



# Global smooth solvability of a parabolic–elliptic nutrient taxis system in domains of arbitrary dimension

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## Abstract

This paper deals with the nutrient taxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ 0 = \Delta v - uv - \mu v + r(x, t), \end{cases}$$

in a bounded domain  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 1$ , with smooth boundary, where  $\mu \geq 0$  is a parameter and  $r \in C^1(\overline{\Omega} \times [0, \infty))$  is a given nonnegative function.

It is shown that for any prescribed initial data  $u_0 \in W^{1,\infty}(\Omega)$  with  $u_0 > 0$  in  $\overline{\Omega}$ , the corresponding Neumann initial–boundary problem admits a global classical solution. With regard to qualitative aspects, it is moreover, inter alia, seen that if  $r$  additionally satisfies

$$\int_t^{t+1} \int_{\Omega} |\nabla \sqrt{r}|^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

then in the large time limit the solution component  $u$  stabilizes toward the constant  $\frac{1}{|\Omega|} \int_{\Omega} u_0$  with respect to the norm in  $L^1(\Omega)$ , and that if furthermore

$$\sup_{t>0} \|r(\cdot, t)\|_{L^q(\Omega)} < \infty$$

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for some  $q \geq 1$  fulfilling  $q > \frac{n}{2}$ , then  $u$  is uniformly bounded.

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## 1. Introduction

Chemotaxis, the biased movement of individuals along concentration gradients of a chemical, is known as a universal and fundamental migration mechanism in numerous biological contexts, having attracted both experimentalists and theoretical researchers inter alia due to its assured potential to generate aggregation patterns in several relevant situations. The corresponding dynamical prospects seem best understood in cases when an attractive signal is produced by individuals of the respective population themselves, as prototypically described by the celebrated Keller–Segel system [13]

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v + u - v, \end{cases} \quad (1.1)$$

for the unknown cell density  $u = u(x, t)$  and signal concentration  $v = v(x, t)$ . Indeed, known results on the occurrence of exploding solutions in two- and higher-dimensional domains illustrate the drastic extent of destabilization clearly due to the introduction of the cross-diffusive term  $-\nabla \cdot (u \nabla v)$  to the otherwise linear system (1.1) ([8] and [25]).

In comparison to this, the knowledge seems much less developed in the less reinforced case when the attractive signal, instead of being produced by individuals, is rather consumed by the latter. This typically occurs in nutrient-directed motion of very primitive cells, such as in populations of aerobic bacteria like *Bacillus subtilis* ([5]), but apart from that such nutrient taxis mechanisms seem to play an important role in predator–prey interactions in which predators orient their movement toward regions of increasing prey concentration ([12], [15], [19], [4]). As corresponding relatives of (1.1), in addition potentially accounting for spontaneous nutrient decay and for external reproduction of the latter, parabolic problems of the form

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = d \Delta v - \lambda u v - \mu v + r(x, t), \end{cases} \quad (1.2)$$

have been introduced as theoretical descriptions in such situations ([5], [23], [19]).

Due to the essentially dissipative character of the influence exerted by cells on the signal concentration through such consumption mechanisms, it might be expected that in comparison to (1.1), the nutrient taxis system (1.2) should exhibit some considerably stronger tendency toward relaxation, and particularly be unable to spontaneously generate extreme aggregates in the sense of exploding solutions. At the level of rigorous mathematical analysis, this could in fact be confirmed for two-dimensional versions of (1.2) for which indeed in quite general frameworks results on global existence of smooth solutions are available, thus in particular ruling out any blow-up phenomenon (cf. [20], [27], [11], [22] and [26], for example). In three-dimensional

settings, however, corresponding findings on smooth solvability seem yet lacking even in the simplest case when  $r \equiv 0$ , although, after all, for associated Neumann-type initial–boundary value problems certain global weak solutions could be constructed which become smooth and classical at least eventually ([20]).

**Main results.** The purpose of the present work consists in revealing that with regard to rigorous verifiability of the intuitive conjecture that the behavior in nutrient taxis systems should be significantly more dissipation-dominated than in (1.1), the situation becomes much more favorable when the fully parabolic model (1.2) is simplified according to a standard assumption on quasi-stationarity of the chemical concentration. In fact, modeling hypotheses of this form, justifiable in any situation in which chemicals diffuse substantially faster than individuals, have enhanced accessibility to tools from mathematical analysis in numerous studies on corresponding parabolic–elliptic variants of (1.1) (see [10], [16], [17], for instance), thereby partially even opening perspectives for quite refined insight (see e.g. [21], [2] for some recent examples). Accordingly assuming that the attractant in (1.2) diffuses much faster than the population, we shall henceforth consider the parabolic–elliptic nutrient taxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, \ t > 0, \\ 0 = \Delta v - uv - \mu v + r(x, t), & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1.3)$$

where  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 1$ , is a bounded domain with smooth boundary, where  $\mu \geq 0$  is a fixed parameter, and where  $r$  and  $u_0$  are suitably regular given functions on  $\Omega \times (0, \infty)$  and on  $\Omega$ , respectively.

Our first result then states that under mild assumptions on these ingredients, globally defined smooth solutions can always be found, hence ruling out any blow-up phenomenon.

**Theorem 1.1.** *Let  $n \geq 1$  and  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary, and let  $\mu \geq 0$  and  $r \in C^1(\overline{\Omega} \times [0, \infty))$  be nonnegative. Then for any choice of  $u_0 \in W^{1,\infty}(\Omega)$  such that  $u_0 > 0$  in  $\overline{\Omega}$ , the problem (1.3) admits a global classical solution  $(u, v)$ , uniquely determined by the inclusions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) & \text{and} \\ v \in C^{2,0}(\overline{\Omega} \times (0, \infty)), \end{cases} \quad (1.4)$$

for which furthermore  $u > 0$  and  $v \geq 0$  in  $\overline{\Omega} \times [0, \infty)$ .

Beyond this addressing the question how far (1.3) really reflects relaxation, we shall next study the large time behavior of the respective first solution component. In this regard, the following second among our main results reveals that indeed  $u$  approaches its conserved spatial average in the large time limit, provided that the external reproduction rate  $r$  in (1.3) satisfies a condition which, surprisingly, exclusively requires some decay of the gradient  $\nabla \sqrt{r}$ , rather than of  $r$  itself; in fact, we find it worth emphasizing that the following statement asserts stabilization of  $u$  in numerous cases in which  $r$  itself is unbounded, and in which, as we shall see in Proposition 1.3 below, even in some situations in which the signal concentration  $v$  definitely must be unbounded. Here and throughout the sequel, we use the notation  $\overline{\psi} := \frac{1}{|\Omega|} \int_{\Omega} \psi$  for  $\psi \in L^1(\Omega)$ .

