} h_\varepsilon(x-y) f_j(y)\, dy = 0$$

and hence, for almost every $x \in \mathbb{R}^n$,

$$F(x) = \lim_{j\to\infty} I_\alpha f_j(x) = \varepsilon^\alpha \lim_{j\to\infty} g_\varepsilon * f_j(x) + \varepsilon^\alpha \lim_{j\to\infty} h_\varepsilon * f_j(x) = \varepsilon^\alpha \lim_{j\to\infty} g_\varepsilon * f_j(x).$$

Further, for any $\varphi \in L_0^{p',\lambda}(\mathbb{R}^n)$ with $\|\varphi\|_{L^{p',\lambda}(\mathbb{R}^n)} \le 1$, applying the Fatou lemma and the Fubini theorem, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \lim_{j\to\infty} (g_\varepsilon * f_j)(x)\varphi(x)\, dx\right| &\le \liminf_{j\to\infty} \int_{\mathbb{R}^n} |(g_\varepsilon * f_j)(x)\varphi(x)|\, dx \\
&\le \liminf_{j\to\infty} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |g_\varepsilon(y)||f_j(x-y)|\, dy\right] |\varphi(x)|\, dx \\
&\le \liminf_{j\to\infty} \int_{\mathbb{R}^n} |g_\varepsilon(y)| \int_{\mathbb{R}^n} |f_j(x-y)||\varphi(x)|\, dx\, dy \\
&= \liminf_{j\to\infty} \int_{\mathbb{R}^n} |g_\varepsilon(y)| \int_{\mathbb{R}^n} |f_j(z)||\varphi(y+z)|\, dz\, dy
\end{aligned}$$

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$$\le \liminf_{j\to\infty} \int_{\mathbb{R}^n} |g_\varepsilon(y)|\, \|f_j\|_{H^{p,\lambda}(\mathbb{R}^n)} \|\varphi(y+\cdot)\|_{L^{p',\lambda}(\mathbb{R}^n)}\, dy$$

$$\le \sup_{j\in\mathbb{N}} \|f_j\|_{H^{p,\lambda}(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}.$$

Therefore, after taking supremum over all $\varphi \in L_0^{p',\lambda}(\mathbb{R}^n)$ with $\|\varphi\|_{L^{p',\lambda}(\mathbb{R}^n)} \le 1$, we derive that

$$\|F\|_{H^{p,\lambda}(\mathbb{R}^n)} \sim \varepsilon^\alpha \left\| \lim_{j\to\infty} g_\varepsilon * f_j \right\|_{H^{p,\lambda}(\mathbb{R}^n)} \lesssim \varepsilon^\alpha \sup_{j\in\mathbb{N}} \|f_j\|_{H^{p,\lambda}(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} \to 0$$

as $\varepsilon \to 0$, which implies that $F = 0$ almost everywhere on $\mathbb{R}^n$. This finishes the proof of Lemma 4.6. $\square$

Denote by $C^\infty(\mathbb{R}_+)$ the set of all infinitely differentiable functions on $\mathbb{R}_+ := [0, \infty)$. Applying Lemma 4.6, we show the following analogue of [7, Lemma 3], especially the condition $\phi \in C_c^\infty(\mathbb{R}_+)$ in [7, Lemma 3] is now relaxed to $\phi \in C^\infty(\mathbb{R}_+)$.

Lemma 4.7. *Let $\alpha \in (0, n)$ be a non-integer, $\lambda \in (0, n)$ and p satisfy (1.12). Assume that $\phi \in C^\infty(\mathbb{R}_+)$ satisfies*

$$|t^{j-1}\phi^{(j)}(t)| \le L, \qquad \forall\, t \in \mathbb{R}_+,\ \forall\, j \in \{0, 1, \dots, \lceil\alpha\rceil\}, \tag{4.4}$$

where L is a positive constant independent of t. Then, for any non-negative function $f \in C_c^\infty(\mathbb{R}^n)$, there exists a function $g \in H^{p,\lambda}(\mathbb{R}^n)$ such that $\phi(I_\alpha f) = I_\alpha g$ outside a set of vanishing $\mathcal{R}_{\alpha,p,\lambda}$-capacity on $\mathbb{R}^n$ and

$$\|g\|_{H^{p,\lambda}(\mathbb{R}^n)} \le C\|f\|_{H^{p,\lambda}(\mathbb{R}^n)},$$

where C is a positive constant depending only on α, p, n and L.

