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Global existence and asymptotic behavior in a two-species chemotaxis system with logistic source ☆

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Abstract

In this work we consider the two-species chemotaxis system with logistic source. We present the global existence of generalized solutions under appropriate regularity assumptions on the initial data. In addition, the asymptotic behavior of the solutions is studied, and our results generalize and improve some well-known results in the literature.

MSC: 35D30; 35A01; 35Q92; 35B40; 92C17

Keywords: Chemotaxis; Lotka-Volterra competitive kinetics; Global existence; Asymptotic stability

1. Introduction

In this paper, we concerned with the two-species chemotaxis system with Lotka-Volterra competitive kinetics:

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$$\begin{cases} u_t = d_1 \Delta u - \nabla \cdot (u\chi_1(w)\nabla w) + \mu_1 u(1-u-a_1 v), & x \in \Omega, \quad t>0, \\ v_t = d_2 \Delta v - \nabla \cdot (v\chi_2(w)\nabla w) + \mu_2 v(1-v-a_2 u), & x \in \Omega, \quad t>0, \\ w_t = d_3 \Delta w - (\alpha u + \beta v) w, & x \in \Omega, \quad t>0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \ t>0, \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x), \ w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial \nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$, u and v represent the population densities of two species and w denotes the concentration of the oxygen. $d_1, d_2, d_3, \mu_1, \mu_2, a_1, a_2, \alpha, \beta$ are positive constants, initial data u_0, v_0, w_0 are known functions satisfying

$$(u_0, v_0, w_0) \in W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \text{ are nonnegative with } u_0 \not\equiv 0 \text{ and } v_0 \not\equiv 0. \tag{1.2}$$

χ_i $(i = 1, 2)$ fulfilling

$$\chi_i \in C^{1+\vartheta}([0, \infty)) \text{ for some } \vartheta > 0 \text{ and } \chi_i > 0 \text{ as well as } \chi_i(0) > 0 \ (i = 1, 2). \tag{1.3}$$

Chemotaxis is the directed movement of cells or organisms in response to the gradients of concentration of the chemical stimuli. It plays fundamental roles in various biological processes including embryonic development, wound healing, and disease progression. Chemotaxis is also crucial for many aspects of behavior, including locating food sources (such as the fruit fly Drosophila melanogaster navigates up gradients of attractive odors during food location), avoidance of predators and attracting mates (such as male moths follow pheromone gradients released by the female during mate location), slime mold formation, angiogenesis in tumor progression and primitive steak formation. The pioneering works of the chemotaxis model was introduced by Keller and Segel in [13], describing the aggregation of cellular slime mold toward a higher concentration of a chemical signal, which reads

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \quad t > 0. \end{cases} \tag{1.4}$$

The mathematical analysis of (1.4) and the variants thereof mainly concentrates on the boundedness and blow-up of the solutions [5,9,27,36,38,43]. As the blow-up has not been observed in the real biological process, many mechanisms, such as nonlinear porous medium diffusion, saturation effect, logistic source may avoid the blow-up of solutions [10,12,24,30,34,60]. In the past few decades, the system (1.4) has attracted extensive attention. For a helpful overview of many models arising out of this fundamental description we refer to the survey [2,6,8].

Keller and Segel [14] introduced a phenomenological model of wave-like solution behavior without any type of cell kinetics, a prototypical version of which is given by:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, \quad t > 0, \end{cases} \tag{1.5}$$

where u represents the density of bacteria and v denotes the concentration of the nutrient. The second equation models consumption of the signal. In the first equation, the chemotactic sensitivity is determined according to the Weber-Fechner law, which says that the chemotactic sensitivity is proportional to the reciprocal of signal density. Winkler [47] proved that if initial data satisfying appropriate regularity assumptions, the system (1.5) possesses at least one global generalized solution in two dimensional bounded domains. Moreover, he took into account asymptotic behavior of solutions to the system (1.5), and proved that $v(\cdot,t) \overset{*}{\rightharpoonup} 0$ in $L^\infty(\Omega)$ and $v(\cdot,t) \to 0$ in $L^p(\Omega)$ as $t \to \infty$ provided $\int_\Omega u_0 \le m$, $-\int_\Omega \ln\left(\frac{v_0}{\|v_0\|_{L^\infty(\Omega)}}\right) \le M$, where m, M are positive constants. The same author [49] showed that the system (1.5) admits at least one global renormalized solution which is radially symmetric if initial data (u_0, v_0) are radially symmetric and $\Omega := B_R(0) \subset \mathbb{R}^N$ with $R > 0$. Wang et al. [41] proved that the system (1.5) admits a unique global solution if initial data are appropriate small and that the second equation of Δv is replaced by $\varepsilon \Delta v$ with $\varepsilon \ge 0$ in whole space. When the system (1.5) has a logistic source $ru - \mu u^2$, Lankeit and Lankeit [17] showed that the system (1.5) possesses a global generalized solution for any $\chi \ge 0$, $r \ge 0$ and $\mu > 0$. When Δu is replaced by Δu^m $(m \ge 1)$, Lankeit [16] proved that if $m > 1 + \frac{N}{4}$, the system (1.5) admits a global classical solution or global locally bounded weak solution. When v does not stand for a nutrient be consumed but for a signalling substance produced by the bacteria themselves, which is given by:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \quad t > 0. \end{cases} \tag{1.6}$$

When $\frac{1}{v}$ is replaced by $\chi(v)$, $\chi(v) \le \frac{\chi_0}{(1+\alpha v)^k}$ with some $\chi_0 > 0$, $\alpha > 0$ and $k > 1$, Winkler [42] proved that for any choice of appropriate initial data, the system (1.6) possesses a unique global classical solution that is bounded in $\Omega \times (0, \infty)$ for $N \ge 1$. Stinner and Winkler [35] showed that for any $\lambda \in (0, \min\{1, \frac{1}{\chi^2}\})$, the system (1.6) admits at least one couple (u, v) of nonnegative functions defined in $\Omega \times (0, \infty)$ such that (u, v) is a global weak power-λ solution of (1.6) for $N \ge 2$. Recently, Winkler and Yokota [52] proved that the system (1.6) possesses a uniquely determined global classical solution if $\chi \in (0, \chi_0]$ and $\chi^2 \le \delta$, where $\chi_0 \in (0, \sqrt{\frac{2}{N}})$, $\delta > 0$ are constants. Furthermore, the solution of (1.6) converges to the homogeneous steady state $(\bar{u}_0, \bar{u}_0)$ at an exponential rate with respect to the norm in $(L^\infty(\Omega))^2$ as $t \to \infty$, where $\bar{u}_0 = \frac{1}{|\Omega|} \int_\Omega u_0$. Lankeit and Winkler [18] introduced an apparently novel type of generalized solution, and proved that under the hypothesis that

$$\chi < \begin{cases} \infty & \text{if } N = 2, \\ \sqrt{8} & \text{if } N = 3, \\ \frac{N}{N-2} & \text{if } N \ge 4, \end{cases}$$

for all initial data satisfying suitable assumptions on regularity and positivity, an associated no-flux initial-boundary value problem admits a globally defined generalized solution. This solution inter alia has the property that $u \in L^1_{loc}(\overline{\Omega} \times [0, \infty))$.

To further understand the development of system (1.1), it is necessary to review the related literature in this direction. Being distinctive from the model (1.1), the following two-species

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chemotaxis system with Lotka-Volterra competitive kinetics, in this system, the signal is produced rather than consumed by the cells:

$$\begin{cases} u_t = \nabla\cdot(D_1(u)\nabla u) - \nabla\cdot(uS_1(u)\cdot\nabla w) + \mu_1 u(1-u-a_1 v), & x\in\Omega,\ \ t>0,\\ v_t = \nabla\cdot(D_2(v)\nabla v) - \nabla\cdot(vS_2(v)\cdot\nabla w) + \mu_2 v(1-v-a_2 u), & x\in\Omega,\ \ t>0,\\ w_t = \Delta w - w + \alpha u + \beta v, & x\in\Omega,\ \ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x), & x\in\Omega. \end{cases} \tag{1.7}$$

In the two-dimensional case, Bai and Winkler [1] obtained global existence of solution to the system (1.7) when $D_1(u) = D_2(v) = 1$, $S_1(u) = \chi_1$, $S_2(v) = \chi_2$. Moreover, they also took into consideration an asymptotic behavior of solutions for the system (1.7), and proved that if $a_1, a_2 \in (0, 1)$ and μ_1, μ_2 are sufficiently large, then any global bounded solution exponentially converges to $(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}, \frac{\alpha(1-a_1)+\beta(1-a_2)}{1-a_1a_2})$ as $t\to\infty$; If $a_1 > 1 > a_2 > 0$ and μ_2 is sufficiently large, then any global bounded solution exponentially converges to $(0, 1, \beta)$ as $t\to\infty$; If $a_1 = 1 > a_2 > 0$ and μ_2 is sufficiently large, then any global bounded solution algebraically converges to $(0, 1, \beta)$ as $t\to\infty$. Mizukami [25] extended this result and showed that the system (1.7) admits a unique global uniformly bounded classical solution if $D_1(u) = d_1$, $D_2(u) = d_2$, $-w + \alpha u + \beta v$ is replaced by $h(u, v, w)$ and $S_1(u)$, $S_2(v)$ are replaced by $\chi_i(w)$, $i = 1, 2$, respectively, and there are some positive constants r, δ, M, p, k_i, $i = 1, 2$ such that $\frac{\partial h}{\partial u}(u, v, w) \ge 0$, $\frac{\partial h}{\partial v}(u, v, w) \ge 0$, $\frac{\partial h}{\partial w}(u, v, w) \le -r$, $|h(u, v, w) + \delta w| \le M(1 + u + v)$, $-\chi_i(w)h(0, 0, w) \le k_i$, $2d_id_3\chi_i'(w) + (cd_3 - d_i)p + \sqrt{(d_3 - d_i)^2p^2 + 4d_id_3p}[\chi_i(w)]^2 \le 0$. In the three-dimensional case, Lin and Mu [20] proved that if μ_1 and μ_2 are large enough to obtain similar results. In the high dimensional case, Lin et al. [21] and Zhang and Li [55] obtained unique global classical bounded solution for appropriate conditions, respectively. However, the asymptotic behavior of solutions is not involved. When the system has a logistic source, while the two-species do not influenced each other, in other words, the competitive kinetics terms $\mu_1u(1 - u - a_1v)$ and $\mu_2v(1 - a_2u - v)$ are replaced by $\mu_1u(1 - u)$ and $\mu_2v(1 - v)$, respectively, Negreanu and Tello [28,29] separately claimed that the system (1.7) has unique uniformly bounded solution with $w_t = \varepsilon\Delta w + h(u, v, w)$, $\varepsilon \in [0, 1)$, Mizukami and Yokota [26] removed the restriction of $\varepsilon \in [0, 1)$ to obtain similar results. When $D_1(u) = D_2(v) = 1$, $S_1(u) = \chi_1$, $S_2(v) = \chi_2$, Zhang and Li [56] proved that the system (1.7) admits a unique global bounded classical solution if μ_1 and μ_2 are large enough in bounded convex domain. They also proved that the system (1.7) possesses at least one global weak solution for any $\mu_1 > 0$ and $\mu_2 > 0$. When $D_1(u) = D_2(v) = 1$, $S_1(u)$, $S_2(v)$ are replaced by χ_i $(i = 1, 2)$, respectively, Li and Wang [19] showed that if

$$\frac{\mu_1}{\alpha} > \frac{(9\chi_1^2 + 3\chi_2^2)(\sqrt{5}+\sqrt{2})}{2\sqrt{\chi_1^2+\chi_2^2}}, \quad \frac{\mu_2}{\beta} > \frac{(9\chi_2^2 + 3\chi_1^2)(\sqrt{5}+\sqrt{2})}{2\sqrt{\chi_1^2+\chi_2^2}},$$

the system (1.7) possesses a unique global bounded classical solution in the there dimensional case. Very recently, Jin et al. [11] proved that system (1.7) possesses a unique global classical solution if $D_1(u)$, $D_2(v)$ are replaced by $d_1(w)$, $d_2(w)$, $S_1(u)$, $S_2(v)$ are replaced by $\chi_1(w)$, $\chi_2(w)$, respectively, $d_i(w)$, $\chi_i(w) > 0$ for all $w \ge 0$, $d_i'(w) < 0$ and $\lim_{w\to\infty} d_i(w) = 0$ and $\lim_{w\to\infty} \frac{\chi_i(w)}{d_i(w)}$,

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$\lim\limits_{w\to\infty}\frac{d_i'(w)}{d_i(w)}$ exist, $i=1,2$. When the third equation degenerates into an elliptic equation, Lin et al. [22] obtained global existence of solution to the system (1.7) with $D_1(u)=D_2(v)=1$, $S_1(u)=\chi_1$, $S_2(v)=\chi_2$. Moreover, they also reckoned on asymptotic behavior of solutions to the system (1.7).