**Theorem 1.2.** Let  $\mu \geq 0$ , and suppose that  $r \in C^1(\overline{\Omega} \times [0, \infty))$  is a nonnegative function fulfilling

$$\sqrt{r} \in L^2_{loc}([0, \infty); W^{1,2}(\Omega)) \quad (1.5)$$

as well as

$$\int_t^{t+1} \int_{\Omega} |\nabla \sqrt{r}|^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (1.6)$$

Then for any  $u_0 \in W^{1,\infty}(\Omega)$  which is such that  $u_0 > 0$  in  $\overline{\Omega}$ , the solution  $(u, v)$  of (1.3) satisfies

$$u(\cdot, t) \rightarrow \bar{u}_0 \quad \text{in } L^1(\Omega) \quad \text{as } t \rightarrow \infty \quad (1.7)$$

and that

$$\int_t^{t+1} \|\nabla v^{\frac{1}{4}}(\cdot, s)\|_{L^2(\Omega)}^4 ds \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (1.8)$$

If furthermore there exists  $q \geq 1$  such that  $q > \frac{n}{2}$  and

$$\sup_{t>0} \|r(\cdot, t)\|_{L^q(\Omega)} < \infty, \quad (1.9)$$

then  $u$  belongs to  $L^\infty(\Omega \times (0, \infty))$ , and we even have

$$u(\cdot, t) \rightarrow \bar{u}_0 \quad \text{in } L^p(\Omega) \quad \text{for all } p \in [1, \infty) \quad \text{and} \quad u(\cdot, t) \xrightarrow{*} \bar{u}_0 \quad \text{in } L^\infty(\Omega) \quad \text{as } t \rightarrow \infty. \quad (1.10)$$

Not only to underline the above, let us finally add some qualitative information on how the large time behavior of  $v$  may be influenced by boundedness properties of  $r$ , and by positivity assumptions of  $\mu$ .

**Proposition 1.3.** Suppose that  $\mu \geq 0$ , that  $u_0 \in W^{1,\infty}(\Omega)$  is positive in  $\overline{\Omega}$ , and that  $r \in C^1(\overline{\Omega} \times [0, \infty))$  is nonnegative, and let  $(u, v)$  denote the corresponding solution of (1.3).

i) We have

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \geq \frac{1}{\int_{\Omega} u_0 + \mu|\Omega|} \cdot \int_{\Omega} r(\cdot, t) \quad \text{for all } t > 0. \quad (1.11)$$

In particular, if  $(r(\cdot, t))_{t>0}$  is unbounded in  $L^1(\Omega)$ , then

$$\limsup_{t \rightarrow \infty} \|v(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \quad (1.12)$$

ii) If  $\mu > 0$  and

$$\sup_{t>0} \|r(\cdot, t)\|_{L^q(\Omega)} < \infty \quad \text{for some } q \geq 1 \text{ such that } q > \frac{n}{2}, \quad (1.13)$$

then there exists  $C > 0$  such that

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t > 0. \quad (1.14)$$

## 2. Global existence of classical solutions. Proof of Theorem 1.1

Unfortunately, mainly due to lacking strict positivity of the Neumann Laplacian on  $\Omega$  which exclusively forms the linear part in the second equation at least in the case  $\mu = 0$  to be explicitly included in our analysis, the system (1.3) apparently fails to fall among any class of taxis-type systems for which standard approaches more or less directly yield local existence and convenient extensibility criteria, typically ensuring that prolongation of a solution is possible once, for instance, the spatial  $L^\infty$  norm of its first component remains bounded (cf. e.g. [3] for a detailed demonstration in the context of a parabolic–elliptic system simplifying a variant of (1.1)). In fact, to warrant invertibility, after all, of the Helmholtz operator  $\varphi \mapsto -\Delta\varphi + u\varphi$ , it seems in order to invest some appropriate positivity properties of the function  $u$  which, in turn, forms part of the unknown solution. Pursuing an approach which on the one hand we find mathematically convenient in this regard, and which on the other hand will turn out to be sufficient for our purposes, relying on our overall assumption on positivity of  $u_0$  we shall first derive the following result on local existence, additionally providing an extensibility criterion involving a positivity property of  $u$ .

**Lemma 2.1.** *Let  $\mu \geq 0$  and  $r \in C^1(\bar{\Omega} \times [0, \infty))$  be nonnegative, and suppose that  $u_0 \in W^{1,\infty}(\Omega)$  is such that  $u_0 > 0$  in  $\bar{\Omega}$ . Then there exist  $T_{\max} \in (0, \infty]$  and a uniquely determined pair  $(u, v)$  of functions*

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) & \text{and} \\ v \in C^{2,0}(\bar{\Omega} \times (0, T_{\max})), \end{cases} \quad (2.1)$$

with  $u > 0$  and  $v \geq 0$  in  $\bar{\Omega} \times [0, T_{\max})$ , such that  $(u, v)$  solves (1.3) in the classical sense in  $\Omega \times (0, T_{\max})$ , and that

$$\text{if } T_{\max} < \infty, \text{ then } \liminf_{t \nearrow T_{\max}} \inf_{x \in \Omega} u(x, t) = 0. \quad (2.2)$$

Moreover,

$$\int_{\Omega} u(\cdot, t) = \int_{\Omega} u_0 \quad \text{for all } t \in (0, T_{\max}). \quad (2.3)$$

**Proof.** In order to create an appropriate fixed point framework, let us first recall a standard result from elliptic regularity theory ([7]) to fix  $\theta_1 \in (0, 1)$  and  $c_1 > 0$  such that

$$\|\varphi\|_{C^{1+\theta_1}(\bar{\Omega})} \leq c_1 \|\Delta\varphi + \varphi\|_{L^\infty(\Omega)} \quad \text{for all } \varphi \in C^2(\bar{\Omega}) \text{ such that } \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega, \quad (2.4)$$

and introduce the positive numbers

$$\eta := \min \left\{ 1, \inf_{x \in \Omega} u_0(x) \right\} \quad \text{and} \quad M := 2\|u_0\|_{L^\infty(\Omega)} \quad (2.5)$$

as well as

$$\delta := \frac{\eta}{2} \quad \text{and} \quad R := \|r\|_{L^\infty(\Omega \times (0,1))}. \quad (2.6)$$

Then writing

$$c_2 := \frac{c_1 \cdot (M + \mu + 1) \cdot R}{\delta + \mu} + c_1 R \quad (2.7)$$

and

$$T := \min \left\{ 1, \frac{\delta + \mu}{(1 + \mu)R} \ln 2, \frac{1}{R} \ln 2 \right\}, \quad (2.8)$$

by means of parabolic regularity theory ([18]) we obtain  $\theta_2 \in (0, 1)$  and  $c_3 > 0$  such that whenever  $a \in C^0(\bar{\Omega} \times (0, T); \mathbb{R}^n)$ ,  $b \in C^0(\bar{\Omega} \times (0, T))$  and  $z \in C^0(\bar{\Omega} \times [0, T]) \cap C^{2,1}(\bar{\Omega} \times (0, T))$  are such that

$$\|a\|_{L^\infty(\Omega \times (0, T))} \leq c_2 \quad \text{and} \quad \|b\|_{L^\infty(\Omega \times (0, T))} \leq \frac{(M + \mu)R}{\delta + \mu} + R \quad (2.9)$$

and that

$$\begin{cases} z_t = \Delta z + a(x, t) \cdot \nabla z + b(x, t)z, & x \in \Omega, \ t \in (0, T), \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \ t \in (0, T), \\ z(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (2.10)$$

we have

$$\|z\|_{C^{\theta_2, \frac{\theta_2}{2}}(\bar{\Omega} \times [0, T])} \leq c_3. \quad (2.11)$$