Proof. Let us first show the conclusion of Lemma 4.7 with an additional assumption $\phi \in C_c^\infty(\mathbb{R}_+)$. Since $f \ge 0$ and $f \in C_c^\infty(\mathbb{R}^n)$, it follows that $0 \le I_\alpha f \in C^\infty(\mathbb{R}^n)$ and $\phi(I_\alpha f) \in C_c^\infty(\mathbb{R}^n)$. By this and [16, p. 74], we know that there exists $g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\phi(I_\alpha f) = I_\alpha g$ pointwise on $\mathbb{R}^n$. As was proved in [7, (10)], one has

$$T_\alpha g(x) \lesssim \mathcal{M}f(x) + T_\alpha f(x), \qquad \forall\, x \in \mathbb{R}^n.$$

Recall that $\mathcal{M}$ is bounded on $H^{p,\lambda}(\mathbb{R}^n)$ whenever $p \in (1, \infty)$ and $\lambda \in (0, n)$; see [13, Lemma 2.12]. With this and Remark 3.2, we find that $g \in H^{p,\lambda}(\mathbb{R}^n)$ and

$$\|g\|_{H^{p,\lambda}(\mathbb{R}^n)} \lesssim \|T_\alpha g\|_{H^{p,\lambda}(\mathbb{R}^n)} \lesssim \|\mathcal{M}f + T_\alpha f\|_{H^{p,\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p,\lambda}(\mathbb{R}^n)}.$$

Now, we assume that $\phi \in C^\infty(\mathbb{R}_+)$ satisfies (4.4), but has no compact support. By [7, p. 264], we know that there exists $\{\phi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}_+)$ satisfying (4.4) uniformly and

$$\lim_{j\to\infty} \phi_j(t) = \phi(t), \qquad \forall\, t \in (0, \infty).$$

According to the above already proved fact, we find $\{g_j\}_{j\in\mathbb{N}} \subset L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that

$$\phi_j(I_\alpha f)(x) = I_\alpha g_j(x), \qquad \forall\, x \in \mathbb{R}^n,\ \forall\, j \in \mathbb{N},$$

and

$$\sup_{j\in\mathbb{N}} \|g_j\|_{H^{p,\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p,\lambda}(\mathbb{R}^n)}.$$

Notice that

$$\lim_{j\to\infty} I_\alpha g_j(x) = \lim_{j\to\infty} \phi_j(I_\alpha f)(x) = \phi(I_\alpha f)(x), \qquad \forall\, x \in \mathbb{R}^n.$$

By these and Lemma 4.6, we find that there exists $g \in H^{p,\lambda}(\mathbb{R}^n)$ such that, for almost every $x \in \mathbb{R}^n$,

$$I_\alpha g(x) = \lim_{j\to\infty} I_\alpha g_j(x) = \phi(I_\alpha f)(x) \tag{4.5}$$

and

$$\|g\|_{H^{p,\lambda}(\mathbb{R}^n)} \le \sup_{j\in\mathbb{N}} \|g_j\|_{H^{p,\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p,\lambda}(\mathbb{R}^n)}.$$

For any $\varepsilon \in (0,\infty)$, define q_ε as in (4.2). Then it follows from (4.5) that

$$q_\varepsilon * I_\alpha g(x) = q_\varepsilon * \phi(I_\alpha f)(x), \qquad \forall\, x \in \mathbb{R}^n,\ \forall\, \varepsilon \in (0,\infty).$$