Compared with system (1.7), the mathematical analysis of two-species chemotaxis-competition system with two signals points to the necessity of a strengthened. The model is given by:

$$\begin{cases} u_t=\nabla\cdot(D_1(u)\nabla u)-\nabla\cdot(S_1(u)\cdot\nabla v)+\mu_1u(1-u^{\alpha_1}-a_1w), & x\in\Omega,\ \ t>0,\\ \tau v_t=\Delta v-v+w^{\gamma_1}, & x\in\Omega,\ \ t>0,\\ w_t=\nabla\cdot(D_2(w)\nabla w)-\nabla\cdot(S_2(w)\cdot\nabla z)+\mu_2w(1-w^{\alpha_2}-a_2u), & x\in\Omega,\ \ t>0,\\ \tau z_t=\Delta z-z+u^{\gamma_2}, & x\in\Omega,\ \ t>0,\\ (D_1(u)\nabla u-S_1(u)\cdot\nabla v)\cdot\nu=\frac{\partial w}{\partial\nu}=0, & x\in\partial\Omega,\ t>0,\\ (D_2(w)\nabla w-S_2(w)\cdot\nabla z)\cdot\nu=\frac{\partial z}{\partial\nu}=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x),\ z(x,0)=z_0(x), & x\in\Omega. \end{cases} \tag{1.8}$$

When $\tau=1$, $D_1(u)=D_2(v)=1$, $S_1(u)=\chi_1$, $S_2(w)=\chi_2$, $\alpha_1=\alpha_2=\gamma_1=\gamma_2\equiv1$, Black [3] showed that system (1.8) possesses a global classical bounded solution in two-dimensional, and further discussed the asymptotic behavior of global bounded solutions in high dimensional cases. When $D_1(u)=D_2(v)=1$, $S_1(w)=S_2(w)=\chi_i>0$, $i=1,2$, $\gamma_1=\gamma_2\equiv1$, $\mu_1=\mu_2\equiv0$, Yu et al. [53] proved that if $m_1m_2-2\pi(\frac{m_1}{\chi_1}+\frac{m_2}{\chi_2})>0$, there are finite time blow-up solutions to the system (1.8) with $m_1=\int_\Omega u_0$ and $m_2=\int_\Omega w_0$, while the global boundedness of solutions is furthermore established under the condition that $\max\{m_1,m_2\}<4\pi$. When $\tau=0$, $\mu_1=\mu_2\equiv0$, $D_i(u)\geq C_{D_i}(1+u)^{m_i-1}$, $S_i(u)\leq C_{S_i}u^{q_i}$, $(i=1,2)$, Zheng [59] proved that if one of the following cases holds:

$$\begin{cases} \chi_1<0,\ \ q_1<m_1+\frac{2}{N}\ \text{ and }\ q_2<m_2+\frac{2}{N}-\frac{(N-2)_+}{N};\\ \chi_1=\chi_2<0,\ \ q_1<m_1+\frac{2}{N}-\frac{(N-2)_+}{N}\ \text{ and }\ q_2<m_2+\frac{2}{N}-\frac{(N-2)_+}{N};\\ \chi_1=\chi_2>0,\ \ q_1<m_1+\frac{2}{N}-1\ \text{ and }\ q_2<m_2+\frac{2}{N}-1, \end{cases}$$

where $(N-2)_+$ denotes the positive part of $(N-2)$, the system (1.8) admits a unique and uniformly bounded global classical solution. When $D_1(u)=D_2(w)=1$, $S_1(u)=\chi_1u$, $S_2(w)=\chi_2w$, $\alpha_1=\alpha_2=\gamma_1=\gamma_2\equiv1$, Zhang et al. [57] showed that if $\chi_1\chi_2<\mu_1\mu_2$, the system (1.8) possesses a unique positive classical bounded solution. Moreover, when $\chi_i<a_i\mu_i$ $(i=1,2)$, it is shown that such solution stabilizes to spatially homogeneous equilibria (N_1,N_2,N_2,N_1) in the large time limit, where $N_1=\frac{1-a_1}{1-a_1a_2}$, $N_2=\frac{1-a_2}{1-a_1a_2}$. When $a_1>1>a_2\geq0$, $\chi_i\geq0$, $\mu_i>0$, Zhang [54] further proved that the unique nonnegative classical solution $(u,v,w,z)\to(0,1,1,0)$ if $\chi_1\chi_2<\mu_1\mu_2$, $\chi_1\leq a_1\mu_1$ and $\chi_2<\mu_2$. Very recently, when $\tau=1$, $D_i(s)\geq C_{D_i}s^{-m_i}$, $S_i(s)\leq C_{S_i}s^{n_i}$, Ren and Liu [31] showed that if

$$n_1 < \begin{cases} \max\left\{-m_1 + \frac{2}{N}, \ \frac{\alpha_1 - m_1}{2} + \frac{1}{N}\right\}, & \text{if } 0 < \alpha_1 \le \frac{(N+2)\gamma_2}{\gamma_1} - 1, \\ \max\left\{-m_1 + \frac{3(\alpha_1+1)}{2(N+2)\gamma_2 - 3}, \ \frac{1-m_1}{2} + L_1\right\}, & \text{if } \frac{(N+2)\gamma_2}{\gamma_1} - 1 < \alpha_1 < \frac{2(N+2)\gamma_2}{3} - 1, \\ \max\left\{-m_1 + 1 + \frac{2}{N} - M_1, \ \frac{1-m_1}{2} + L_1\right\}, & \text{if } \frac{2(N+2)\gamma_2}{3} - 1 \le \alpha_1 < (N+2)\gamma_2 - 1, \\ \max\left\{-m_1 + 1 + \frac{1}{N}, \ \frac{-m_1+\alpha_1+2}{2}\right\}, & \text{if } \alpha_1 \ge (N+2)\gamma_2 - 1 \end{cases}$$

as well as

$$n_2 < \begin{cases} \max\left\{-m_2 + \frac{2}{N}, \ \frac{\alpha_2 - m_2}{2} + \frac{1}{N}\right\}, & \text{if } 0 < \alpha_2 \le \frac{(N+2)\gamma_1}{\gamma_2} - 1, \\ \max\left\{-m_2 + \frac{3(\alpha_2+1)}{2(N+2)\gamma_1 - 3}, \ \frac{1-m_2}{2} + L_2\right\}, & \text{if } \frac{(N+2)\gamma_1}{\gamma_2} - 1 < \alpha_2 < \frac{2(N+2)\gamma_1}{3} - 1, \\ \max\left\{-m_2 + 1 + \frac{2}{N} - M_2, \ \frac{1-m_2}{2} + L_2\right\}, & \text{if } \frac{2(N+2)\gamma_1}{3} - 1 \le \alpha_2 < (N+2)\gamma_1 - 1, \\ \max\left\{-m_2 + 1 + \frac{1}{N}, \ \frac{-m_2+\alpha_2+2}{2}\right\}, & \text{if } \alpha_2 \ge (N+2)\gamma_1 - 1, \end{cases}$$

where $L_1 := \frac{(\alpha_1+1)(N+2\gamma_1+2\gamma_2)}{2(N+2)\gamma_2}$, $M_1 := \frac{1}{\alpha_1+1}(\frac{(N+2)\gamma_2}{\gamma_1} - 1)$, $L_2 := \frac{(\alpha_2+1)(N+2\gamma_2+2\gamma_1)}{2(N+2)\gamma_1}$ and $M_2 := \frac{1}{\alpha_2+1}(\frac{(N+2)\gamma_1}{\gamma_2} - 1)$. Then for any choice of the initial data, the system (1.8) possesses at least one global weak solution. When $\tau = 0$, they proved that the system (1.8) possesses at least one non-negative global weak solution if (1) $\alpha_1 > n_1 - 1$, $\alpha_2 > n_2 - 1$: (i) $\alpha_2 > \alpha_1$, $\alpha_1 - n_1 + 1 < \gamma_1 < \frac{\alpha_2}{\alpha_1}(\alpha_1 - n_1 + 1)$ or $\frac{\alpha_1}{\alpha_2}(\alpha_2 - n_2 + 1) < \gamma_2 < \alpha_2 - n_2 + 1$, (ii) $\alpha_1 > \alpha_2$, $\frac{\alpha_2}{\alpha_1}(\alpha_1 - n_1 + 1) < \gamma_1 < \alpha_1 - n_1 + 1$, $\alpha_2 - n_2 + 1 < \gamma_2 < \frac{\alpha_1}{\alpha_2}(\alpha_2 - n_2 + 1)$; (2) $1 - m_1 - n_1 > 0$, $1 - m_2 - n_2 > 0$: (i) $m_2 > m_1$, $\frac{m_2}{m_1}(1 - m_1 - n_1) < \gamma_1 < 1 - m_1 - n_1$, $1 - m_2 - n_2 < \gamma_2 < \frac{m_1}{m_2}(1 - m_2 - n_2)$, (ii) $m_1 > m_2$, $1 - m_1 - n_1 < \gamma_1 < \frac{m_2}{m_1}(1 - m_1 - n_1)$, $\frac{m_1}{m_2}(1 - m_2 - n_2) < \gamma_2 < 1 - m_2 - n_2$. In addition, they also took into account asymptotic behavior of solutions to the system (1.8). When $\tau = 1$, $\alpha_1 = \alpha_2 = \gamma_1 = \gamma_2 \equiv 1$, $S_1(u) \le C_{S_1}(1+u)^{n_1}$, $S_2(w) \le C_{S_2}(1+w)^{n_2}$, the solution of (1.8) has the following properties: (i) Let $a_1, a_2 \in (0, 1)$, under the condition that there exists $c_1, c_2 > 1$ such that

$$\frac{\mu_1}{C_{S_1}^2} > \frac{c_1}{16C_{D_1}} \frac{a_2}{a_1} \frac{(1-a_1)}{(1-a_2)} \frac{1}{(1-a_1a_2)} \quad \text{and} \quad \frac{\mu_2}{C_{S_2}^2} > \frac{c_2}{16C_{D_2}} \frac{a_1}{a_2} \frac{(1-a_2)}{(1-a_1)} \frac{1}{(1-a_1a_2)},$$

where $c_i = \max\left\{1, (1 + \|\phi\|_{L^\infty[0, \|\phi_0\|_{L^\infty(\Omega)}+1]})^{1+m_i+n_i}\right\}$ $(i = 1, 2)$, $\phi = u$ if $i = 1$ and $\phi = w$ if $i = 2$, then $(u, v, w, z) \to (N_1, N_2, N_2, N_1)$ in $L^\infty(\Omega)$ as $t \to \infty$, where $N_1 := \frac{1-a_1}{1-a_1a_2}$, $N_2 := \frac{1-a_2}{1-a_1a_2}$; (ii) Let $a_1 \ge 1 > a_2 > 0$, under the condition that there exists $c_2 > 1$ such that $\frac{\mu_2}{C_{S_2}^2} > \frac{c_2}{8C_{D_2}a_2(1-a_2)}$, where c_2 is defined as (i), then $(u, v, w, z) \to (0, 1, 1, 0)$ in $L^\infty(\Omega)$ as $t \to \infty$. When $\tau = 0$, $\alpha_1 = \alpha_2 = \gamma_1 = \gamma_2 \equiv 1$, $S_1(u) \le C_{S_1}(1+u)^{n_1}$, $S_2(w) \le C_{S_2}(1+w)^{n_2}$, the solution of (1.8) has the following properties: (i) Let $a_1, a_2 \in (0, 1)$, under the condition that there exist $c_1, c_2 > 1$, $\beta_1, \beta_2 \in (0, 1)$ such that $\beta_1\beta_2 > a_1a_2$ and $\mu_1 > \frac{a_2c_1N_1C_{S_1}^2}{16C_{D_1}a_1(1-\beta_2)}$ as well as $\mu_2 > \frac{a_1c_2N_2C_{S_2}^2}{16C_{D_2}a_2(1-\beta_1)}$, then $(u, v, w, z) \to (N_1, N_2, N_2, N_1)$ in $L^\infty(\Omega)$ as $t \to \infty$; (ii) Let

$a_1 \geq 1 > a_2 > 0$, under the condition that there exist $\beta_3 \in (0,1)$ and $c_2 > 1$ such that $\beta_3 > a_1 a_2$ and $\mu_2 > \frac{a_1 c_2^2 C_{S_2}^2}{16 a_2 C_{D_2}(1-\beta_3)}$, then $(u, v, w, z) \to (0, 1, 1, 0)$ in $L^\infty(\Omega)$ as $t \to \infty$.

Wang et al. [39] proved that the system (1.1) admits a unique global bounded classical solution if one of the following cases holds: for $i = 1, 2$,

$$(1)\ \chi_i(w) = \chi_{0,i} > 0 \ \text{ and } \ \|w_0\|_{L^\infty(\Omega)} < \frac{\pi\sqrt{d_i d_3}}{\sqrt{N+1}\chi_{0,i}} - \frac{2\sqrt{d_i d_3}}{\sqrt{N+1}\chi_{0,i}} arctan \frac{d_i - d_3}{2}\sqrt{\frac{N+1}{d_i d_3}};$$

$$(2)\ 0 < \|w_0\|_{L^\infty(\Omega)} < \frac{d_3}{3(N+1)\|\chi_i\|_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}} \min\{\frac{2d_i}{d_i + d_3}, 1\}.$$

Moreover, it is shown that such solution stabilizes to spatially homogeneous state in the sense that: (1) If $a_1, a_2 \in (0,1)$ and $u_0, v_0 \neq 0$, (u, v, w) exponentially converges to $(\frac{1-a_1}{1-a_1 a_2}, \frac{1-a_2}{1-a_1 a_2}, 0)$; (2) If $a_1 > 1 > a_2 > 0$ and $v_0 \neq 0$, (u, v, w) exponentially converges to $(0, 1, 0)$; (3) If $a_1 = 1 > a_2 > 0$ and $v_0 \neq 0$, (u, v, w) algebraically converges to $(0, 1, 0)$. When system (1.1) without Lotka-Volterra competitive kinetics, $\chi_1(w) \equiv \chi_1$ and $\chi_2(w) \equiv \chi_2$, Zhang and Tao [58] showed that the system possesses a unique global classical solution that is uniformly bounded if $\max\{\chi_1, \chi_2\}\|w(x, 0)\|_{L^\infty(\Omega)} < \sqrt{\frac{2}{N}}\pi$. Furthermore, they also considered asymptotic behavior of solutions to the system (1.1) and proved that $(u, v, w) \to (\frac{1}{|\Omega|}\int_\Omega u_0, \frac{1}{|\Omega|}\int_\Omega v_0, 0)$ in L^∞ norm as $t \to \infty$ uniformly with respect $x \in \Omega$. For two-species chemotaxis-Navier-Stokes system with Lotka-Volterra competitive kinetics, we recommend that readers refer to the literature [7].

Throughout the above analysis, compared with two-species chemotaxis system (1.7) with Lotka-Volterra competitive kinetics where the signal is produced or two-species chemotaxis-competition system (1.8) with two signals, it is so fragmentary that two-species chemotaxis system (1.1) with Lotka-Volterra competitive kinetics where the signal is consumed. To the best of our knowledge, the global generalized solution of system (1.1) remains under-explored. Motivated by the arguments in previous studies [4,32,39,40,46,51,58], we mainly revealed that the system (1.1) has a global generalized solutions. In addition, the asymptotic behavior of the solution was well addressed. Theorem 3.1 partially generalizes and improves Theorem 1.2 in [39] and Theorem 1.1 in [58], Theorem 3.2 partially generalizes and improves Theorem 1.2 in [58].

In this paper, we use symbols C_i and c_i $(i = 1, 2, \cdots)$ as some generic positive constants which may vary in the context. For simplicity, $u(x, t)$ is written as u, the integral $\int_\Omega u(x)dx$ is written as $\int_\Omega u(x)$.

The present paper has the following layout. In Section 2, we summarize a useful lemma in order to prove the main results. In Section 3, we give some fundamental estimates for the solution to the system (3.5) prove Theorem 3.1, and then construct a functional to prove Theorem 3.2.