Then taking any  $\theta \in (0, \theta_2)$ , in the closed convex subset

$$S := \left\{ \varphi \in X \mid \delta \leq \varphi \leq M \text{ in } \bar{\Omega} \times [0, T] \right\}$$

of the Banach space  $X := C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [0, T])$ , given  $\bar{u} \in S$  we use that  $\mu \geq 0$ , and that  $r(\cdot, t) \geq 0$  and  $\delta \leq \bar{u}(\cdot, t) \leq M$  in  $\bar{\Omega}$  for all  $t \in (0, T)$ , to see employing the Lax–Milgram lemma that for any such  $t$  the problem

$$\begin{cases} -\Delta v + \bar{u}(x, t)v + \mu v = r(x, t), & x \in \Omega, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \quad (2.12)$$

admits a uniquely determined weak solution  $v(\cdot, t) \in W^{1,2}(\Omega)$ . By standard bootstrap arguments relying on the inclusions  $\bar{u}(\cdot, t) \in C^\theta(\bar{\Omega})$  and  $r(\cdot, t) \in C^1(\bar{\Omega})$  for  $t \in (0, T)$ , it can moreover readily be verified that  $v(\cdot, t)$  actually belongs to  $C^{2+\theta}(\bar{\Omega})$  and satisfies (2.12) in the classical sense for all  $t \in (0, T)$ , whence the classical comparison principle ([7, Theorem 3.5]) becomes applicable so as to ensure that

$$0 \leq v(\cdot, t) \leq \frac{R}{\delta + \mu} \quad \text{in } \Omega \quad \text{for all } t \in (0, T), \quad (2.13)$$

because for  $\underline{v} := 0$  and  $\bar{v} := \frac{R}{\delta + \mu}$  we have  $-\Delta \underline{v} + \bar{u}(\cdot, t)\underline{v} + \mu \underline{v} - r(\cdot, t) = -r(\cdot, t) \leq 0$  and

$$-\Delta \bar{v} + \bar{u}(\cdot, t)\bar{v} + \mu \bar{v} - r(\cdot, t) = \left(\bar{u}(\cdot, t) + \mu\right) \cdot \frac{R}{\delta + \mu} - r(\cdot, t) \geq R - r(\cdot, t) \geq 0$$

in  $\Omega$  for all  $t \in (0, T)$  due to the definitions of  $S$  and  $R$  and the fact that  $T \leq 1$ .

Now for  $t \in (0, T)$  and  $s \in (0, T)$ , (2.12) implies that  $w := v(\cdot, t) - v(\cdot, s)$  satisfies

$$-\Delta w + \left(\bar{u}(\cdot, t) + \mu\right)w = -\left(\bar{u}(\cdot, t) - \bar{u}(\cdot, s)\right) \cdot v(\cdot, s) + r(\cdot, t) - r(\cdot, s) \quad \text{in } \Omega,$$

which upon testing by  $w$  any invoking Young's inequality yields

$$\begin{aligned} \int_{\Omega} |\nabla w|^2 + (\delta + \mu) \int_{\Omega} w^2 &\leq \int_{\Omega} |\nabla w|^2 + \int_{\Omega} \left(\bar{u}(\cdot, t) + \mu\right) w^2 \\ &= - \int_{\Omega} \left(\bar{u}(\cdot, t) - \bar{u}(\cdot, s)\right) v(\cdot, s) w + \int_{\Omega} \left(r(\cdot, t) - r(\cdot, s)\right) w \\ &\leq \frac{\delta + \mu}{2} \int_{\Omega} w^2 + \frac{1}{\delta + \mu} \int_{\Omega} \left(\bar{u}(\cdot, t) - \bar{u}(\cdot, s)\right)^2 v^2(\cdot, s) \\ &\quad + \frac{1}{\delta + \mu} \int_{\Omega} \left(r(\cdot, t) - r(\cdot, s)\right)^2 \\ &\leq \frac{\delta + \mu}{2} \int_{\Omega} w^2 + \frac{R^2}{(\delta + \mu)^3} \int_{\Omega} \left(\bar{u}(\cdot, t) - \bar{u}(\cdot, s)\right)^2 \\ &\quad + \frac{1}{\delta + \mu} \int_{\Omega} \left(r(\cdot, t) - r(\cdot, s)\right)^2 \end{aligned}$$

because of (2.13). As clearly  $\bar{u}$  and  $r$  belong to  $C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [0, T])$ , we therefore obtain  $c_4 = c_4(\bar{u}) > 0$  such that

$$\|v(\cdot, t) - v(\cdot, s)\|_{W^{1,2}(\Omega)} \leq c_4 |t - s|^{\frac{\theta}{2}} \quad \text{for all } t \in (0, T) \text{ and } s \in (0, T),$$

so that since furthermore (2.4), (2.12) and (2.13) ensure that

$$\begin{aligned} \|v(\cdot, t)\|_{C^{1+\theta_1}(\bar{\Omega})} &\leq c_1 \|v(\cdot, t) - \bar{u}(\cdot, t)v(\cdot, t) - \mu v(\cdot, t) + r(\cdot, t)\|_{L^\infty(\Omega)} \\ &\leq c_1 \cdot \left\{ \frac{R}{\delta + \mu} + M \cdot \frac{R}{\delta + \mu} + \mu \cdot \frac{R}{\delta + \mu} + R \right\} \\ &= c_2 \quad \text{for all } t \in (0, T), \end{aligned} \quad (2.14)$$

by interpolation we easily find  $\theta_3 = \theta_3(\bar{u}) \in (0, 1)$  such that  $v \in C^{1+\theta_3, \theta_3}(\bar{\Omega} \times [0, T])$ . Accordingly, standard parabolic Schauder theory ([14]) guarantees the existence of a classical solution  $u \in C^0(\bar{\Omega} \times [0, T]) \cap C^{2,1}(\bar{\Omega} \times (0, T))$  of

$$\begin{cases} u_t = \Delta u - \nabla v \cdot \nabla u - u^2 v - \mu u v + r(x, t)u, & x \in \Omega, \ t \in (0, T), \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \ t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (2.15)$$

which we claim to actually be an element of  $S$ .