On the one hand, by Lemma 4.5, there exists a subset $E^* \subset \mathbb{R}^n$ such that $\mathcal{R}_{\alpha,p,\lambda}(E^*) = 0$ and

$$\lim_{\varepsilon\to 0} |q_\varepsilon * I_\alpha g(x) - I_\alpha g(x)| \le \lim_{\varepsilon\to 0} \fint_{B(\vec{0}_n,1)} |I_\alpha g(x+\varepsilon y) - I_\alpha g(x)|\, dy = 0, \quad \forall\, x \in \mathbb{R}^n \setminus E^*.$$

On the other hand, by the mean value theorem and (4.4) as well as Lemma 4.5, there exists another subset $E^{**} \subset \mathbb{R}^n$ such that $\mathcal{R}_{\alpha,p,\lambda}(E^{**}) = 0$ and that, for any $x \in \mathbb{R}^n \setminus E^{**}$,

$$\begin{aligned}
\lim_{\varepsilon\to 0} |q_\varepsilon * \phi(I_\alpha f)(x) - \phi(I_\alpha f)(x)| &\le \lim_{\varepsilon\to 0} \fint_{B(\vec{0}_n,1)} |\phi(I_\alpha f)(x+\varepsilon y) - \phi(I_\alpha f)(x)|\, dy \\
&\le L \lim_{\varepsilon\to 0} \fint_{B(\vec{0}_n,1)} |I_\alpha f(x+\varepsilon y) - I_\alpha f(x)|\, dy \\
&= 0.
\end{aligned}$$

Combining the last three formulae, we obtain

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$$I_\alpha g(x)=\phi(I_\alpha f)(x), \qquad \forall x\in\mathbb{R}^n\setminus(E^*\cup E^{**}).$$

Also, by Lemma 4.1(ii), we have $\mathcal{R}_{\alpha,p,\lambda}(E^*\cup E^{**})=0$, which completes the proof of Lemma 4.7. □

Via an argument similar to that used in the proof of the upper bound estimate of Theorem 1.2, we have the following boundedness result for a variant of the operator T_α.

Lemma 4.8. *Let $\alpha\in(0,n)$ be a non-integer, $p\in(1,\infty)$ and $\lambda\in(0,n)$. Then there exists a constant $\sigma_0\in(1,p)$, depending only on α, n, p and λ, such that, for any $s\in(1,\sigma_0)$ and $f\in H^{p,\lambda}(\mathbb{R}^n)$,*

$$T_\alpha^s f(x):=\left\{\int_0^\infty\left[\int_{B(\vec{0}_n,1)}|p_{r,y}*f(x)|^s\,dy\right]^{2/s}\frac{dr}{r^{1+2\alpha}}\right\}^{1/2}$$

is well defined for almost every $x\in\mathbb{R}^n$. Moreover, there exists a positive constant C such that

$$\|T_\alpha^s f\|_{H^{p,\lambda}(\mathbb{R}^n)}\le C\|f\|_{H^{p,\lambda}(\mathbb{R}^n)}, \qquad \forall f\in H^{p,\lambda}(\mathbb{R}^n).$$

Proof. Given any $s\in(1,\infty)$, if we have already proved that, for any $f\in C_c^\infty(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$\mathcal{M}^\sharp(T_\alpha^s f)(x)\lesssim\mathcal{M}f(x)+\mathcal{M}^2f(x)+[\mathcal{M}(|\mathcal{M}f|^s)(x)]^{1/s}+[\mathcal{M}(|f|^s)(x)]^{1/s}, \tag{4.6}$$

then the conclusion of Lemma 4.8 follows from the same argument as that used in Section 3.2 whenever $s\in(1,\sigma_0)$. In particular, σ_0 is the same as the one appeared in Section 3.2, which depends only on α, n, p and λ.