2. Preliminaries

In this section, we state a useful lemma to prove the main results in Section 3.

Lemma 2.1 *(Gagliardo-Nirenberg interpolation inequality). ([23]) Let* $0 < \theta \leq p \leq \frac{2N}{(N-2)_+}$. *Then there exists positive constant* C_{GN} *such that for all* $u \in W^{1,2}(\Omega) \cap L^\theta(\Omega)$,

$$\|u\|_{L^p(\Omega)} \leq C_{GN}(\|\nabla u\|^a_{L^2(\Omega)}\|u\|^{1-a}_{L^\theta(\Omega)} + \|u\|_{L^\theta(\Omega)})$$

is valid with $a = \frac{\frac{N}{\theta} - \frac{N}{p}}{1 - \frac{N}{2} + \frac{N}{\theta}} \in (0, 1)$*, where* $(N-2)_+$ *denotes the positive part of* $(N-2)$.

3. Dynamics for an two-species chemotaxis system with consumption of chemoattractant

3.1. Global existence to a two-species chemotaxis system

In this subsection, we consider the global existence of two-species chemotaxis system with consumption of chemoattractant. Let us first introduce the concept of generalized solution, which can be found in [46], and also in [4,32,40,51]. Then, we state the main result of this subsection.

Definition 3.1. Let $T > 0$. A triplet (u, v, w) of functions

$$u \in L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \quad v \in L^1_{loc}(\overline{\Omega} \times [0,\infty)),$$
$$w \in L^\infty_{loc}(\overline{\Omega} \times [0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega))$$

such that

$$\nabla \ln(v+1) \text{ and } u\nabla w \text{ belong to } L^2_{loc}(\overline{\Omega} \times [0,\infty); \mathbb{R}^N)$$

will be called a generalized solution to (1.1) if for all $\psi \in C_0^\infty(\overline{\Omega} \times [0, T))$ the identities

$$-\int_0^T\int_\Omega u\psi_t - \int_\Omega u_0\psi(\cdot,0) = -d_1\int_0^T\int_\Omega \nabla u\cdot\nabla\psi + \int_0^T\int_\Omega u\chi_1(w)\nabla w\cdot\nabla\psi$$
$$+\int_0^T\int_\Omega \mu_1 u(1-u-a_1v)\psi \tag{3.1}$$

and

$$-\int_0^T\int_\Omega w\psi_t - \int_\Omega w_0\psi(\cdot,0) = -d_3\int_0^T\int_\Omega \nabla w\cdot\nabla\psi - \int_0^T\int_\Omega (\alpha u + \beta v)w\psi \tag{3.2}$$

hold, if for all nonnegative $\psi \in C_0^\infty(\overline{\Omega} \times [0, T))$ the inequality

$$-\int_0^T\int_\Omega \ln(v+1)\psi_t - \int_\Omega \ln(v_0+1)\psi(\cdot,0)$$
$$\geq d_2\int_0^T\int_\Omega |\nabla\ln(v+1)|^2\psi - d_2\int_0^T\int_\Omega \nabla\ln(v+1)\cdot\nabla\psi + \int_0^T\int_\Omega \frac{v}{v+1}\chi_2(w)\nabla w\cdot\nabla\psi$$

$$-\int_0^T\int_\Omega \frac{v}{v+1}\chi_2(w)(\nabla w\cdot\nabla\ln(v+1))\psi+\mu_2\int_0^T\int_\Omega \frac{v\psi}{v+1}(1-v-a_2u) \tag{3.3}$$

is valid, and if

$$\int_\Omega v(\cdot,t)\le\mu_2\int_\Omega v(1-v-a_2u)\quad \text{for } a.e.\ t>0. \tag{3.4}$$

A global generalized solution of (1.1) is a triplet (u,v,w) of functions defined in $\Omega\times(0,\infty)$ which is a generalized solution to (1.1) in $\Omega\times(0,T)$ for all $T>0$.

Remark 3.1. Analogous to the reasoning in [17, Theorem 2.5], multiplying the inequality $v_t\ge d_2\Delta v-\nabla\cdot(v\chi_2(w)\nabla w)+\mu_2 v(1-v-a_2u)$ by $\frac{\psi}{v+1}$, the above definition in accordance with classical solution if (u,v,w) is a global very weak solution in the above sense which additionally fulfills $(u,v,w)\in(C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)))^3$, then (u,v,w) already solves (1.1) classically in $\Omega\times(0,\infty)$.

Now, we state our main result in this subsection.

Theorem 3.1. *Let $\Omega\subset\mathbb{R}^N$ $(N\ge1)$ be a bounded domain with smooth boundary, $d_2>0$, $\mu_1>0$, $\mu_2>0$, $a_1>0$, $a_2>0$, $\alpha>0$, $\beta>0$, $d_1,d_3>0$, χ_i $(i=1,2)$ fulfill (1.3) and initial data (u_0,v_0,w_0) satisfy (1.2) as well as*

$$\sup_{t>0}\|w_0\|_{L^\infty(\Omega)}<\bar{w},$$

where the positive constant is given by

$$\bar{w}:=\frac{1}{\|\chi_1\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}}\cdot\begin{cases}\frac{d_3-d_1+\sqrt{(d_3-d_1)^2+d_1d_3}}{2} & \text{if } d_3-d_1\le\frac{1+\sqrt{13}}{6}d_1,\\ \frac{2d_1d_3(d_3-d_1)}{(d_3-d_1)^2+d_1d_3} & \text{if } \frac{1+\sqrt{13}}{6}d_1<d_3-d_1\le\frac{1+\sqrt{5}}{2}d_1,\\ \sqrt{d_1d_3} & \text{if } d_3-d_1>\frac{1+\sqrt{5}}{2}d_1.\end{cases}$$

Then there are nonnegative functions

$$\begin{cases}u\in L^\infty((0,\infty);L^1(\Omega))\cap L^2_{loc}([0,\infty);W^{1,2}(\Omega)),\\ v\in L^\infty((0,\infty);L^1(\Omega)),\\ w\in L^\infty(\Omega\times(0,\infty))\cap L^2_{loc}([0,\infty);W^{1,2}(\Omega))\end{cases}$$

such that

$$\int_\Omega u(\cdot,t)\le m_1:=\max\left\{\int_\Omega u_0,\ \frac{|\Omega|}{\mu_1}\right\}\ \text{for } a.e.\ t>0,$$

$$\int_\Omega v(\cdot,t) \le m_2 := \max\Big\{\int_\Omega v_0,\ \frac{|\Omega|}{\mu_2}\Big\} \ \textit{for a.e. } t>0,$$

and (u, v, w) *is a global generalized solution of (1.1) in the sense of Definition 3.1.*

Remark 3.2. Theorem 3.1 partially generalizes and improves Theorem 1.2 in [39] and Theorem 1.1 in [58].

In order to construct such generalized solutions by an approximation procedure, we introduce the following regularized problems

$$\begin{cases} u_{\varepsilon t} = d_1\Delta u_\varepsilon - \nabla\cdot(u_\varepsilon\chi_{1\varepsilon}(w_\varepsilon)\cdot\nabla w_\varepsilon) + \mu_1 u_\varepsilon(1-u_\varepsilon-a_1 v_\varepsilon), & x\in\Omega,\ \ t>0,\\ v_{\varepsilon t} = d_2\Delta v_\varepsilon - \nabla\cdot(v_\varepsilon\chi_{2\varepsilon}(w_\varepsilon)\cdot\nabla w_\varepsilon) + \mu_2 v_\varepsilon(1-v_\varepsilon-a_2 u_\varepsilon), & x\in\Omega,\ \ t>0,\\ w_{\varepsilon t} = d_3\Delta w_\varepsilon - \frac{(\alpha u_\varepsilon+\beta v_\varepsilon)w_\varepsilon}{1+\varepsilon(\alpha u_\varepsilon+\beta v_\varepsilon)w_\varepsilon}, & x\in\Omega,\ \ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0)=u_0(x),\ v_\varepsilon(x,0)=v_0(x),\ w_\varepsilon(x,0)=w_0(x), & x\in\Omega \end{cases} \tag{3.5}$$

for $\varepsilon\in(0,1)$. All of these problems (3.5) are indeed globally solvable in the classical sense.

Lemma 3.1. *Let* $\Omega\subset\mathbb{R}^N$ $(N\ge 1)$ *be a bounded domain with smooth boundary, let* $d_1>0$, $d_2>0$, $d_3>0$, $\mu_1>0$, $\mu_2>0$, $a_1>0$, $a_2>0$, $\alpha>0$, $\beta>0$, χ_i $(i=1,2)$ *fulfill (1.3) and initial data* (u_0, v_0, w_0) *satisfy (1.2). Then for each* $\varepsilon\in(0,1)$, *there exist functions*

$$(u_\varepsilon, v_\varepsilon, w_\varepsilon)\in\Big(\bigcap_{q>N} C^0([0,\infty); W^{1,q}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))\Big)^3,$$

and that $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *solves (3.5) classically. Furthermore,*

$$\int_\Omega u_\varepsilon(\cdot,t)\le m_1 := \max\Big\{\int_\Omega u_0,\ \frac{|\Omega|}{\mu_1}\Big\} \ \textit{for all } t>0, \tag{3.6}$$

$$\int_\Omega v_\varepsilon(\cdot,t)\le m_2 := \max\Big\{\int_\Omega v_0,\ \frac{|\Omega|}{\mu_2}\Big\} \ \textit{for all } t>0 \tag{3.7}$$

and

$$\int_0^T\int_\Omega u_\varepsilon^2 \le m_1 T + \frac{1}{\mu_1}\int_\Omega u_0 \ \textit{for all } T>0, \tag{3.8}$$

$$\int_0^T\int_\Omega v_\varepsilon^2 \le m_2 T + \frac{1}{\mu_2}\int_\Omega v_0 \ \textit{for all } T>0. \tag{3.9}$$

Proof. Local existence, up to a maximal existence time $T_{\max} \in (0, \infty]$ can be seen by means of well-established fixed point arguments [40,44,58], where if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} (\|u_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|w_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)}) = \infty \tag{3.10}$$

for all $q > N$. Moreover, using strong maximum principle assure positivity of u_ε, v_ε, and non-negativity of w_ε, integrating the first equation in (3.5) over Ω, we have

$$\frac{d}{dt}\int_\Omega u_\varepsilon + \mu_1 \int_\Omega u_\varepsilon^2 \le \mu_1 \int_\Omega u_\varepsilon. \tag{3.11}$$

By the elementary inequality, we obtain $2u_\varepsilon \le u_\varepsilon^2 + 1$, substituting these facts into (3.11), which yields

$$\frac{d}{dt}\int_\Omega u_\varepsilon + 2\mu_1 \int_\Omega u_\varepsilon \le \mu_1 \int_\Omega u_\varepsilon + |\Omega|,$$

it immediately derives that

$$\int_\Omega u_\varepsilon(\cdot,t) \le m_1 := \max\Big\{\int_\Omega u_0, \frac{|\Omega|}{\mu_1}\Big\} \quad \text{for all } t \in (0, T_{\max}).$$

Similarly, we have

$$\int_\Omega v_\varepsilon(\cdot,t) \le m_2 := \max\Big\{\int_\Omega v_0, \frac{|\Omega|}{\mu_2}\Big\} \quad \text{for all } t \in (0, T_{\max}).$$

Moreover, if we let $\tau := \{1, \frac{T_{\max}}{2}\}$, then for each $t \in [0, T_{\max} - \tau)$, integrating of (3.11) over $(t, t+\tau)$, we obtain

$$\int_t^{t+\tau}\int_\Omega u_\varepsilon^2 \le m_1\tau + \frac{1}{\mu_1}\int_\Omega u_0 \quad \text{for all } t \in [0, T_{\max} - \tau). \tag{3.12}$$

Likewise,

$$\int_t^{t+\tau}\int_\Omega v_\varepsilon^2 \le m_2\tau + \frac{1}{\mu_2}\int_\Omega v_0 \quad \text{for all } t \in [0, T_{\max} - \tau). \tag{3.13}$$

If $T_{\max}$ was finite, applying maximal Sobolev regularity theory [15,33] to the third equation in (3.5), we readily conclude that $w_\varepsilon \in L^\infty((0, T_{\max}); W^{1,\infty}(\Omega)) \cap L^p((0, T_{\max}); W^{2,p}(\Omega))$ for all $p \in (1, \infty)$. Analogously, we obtain $u_\varepsilon, v_\varepsilon \in L^\infty((0, T_{\max}); W^{1,\infty}(\Omega)) \cap L^p((0, T_{\max}); W^{2,p}(\Omega))$, complies with (3.10), we know that $T_{\max} = \infty$, and thus $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is global classical solution. This completes the proof. □

Lemma 3.2. *Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the solution of (3.5). Then*

$$\sup_{\varepsilon\in(0,1)}\sup_{t>0}\|w_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}<\bar{w}\ \text{ for all } t>0. \tag{3.14}$$

Proof. Thanks to the nonnegativity of $u_\varepsilon, v_\varepsilon, w_\varepsilon$ and positivity of α, β, along with the maximum principle and the conditions of Theorem 3.1, we immediately obtain (3.14). This completes the proof. □

Inspired by [37,45,48,51], we shall derive the following lemma.

Lemma 3.3. *Assume that the conditions of Theorem 3.1 hold. Then there exists $\bar{p}>4$ such that for all $p\in[2,\bar{p}]$, it can be find $r>0$, $\eta>\bar{w}$ and $C_1, C_2>0$ such that for each $\varepsilon\in(0,1)$*

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p(\eta-w_\varepsilon)^{-r}+C_1\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2+C_1\int_\Omega u_\varepsilon^p|\nabla w_\varepsilon|^2\le C_2\int_\Omega u_\varepsilon^p(\eta-w_\varepsilon)^{-r} \tag{3.15}$$

for all $t>0$.