Indeed, letting  $\underline{u}(x, t) := \underline{y}(t)$ ,  $(x, t) \in \bar{\Omega} \times [0, \infty)$ , with  $\underline{y} \in C^1([0, \infty))$  denoting the solution of

$$\begin{cases} \underline{y}'(t) = -\frac{R}{\delta + \mu} \cdot \underline{y}^2(t) - \frac{\mu R}{\delta + \mu} \cdot \underline{y}(t), & t > 0, \\ \underline{y}(0) = \eta, \end{cases}$$

we see that since  $\eta \leq u_0$  in  $\bar{\Omega}$  by (2.5), the comparison principle asserts that  $u \geq \underline{u}$  in  $\bar{\Omega} \times [0, T]$ . As (2.5) moreover ensures that  $\eta \leq 1$ ,  $\underline{y}$  can conveniently be estimated from below by using that thus  $\underline{y} \leq \eta \leq 1$  and hence  $\underline{y}'(t) \geq -\frac{(1+\mu)R}{\delta + \mu} \cdot \underline{y}(t)$  for all  $t > 0$ , which namely warrants that

$$\underline{y}(t) \geq \eta e^{-\frac{(1+\mu)R}{\delta + \mu} \cdot t} \geq \eta e^{-\frac{(1+\mu)R}{\delta + \mu} \cdot T} \geq \frac{\eta}{2} \quad \text{for all } t \in (0, T)$$

according to the second restriction implied by (2.8). In consequence,

$$u(x, t) \geq \underline{y}(t) \geq \frac{\eta}{2} = \delta \quad \text{for all } x \in \Omega \text{ and } t \in (0, T) \quad (2.16)$$

due to (2.6).

To achieve a pointwise upper estimate, we let  $\bar{u}(x, t) := \bar{y}(t) := \|u_0\|_{L^\infty(\Omega)} e^{Rt}$  for  $(x, t) \in \bar{\Omega} \times [0, \infty)$  and observe that by nonnegativity of  $u$ ,  $v$  and  $\mu$ , a second application of the comparison principle ensures that  $u \leq \bar{u}$  in  $\bar{\Omega} \times [0, T]$ , and that therefore

$$u(x, t) \leq \|u_0\|_{L^\infty(\Omega)} e^{RT} = \frac{M}{2} \cdot e^{RT} \leq M \quad \text{for all } x \in \Omega \text{ and } t \in (0, T) \quad (2.17)$$

according to (2.5) and the third implication contained in (2.8).

Now the inequalities in (2.16) and (2.17) enable us to apply (2.10) to  $a := -\nabla v$ ,  $b := -uv - \mu v + r$  and  $z := u$ , which indeed satisfy



$$\|a(\cdot, t)\|_{L^\infty(\Omega)} \leq \|\nabla v(\cdot, t)\|_{C^{1+\theta_1}(\bar{\Omega})} \leq c_2 \quad \text{for all } t \in (0, T)$$

and

$$\|b(\cdot, t)\|_{L^\infty(\Omega)} \leq M \cdot \frac{R}{\delta + \mu} + \mu \cdot \frac{R}{\delta + \mu} + R \quad \text{for all } t \in (0, T).$$

Consequently, (2.11) states that  $u$  in fact belongs to  $C^{\theta_2, \frac{\theta_2}{2}}(\bar{\Omega} \times [0, T])$  with

$$\|u\|_{C^{\theta_2, \frac{\theta_2}{2}}(\bar{\Omega} \times [0, T])} \leq c_3. \quad (2.18)$$

In conjunction with (2.16) and (2.17), this shows that if we let  $F\bar{u} := u$ , then indeed  $F$  maps  $S$  into itself, whereas the quantitative estimate in (2.18) along with the Arzelà–Ascoli theorem and our restriction that  $\theta < \theta_2$  ensures that  $\overline{FS}$  is compact in  $X$ . Since it can readily be shown that  $F$  furthermore is continuous, it thus follows from Schauder’s theorem that  $F$  possesses a fixed point  $u \in S$  which, along with  $v$  as accordingly determined through (2.12), clearly yields a classical solution of (1.3) in  $\Omega \times (0, T)$ . As our above choice of  $T$  only depends on  $u_0$  and  $r$  through  $\delta = \frac{1}{2} \min\{1, \inf_{\Omega} u_0\}$  and  $R = \sup_{\Omega \times (0, 1)} r$ , a standard argument thereupon warrants extensibility up to some maximal  $T_{\max} \in (0, \infty]$  fulfilling (2.2).

The claimed uniqueness property can be verified in a straightforward manner, adapting the reasonings e.g. from [9] or [3], so that we may omit giving details on this here. Finally, (2.3) can be obtained upon direct integration of the first equation in (1.3).  $\square$

In order to warrant global extensibility by ruling out the second alternative in (2.2), by means of another maximum principle-based argument we can establish temporally local pointwise lower bounds on  $u$  as follows.

**Lemma 2.2.** *Under the assumptions from Lemma 2.1, for any  $T > 0$  one can find  $C(T) > 0$  such that*

$$u(x, t) \geq C(T) \quad \text{for all } x \in \bar{\Omega} \text{ and } t \in [0, \min\{T, T_{\max}\}). \quad (2.19)$$

**Proof.** Given  $T > 0$ , we let

$$R(T) := \|r\|_{L^\infty(\Omega \times (0, T))} \quad (2.20)$$

and fix  $\alpha > 0$  suitably large such that

$$\alpha > R(T). \quad (2.21)$$

We furthermore rely on the presupposed positivity of  $u_0$  in  $\bar{\Omega}$  in fixing  $\eta > 0$  small enough fulfilling

$$\eta < \inf_{x \in \Omega} u_0(x), \quad (2.22)$$

and thereupon define

$$z(x, t) := u(x, t) - \eta e^{-\alpha t}, \quad x \in \overline{\Omega}, \quad t \in [0, T_{\max}). \quad (2.23)$$

Then  $z$  belongs to  $C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$  with

$$z(x, 0) = u_0(x) - \eta > 0 \quad \text{for all } x \in \overline{\Omega} \quad (2.24)$$

by (2.22), and with

$$\frac{\partial z(x, t)}{\partial \nu} = 0 \quad \text{for all } x \in \partial\Omega \text{ and } t \in (0, T_{\max}) \quad (2.25)$$

according to (1.3), and from (1.3) we moreover obtain that

$$\begin{aligned} z_t &= u_t + \alpha \eta e^{-\alpha t} \\ &= \Delta u - \nabla v \cdot \nabla u - u^2 v - \mu u v + u r + \alpha \eta e^{-\alpha t} \\ &= \Delta z - \nabla v \cdot \nabla z - (z + \eta e^{-\alpha t})^2 v - \mu(z + \eta e^{-\alpha t})v + (z + \eta e^{-\alpha t})r + \alpha \eta e^{-\alpha t} \\ &\quad \text{for all } x \in \overline{\Omega} \text{ and } t \in (0, T_{\max}). \end{aligned} \quad (2.26)$$