Now, we fix $x_0\in\mathbb{R}^n$ and validate (4.6) for $\mathcal{M}^\sharp(T_\alpha^s f)(x_0)$. According to the proof of Lemma 3.1, we only need to show that, for any given ball $B\subset\mathbb{R}^n$ containing x_0,

$$\begin{aligned}&\fint_B|T_\alpha^s f(x)-T_\alpha^s f_2(x_B)|\,dx\\ &\quad\lesssim\mathcal{M}f(x_0)+\mathcal{M}^2f(x_0)+[\mathcal{M}(|\mathcal{M}f|^s)(x_0)]^{1/s}+[\mathcal{M}(|f|^s)(x_0)]^{1/s},\end{aligned} \tag{4.7}$$

where $f=f_1+f_2$ with $f_1:=f\mathbf{1}_{8B}$ and $f_2:=f-f_1$, and x_B is a point in B satisfying $T_\alpha^s f_2(x_B)<\infty$. The existence of such a point x_B is guaranteed by Lemma 2.1(iii).

Notice that (3.2) remains true, but with T_α therein replaced by T_α^s, which implies that

$$\fint_B|T_\alpha^s f(x)-T_\alpha^s f_2(x_B)|\,dx\le\fint_B|T_\alpha^s f_1(x)|\,dx+\fint_B|T_\alpha^s f_2(x)-T_\alpha^s(f_2)(x_B)|\,dx.$$

By the Hölder inequality and the fact that T_α^s is bounded on $L^\sigma(\mathbb{R}^n)$ for any given $\sigma\in(1,\infty)$ (see [7, Remark in p. 262]), we obtain

$$\fint_B |T_\alpha^s f_1(x)|\,dx \le \left[\fint_B |T_\alpha^s f_1(x)|^\sigma\,dx\right]^{1/\sigma} \lesssim \left[\frac{1}{|B|}\int_{\mathbb{R}^n} |f_1(x)|^\sigma\,dx\right]^{1/\sigma} \lesssim [\mathcal{M}(|f|^\sigma)(x_0)]^{1/\sigma},$$

which is as desired if we take $\sigma := s$. Thus, the proof of (4.7) falls into estimating

$$\left|T_\alpha^s f_2(x) - T_\alpha^s f_2(x_B)\right| \lesssim \mathcal{M}f(x_0) + \mathcal{M}^2 f(x_0) + [\mathcal{M}(|\mathcal{M}f|^s)(x_0)]^{1/s}, \qquad \forall\, x \in B. \quad (4.8)$$

Similarly to the estimation of (3.3), we use $s \in (1,\infty)$ and the Minkowski inequality to write

$$\begin{aligned}
&\left|T_\alpha^s f_2(x) - T_\alpha^s f_2(x_B)\right| \\
&\quad \le \left\{\int_0^\infty \left[\int_{B(\vec{0}_n,1)} \left\{\int_{\substack{|z-x_B|\ge 5\max\{r,r_0\}\\ |z-x|\ge 5\max\{r,r_0\}}} |p_{r,y}(x-z) - p_{r,y}(x_B-z)||f(z)|\,dz\right\}^s dy\right]^{2/s} \frac{dr}{r^{1+2\alpha}}\right\}^{1/2} \\
&\qquad + \left\{\int_0^\infty \left[\int_{B(\vec{0}_n,1)} \left\{\int_{5r_0\le|z-x|<7r} \cdots\,dz\right\}^s dy\right]^{2/s} \frac{dr}{r^{1+2\alpha}}\right\}^{1/2} \\
&\quad =: \widetilde{\mathrm{Z}}_1(x) + \widetilde{\mathrm{Z}}_2(x).
\end{aligned}$$