Proof. Differentiating $\int_\Omega u_\varepsilon^p(\eta-w_\varepsilon)^{-r}$ and by a straightforward calculation, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p(\eta-w_\varepsilon)^{-r}\\
&=p\int_\Omega u_\varepsilon^{p-1}(\eta-w_\varepsilon)^{-r}[d_1\Delta u_\varepsilon-\nabla\cdot(u_\varepsilon\chi_{1\varepsilon}(w_\varepsilon)\cdot\nabla w_\varepsilon)+\mu_1u_\varepsilon(1-u_\varepsilon-a_1v_\varepsilon)]\\
&\quad+r\int_\Omega u_\varepsilon^p(\eta-w_\varepsilon)^{-r-1}\Big[d_3\Delta w_\varepsilon-\frac{(\alpha u_\varepsilon+\beta v_\varepsilon)w_\varepsilon}{1+\varepsilon(\alpha u_\varepsilon+\beta v_\varepsilon)w_\varepsilon}\Big]\\
&\le-d_1p(p-1)\int_\Omega u_\varepsilon^{p-2}(\eta-w_\varepsilon)^{-r}|\nabla u_\varepsilon|^2-d_1pr\int_\Omega u_\varepsilon^{p-1}(\eta-w_\varepsilon)^{-r-1}\nabla u_\varepsilon\cdot\nabla w_\varepsilon\\
&\quad+p(p-1)\int_\Omega u_\varepsilon^{p-1}(\eta-w_\varepsilon)^{-r}\chi_{1\varepsilon}(w_\varepsilon)\nabla u_\varepsilon\cdot\nabla w_\varepsilon+pr\int_\Omega u_\varepsilon^p(\eta-w_\varepsilon)^{-r-1}\chi_{1\varepsilon}(w_\varepsilon)|\nabla w_\varepsilon|^2\\
&\quad+\mu_1p\int_\Omega u_\varepsilon^p(\eta-w_\varepsilon)^{-r}-d_3pr\int_\Omega u_\varepsilon^{p-1}(\eta-w_\varepsilon)^{-r-1}\nabla u_\varepsilon\cdot\nabla w_\varepsilon\\
&\quad-r(r+1)d_3\int_\Omega u_\varepsilon^p(\eta-w_\varepsilon)^{-r-2}|\nabla w_\varepsilon|^2\\
&\le-d_1p(p-1)\int_\Omega u_\varepsilon^{p-2}(\eta-w_\varepsilon)^{-r}|\nabla u_\varepsilon|^2+\mu_1p\int_\Omega u_\varepsilon^p(\eta-w_\varepsilon)^{-r}
\end{aligned}$$

$$+\int_\Omega [-pr(d_1+d_3)+p(p-1)(\eta-w_\varepsilon)\chi_{1\varepsilon}(w_\varepsilon)]u_\varepsilon^{p-1}(\eta-w_\varepsilon)^{-r-1}\nabla u_\varepsilon\cdot\nabla w_\varepsilon$$
$$-\int_\Omega [r(r+1)d_3-pr\|\chi_{1\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}(\eta-w_\varepsilon)]u_\varepsilon^{p}(\eta-w_\varepsilon)^{-r-2}|\nabla w_\varepsilon|^2 \tag{3.16}$$

for all $t>0$. By the Young's inequality there exists $\delta>0$ such that

$$\int_\Omega [-pr(d_1+d_3)+p(p-1)(\eta-w_\varepsilon)\chi_{1\varepsilon}(w_\varepsilon)]u_\varepsilon^{p-1}(\eta-w_\varepsilon)^{-r-1}\nabla u_\varepsilon\cdot\nabla w_\varepsilon$$
$$\le\int_\Omega \frac{[-pr(d_1+d_3)+p(p-1)(\eta-w_\varepsilon)\|\chi_{1\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}]^2}{4d_1p(p-1)\delta}u_\varepsilon^{p}(\eta-w_\varepsilon)^{-r-2}|\nabla w_\varepsilon|^2$$
$$+d_1p(p-1)\delta\int_\Omega u_\varepsilon^{p-2}(\eta-w_\varepsilon)^{-r}|\nabla u_\varepsilon|^2. \tag{3.17}$$

We claim that there exists constant $c_1>0$ such that for all $\varepsilon\in(0,1)$,

$$r(r+1)d_3-pr\|\chi_{1\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}(\eta-w_\varepsilon)$$
$$-\frac{[-pr(d_1+d_3)+p(p-1)(\eta-w_\varepsilon)\|\chi_{1\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}]^2}{4d_1p(p-1)\delta}\ge c_1. \tag{3.18}$$

It is clear to see that (3.18) to be equivalent to

$$p(p-1)^2(\eta-w_\varepsilon)^2\|\chi_{1\varepsilon}\|^2_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}$$
$$+[p(d_1+d_3)^2-4(p-1)d_1d_3\delta]r^2-4(p-1)d_1d_3r\delta$$
$$-2p(p-1)r\|\chi_{1\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}(\eta-w_\varepsilon)(d_1+d_3-2\delta)\le -4c_1d_1(p-1)\delta.$$

Letting $r=(p-1)\theta$, $\theta\in[0,1]$, we obtain

$$p(\eta-w_\varepsilon)^2\|\chi_{1\varepsilon}\|^2_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}-2p\theta\|\chi_{1\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}(\eta-w_\varepsilon)(d_1+d_3-2d_1\delta)$$
$$+[p(d_1+d_3)^2-4(p-1)d_1d_3\delta]\theta^2-4d_1d_3\theta\delta\le-\frac{c_1}{2}.$$

Picking $\delta\in(0,1)$ suitably close to 1, we have

$$p(\eta-w_\varepsilon)^2\|\chi_{1\varepsilon}\|^2_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}-2p\theta\|\chi_{1\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}(\eta-w_\varepsilon)(d_3-d_1)$$
$$+[p(d_1-d_3)^2+4d_1d_3]\theta^2-4d_1d_3\theta\le-c_1.$$

For simplicity, let $\phi=(\eta-w_\varepsilon)\|\chi_{1\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}$, $s=w_\varepsilon$, there exists $\bar{s}\in(0,\bar{w})$ such that $s\le\bar{s}$ for all $(x,t)\in\Omega\times(0,\infty)$ and each $\varepsilon\in(0,1)$, we just have to prove

$$\Pi_{d_1,d_3,p,\theta}(\phi) := p\phi^2 - 2p\theta\phi(d_3 - d_1) + [p(d_1 - d_3)^2 + 4d_1d_3]\theta^2 - 4d_1d_3\theta < 0,$$

by simple calculation, we readily conclude that $\Pi_{d_1,d_3,p,\theta} < 0$ if and only if $\phi \in (\phi_1, \phi_2)$, where

$$\phi_1 = \theta(d_3 - d_1) - 2\sqrt{\frac{d_1d_3\theta(1-\theta)}{p}}, \quad \phi_2 = \theta(d_3 - d_1) + 2\sqrt{\frac{d_1d_3\theta(1-\theta)}{p}}.$$

By some calculation, we know that

$$\phi_1 \le 0 \Longleftrightarrow \begin{cases} d_3 \le d_1, \ \ p > 1 \ \text{ and } \ \theta \in [0,1] \ \text{ or} \\ d_3 > d_1, \ \ p > 1 \ \text{ and } \ \theta \in [0,\theta^*], \end{cases}$$

where $\theta^* = \frac{4d_1d_3}{p(d_1-d_3)^2+4d_1d_3}$, and ϕ_2 obtain the maximum at $\theta = \frac{1}{2} + \frac{d_3-d_1}{2\sqrt{(d_3-d_1)^2+\frac{4d_1d_3}{p}}}$, that is,

$$\phi_{2\max} = \phi_2(\theta) = \frac{d_3 - d_1 + \sqrt{(d_3-d_1)^2 + \frac{4d_1d_3}{p}}}{2}.$$

The following details are quite a few elementary calculations, we left it out, so (3.18) is valid. Owing to $\eta > \bar{w}$, we obtain

$$(1-\epsilon)d_1p(p-1)\int_\Omega u_\varepsilon^{p-2}(\eta - w_\varepsilon)^{-r}|\nabla u_\varepsilon|^2 \ge (1-\epsilon)d_1p(p-1)\eta^{-r}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 \quad (3.19)$$

as well as

$$c_1\int_\Omega u_\varepsilon^p(\eta - w_\varepsilon)^{-r-2}|\nabla w_\varepsilon|^2 \ge c_1\eta^{-r-2}\int_\Omega u_\varepsilon^p|\nabla w_\varepsilon|^2. \quad (3.20)$$

Together with (3.17), (3.19) and (3.20) inserted into (3.16), this shows that (3.15) holds. This completes the proof. □

Lemma 3.4. *Suppose that the conditions of Theorem 3.1 hold. Then there exists $\bar{p} > 4$ such that for all $T > 0$ and each $\varepsilon \in (0,1)$ one can find $C(T) > 0$ satisfying*

$$\int_\Omega u_\varepsilon^{\bar{p}}(\cdot,t) \le C(T) \quad \textit{for all } t > 0 \quad (3.21)$$

as well as

$$\int_0^T\int_\Omega |\nabla u_\varepsilon|^2 \le C(T) \quad (3.22)$$

and

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$$\int_0^T \int_\Omega u_\varepsilon^2 |\nabla w_\varepsilon|^2 \le C(T). \tag{3.23}$$

Proof. We know from Lemma 3.3 that there are constants $c_1, c_2 > 0$ such that for each $\varepsilon \in (0, 1)$,

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p(\eta - w_\varepsilon)^{-r} + c_1(p)\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + c_1(p)\int_\Omega u_\varepsilon^p|\nabla w_\varepsilon|^2 \le c_2(p)\int_\Omega u_\varepsilon^p(\eta - w_\varepsilon)^{-r} \tag{3.24}$$

for all $t > 0$. Integrating (3.24) from 0 to T, we have

$$\int_\Omega u_\varepsilon^p(\eta - w_\varepsilon)^{-r} \le c_3(p, T) := \Big(\int_\Omega u_0^p(\eta - w_0)^{-r}\Big)e^{c_2(p)T} \tag{3.25}$$

for all $t \in (0, T)$ and each $\varepsilon \in (0, 1)$, as well as

$$\begin{aligned} c_1(p)\int_0^T\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + c_1(p)\int_0^T\int_\Omega u_\varepsilon^p|\nabla w_\varepsilon|^2 &\le c_2(p)\int_0^T\int_\Omega u_\varepsilon^p(\eta - w_\varepsilon)^{-r} \\ &\le c_2(p)c_3(p, T)T \end{aligned} \tag{3.26}$$

for all $\varepsilon \in (0, 1)$. We choose $p = \bar{p}$, (3.21) follows from (3.25), here we have been used the fact that $(\eta - w_\varepsilon)^{-r} \ge \eta^{-r}$ in $\Omega \times (0, \infty)$, whereas evaluating (3.26) for $p = 2$ and $p = 4$, respectively, shows that (3.22) as well as (3.23) are valid. This completes the proof. □

Lemma 3.5. *Suppose that the conditions of Theorem 3.1 hold. Then for all $T > 0$ there exists $C > 0$ such that*

$$\int_0^T\int_\Omega |\nabla w_\varepsilon|^2 \le C \quad \textit{for all } \varepsilon \in (0, 1). \tag{3.27}$$

Proof. Multiplying the third equation in (3.5) by w_ε, integrating by parts and using the nonnegativity of α and β as well as Lemma 3.2 to compute

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w_\varepsilon^2 + d_3\int_\Omega |\nabla w_\varepsilon|^2 = -\int_\Omega \frac{(\alpha u_\varepsilon + \beta v_\varepsilon)w_\varepsilon^2}{1 + \varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon)w_\varepsilon} \le 0 \ \text{ for all } t > 0, \tag{3.28}$$

from which (3.27) follows by integration. This completes the proof. □

Lemma 3.6. *Assume that the conditions of Theorem 3.1 hold. Then for all $T > 0$ there exists $C(T) > 0$ such that for all $\varepsilon \in (0, 1)$,*

$$\int_0^T \int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} \le C(T). \tag{3.29}$$

Proof. Multiplying the second equation in (3.5) by $\frac{1}{v_\varepsilon+1}$, integrating by parts and using Young's inequality, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \ln(v_\varepsilon+1) &= \int_\Omega \frac{1}{v_\varepsilon+1}[d_2\Delta v_\varepsilon - \nabla\cdot(v_\varepsilon \chi_{2\varepsilon}(w_\varepsilon)\cdot\nabla w_\varepsilon) + \mu_2 v_\varepsilon(1-v_\varepsilon-a_2u_\varepsilon)] \\
&= d_2\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} - \int_\Omega \frac{v_\varepsilon}{(v_\varepsilon+1)^2}\chi_{2\varepsilon}(w_\varepsilon)\nabla v_\varepsilon\cdot\nabla w_\varepsilon \\
&\quad + \mu_2\int_\Omega \frac{v_\varepsilon}{v_\varepsilon+1}(1-v_\varepsilon-a_2u_\varepsilon) \\
&\ge \frac{d_2}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} - \frac{1}{2d_2}\|\chi_{2\varepsilon}\|^2_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}\int_\Omega \frac{v_\varepsilon^2}{(v_\varepsilon+1)^2}|\nabla w_\varepsilon|^2 \\
&\quad + \mu_2\int_\Omega \frac{v_\varepsilon}{v_\varepsilon+1} - a_2\mu_2\int_\Omega \frac{u_\varepsilon v_\varepsilon}{v_\varepsilon+1} - \mu_2\int_\Omega \frac{v_\varepsilon^2}{v_\varepsilon+1} \\
&\ge \frac{d_2}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} - \frac{1}{2d_2}\|\chi_{2\varepsilon}\|^2_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}\int_\Omega |\nabla w_\varepsilon|^2 \\
&\quad - a_2\mu_2\int_\Omega u_\varepsilon - \mu_2\int_\Omega v_\varepsilon.
\end{aligned} \tag{3.30}$$

We apply Lemma 3.1 and integrate (3.30) to obtain that for arbitrary $T>0$ and each $\varepsilon\in(0,1)$,

$$\begin{aligned}
\frac{d_2}{2}\int_0^T\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} &\le \int_\Omega \ln(v_\varepsilon(\cdot,T)+1) - \int_\Omega \ln(v_0+1) + \mu_2(a_2c_1+c_2) \\
&\quad + \frac{1}{2d_2}\|\chi_{2\varepsilon}\|^2_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}\int_0^T\int_\Omega |\nabla w_\varepsilon|^2,
\end{aligned}$$

where $c_1, c_2 > 0$ are constants, combined with Lemma 3.5, we immediately get (3.29). This completes the proof. □

Lemma 3.7. *Suppose that the conditions of Theorem 3.1 hold. Then for all $T>0$ there is a constant $C(T)>0$ such that for all $\varepsilon\in(0,1)$,*

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$$\int_0^T \int_\Omega \frac{(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon \ln(v_\varepsilon + 1)}{1 + \varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon} \le C(T). \tag{3.31}$$