Now writing  $\widehat{T} := \min\{T, T_{\max}\}$ , we see that (2.24) ensures that

$$t_0 := \sup \left\{ t_\star \in (0, \widehat{T}) \mid z(x, t) > 0 \text{ for all } x \in \overline{\Omega} \text{ and } t \in [0, t_\star] \right\} \quad (2.27)$$

is well-defined and positive, and to verify that actually  $t_0 = \widehat{T}$ , assuming the contrary we could find  $x_0 \in \overline{\Omega}$  such that  $z(x_0, t_0) = 0$ , because  $z$  is continuous in  $\overline{\Omega} \times [0, \widehat{T})$ . As thus  $z(x_0, t_0) = \min_{x \in \overline{\Omega}} z(x, t_0)$ , from (2.25) it follows that in both cases  $x_0 \in \Omega$  and  $x_0 \in \partial\Omega$  we furthermore know that  $\nabla z(x_0, t_0) = 0$ , and that hence the inclusion  $z(\cdot, t_0) \in C^2(\overline{\Omega})$  enables us to infer that  $\Delta z(x_0, t_0) \geq 0$ . Since apart from that the nonnegativity of  $z$  in  $\overline{\Omega} \times [0, t_0]$ , as implied by (2.27), warrants that  $z_t(x_0, t_0) \leq 0$ , using that (2.26) is especially valid in all of  $\overline{\Omega} \times \{t_0\}$  we may conclude from the latter that

$$0 \geq z_t(x_0, t_0) \geq -\eta^2 e^{-2\alpha t_0} v(x_0, t_0) - \mu \eta e^{-\alpha t_0} v(x_0, t_0) + \eta e^{-\alpha t_0} r(x_0, t_0) + \alpha \eta e^{-\alpha t_0}. \quad (2.28)$$

To derive a contradiction from this, we now use the continuity of  $v(\cdot, t_0)$  in picking  $x_1 \in \overline{\Omega}$  such that  $v(x_1, t_0) = \max_{x \in \overline{\Omega}} v(x, t_0)$ , and similarly to the above consideration note that since also  $\frac{\partial v(x, t_0)}{\partial \nu} = 0$  for all  $x \in \partial\Omega$  by (1.3), regardless of whether  $x_1 \in \Omega$  or  $x_1 \in \partial\Omega$  we have  $\nabla v(x_1, t_0) = 0$  and  $\Delta v(x_1, t_0) \leq 0$ , through (1.3) implying that

$$\begin{aligned} 0 &\geq \Delta v(x_1, t_0) = u(x_1, t_0)v(x_1, t_0) + \mu v(x_1, t_0) - r(x_1, t_0) \\ &\geq u(x_1, t_0)v(x_1, t_0) + \mu v(x_1, t_0) - R(T) \end{aligned} \quad (2.29)$$

due to (2.20). But once more by nonnegativity of  $z$  on  $\overline{\Omega} \times [0, t_0]$ , from (2.23) we particularly infer that  $u(x_1, t_0) \geq \eta e^{-\alpha t_0}$ , and thus from (2.29) we obtain that

$$v(x_1, t_0) \leq \frac{R(T)}{\eta e^{-\alpha t_0} + \mu}.$$

As  $v(x_0, t_0) \leq v(x_1, t_0)$  by definition of  $x_1$ , this enables us to draw from (2.28) and (2.21) the absurd conclusion that

$$\begin{aligned} 0 &\geq -\left(\eta^2 e^{-2\alpha t_0} + \mu \eta e^{-\alpha t_0}\right) \cdot \frac{R(T)}{\eta e^{-\alpha t_0} + \mu} + r(x_0, t_0) \eta e^{-\alpha t_0} + \alpha \eta e^{-\alpha t_0} \\ &= -R \eta e^{-\alpha t_0} + r(x_0, t_0) \eta e^{-\alpha t_0} + \alpha \eta e^{-\alpha t_0} \\ &\geq (\alpha - R) \eta e^{-\alpha t_0} \\ &> 0, \end{aligned}$$

because  $r$  is nonnegative. In consequence, we indeed must have  $t_0 = \widehat{T}$  and hence  $z > 0$  in  $\overline{\Omega} \times [0, \widehat{T})$ , which in view of (2.23) establishes (2.19) with  $C(T) := \eta e^{-\alpha T}$ .  $\square$

Thereby our result on global classical solvability becomes evident.

**Proof of Theorem 1.1.** The claim directly follows by combining Lemma 2.1 with Lemma 2.2.  $\square$

### 3. Stabilization. Proof of Theorem 1.2

In preparation of our energy-based analysis of the large time behavior in the first solution component, let us state the following quite elementary functional inequality.

**Lemma 3.1.** *Let  $\varphi \in C^2(\overline{\Omega})$  be such that  $\varphi > 0$  in  $\overline{\Omega}$  and  $\frac{\partial \varphi}{\partial \nu} = 0$  on  $\partial\Omega$ . Then*

$$\int_{\Omega} |\nabla \varphi^{\frac{1}{4}}|^2 \leq \frac{\sqrt{|\Omega|}}{8} \cdot \left\{ \int_{\Omega} \frac{|\Delta \varphi|^2}{\varphi} \right\}^{\frac{1}{2}}. \quad (3.1)$$

**Proof.** We only need to integrate by parts and employ the Cauchy–Schwarz inequality to see that indeed

$$\begin{aligned} \int_{\Omega} |\nabla \varphi^{\frac{1}{4}}|^2 &= \frac{1}{16} \int_{\Omega} \varphi^{-\frac{3}{2}} \nabla \varphi \cdot \nabla \varphi \\ &= -\frac{1}{8} \int_{\Omega} \nabla \varphi^{-\frac{1}{2}} \cdot \nabla \varphi \\ &= \frac{1}{8} \int_{\Omega} \varphi^{-\frac{1}{2}} \Delta \varphi \\ &\leq \frac{1}{8} |\Omega|^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} \varphi^{-1} |\Delta \varphi|^2 \right\}^{\frac{1}{2}} \end{aligned}$$

for any such  $\varphi$ .  $\square$

Now the key observation toward our derivation of Theorem 1.2 consists in the following detection of an energy-like structure in (1.3).

**Lemma 3.2.** *Whenever  $\mu \geq 0$  and  $u_0 \in W^{1,\infty}(\Omega)$  is positive in  $\overline{\Omega}$ , for any choice of  $0 \leq r \in C^1(\overline{\Omega} \times [0, \infty))$  fulfilling (1.5), the solution  $(u, v)$  of (1.3) satisfies*

$$\frac{d}{dt} \int_{\Omega} u \ln u + \int_{\Omega} \frac{|\nabla u|^2}{u} + \frac{64}{|\Omega|} \cdot \left\{ \int_{\Omega} |\nabla v^{\frac{1}{4}}|^2 \right\}^2 \leq \int_{\Omega} |\nabla \sqrt{r}|^2 \quad \text{for all } t > 0. \quad (3.2)$$

**Proof.** Using the positivity of  $u$  in  $\overline{\Omega} \times (0, \infty)$ , by means of the first equation in (1.3) we compute

$$\frac{d}{dt} \int_{\Omega} u \ln u + \int_{\Omega} \frac{|\nabla u|^2}{u} = \int_{\Omega} \nabla u \cdot \nabla v \quad \text{for all } t > 0, \quad (3.3)$$

which trivially asserts (3.2) for each  $t \in S$  with  $S := \{t > 0 \mid r(\cdot, t) \equiv 0 \text{ in } \Omega\}$ , because the second equation in (1.3) implied that  $v(\cdot, t) \equiv 0$  in  $\Omega$  for any such  $t$ .