By an argument similar to that used in the estimation of Z_1 in the proof of Lemma 3.1, we obtain

$$\widetilde{\mathrm{Z}}_1(x) \lesssim \mathcal{M}f(x_0).$$

For $\widetilde{\mathrm{Z}}_2$, following the estimation of Z_2 in the proof of Lemma 3.1, we write

$$\begin{aligned}
\widetilde{\mathrm{Z}}_2(x) &\lesssim \left\{\int_{\frac{5}{7}r_0}^\infty \left[\int_{B(\vec{0}_n,1)} \left\{\int_{\substack{5r_0\le|z-x|<7r\\ |x-z+ry|<4r_0}} \left||x-z+ry|^{\alpha-n} - |x_B-z+ry|^{\alpha-n}\right|\right.\right.\right. \\
&\qquad \left.\left.\left. \times |f(z)|\,dz\right\}^s dy\right]^{2/s} \frac{dr}{r^{1+2\alpha}}\right\}^{1/2} \\
&\quad + \left\{\int_{\frac{5}{7}r_0}^\infty \left[\int_{B(\vec{0}_n,1)} \left\{\int_{\substack{5r_0\le|z-x|<7r\\ |x-z+ry|\ge 4r_0}} \cdots\,dz\right\}^s dy\right]^{2/s} \frac{dr}{r^{1+2\alpha}}\right\}^{1/2} \\
&\quad + \left\{\int_{\frac{5}{7}r_0}^\infty \left[\int_{B(\vec{0}_n,1)} \left\{\int_{5r_0\le|z-x|<7r} \sum_{|\beta|\le M} r_0 r^{|\beta|}|x-z|^{\alpha-n-|\beta|-1}\right.\right.\right. \\
&\qquad \left.\left.\left. \times |f(z)|\,dz\right\}^s dy\right]^{2/s} \frac{dr}{r^{1+2\alpha}}\right\}^{1/2} \\
&=: \widetilde{\mathrm{Z}}_{2,1}(x) + \widetilde{\mathrm{Z}}_{2,2}(x) + \widetilde{\mathrm{Z}}_{2,3}(x).
\end{aligned}$$

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The treatment of $\mathrm{Z}_{2,3}$ in the proof of Lemma 3.1 directly gives

$$\widetilde{\mathrm{Z}}_{2,3}(x) \lesssim \mathcal{M}^2 f(x_0).$$

Instead of (3.4) in the estimation of $\mathrm{Z}_{2,1}$, we now obtain

$$\begin{aligned}
\widetilde{\mathrm{Z}}_{2,1}(x) &\lesssim r_0^\alpha \left\{ \int_{\frac{5}{7}r_0}^\infty \left[\int_{B(\vec{0}_n,1)} \{\mathcal{M}f(x_0+ry)+\mathcal{M}f(x_B+ry)\}^s\,dy \right]^{2/s} \frac{dr}{r^{1+2\alpha}} \right\}^{1/2} \\
&\lesssim r_0^\alpha \left\{ \int_{\frac{5}{7}r_0}^\infty \left[r^{-n} \int_{B(x_0,r)} \{\mathcal{M}f(u)\}^s\,du + r^{-n} \int_{B(x_B,r)} \{\mathcal{M}f(u)\}^s\,du \right]^{2/s} \frac{dr}{r^{1+2\alpha}} \right\}^{1/2} \\
&\lesssim [\mathcal{M}(|\mathcal{M}f|^s)(x_0)]^{1/s}.
\end{aligned}$$

Similarly, we also have

$$\widetilde{\mathrm{Z}}_{2,2}(x) \lesssim [\mathcal{M}(|\mathcal{M}f|^s)(x_0)]^{1/s}.$$

Altogether, we obtain (4.8), which completes the proof of Lemma 4.8. □

Proof of Theorem 1.3. Without loss of generality, we may assume that

$$0 \le f \in C_c^\infty(\mathbb{R}^n).$$

Due to Remark 4.4, we only need to show Theorem 1.3 under an additional assumption that p satisfies (1.12). To this end, let ϕ be an infinitely differentiable increasing function on $\mathbb{R}$ such that

$$\phi(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0], \\ 1 & \text{if } t \in [1,\infty). \end{cases}$$