Proof. From the second and third equation in (3.5), we calculate

$$\begin{aligned}
\frac{d}{dt}\int_\Omega w_\varepsilon \ln(v_\varepsilon+1) &= \int_\Omega \frac{w_\varepsilon}{v_\varepsilon+1}[d_2\Delta v_\varepsilon - \nabla\cdot(v_\varepsilon \chi_{2\varepsilon}(w_\varepsilon)\cdot\nabla w_\varepsilon) + \mu_2 v_\varepsilon(1 - v_\varepsilon - a_2 u_\varepsilon)] \\
&\quad + \int_\Omega \Big[d_3\Delta w_\varepsilon - \frac{(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon}{1 + \varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon}\Big]\ln(v_\varepsilon+1) \\
&= -d_2\int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon+1}\cdot\nabla w_\varepsilon - \int_\Omega \frac{(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon}{1 + \varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon}\ln(v_\varepsilon+1) \\
&\quad - d_3\int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon+1}\cdot\nabla w_\varepsilon + d_2\int_\Omega \frac{w_\varepsilon}{(v_\varepsilon+1)^2}|\nabla v_\varepsilon|^2 \\
&\quad + \int_\Omega \frac{v_\varepsilon \chi_{2\varepsilon}(w_\varepsilon)}{v_\varepsilon+1}|\nabla w_\varepsilon|^2 - \int_\Omega \frac{v_\varepsilon w_\varepsilon}{(v_\varepsilon+1)^2}\chi_{2\varepsilon}(w_\varepsilon)\nabla v_\varepsilon\cdot\nabla w_\varepsilon \\
&\quad + \mu_2\int_\Omega \frac{v_\varepsilon w_\varepsilon}{v_\varepsilon+1}(1 - v_\varepsilon - a_2 u_\varepsilon) \quad \text{for all } t > 0.
\end{aligned} \tag{3.32}$$

Using Young's inequality, we obtain

$$-(d_2+d_3)\int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon+1}\cdot\nabla w_\varepsilon \le \int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} + \frac{(d_2+d_3)^2}{4}\int_\Omega |\nabla w_\varepsilon|^2, \tag{3.33}$$

$$d_2\int_\Omega \frac{w_\varepsilon}{(v_\varepsilon+1)^2}|\nabla v_\varepsilon|^2 \le d_2\bar{w}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2}, \tag{3.34}$$

$$\int_\Omega \frac{v_\varepsilon \chi_{2\varepsilon}(w_\varepsilon)}{v_\varepsilon+1}|\nabla w_\varepsilon|^2 \le \|\chi_{2\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}\int_\Omega |\nabla w_\varepsilon|^2 \tag{3.35}$$

as well as

$$\begin{aligned}
& -\int_\Omega \frac{v_\varepsilon w_\varepsilon}{(v_\varepsilon+1)^2}\chi_{2\varepsilon}(w_\varepsilon)\nabla v_\varepsilon\cdot\nabla w_\varepsilon \\
& \le \bar{w}\|\chi_{2\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}\int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon+1}\cdot\nabla w_\varepsilon \\
& \le \int_\Omega |\nabla w_\varepsilon|^2 + \frac{1}{4}\bar{w}^2\|\chi_{2\varepsilon}\|^2_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2}
\end{aligned} \tag{3.36}$$

and

$$\mu_2 \int_\Omega \frac{v_\varepsilon w_\varepsilon}{v_\varepsilon + 1}(1 - v_\varepsilon - a_2 u_\varepsilon) \le \mu_2 \bar{w}|\Omega| \tag{3.37}$$

for all $t > 0$. From (3.32)–(3.37) we obtain after an integration that for all $T > 0$,

$$\begin{aligned}\int_0^T\int_\Omega \frac{(\alpha u_\varepsilon + \beta v_\varepsilon)w_\varepsilon \ln(v_\varepsilon+1)}{1+\varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon)w_\varepsilon} \le{}& (1 + d_2\bar{w} + \frac{1}{4}\bar{w}^2\|\chi_{2\varepsilon}\|^2_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])})\int_0^T\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} \\ &+(\frac{(d_2+d_3)^2}{4} + \|\chi_{2\varepsilon}\|^2_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])} + 1)\int_\Omega |\nabla w_\varepsilon|^2 \\ &+ \int_\Omega w_0 \ln(v_0+1) + \mu_2\bar{w}|\Omega|\end{aligned}$$

for all $\varepsilon \in (0, 1)$. By Lemmas 3.5 and 3.6 results in (3.31). This completes the proof. □

Inspired by [40,46,51], we shall derive the following lemma.

Lemma 3.8. *Assume that the conditions of Theorem 3.1 hold and let $T > 0$. Then*

$$\left\{\frac{(\alpha u_\varepsilon + \beta v_\varepsilon)w_\varepsilon}{1+\varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon)w_\varepsilon}\right\}_{\varepsilon\in(0,1)} \quad \textit{is uniformly integrable over } \Omega \times (0, T). \tag{3.38}$$

Proof. For given $T > 0$, in accordance with Lemmas 3.1 and 3.7, there are some constants $c_1, c_2 > 0$ such that

$$\int_0^T\int_\Omega u_\varepsilon^2 \le c_1 \quad \text{for all } \varepsilon \in (0, 1) \tag{3.39}$$

and

$$\int_0^T\int_\Omega \frac{(\alpha u_\varepsilon + \beta v_\varepsilon)w_\varepsilon \ln(v_\varepsilon+1)}{1+\varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon)w_\varepsilon} \le c_2 \quad \text{for all } \varepsilon \in (0, 1). \tag{3.40}$$

Given $\zeta > 0$, we can choose $L > 0$ appropriate large fulfilling

$$\frac{c_2}{\ln(L+1)} \le \frac{\zeta}{3} \tag{3.41}$$

and $\sigma > 0$ appropriate small such that

$$\beta\bar{w}L\sigma \le \frac{\zeta}{3}, \quad \alpha\bar{w}\sqrt{c_1\sigma} \le \frac{\zeta}{3}. \tag{3.42}$$

If $\Lambda \subset \Omega \times (0, T)$ is an arbitrary measurable set satisfying $|\Lambda| \le \sigma$, we have

$$\begin{aligned}
&\iint\limits_{\Lambda} \frac{(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon}{1 + \varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon} \\
&= \iint\limits_{\Lambda \cap \{v_\varepsilon \ge L\}} \frac{(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon}{1 + \varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon} + \iint\limits_{\Lambda \cap \{v_\varepsilon < L\}} \frac{(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon}{1 + \varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon} \\
&\le \frac{1}{\ln(L+1)} \iint\limits_{\Lambda \cap \{v_\varepsilon \ge L\}} \frac{(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon \ln(v_\varepsilon + 1)}{1 + \varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon} + \alpha \bar{w} \iint\limits_{\Lambda} u_\varepsilon + \beta \bar{w} L |\Lambda \cap \{v_\varepsilon < L\}| \\
&\le \frac{c_2}{\ln(L+1)} + \beta \bar{w} L \sigma + \alpha \bar{w} |\Lambda|^{\frac{1}{2}} \Big(\int_0^T \int_\Omega u_\varepsilon^2 \Big)^{\frac{1}{2}} \\
&\le \frac{\zeta}{3} + \frac{\zeta}{3} + \frac{\zeta}{3} = \zeta \quad \text{for all } \varepsilon \in (0, 1),
\end{aligned}$$

thanks to $\zeta > 0$ was arbitrary, this clearly yields (3.31). This completes the proof. □

Lemma 3.9. *Suppose that the conditions of Theorem 3.1 hold, and let $m \in \mathbb{N}$ be such that $m > \frac{N}{2}$. Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^T \|u_{\varepsilon t}(\cdot, t)\|^2_{(W^{1,2}(\Omega))^*} dt \le C(T) \quad \text{for all } \varepsilon \in (0, 1) \tag{3.43}$$

as well as

$$\int_0^T \|\partial t \ln(v_\varepsilon(\cdot, t) + 1)\|_{(W^{m,2}(\Omega))^*} dt \le C(T) \quad \text{for all } \varepsilon \in (0, 1) \tag{3.44}$$

and

$$\int_0^T \|w_{\varepsilon t}(\cdot, t)\|_{(W^{m,2}(\Omega))^*} dt \le C(T) \quad \text{for all } \varepsilon \in (0, 1). \tag{3.45}$$

Proof. Given $t > 0$ and $\varphi \in C_0^\infty(\overline{\Omega} \times [0, T))$. Multiplying the first equation in (3.5) by φ, integrating by parts and using Hölder inequality, for all $\varepsilon \in (0, 1)$, we have

$$\begin{aligned}
\int_\Omega u_{\varepsilon t}(\cdot, t) \varphi &= -d_1 \int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi + \int_\Omega u_\varepsilon \chi_{1\varepsilon}(w_\varepsilon) \nabla w_\varepsilon \cdot \nabla \varphi + \mu_1 \int_\Omega u_\varepsilon (1 - u_\varepsilon - a_1 v_\varepsilon) \varphi \\
&\le -d_1 \int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi + \|\chi_{1\varepsilon}\|_{L^\infty([0, \|w_0\|_{L^\infty(\Omega)}])} \int_\Omega u_\varepsilon \nabla w_\varepsilon \cdot \nabla \varphi
\end{aligned}$$

$$
\begin{aligned}
&+\mu_1 \int_\Omega u_\varepsilon(1-u_\varepsilon)\varphi \\
&\le d_1\|\nabla u_\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} + \|\chi_{1\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}\|u_\varepsilon\nabla w_\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} \\
&+\mu_1\|u_\varepsilon\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} + \mu_1\|u_\varepsilon^2\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}.
\end{aligned}
$$

By the Young's inequality, we obtain

$$
\begin{aligned}
\|u_{\varepsilon t}(\cdot,t)\|^2_{(W^{1,2}(\Omega))^*} \le \Big(&2d_1^2\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + 2\|\chi_{1\varepsilon}\|^2_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}\|u_\varepsilon\nabla w_\varepsilon\|^2_{L^2(\Omega)} \\
&+2\mu_1^2\|u_\varepsilon\|^2_{L^2(\Omega)} + 2\mu_1^2\|u_\varepsilon^2\|^2_{L^2(\Omega)}\Big)\|\varphi\|^2_{W^{1,2}(\Omega)}.
\end{aligned}
\tag{3.46}
$$

We apply Lemmas 3.1, 3.4 and integrate (3.46) to conclude that

$$
\int_0^T \|u_{\varepsilon t}(\cdot,t)\|^2_{(W^{1,2}(\Omega))^*}dt \le C(T) \quad \text{for all } \varepsilon\in(0,1).
$$

Similarly, multiplying the second equation in (3.5) by $\frac{\varphi}{v_\varepsilon(\cdot,t)+1}$, integrating by parts and using Young's inequality, it immediately derives that for all $t>0$,

$$
\begin{aligned}
&\int_\Omega \partial t \ln(v_\varepsilon(\cdot,t)+1)\varphi \\
&= \int_\Omega \frac{\varphi}{v_\varepsilon+1}[d_2\Delta v_\varepsilon - \nabla\cdot(v_\varepsilon\chi_{2\varepsilon}(w_\varepsilon)\cdot\nabla w_\varepsilon) + \mu_2 v_\varepsilon(1-v_\varepsilon-a_2u_\varepsilon)] \\
&\le d_2\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2}\varphi - d_2\int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon+1}\cdot\nabla\varphi - \int_\Omega \frac{v_\varepsilon\varphi}{v_\varepsilon+1}\chi_{2\varepsilon}(w_\varepsilon)\nabla w_\varepsilon\cdot\frac{\nabla v_\varepsilon}{v_\varepsilon+1} \\
&+\int_\Omega \frac{v_\varepsilon}{v_\varepsilon+1}\chi_{2\varepsilon}(w_\varepsilon)\nabla w_\varepsilon\cdot\nabla\varphi + \mu_2\int_\Omega v_\varepsilon - \mu_2\int_\Omega v_\varepsilon^2 \\
&\le d_2\|\varphi\|_{L^\infty(\Omega)}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} + d_2\Big(\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} + \frac14\Big)\|\nabla\varphi\|_{L^2(\Omega)} \\
&+\|\chi_{2\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}\Big(\frac14\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} + \int_\Omega |\nabla w_\varepsilon|^2\Big)\|\varphi\|_{L^\infty(\Omega)} \\
&+\|\chi_{2\varepsilon}\|_{L^\infty([0,\|w_0\|_{L^\infty(\Omega)}])}\Big(\int_\Omega |\nabla w_\varepsilon|^2 + \frac14\Big)\|\nabla\varphi\|_{L^2(\Omega)} + \frac14\mu_2|\Omega|
\end{aligned}
$$

for all $\varepsilon\in(0,1)$. Since $m>\frac{N}{2}$, by the embedding theorem, we immediately find that $W^{m,2}(\Omega)\hookrightarrow L^\infty(\Omega)$, there is a constant $c_1>0$ such that for all $t>0$ and each $\varepsilon\in(0,1)$,

$$\|\partial t \ln(v_{\varepsilon t}(\cdot,t)+1)\|_{(W^{m,2}(\Omega))^*} \le c_1\Big(\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} + \int_\Omega |\nabla w_\varepsilon|^2 + 1\Big),$$

and thus, (3.44) follows from Lemmas 3.5 and 3.6. Then, multiplying the third equation in (3.5) by φ and using Young's inequality, we have

$$\begin{aligned}\int_\Omega w_{\varepsilon t}(\cdot,t)\varphi &= -d_3\int_\Omega \nabla w_\varepsilon\cdot\nabla\varphi - \int_\Omega \frac{(\alpha u_\varepsilon+\beta v_\varepsilon)w_\varepsilon}{1+\varepsilon(\alpha u_\varepsilon+\beta v_\varepsilon)w_\varepsilon}\varphi \\ &\le d_3(\|\nabla w_\varepsilon\|^2_{L^2(\Omega)}+\frac{1}{4})\|\nabla\varphi\|_{L^2(\Omega)} + \bar{w}(\alpha\|u_\varepsilon\|_{L^1(\Omega)}+\beta\|v_\varepsilon\|_{L^1(\Omega)})\|\varphi\|_{L^\infty(\Omega)}.\end{aligned}$$

By continuity of embedding $W^{m,2}(\Omega)\hookrightarrow L^\infty(\Omega)$, combined with Lemma 2.1, there is a constant $c_2>0$ such that for all $t>0$ and each $\varepsilon\in(0,1)$,

$$\|w_{\varepsilon t}(\cdot,t)\|_{(W^{m,2}(\Omega))^*} \le c_2\Big(\int_\Omega |\nabla w_\varepsilon|^2+1\Big),$$

and thus, (3.45) follows from Lemma 3.5. This completes the proof. □

Now, we are preparing to extract a suitable sequence of number ε along with the respective solutions approach a limit in appropriate topologies.