If  $t \in (0, \infty) \setminus S$ , then according to the nonnegativity of  $\mu$  and  $r(\cdot, t)$  and the positivity of  $u(\cdot, t)$ , a strong maximum principle ([7]) warrants that  $v(\cdot, t) > 0$  in  $\overline{\Omega}$ , whence we may multiply the second equation in (1.3) by  $\frac{\Delta v(\cdot, t)}{v(\cdot, t)}$  to see upon integrating by parts and employing Young's inequality that

$$\begin{aligned} \int_{\Omega} \frac{|\Delta v|^2}{v} &= \int_{\Omega} u \Delta v + \mu \int_{\Omega} \Delta v - \int_{\Omega} r \frac{\Delta v}{v} \\ &= - \int_{\Omega} \nabla u \cdot \nabla v - \int_{\Omega} r \frac{|\nabla v|^2}{v^2} + \int_{\Omega} \nabla r \cdot \frac{\nabla v}{v} \\ &= - \int_{\Omega} \nabla u \cdot \nabla v - \int_{\Omega} r \frac{|\nabla v|^2}{v^2} + 2 \int_{\Omega} \sqrt{r} \nabla \sqrt{r} \cdot \frac{\nabla v}{v} \\ &\leq - \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} |\nabla \sqrt{r}|^2 \quad \text{for all } t \in (0, \infty) \setminus S. \end{aligned}$$

Adding this to (3.3) shows that

$$\frac{d}{dt} \int_{\Omega} u \ln u + \int_{\Omega} \frac{|\nabla u|^2}{u} + \int_{\Omega} \frac{|\Delta v|^2}{v} \leq \int_{\Omega} |\nabla \sqrt{r}|^2 \quad \text{for all } t \in (0, \infty) \setminus S$$

and thereby yields (3.2) also for all these  $t$ , because according to Lemma 3.1 we have

$$\int_{\Omega} \frac{|\Delta v|^2}{v} \geq \frac{64}{|\Omega|} \cdot \left\{ \int_{\Omega} |\nabla v^{\frac{1}{4}}|^2 \right\}^2$$

for all  $t \in (0, \infty) \setminus S$ .  $\square$

In appropriately exploiting the latter under suitable assumptions on temporal decay of  $\nabla\sqrt{r}$ , we shall make use of the following basic statement on decay in a linear inhomogeneous ODI.

**Lemma 3.3.** *Let  $y \in C^1([0, \infty))$  and  $h \in L^1_{loc}([0, \infty))$  both be nonnegative, and suppose that*

$$\int_t^{t+1} h(s) ds \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

*and that there exists  $\lambda > 0$  such that*

$$y'(t) + \lambda y(t) \leq h(t) \quad \text{for all } t > 0.$$

*Then*

$$y(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

**Proof.** A proof of this can be found in [6, Lemma 4.6].  $\square$

By means of a standard logarithmic Sobolev inequality and a Csiszár–Kullback inequality, through Lemma 3.3 the inequality from Lemma 3.2 indeed can be seen to imply the stabilization properties claimed in Theorem 1.2.

**Lemma 3.4.** *Let  $\mu \geq 0$ , and suppose that  $r \in C^1(\overline{\Omega} \times [0, \infty))$  is a nonnegative function fulfilling (1.5) as well as (1.6). Then for any choice of  $0 < u_0 \in W^{1,\infty}(\Omega)$ , the solution  $(u, v)$  of (1.3) has the properties (1.7) and (1.8).*

**Proof.** According to a well-known logarithmic Sobolev inequality ([1]), there exists  $c_1 > 0$  such that

$$\int_{\Omega} \varphi^2 \ln \frac{\varphi^2}{\frac{1}{|\Omega|} \int_{\Omega} \varphi^2} \leq c_1 \int_{\Omega} |\nabla \varphi|^2 \quad \text{for all } \varphi \in C^1(\overline{\Omega}), \quad (3.4)$$

and a Csiszár–Kullback type inequality ([24]) provides  $c_2 > 0$  with the property that

$$\|\psi - \bar{\psi}\|_{L^1(\Omega)}^2 \leq c_2 \bar{\psi} \int_{\Omega} \psi \ln \frac{\psi}{\bar{\psi}} \quad \text{for all } \psi \in C^0(\overline{\Omega}) \text{ such that } \psi > 0 \text{ in } \overline{\Omega}. \quad (3.5)$$

Thanks to (2.3), an application of (3.4) to  $\varphi := \sqrt{u(\cdot, t)}$  for  $t > 0$  thus shows that

$$\int_{\Omega} u \ln \frac{u}{\bar{u}_0} \leq \frac{c_1}{4} \int_{\Omega} \frac{|\nabla u|^2}{u} \quad \text{for all } t > 0,$$

whence due to (3.2),  $y(t) := \int_{\Omega} u(\cdot, t) \ln \frac{u(\cdot, t)}{\bar{u}_0}$ ,  $t \geq 0$ , as well as  $g(t) := \frac{64}{|\Omega|} \left\{ \int_{\Omega} |\nabla v^{\frac{1}{4}}(\cdot, t)|^2 \right\}^2$  and  $h(t) := \int_{\Omega} |\nabla \sqrt{r(\cdot, t)}|^2$ ,  $t > 0$ , satisfy

$$y'(t) + \frac{4}{c_1} y(t) + g(t) \leq h(t) \quad \text{for all } t > 0. \quad (3.6)$$

As a consequence of our hypothesis (1.6), through Lemma 3.3 this firstly implies that

$$y(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (3.7)$$

which in view of (3.5) immediately yields (1.7).

Thereafter going back to (3.6) once more, upon an integration thereof we obtain that

$$\int_t^{t+1} g(s) ds \leq y(t) + \int_t^{t+1} h(s) ds \quad \text{for all } t > 0,$$

so that combining (3.7) with, again, (1.6) entails (1.8).  $\square$

In order to complete our verification of Theorem 1.2 by deriving the boundedness statement formulated therein, let us suitably adapt the celebrated Moser-type iterative argument to the present situation.