For any $j \in \mathbb{Z}$, let $\phi_j(\cdot) := 2^j \phi(2^{2-j}\cdot -1)$. Clearly, the sequence $\{\phi_j\}_{j\in\mathbb{Z}}$ satisfies (4.4) uniformly. For any $j \in \mathbb{Z}$, we apply Lemma 4.7 to find a function $g_j \in H^{p,\lambda}(\mathbb{R}^n)$ such that $\phi_j(I_\alpha f) = I_\alpha g_j$ outside a set of vanishing $\mathcal{R}_{\alpha,p,\lambda}$-capacity on $\mathbb{R}^n$ and $\sup_{j\in\mathbb{Z}} \|g_j\|_{H^{p,\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p,\lambda}(\mathbb{R}^n)}$. From these and Lemma 4.1(i), it follows that

$$\begin{aligned}
\int_0^\infty \mathcal{R}_{\alpha,p,\lambda}\left(\{x\in\mathbb{R}^n:\ I_\alpha f(x) > t\}\right) dt^p &\lesssim \sum_{j\in\mathbb{Z}} 2^{jp} \mathcal{R}_{\alpha,p,\lambda}\left(\left\{x\in\mathbb{R}^n:\ I_\alpha f(x) > 2^j\right\}\right) \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{jp} \mathcal{R}_{\alpha,p,\lambda}\left(\left\{x\in\mathbb{R}^n:\ \phi_j(I_\alpha f(x)) > 2^j\right\}\right)
\end{aligned}$$

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$$
\begin{aligned}
&\sim \sum_{j\in\mathbb{Z}} 2^{jp}\mathcal{R}_{\alpha,p,\lambda}\left(\left\{x\in\mathbb{R}^n:\ I_\alpha g_j(x)>2^j\right\}\right)\\
&\lesssim \sum_{j\in\mathbb{Z}} \|g_j\|^p_{H^{p,\lambda}(\mathbb{R}^n)},
\end{aligned}
$$

where the last step is due to Lemma 4.2. Invoking Theorem 1.2, we obtain the desired result of Theorem 1.3, provided that

$$
\sum_{j\in\mathbb{Z}} \|T_\alpha g_j\|^p_{H^{p,\lambda}(\mathbb{R}^n)} \lesssim \|f\|^p_{H^{p,\lambda}(\mathbb{R}^n)}. \tag{4.9}
$$

Now we show (4.9). On the one hand, we have

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} \|T_\alpha g_j\|^p_{H^{p,\lambda}(\mathbb{R}^n)} &= \sum_{j\in\mathbb{Z}} \inf_\omega \int_{\mathbb{R}^n} |T_\alpha g_j(x)|^p[\omega(x)]^{1-p}\,dx\\
&\le \inf_\omega \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |T_\alpha g_j(x)|^p[\omega(x)]^{1-p}\,dx,
\end{aligned}
$$

where the infimum is taken over all non-negative functions ω on $\mathbb{R}^n$ satisfying (1.2). On the other hand, we deduce from [7, (18)] that

$$
\sum_{j\in\mathbb{Z}} |T_\alpha g_j(x)|^p \lesssim [\mathcal{M}f(x)+T^s_\alpha f(x)]^p, \qquad \forall\, x\in\mathbb{R}^n,
$$

where s can be any number in $(1, \frac{n}{n-\alpha})$. By these, (1.4), Lemma 4.8 and the boundedness of $\mathcal{M}$ on $H^{p,\lambda}(\mathbb{R}^n)$ (see [13, Theorem 2.12]), we conclude that

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} \|T_\alpha g_j\|^p_{H^{p,\lambda}(\mathbb{R}^n)} &\lesssim \|\mathcal{M}f+T^s_\alpha f\|^p_{H^{p,\lambda}(\mathbb{R}^n)}\\
&\lesssim \|\mathcal{M}f\|^p_{H^{p,\lambda}(\mathbb{R}^n)} + \|T^s_\alpha f\|^p_{H^{p,\lambda}(\mathbb{R}^n)}\\
&\lesssim \|f\|^p_{H^{p,\lambda}(\mathbb{R}^n)}.
\end{aligned}
$$

This finishes the proof of (4.9) and hence of Theorem 1.3. $\square$

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