Lemma 3.10. *Assume that the conditions of Theorem 3.1 hold. Then there are $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ and nonnegative functions u, v and w defined a.e. in $\Omega\times(0,\infty)$, such that $\varepsilon_j\searrow 0$ as $j\to\infty$, that with some $\bar{p}>0$ we have $u\in L^\infty((0,\infty);L^{\bar{p}}(\Omega))\cap L^2_{loc}([0,\infty);W^{1,2}(\Omega))$, $v\in L^\infty((0,\infty);L^1(\Omega))$ and $w\in L^\infty(\Omega\times(0,\infty))\cap L^2_{loc}([0,\infty);W^{1,2}(\Omega))$, that $\nabla\ln(v+1)$ and $u\nabla w$ belong to $L^2_{loc}([0,\infty);W^{1,2}(\Omega))$, and that*

$$u_\varepsilon\to u \quad in\ L^2_{loc}(\overline{\Omega}\times[0,\infty))\ and\ a.e.\ in\ \Omega\times(0,\infty), \tag{3.47}$$

$$u_\varepsilon(\cdot,t)\to u(\cdot,t) \quad in\ L^2(\Omega)\ for\ a.e.\ t>0, \tag{3.48}$$

$$\nabla u_\varepsilon\rightharpoonup\nabla u \quad in\ L^2_{loc}(\overline{\Omega}\times[0,\infty)) \tag{3.49}$$

and

$$v_\varepsilon\to v \quad a.e.\ in\ \Omega\times(0,\infty), \tag{3.50}$$

$$\ln(v_\varepsilon+1)\rightharpoonup\ln(v+1) \quad in\ L^2_{loc}([0,\infty);W^{1,2}(\Omega)) \tag{3.51}$$

as well as

$$w_\varepsilon\to w \quad in\ L^2_{loc}(\overline{\Omega}\times[0,\infty))\ and\ a.e.\ in\ \Omega\times(0,\infty), \tag{3.52}$$

$$w_\varepsilon(\cdot,t)\to w(\cdot,t) \quad in\ L^2(\Omega)\ for\ a.e.\ t>0, \tag{3.53}$$

$$w_\varepsilon\overset{*}{\rightharpoonup} w \quad in\ L^\infty(\Omega\times(0,\infty)), \tag{3.54}$$

$$\nabla w_\varepsilon \rightharpoonup \nabla w \quad in\ L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{3.55}$$

$$\frac{(\alpha u_\varepsilon + \beta v_\varepsilon)w_\varepsilon}{1+\varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon)w_\varepsilon} \to (\alpha u + \beta v)w \quad in\ L^1_{loc}(\overline{\Omega} \times [0,\infty)), \tag{3.56}$$

$$u_\varepsilon \nabla w_\varepsilon \rightharpoonup u\nabla w \quad in\ L^2_{loc}(\overline{\Omega} \times [0,\infty)) \tag{3.57}$$

as $\varepsilon = \varepsilon_j \searrow 0$. *Furthermore, the identities (3.1), (3.2) and inequality (3.4) hold for all* $\psi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ *and*

$$\int_\Omega u(\cdot,t) \le \max\Big\{\int_\Omega u_0,\ \frac{|\Omega|}{\mu_1}\Big\} \quad for\ a.e.\ t > 0. \tag{3.58}$$

Proof. According to Lemmas 3.4 and 3.9, Aubin-Lions lemma, it immediately derives that there are $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and a nonnegative function $u \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and that (3.47)–(3.49) hold. Owing to (3.21) and Fatou's lemma, we know that $u \in L^\infty((0,\infty); L^{\bar{p}}(\Omega))$, (3.58) is a consequence of (3.6) when combined with (3.47), similar to (3.58), we also get (3.4). Analogously, according to Lemma 3.6 3.6, choose any integer $m > \frac{N}{2}$, combined with (3.7) and Lemma 3.9, we have $(\ln(v_\varepsilon + 1))_{\varepsilon\in(0,1)}$ is bounded in $L^2([0,T); W^{1,2}(\Omega))$ and $(\partial_t \ln(v_\varepsilon + 1))_{\varepsilon\in(0,1)}$ is bounded in $L^2([0,T); (W^{m,2}(\Omega))^*)$ for all $T > 0$, applying Aubin-Lions lemma once more, there exists subsequence such that (3.50) and (3.51) hold. Then, together with Lemmas 3.2, 3.5 and 3.9, we have $(w_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2([0,T); W^{1,2}(\Omega))$ and $(w_{\varepsilon t})_{\varepsilon\in(0,1)}$ is bounded in $L^2([0,T); (W^{m,2}(\Omega))^*)$ for all $T > 0$, using Aubin-Lions lemma again, there is a subsequence such that (3.52), (3.53) and (3.55) hold. The combination of (3.47) and (3.55), we immediately find that (3.57) holds. From Lemma 3.2 and Banach-Alaoglu theorem, we readily conclude that (3.54) is valid. Finally, in accordance with Lemma 3.8 and Vitali convergence theorem, combined with (3.47), (3.50), (3.52), the convergence of (3.56) holds. (3.1) and (3.2) can be obtained by means of (3.49), (3.52) and (3.55)–(3.57), the details similar to our recent work [23], we omit giving details on this here. This completes the proof. □

Inspired by [46] and also [4,32,40], we shall derive the following lemma.

Lemma 3.11. *Assume that the conditions of Theorem 3.1 hold, and let* $(\varepsilon_j)_{j\in\mathbb{N}}$ *and* w *be as in Lemma 3.10. Then for all* $T > 0$,

$$\nabla w_\varepsilon \to \nabla w \quad in\ L^2(\Omega \times (0,T)) \quad as\ \varepsilon = \varepsilon_j \searrow 0. \tag{3.59}$$

Proof. Fixed $T > 0$, from Lemma 3.10, we can find $t_* > T$ such that t_* is a Lebesgue point of $0 < t \mapsto \int_\Omega w^2(\cdot,t)$, and that moreover

$$\int_\Omega w_\varepsilon^2(\cdot,t_*) \to \int_\Omega w^2(\cdot,t_*) \ \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{3.60}$$

Given any $t_* > 0$ and $\delta \in (0,1)$, we let

$$\rho_\delta(t) := \begin{cases} 1, & t \in [0, t_*], \\ \frac{t_*+\delta-t}{\delta}, & t \in (t_*, t_*+\delta), \\ 0, & t \in [t_*+\delta, \infty) \end{cases}$$

and

$$\bar{w}(x,t) := \begin{cases} w(x,t), & (x,t) \in \Omega \times (0,\infty), \\ w_0(x), & (x,t) \in (-\infty, 0]. \end{cases}$$

Then for all $\delta \in (0,1)$ and each $h \in (0,\delta)$, we introduce

$$\psi(x,t) := \rho_\delta(t) \cdot (A_h \bar{w})(x,t), \quad (x,t) \in \Omega \times (0,\infty),$$

where $(A_h\bar{w})(x,t) := \frac{1}{h}\int_{t-h}^t \bar{w}(x,s)ds$, $(x,t) \in \Omega \times (0,\infty)$. Then $\psi \in L^\infty(\Omega \times (0,\infty)) \cap L^2((0,\infty); W^{1,2}(\Omega))$ with $\psi_t \in L^\infty(\Omega \times (0,\infty))$, and in addition ψ is supported in $\overline{\Omega} \times [0, t_*+1]$. Thus, we now insert ψ into (3.2) to obtain

$$\begin{aligned} J(\delta,h) &:= -d_3 \int_0^\infty \int_\Omega \rho_\delta(t) \nabla w(x,t) \cdot \nabla(A_h\bar{w})(x,t)dxdt \\ &= -\int_0^\infty \int_\Omega \rho'_\delta(t) w(x,t)(A_h\bar{w})(x,t)dxdt - \int_\Omega w_0^2(x) \\ &\quad - \int_0^\infty \int_\Omega \rho_\delta(t) w(x,t) \cdot \frac{1}{h}(\bar{w}(x,t) - \bar{w}(x,t-h))dxdt \\ &\quad + \int_0^\infty \int_\Omega \rho_\delta(t)(\alpha u(x,t) + \beta v(x,t)) w(x,t)(A_h\bar{w})(x,t)dxdt \\ &=: J_1(\delta,h) + J_2(\delta,h) + J_3(\delta,h) + J_4(\delta,h) \end{aligned} \tag{3.61}$$

for all $\delta \in (0,1)$ and each $h \in (0,\delta)$. Applying Lemma A.2(a) in [46], we obtain

$$\nabla(A_h\bar{w}) = A_h \nabla\bar{w} \rightharpoonup \nabla\bar{w} = \nabla w \quad \text{in } L^2(\Omega \times (0, t_*+1)) \text{ as } h \searrow 0,$$

and thus

$$J(\delta,h) \to -d_3 \int_0^{t_*+1} \int_\Omega \rho_\delta(t) |\nabla w|^2 dxdt \quad \text{as } h \searrow 0. \tag{3.62}$$

Similarly, combined with Lemma A.2(b) in [46] and $\bar{w} \in L^\infty(\Omega \times (-1, t_*+1))$ assures that

$$A_h\bar{w} \overset{*}{\rightharpoonup} \bar{w} = w \quad \text{in } L^\infty(\Omega\times(0,t_*+1)) \quad \text{as } h\searrow 0,$$

and thus

$$J_1(\delta,h) \to -\int_0^{t_*+1}\int_\Omega \rho_\delta'(t)w^2dxdt \quad \text{as } h\searrow 0. \tag{3.63}$$

Using Young's inequality, we have

$$\begin{aligned}
J_3(\delta,h) &= -\frac{1}{h}\int_0^\infty\int_\Omega \rho_\delta(t)\bar{w}^2(x,t)dxdt + \frac{1}{h}\int_0^\infty\int_\Omega \rho_\delta(t)\bar{w}(x,t)\bar{w}(x,t-h)dxdt \\
&\le -\frac{1}{2h}\int_0^\infty\int_\Omega \rho_\delta(t)\bar{w}^2(x,t)dxdt + \frac{1}{2h}\int_0^\infty\int_\Omega \rho_\delta(t)\bar{w}^2(x,t-h)dxdt \\
&= \frac{1}{2}\int_0^\infty\int_\Omega \frac{\rho_\delta(\sigma+h)-\rho_\delta(\sigma)}{h}w^2(x,\sigma)dxd\sigma + \frac{1}{2}\int_\Omega w_0^2(x) \\
&=: J_{31}(\delta,h) + J_{32}(\delta,h) \quad \text{for all } \delta\in(0,1) \text{ and } h\in(0,\delta).
\end{aligned}$$

By the dominated convergence theorem,

$$J_{31}(\delta,h) \to -\frac{1}{2\delta}\int_{t_*}^{t_*+1} w^2(x,\sigma)dxd\sigma \quad \text{as } h\searrow 0,$$

here we have been used the fact that

$$\rho_\delta' \equiv -\frac{1}{\delta} \ \text{ in } (t_*,t_*+\delta) \ \text{ and } \ \rho_\delta' \equiv 0 \ \text{ in } (0,t_*)\cup(t_*+\delta,\infty),$$

and thus

$$\begin{aligned}
d_3\int_0^{t_*+\delta}\int_\Omega \rho_\delta(t)|\nabla w(x,t)|^2dxdt \ge{}& -\frac{1}{2\delta}\int_{t_*}^{t_*+\delta} w^2(x,t)dxdt + \frac{1}{2}\int_\Omega w_0^2(x) \\
&- \int_0^\infty\int_\Omega \rho_\delta(t)(\alpha u(x,t)+\beta v(x,t))w^2(x,t)dxdt
\end{aligned}$$

for all $\delta\in(0,1)$, when combined with Lebesgue theorem and dominated convergence theorem, implies that

$$d_3 \iint_{\Omega}^{t_*} |\nabla w|^2 dxdt \geq -\frac{1}{2}\int_{\Omega} w^2(\cdot,t_*) + \frac{1}{2}\int_{\Omega} w_0^2 - \int_0^{t_*}\int_{\Omega} (\alpha u + \beta v) w^2 dxdt. \tag{3.64}$$

The combination of (3.60), (3.52), (3.54) and (3.56), we obtain

$$\begin{aligned} & -\frac{1}{2}\int_{\Omega} w^2(\cdot,t_*) + \frac{1}{2}\int_{\Omega} w_0^2 - \int_0^{t_*}\int_{\Omega} (\alpha u + \beta v) w^2 dxdt \\ &= \lim_{\varepsilon=\varepsilon_j\searrow 0}\Big(-\frac{1}{2}\int_{\Omega} w_\varepsilon^2(\cdot,t_*) + \frac{1}{2}\int_{\Omega} w_0^2 - \int_0^{t_*}\int_{\Omega} \frac{(\alpha u_\varepsilon + \beta v_\varepsilon)w_\varepsilon}{1+\varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon)w_\varepsilon}\cdot w_\varepsilon\Big) \\ &= \lim_{\varepsilon=\varepsilon_j\searrow 0} d_3 \iint_{\Omega}^{t_*} |\nabla w_\varepsilon|^2 dxdt. \end{aligned}$$

From (3.64) we infer that

$$\iint_{\Omega}^{t_*} |\nabla w|^2 dxdt \geq \liminf_{\varepsilon=\varepsilon_j\searrow 0} \iint_{\Omega}^{t_*} |\nabla w_\varepsilon|^2 dxdt,$$

and thus, by (3.55),

$$\nabla w_\varepsilon \to \nabla w \quad \text{in } L^2(\Omega\times(0,t_*))$$

as $\varepsilon = \varepsilon_j \searrow 0$, thanks to $t_* > T$, we immediately obtain (3.59). This completes the proof. □

Lemma 3.12. *Suppose that the conditions of Theorem 3.1 hold, and let $(\varepsilon_j)_{j\in\mathbb{N}}$ and u be as in Lemma 3.10. Then for all $T > 0$,*

$$\nabla u_\varepsilon \to \nabla u \quad \textit{in } L^2(\Omega\times(0,T)) \ \ \varepsilon = \varepsilon_j \searrow 0. \tag{3.65}$$

Proof. The proof is similar to Lemma 3.11, to avoid repetition, we omit giving details on this here. □

Finally, we prove the main theorem.