**Lemma 3.5.** *Assume that  $\mu \geq 0$  and that  $u_0 \in W^{1,\infty}(\Omega)$  is positive in  $\bar{\Omega}$ , and suppose that  $r \in C^1(\bar{\Omega} \times [0, \infty))$  is nonnegative and such that (1.9) is valid with some  $q \geq 1$  such that  $q > \frac{n}{2}$ . Then there exists  $C > 0$  such that for the solution of (1.3) we have*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t > 0. \quad (3.8)$$

**Proof.** For nonnegative integers  $k$ , we let  $p_k := 2^k$  as well as

$$M_k(T) := \sup_{t \in (0, T)} \int_{\Omega} u^{p_k}(\cdot, t) \quad \text{and} \quad M_k := \sup_{T > 0} M_k(T), \quad (3.9)$$

noting that then  $M_k(T)$  is finite for each  $T > 0$ , and that

$$M_0(T) = \int_{\Omega} u_0 \quad \text{for all } T > 0 \quad (3.10)$$

by (2.3). In order to estimate  $M_k(T)$  for  $k \geq 1$ , we integrate by parts in (1.3) to obtain

$$\begin{aligned} & \frac{1}{p_k} \frac{d}{dt} \int_{\Omega} u^{p_k} + (p_k - 1) \int_{\Omega} u^{p_k-2} |\nabla u|^2 \\ &= (p_k - 1) \int_{\Omega} u^{p_k-1} \nabla u \cdot \nabla v \\ &= \frac{p_k - 1}{p_k} \int_{\Omega} \nabla u^{p_k} \cdot \nabla v \end{aligned}$$

$$\begin{aligned}
&= -\frac{p_k - 1}{p_k} \int_{\Omega} u^{p_k} \Delta v \\
&= -\frac{p_k - 1}{p_k} \int_{\Omega} u^{p_k+1} v - \mu \cdot \frac{p_k - 1}{p_k} \int_{\Omega} u^{p_k} v + \frac{p_k - 1}{p_k} \int_{\Omega} u^{p_k} r \\
&\leq \frac{p_k - 1}{p_k} \int_{\Omega} u^{p_k} r \quad \text{for all } t > 0,
\end{aligned} \tag{3.11}$$

where according to our assumption (1.9), using the Hölder inequality we see that with some  $c_1 > 0$  we have

$$\begin{aligned}
\frac{p_k - 1}{p_k} \int_{\Omega} u^{p_k} r &\leq \|u^{\frac{p_k}{2}}\|_{L^{2q'}(\Omega)}^2 \|r\|_{L^q(\Omega)} \\
&\leq c_1 \|u^{\frac{p_k}{2}}\|_{L^{2q'}(\Omega)}^2 \quad \text{for all } t > 0
\end{aligned}$$

with  $q' \in (1, \infty]$  taken in such a way that  $\frac{1}{q} + \frac{1}{q'} = 1$ . As  $p_k - 1 \geq \frac{p_k}{2}$  for all  $k \geq 1$ , from (3.11) we thus obtain the inequality

$$\frac{d}{dt} \int_{\Omega} u^{p_k} + 2 \int_{\Omega} |\nabla u^{\frac{p_k}{2}}|^2 \leq c_1 p_k \|u^{\frac{p_k}{2}}\|_{L^{2q'}(\Omega)}^2 \quad \text{for all } t > 0, \tag{3.12}$$

where thanks to our restrictions on  $q$  we have  $(n-2) \cdot 2q' < 2n$ , so that the Gagliardo–Nirenberg inequality becomes applicable so as to provide  $c_2 > 0$  such that whenever  $k \geq 1$ ,

$$\begin{aligned}
c_1 p_k \|u^{\frac{p_k}{2}}\|_{L^{2q'}(\Omega)}^2 &\leq c_2 p_k \|\nabla u^{\frac{p_k}{2}}\|_{L^2(\Omega)}^{2a} \|u^{\frac{p_k}{2}}\|_{L^1(\Omega)}^{2(1-a)} + c_2 p_k \|u^{\frac{p_k}{2}}\|_{L^1(\Omega)}^2 \\
&\leq c_2 p_k M_{k-1}^{2(1-a)}(T) \|\nabla u^{\frac{p_k}{2}}\|_{L^2(\Omega)}^{2a} + c_2 p_k M_{k-1}^2(T) \quad \text{for all } t \in (0, T)
\end{aligned}$$

with  $a := \frac{n(q+1)}{(n+2)q} \in (0, 1)$ . In light of Young's inequality and the observation that  $p_k \leq p_k^{\frac{1}{1-a}}$  due to the fact that  $\frac{1}{1-a} > 1$ , we thus obtain  $c_3 > 0$  with the property that for all  $k \geq 1$ ,

$$\begin{aligned}
c_1 p_k \|u^{\frac{p_k}{2}}\|_{L^{2q'}(\Omega)}^2 &\leq \|\nabla u^{\frac{p_k}{2}}\|_{L^2(\Omega)}^2 + c_3 p_k^{\frac{1}{1-a}} M_{k-1}^2(T) + c_2 p_k M_{k-1}^2(T) \\
&\leq \|\nabla u^{\frac{p_k}{2}}\|_{L^2(\Omega)}^2 + (c_2 + c_3) p_k^{\frac{1}{1-a}} M_{k-1}^2(T) \quad \text{for all } t \in (0, T).
\end{aligned} \tag{3.13}$$

Apart from that, an Ehrling-type inequality associated with the embeddings  $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^1(\Omega)$  asserts the existence of  $c_4 > 0$  such that for all  $k \geq 1$ , since  $p_k^{\frac{1}{1-a}} \geq 1$  we have

$$\begin{aligned}
\|u^{\frac{p_k}{2}}\|_{L^2(\Omega)}^2 &\leq \|\nabla u^{\frac{p_k}{2}}\|_{L^2(\Omega)}^2 + c_4 \|u^{\frac{p_k}{2}}\|_{L^1(\Omega)}^2 \\
&\leq \|\nabla u^{\frac{p_k}{2}}\|_{L^2(\Omega)}^2 + c_4 M_{k-1}^2(T) \\
&\leq \|\nabla u^{\frac{p_k}{2}}\|_{L^2(\Omega)}^2 + c_4 p_k^{\frac{1}{1-a}} M_{k-1}^2(T) \quad \text{for all } t \in (0, T).
\end{aligned}$$

Together with (3.13), this shows that (3.12) entails the inequality

$$\frac{d}{dt} \int_{\Omega} u^{p_k} + \int_{\Omega} u^{p_k} \leq c_5 p_k^{\frac{1}{1-a}} M_{k-1}^2(T) \quad \text{for all } t \in (0, T)$$

with  $c_5 := c_2 + c_3 + c_4$ , so that by a comparison argument,

$$\int_{\Omega} u^{p_k} \leq \max \left\{ \int_{\Omega} u_0^{p_k}, c_5 p_k^{\frac{1}{1-a}} M_{k-1}^2(T) \right\} \quad \text{for all } t \in (0, T).$$

In view of (3.10), through an induction this readily implies that actually all the  $M_k$  as introduced in (3.9) are finite for  $k \geq 1$ , and that

$$M_k \leq \max \left\{ \int_{\Omega} u_0^{p_k}, c_5 p_k^{\frac{1}{1-a}} M_{k-1}^2 \right\} \quad \text{for all } k \geq 1. \quad (3.14)$$

The remaining part is quite standard: If incidentally  $M_k \leq \int_{\Omega} u_0^{p_k}$  for infinitely many  $k \geq 1$ , then we directly infer that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \liminf_{k \rightarrow \infty} M_k^{\frac{1}{p_k}} \leq \liminf_{k \rightarrow \infty} \left\{ \int_{\Omega} u_0^{p_k} \right\}^{\frac{1}{p_k}} = \|u_0\|_{L^\infty(\Omega)} \quad \text{for all } t > 0.$$

Otherwise, (3.14) easily yields  $b > 1$  such that

$$M_k \leq b^k M_{k-1}^2 \quad \text{for all } k \geq 1,$$

which by straightforward induction entails that

$$M_k \leq b^{2^{k+1}-k-2} M_0^{2^k} \quad \text{for all } k \geq 1$$

and thus

$$M_k^{\frac{1}{p_k}} \leq b^{\frac{2^{k+1}-k-2}{2^k}} M_0 \leq b^2 M_0 \quad \text{for all } k \geq 1,$$

by (3.10) implying that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq b^2 \int_{\Omega} u_0 \quad \text{for all } t > 0$$

and thus ensuring boundedness of  $u$  also in this case.  $\square$



Actually, our main results on qualitative behavior in the first solution component have thereby essentially been established already.

**Proof of Theorem 1.2.** The properties (1.7) and (1.8) have precisely been asserted by Lemma 3.4, whereas boundedness of  $u$  under the additional assumption (1.9) has been established in Lemma 3.5. On the basis of (1.7), a derivation of (1.10) can thereafter be achieved by standard arguments involving the dominated convergence theorem and the Banach–Alaoglu theorem.  $\square$

#### 4. Boundedness vs. unboundedness of $v$ . Proof of Proposition 1.3

Let us finally verify the statements on unboundedness vs. boundedness of  $v$ , as claimed in Proposition and forming, especially in their first part, some at least partially remarkable complement to the outcome of Theorem 1.2.

**Proof of Proposition 1.3.** i) Since  $\frac{\partial v}{\partial \nu} = 0$  on  $\partial\Omega \times (0, \infty)$ , an integration of the second equation in (1.3) shows that

$$\int_{\Omega} r = \int_{\Omega} uv + \mu \int_{\Omega} v \leq \|u\|_{L^1(\Omega)} \|v\|_{L^\infty(\Omega)} + \mu |\Omega| \|v\|_{L^\infty(\Omega)} \quad \text{for all } t > 0,$$

which immediately yields (1.11) due to (2.3).

ii) Performing a simplified variant of the Moser-type procedure from Lemma 3.5, for integers  $k \geq 0$  we let  $p_k := 2^k$  and

$$M_k(T) := \max \left\{ 1, \sup_{t \in (0, T)} \int_{\Omega} v^{p_k}(\cdot, t) \right\}, \quad T > 0, \quad (4.1)$$

as well as  $M_k := \sup_{T > 0} M_k(T)$ , and then obtain on testing the second equation in (1.3) by  $v^{p_k-1}$  for  $k \geq 1$  that since  $p_k - 1 \geq \frac{p_k}{2}$  and  $u \geq 0$ ,

$$\begin{aligned} 2 \int_{\Omega} |\nabla v^{\frac{p_k}{2}}|^2 + \mu p_k \int_{\Omega} v^{p_k} &\leq \frac{4(p_k - 1)}{p_k} \int_{\Omega} |\nabla v^{\frac{p_k}{2}}|^2 + \mu p_k \int_{\Omega} v^{p_k} \\ &= p_k \int_{\Omega} r v^{p_k-1} - p_k \int_{\Omega} u v^{p_k} \\ &\leq (p_k - 1) \int_{\Omega} r v^{p_k} + \int_{\Omega} r \quad \text{for all } t > 0 \end{aligned} \quad (4.2)$$

due to Young's inequality. Once more taking  $q' := \frac{q}{q-1}$  if  $q > 1$  and  $q' := \infty$  if  $q = 1$ , by means of the Hölder inequality, our assumption (1.13), the Gagliardo–Nirenberg inequality and again Young's inequality, we see that with some positive constants  $c_1, c_2$  and  $c_3$  and with  $a := \frac{n(q+1)}{(n+2)q} \in (0, 1)$ , for all  $k \geq 1$  we have

$$\begin{aligned}
(p_k - 1) \int_{\Omega} r v^{p_k} + \int_{\Omega} r &\leq p_k \|r\|_{L^q(\Omega)} \|v^{\frac{p_k}{2}}\|_{L^{2q'}(\Omega)}^2 + \|r\|_{L^1(\Omega)} \\
&\leq c_1 p_k \|v^{\frac{p_k}{2}}\|_{L^{2q'}(\Omega)}^2 + c_1 \\
&\leq c_2 p_k \|\nabla v^{\frac{p_k}{2}}\|_{L^2(\Omega)}^{2a} \|v^{\frac{p_k}{2}}\|_{L^1(\Omega)}^{2(1-a)} + c_2 p_k \|v^{\frac{p_k}{2}}\|_{L^1(\Omega)}^2 + c_1 \\
&\leq c_2 p_k M_{k-1}^{2(1-a)}(T) \|\nabla v^{\frac{p_k}{2}}\|_{L^2(\Omega)}^{2a} + c_2 p_k M_{k-1}^2(T) + c_1 \\
&\leq 2 \|\nabla v^{\frac{p_k}{2}}\|_{L^2(\Omega)}^2 + c_3 p_k^{\frac{1}{1-a}} M_{k-1}^2(T) + c_2 p_k M_{k-1}^2(T) + c_1 \\
&\leq 2 \|\nabla v^{\frac{p_k}{2}}\|_{L^2(\Omega)}^2 + (c_1 + c_2 + c_3) p_k^{\frac{1}{1-a}} M_{k-1}^2(T) \quad \text{for all } t \in (0, T),
\end{aligned}$$

because  $1 \leq p_k \leq p_k^{\frac{1}{1-a}}$  and  $M_{k-1}(T) \geq 1$  for  $k \geq 1$  and  $T > 0$ .

From (4.2) we therefore obtain that if we let  $c_4 := \frac{c_1 + c_2 + c_3}{\mu}$ , then

$$\int_{\Omega} v^{p_k} \leq c_4 p_k^{\frac{a}{1-a}} M_{k-1}^2(T) \quad \text{for all } t \in (0, T), \quad (4.3)$$

which firstly, by induction, shows that  $M_k < \infty$  for all  $k \geq 1$  due to the fact that

$$\int_{\Omega} v = \frac{1}{\mu} \cdot \left\{ \int_{\Omega} r - \int_{\Omega} uv \right\} \leq c_5 := \frac{1}{\mu} \|r\|_{L^\infty((0, \infty); L^1(\Omega))} \quad \text{for all } t > 0$$

and hence

$$M_0(T) \leq \max\{1, c_5\} \quad \text{for all } T > 0. \quad (4.4)$$

Thereafter, (4.3) secondly ensures that since  $M_k \geq 1$  by (4.1), we can find  $b > 1$  such that  $M_k \leq b^k M_{k-1}^2$  for all  $k \geq 1$  and that thus

$$M_k^{\frac{1}{p_k}} \leq b^{\frac{2^{k+1}-k-2}{2^k}} M_0 \leq b^2 M_0 \quad \text{for all } k \geq 1,$$

readily implying (1.14) through (4.1) and (4.4).  $\square$

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