The Proof of Theorem 3.1. Since the regularity properties of u, v and w have been proved in Lemma 3.10, we will show (3.3) holds. For fixed an arbitrary nonnegative $\psi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$, multiplying the second equation in (3.5) by $\frac{\psi}{v_\varepsilon+1}$, integrating by parts, we obtain that for all $\varepsilon \in (0,1)$,

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$$d_2 \int_0^\infty \int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2}\psi = -\int_0^\infty \int_\Omega \ln(v_\varepsilon+1)\psi_t - \int_\Omega \ln(v_0+1)\psi(\cdot,0)$$
$$+\int_0^\infty \int_\Omega \frac{\nabla v_\varepsilon \cdot \nabla\psi}{v_\varepsilon+1} - \int_0^\infty \int_\Omega \frac{v_\varepsilon \chi_{2\varepsilon}(w_\varepsilon)}{v_\varepsilon+1}\nabla u_\varepsilon \cdot \nabla\psi$$
$$+\int_0^\infty \int_\Omega \frac{v_\varepsilon \chi_{2\varepsilon}(w_\varepsilon)}{v_\varepsilon+1}(\nabla u_\varepsilon \cdot \frac{\nabla v_\varepsilon}{v_\varepsilon+1})\psi. \tag{3.66}$$

From (3.51) we obtain

$$-\int_0^\infty \int_\Omega \ln(v_\varepsilon+1)\psi_t \to -\int_0^\infty \int_\Omega \ln(v+1)\psi_t \quad \text{as } \varepsilon=\varepsilon_j \searrow 0 \tag{3.67}$$

and

$$\int_0^\infty \int_\Omega \frac{\nabla v_\varepsilon \cdot \nabla\psi}{v_\varepsilon+1} = \int_0^\infty \int_\Omega \nabla \ln(v_\varepsilon+1)\cdot\nabla\psi \to \int_0^\infty \int_\Omega \nabla \ln(v+1)\cdot\nabla\psi \tag{3.68}$$

as $\varepsilon=\varepsilon_j \searrow 0$. By (3.50) and dominated convergence theorem, we have

$$\frac{v_\varepsilon}{v_\varepsilon+1} \to \frac{v}{v+1} \quad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \ \text{ as } \varepsilon=\varepsilon_j \searrow 0, \tag{3.69}$$

combined with (3.49) and (3.52), it is sufficient to ensure that

$$-\int_0^\infty \int_\Omega \frac{v_\varepsilon \chi_{2\varepsilon}(w_\varepsilon)}{v_\varepsilon+1}\nabla u_\varepsilon\cdot\nabla\psi \to -\int_0^\infty \int_\Omega \frac{v\chi_2(w)}{v+1}\nabla u\cdot\nabla\psi \quad \text{as } \varepsilon=\varepsilon_j \searrow 0. \tag{3.70}$$

Since $0 \le \frac{v_\varepsilon}{v_\varepsilon+1} \le 1$ and the fact that

$$\frac{v_\varepsilon}{v_\varepsilon+1} \to \frac{v}{v+1} \quad a.e. \text{ in } \Omega\times(0,\infty)) \ \text{ as } \varepsilon=\varepsilon_j \searrow 0,$$

combined with Lemma A.4 in [46], guarantees that

$$\frac{v_\varepsilon}{v_\varepsilon+1}\nabla u_\varepsilon \to \frac{v}{v+1}\nabla u \quad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \ \text{ as } \varepsilon=\varepsilon_j \searrow 0,$$

and thus,

$$\int_0^\infty \int_\Omega \frac{v_\varepsilon \chi_{2\varepsilon}(w_\varepsilon)}{v_\varepsilon+1}(\nabla u_\varepsilon\cdot\frac{\nabla v_\varepsilon}{v_\varepsilon+1})\psi \to \int_0^\infty \int_\Omega \frac{v\chi_2(w)}{v+1}(\nabla u\cdot\frac{\nabla v}{v+1})\psi \quad \text{as } \varepsilon=\varepsilon_j \searrow 0. \tag{3.71}$$

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By lower semicontinuity of norm in $L^2(\Omega \times (0,\infty); \mathbb{R}^N)$ with respect to weak convergence, we readily infer the validity of (3.3), for more details, we recommend that readers refer to [46, Lemma 8.2]. This completes the proof. $\square$

3.2. Asymptotic behavior of the two-species chemotaxis system with consumption of chemoattractant

In this subsection, inspired by [50], we will consider the large time behavior of the solutions gained above. As first, we show the main theorem.

Theorem 3.2. *Under the condition of Theorem 3.1, there is a null set $\Lambda \subset (0,\infty)$ such that the solution of (1.1) has the following properties:*

(i) Assume that $a_1, a_2 \in (0,1)$. Then

$$\|u(\cdot,t) - N_1\|_{L^1(\Omega)} + \|v(\cdot,t) - N_2\|_{L^1(\Omega)} + \|w(\cdot,t)\|_{L^2(\Omega)} \to 0 \;\; as \; (0,\infty)\backslash\Lambda \ni t \to \infty,$$

where $N_1 = \frac{1-a_1}{1-a_1a_2}$ and $N_2 = \frac{1-a_2}{1-a_1a_2}$.

(ii) Assume that $a_1 \geq 1 > a_2 > 0$. Then

$$\|u(\cdot,t)\|_{L^1(\Omega)} + \|v(\cdot,t) - 1\|_{L^1(\Omega)} + \|w(\cdot,t)\|_{L^2(\Omega)} \to 0 \;\; as \; (0,\infty)\backslash\Lambda \ni t \to \infty.$$

Remark 3.3. Theorem 3.2 partially generalizes and improves Theorem 1.2 in [58].

To get the desire results, we give the following key estimate of stabilization in (1.1) in the case of $a_1, a_2 \in (0,1)$.

Lemma 3.13. *Let $a_1, a_2 \in (0,1)$. Under the assumption of Theorem 3.2 (i), the solution of (1.1) has the property that there are constants $\varrho_1 > 0$ and $C_1 > 0$ such that the nonnegative function E_1 defined by*

$$\begin{aligned} E_1(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t), w_\varepsilon(\cdot,t)) := &\int_\Omega (u_\varepsilon - N_1 - N_1 \ln \frac{u_\varepsilon}{N_1}) \\ &+ \frac{a_1\mu_1}{a_2\mu_2}\int_\Omega (v_\varepsilon - N_2 - N_2 \ln \frac{v_\varepsilon}{N_2}) + \frac{\varrho_1}{2}\int_\Omega w_\varepsilon^2 \end{aligned} \tag{3.72}$$

satisfying

$$\begin{aligned} \frac{d}{dt} E_1(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t), w_\varepsilon(\cdot,t)) + C_1\Big\{ &\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \\ &+ \int_\Omega (u_\varepsilon - N_1)^2 + \int_\Omega (v_\varepsilon - N_2)^2 + \int_\Omega |\nabla w_\varepsilon|^2 \Big\} \leq 0 \end{aligned} \tag{3.73}$$

for all $t > 0$, where $N_1 = \frac{1-a_1}{1-a_1a_2}$ and $N_2 = \frac{1-a_2}{1-a_1a_2}$. Moreover,

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$$E_1(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t), w_\varepsilon(\cdot,t)) \le E_1(u_\varepsilon(\cdot,t_0), v_\varepsilon(\cdot,t_0), w_\varepsilon(\cdot,t_0)) \tag{3.74}$$

for all $t_0 \in [0,t)$ *and each* $\varepsilon \in (0,1)$, *and there exists* $C_2 > 0$ *such that*

$$\int_0^\infty \int_\Omega \Big(\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + (u_\varepsilon - N_1)^2 + (v_\varepsilon - N_2)^2 + |\nabla w_\varepsilon|^2 \Big) \le C_2 \quad \textit{for all } \varepsilon \in (0,1). \tag{3.75}$$

Proof. As the proof is quite a lot of calculation, we will refrain from a detailed of the proof and only sketch the main steps. For more details the readers can be referred to our recent work [31,33]. The proof of nonnegativity for E_1 is similar to Lemma 4.2 in [31], so we omit it. By the straightforward calculation we have

$$\begin{aligned}
\frac{d}{dt}E_1(t) = & \int_\Omega (1 - \frac{N_1}{u_\varepsilon})[d_1 \Delta u_\varepsilon - \nabla \cdot (u_\varepsilon \chi_{1\varepsilon}(w_\varepsilon) \cdot \nabla w_\varepsilon) + \mu_1 u_\varepsilon (1 - u_\varepsilon - a_1 v_\varepsilon)] \\
& + \frac{a_1\mu_1}{a_2\mu_2} \int_\Omega (1 - \frac{N_2}{v_\varepsilon})[d_2 \Delta v_\varepsilon - \nabla \cdot (v_\varepsilon \chi_{2\varepsilon}(w_\varepsilon) \cdot \nabla w_\varepsilon) + \mu_2 v_\varepsilon (1 - v_\varepsilon - a_2 u_\varepsilon)] \\
& + \varrho_1 \int_\Omega w_\varepsilon [d_3 \Delta w_\varepsilon - \frac{(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon}{1 + \varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon}] \\
= & -d_1 N_1 \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + N_1 \int_\Omega \frac{\chi_{1\varepsilon}(w_\varepsilon)}{u_\varepsilon} \nabla u_\varepsilon \cdot \nabla w_\varepsilon + \mu_1 \int_\Omega u_\varepsilon (1 - u_\varepsilon - a_1 v_\varepsilon) \\
& - \mu_1 N_1 \int_\Omega (1 - u_\varepsilon - a_1 v_\varepsilon) - \frac{a_1 \mu_1 d_2 N_2}{a_2 \mu_2} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \\
& + \frac{a_1\mu_1 N_2}{a_2\mu_2} \int_\Omega \frac{\chi_{2\varepsilon}(w_\varepsilon)}{v_\varepsilon} \nabla v_\varepsilon \cdot \nabla w_\varepsilon + \frac{a_1\mu_1}{a_2} \int_\Omega v_\varepsilon (1 - v_\varepsilon - a_2 u_\varepsilon) \\
& \frac{a_1\mu_1 N_2}{a_2} \int_\Omega (1 - v_\varepsilon - a_2 u_\varepsilon) - d_3 \varrho_1 \int_\Omega |\nabla w_\varepsilon|^2 - \varrho_1 \int_\Omega \frac{(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon^2}{1 + \varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon} \\
= & -\mu_1 \int_\Omega (u_\varepsilon - N_1)^2 - 2a_1\mu_1 \int_\Omega (u_\varepsilon - N_1)(v_\varepsilon - N_2) - d_1 N_1 \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} \\
& + N_1 \int_\Omega \frac{\chi_{1\varepsilon}(w_\varepsilon)}{u_\varepsilon} \nabla u_\varepsilon \cdot \nabla w_\varepsilon - \frac{a_1\mu_1}{a_2} \int_\Omega (v_\varepsilon - N_2)^2 - d_3 \varrho_1 \int_\Omega |\nabla w_\varepsilon|^2 \\
& + \frac{a_1\mu_1 N_2}{a_2\mu_2} \int_\Omega \frac{\chi_{2\varepsilon}(w_\varepsilon)}{v_\varepsilon} \nabla v_\varepsilon \cdot \nabla w_\varepsilon - \frac{a_1\mu_1 d_2 N_2}{a_2\mu_2} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \\
& - \varrho_1 \int_\Omega \frac{(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon^2}{1 + \varepsilon(\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon},
\end{aligned} \tag{3.76}$$

here we have been used the fact that $\mu_1 + \mu_1 N_1 = 2\mu_1 N_1 + a_1\mu_1 N_2$, $a_1 N_2 + N_1 = 1$, $a_2 N_1 + N_2 = 1$ and $1 + N_2 = 2N_2 + a_2 N_1$. By the Young's inequality, we obtain

$$N_1 \int_\Omega \frac{\chi_{1\varepsilon}(w_\varepsilon)}{u_\varepsilon} \nabla u_\varepsilon \cdot \nabla w_\varepsilon \le \frac{d_1 N_1}{2} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \frac{N_1 \|\chi_{1\varepsilon}\|^2_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}}{2d_1} \int_\Omega |\nabla w_\varepsilon|^2 \tag{3.77}$$

as well as

$$\begin{aligned} &\frac{a_1\mu_1 N_2}{a_2\mu_2} \int_\Omega \frac{\chi_{2\varepsilon}(w_\varepsilon)}{v_\varepsilon} \nabla v_\varepsilon \cdot \nabla w_\varepsilon \\ &\le \frac{a_1\mu_1 d_2 N_2}{2a_2\mu_2} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + \frac{a_1\mu_1 N_2 \|\chi_{2\varepsilon}\|^2_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}}{2a_2\mu_2 d_2} \int_\Omega |\nabla w_\varepsilon|^2. \end{aligned} \tag{3.78}$$

Collecting (3.76)–(3.78), we thus infer that

$$\begin{aligned} \frac{d}{dt}E_1(t) \le& -\mu_1 \int_\Omega (u_\varepsilon - N_1)^2 - 2a_1\mu_1 \int_\Omega (u_\varepsilon - N_1)(v_\varepsilon - N_2) - \frac{d_1 N_1}{2} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} \\ & - \Big(d_3 \varrho_1 - \frac{N_1 \|\chi_{1\varepsilon}\|^2_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}}{2d_1} - \frac{a_1\mu_1 N_2 \|\chi_{2\varepsilon}\|^2_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}}{2a_2\mu_2 d_2} \Big) \int_\Omega |\nabla w_\varepsilon|^2 \\ & - \frac{a_1\mu_1}{a_2} \int_\Omega (v_\varepsilon - N_2)^2 - \frac{a_1\mu_1 d_2 N_2}{2a_2\mu_2} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}. \end{aligned} \tag{3.79}$$

Similar to Lemma 5.2 in [39], there exists constant $\kappa_1 > 0$ such that

$$\begin{aligned} & -\mu_1 \int_\Omega (u_\varepsilon - N_1)^2 - 2a_1\mu_1 \int_\Omega (u_\varepsilon - N_1)(v_\varepsilon - N_2) - \frac{a_1\mu_1}{a_2} \int_\Omega (v_\varepsilon - N_2)^2 \\ \le& -\kappa_1 \int_\Omega (u_\varepsilon - N_1)^2 - \kappa_1 \int_\Omega (v_\varepsilon - N_2)^2. \end{aligned} \tag{3.80}$$

Letting

$$o_1 = \varrho_1 - \frac{N_1 \|\chi_{1\varepsilon}\|^2_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}}{2d_1 d_3} - \frac{a_1\mu_1 N_2 \|\chi_{2\varepsilon}\|^2_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}}{2a_2\mu_2 d_2 d_3}$$

and choosing

$$\varrho_1 > \frac{N_1 \|\chi_{1\varepsilon}\|^2_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}}{2d_1 d_3} + \frac{a_1\mu_1 N_2 \|\chi_{2\varepsilon}\|^2_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}}{2a_2\mu_2 d_2 d_3},$$

it is clear to see that $o_1 > 0$. Taking

$$C_1 := \min\{\frac{d_1N_1}{2},\ \frac{a_1\mu_1d_2N_2}{2a_2\mu_2},\ o_1,\ \kappa_1\},$$

and thus establishes (3.73). (3.74) and (3.75) are direct results on integrating (3.73) in time. This completes the proof. □

Lemma 3.14. *Assume that the conditions of Lemma 3.13 hold. Then for all $\varsigma > 0$ there exists $T(\varsigma) > 0$ such that for all $\varepsilon \in (0, 1)$,*

$$\|u_\varepsilon(\cdot,t) - N_1\|_{L^1(\Omega)} + \|v_\varepsilon(\cdot,t) - N_2\|_{L^1(\Omega)} + \|w_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le \varsigma \quad \textit{for all } t > T(\varsigma). \tag{3.81}$$

Proof. First of all, we claim that there are some constants $\varsigma, T(\varsigma) > 0$ such that for all $t > T(\varsigma)$,

$$E_1(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t), w_\varepsilon(\cdot,t)) \le \varsigma. \tag{3.82}$$

In reality, in accordance with Lemma 3.13, there is a constant $c_1 > 0$ such that

$$\int_0^\infty\int_\Omega\Big(\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + (u_\varepsilon - N_1)^2 + (v_\varepsilon - N_2)^2 + |\nabla w_\varepsilon|^2\Big) \le c_1 \quad \text{for all } \varepsilon \in (0,1). \tag{3.83}$$

Applying strong maximum principle, we know that $u_\varepsilon > 0$ in $\overline{\Omega} \times (0,\infty)$. Analogous to the reason of Lemma 8.4 in [50], there are some constants $c_2 = c_2(N_1) > 0$ and $c_3 = c_3(N_2) > 0$ such that whenever

$$\|u_\varepsilon(\cdot,t) - N_1\|^2_{L^2(\Omega)} \le \frac{N_1^2|\Omega|}{16}, \quad \|v_\varepsilon(\cdot,t) - N_2\|^2_{L^2(\Omega)} \le \frac{N_2^2|\Omega|}{16}, \tag{3.84}$$

we have

$$\begin{aligned}\int_\Omega (u_\varepsilon(\cdot,t) - N_1 - N_1\ln\frac{u_\varepsilon(\cdot,t)}{N_1}) &\le c_2\Big(\int_\Omega \frac{|\nabla u_\varepsilon(\cdot,t)|^2}{u_\varepsilon^2(\cdot,t)}\Big)^{\frac12} + c_2\int_\Omega (u_\varepsilon(\cdot,t) - N_1)^2\\ &\quad + c_2\Big(\int_\Omega (u_\varepsilon(\cdot,t) - N_1)^2\Big)^{\frac12}\end{aligned} \tag{3.85}$$

and

$$\begin{aligned}\int_\Omega (v_\varepsilon(\cdot,t) - N_2 - N_2\ln\frac{v_\varepsilon(\cdot,t)}{N_2}) &\le c_3\Big(\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon^2(\cdot,t)}\Big)^{\frac12} + c_3\int_\Omega (v_\varepsilon(\cdot,t) - N_2)^2\\ &\quad + c_3\Big(\int_\Omega (v_\varepsilon(\cdot,t) - N_2)^2\Big)^{\frac12}.\end{aligned} \tag{3.86}$$

For given $\varsigma > 0$, we can find γ_i $(i = 1,2,3,4)$ such that

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$$c_2\gamma_1^{\frac{1}{2}} \le \frac{\varsigma}{8}, \quad c_2\gamma_2^{\frac{1}{2}} + c_2\gamma_2 \le \frac{\varsigma}{8}, \quad \gamma_2 \le \frac{N_1^2|\Omega|}{16} \tag{3.87}$$

as well as

$$c_3\gamma_3^{\frac{1}{2}} \le \frac{\varsigma}{8}, \quad c_3\gamma_4^{\frac{1}{2}} + c_3\gamma_4 \le \frac{\varsigma}{8}, \quad \gamma_4 \le \frac{N_2^2|\Omega|}{16} \tag{3.88}$$

and pick $T = T(\varsigma)$ large enough fulfilling

$$T \ge \left\{\frac{c_1}{\gamma_1}, \ \frac{c_1}{\gamma_2}, \ \frac{c_1}{\gamma_3}, \ \frac{c_1}{\gamma_4}, \ \frac{c_1\varrho_1}{\varsigma}\right\}. \tag{3.89}$$

From (3.83), there exists $t_* = t_*(\varepsilon) \in (0, T)$ such that for each $\varepsilon \in (0, 1)$,

$$\int_\Omega \left(\frac{|\nabla u_\varepsilon(\cdot, t_*)|^2}{u_\varepsilon^2(\cdot, t_*)} + \frac{|\nabla v_\varepsilon(\cdot, t_*)|^2}{v_\varepsilon^2(\cdot, t_*)} + (u_\varepsilon(\cdot, t_*) - N_1)^2 + (v_\varepsilon(\cdot, t_*) - N_2)^2 + |\nabla w_\varepsilon(\cdot, t_*)|^2\right) \le \frac{c_1}{T}.$$

In particular, we have

$$\int_\Omega \frac{|\nabla u_\varepsilon(\cdot, t_*)|^2}{u_\varepsilon^2(\cdot, t_*)} \le \gamma_1, \quad \int_\Omega \frac{|\nabla v_\varepsilon(\cdot, t_*)|^2}{v_\varepsilon^2(\cdot, t_*)} \le \gamma_3 \tag{3.90}$$

as well as

$$\int_\Omega (u_\varepsilon(\cdot, t_*) - N_1)^2 \le \gamma_2, \quad \int_\Omega (v_\varepsilon(\cdot, t_*) - N_2)^2 \le \gamma_4 \tag{3.91}$$

and

$$\int_\Omega |\nabla w_\varepsilon(\cdot, t_*)|^2 \le \frac{\varsigma}{4}. \tag{3.92}$$

The combination of (3.84)–(3.92) and use the Poincaré inequality, we obtain

$$\begin{aligned} E_1(u_\varepsilon(\cdot, t_*), v_\varepsilon(\cdot, t_*), w_\varepsilon(\cdot, t_*)) &= \int_\Omega \left(u_\varepsilon(\cdot, t_*) - N_1 - N_1 \ln \frac{u_\varepsilon(\cdot, t_*)}{N_1}\right) + \frac{\varrho_1}{2}\int_\Omega w_\varepsilon^2(\cdot, t_*) \\ &\quad + \int_\Omega \left(v_\varepsilon(\cdot, t_*) - N_2 - N_2 \ln \frac{v_\varepsilon(\cdot, t_*)}{N_2}\right) \\ &\le \frac{\varsigma}{4} + \frac{\varsigma}{2} + \frac{\varsigma}{4} = \varsigma. \end{aligned}$$

For given $\iota > 0$, we fixed $\varsigma > 0$ such that

$$c_4\varsigma^{\frac{1}{2}} + c_4\varsigma \le \frac{\iota}{2}, \quad c_4\varsigma^{\frac{1}{2}} + c_4\varsigma \le \frac{\iota}{2}, \quad (2\varsigma)^{\frac{1}{2}}\varrho_1^{-\frac{1}{2}} \le \frac{\iota}{2}, \tag{3.93}$$

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where $c_4, c_5 > 0$. From (3.82), there is a constant $T = T(\iota) > 0$ such that for all $\varepsilon \in (0, 1)$,

$$\int_\Omega \left(u_\varepsilon(\cdot,t) - N_1 - N_1 \ln \frac{u_\varepsilon(\cdot,t)}{N_1}\right) \leq \varsigma \quad \text{for all } t > T \tag{3.94}$$

as well as

$$\int_\Omega \left(v_\varepsilon(\cdot,t) - N_2 - N_2 \ln \frac{v_\varepsilon(\cdot,t)}{N_2}\right) \leq \varsigma \quad \text{for all } t > T \tag{3.95}$$

and

$$\frac{\varrho_1}{2} \int_\Omega w_\varepsilon^2(\cdot,t) \leq \varsigma \quad \text{for all } t > T, \tag{3.96}$$

which yields

$$\|w_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq (2\varsigma)^{\frac{1}{2}} \varrho_1^{-\frac{1}{2}} \leq \frac{\iota}{2} \quad \text{for all } t > T. \tag{3.97}$$

Similar to Lemma 8.6 in [50], there exists constant $c_4 > 0$ such that

$$\begin{aligned}
&\|u_\varepsilon(\cdot,t) - N_1\|_{L^1(\Omega)} \\
&\leq c_4 \left\{ \int_\Omega \left(u_\varepsilon(\cdot,t) - N_1 - N_1 \ln \frac{u_\varepsilon(\cdot,t)}{N_1}\right)\right\}^{\frac{1}{2}} + c_4 \int_\Omega \left(u_\varepsilon(\cdot,t) - N_1 - N_1 \ln \frac{u_\varepsilon(\cdot,t)}{N_1}\right) \\
&\leq c_4 \varsigma^{\frac{1}{2}} + c_4 \varsigma \leq \frac{\iota}{2} \quad \text{for all } t > T.
\end{aligned}$$

Analogously, we obtain

$$\begin{aligned}
&\|v_\varepsilon(\cdot,t) - N_2\|_{L^1(\Omega)} \\
&\leq c_5 \left\{ \int_\Omega \left(v_\varepsilon(\cdot,t) - N_2 - N_2 \ln \frac{v_\varepsilon(\cdot,t)}{N_2}\right)\right\}^{\frac{1}{2}} + c_5 \int_\Omega \left(v_\varepsilon(\cdot,t) - N_2 - N_2 \ln \frac{v_\varepsilon(\cdot,t)}{N_2}\right) \\
&\leq c_5 \varsigma^{\frac{1}{2}} + c_5 \varsigma \leq \frac{\iota}{2} \quad \text{for all } t > T,
\end{aligned}$$

where $c_5 > 0$ is a constant. This completes the proof. □

Finally, we give the following key estimates of stabilization in (1.1) in the case of $a_1 \geq 1 > a_2$.

Lemma 3.15. *Let $a_1 \geq 1 > a_2$. Under the assumption of Theorem 3.2 (ii), the solution of (1.1) has the property that there are constants $\varrho_2 > 0$ and $C_3 > 0$ such that the nonnegative function E_2 defined by*

$$E_2(u_\varepsilon(\cdot,t),v_\varepsilon(\cdot,t),w_\varepsilon(\cdot,t)):=\int_\Omega u_\varepsilon+\frac{\mu_1}{a_2\mu_2}\int_\Omega(v_\varepsilon-1-\ln v_\varepsilon)+\frac{\varrho_2}{2}\int_\Omega w_\varepsilon^2 \tag{3.98}$$

satisfying

$$\frac{d}{dt}E_2(u_\varepsilon(\cdot,t),v_\varepsilon(\cdot,t),w_\varepsilon(\cdot,t))+C_3\Big\{\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}+\int_\Omega u_\varepsilon^2+\int_\Omega(v_\varepsilon-1)^2+\int_\Omega|\nabla w_\varepsilon|^2\Big\}\le 0 \tag{3.99}$$

for all $t>0$. *Moreover,*

$$E_2(u_\varepsilon(\cdot,t),v_\varepsilon(\cdot,t),w_\varepsilon(\cdot,t))\le E_2(u_\varepsilon(\cdot,t_0),v_\varepsilon(\cdot,t_0),w_\varepsilon(\cdot,t_0)) \tag{3.100}$$

for all $t_0\in[0,t)$ *and each* $\varepsilon\in(0,1)$, *and there exists* $C_4>0$ *such that*

$$\int_0^\infty\int_\Omega\Big(\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}+u_\varepsilon^2+(v_\varepsilon-1)^2+|\nabla w_\varepsilon|^2\Big)\le C_4 \quad \textit{for all } \varepsilon\in(0,1). \tag{3.101}$$

Proof. Similar to Lemma 3.13, by the straightforward calculation we have

$$\begin{aligned}
\frac{d}{dt}E_2(t)&=\mu_1\int_\Omega u_\varepsilon(1-u_\varepsilon-a_1v_\varepsilon)+\varrho_2\int_\Omega w_\varepsilon[d_3\Delta w_\varepsilon-\frac{(\alpha u_\varepsilon+\beta v_\varepsilon)w_\varepsilon}{1+\varepsilon(\alpha u_\varepsilon+\beta v_\varepsilon)w_\varepsilon}]\\
&\quad+\frac{\mu_1}{a_2\mu_2}\int_\Omega(1-\frac{1}{v_\varepsilon})[d_2\Delta v_\varepsilon-\nabla\cdot(v_\varepsilon\chi_{2\varepsilon}(w_\varepsilon)\cdot\nabla w_\varepsilon)+\mu_2v_\varepsilon(1-v_\varepsilon-a_2u_\varepsilon)]\\
&=\mu_1\int_\Omega u_\varepsilon-\mu_1\int_\Omega u_\varepsilon^2-a_1\mu_1\int_\Omega u_\varepsilon v_\varepsilon-d_3\varrho_2\int_\Omega|\nabla w_\varepsilon|^2\\
&\quad-\varrho_2\int_\Omega\frac{(\alpha u_\varepsilon+\beta v_\varepsilon)w_\varepsilon^2}{1+\varepsilon(\alpha u_\varepsilon+\beta v_\varepsilon)w_\varepsilon}-\frac{\mu_1 d_2}{a_2\mu_2}\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}+\frac{\mu_1}{a_2\mu_2}\int_\Omega\frac{\chi_{2\varepsilon}(w_\varepsilon)}{v_\varepsilon}\nabla v_\varepsilon\cdot\nabla w_\varepsilon\\
&\quad+\frac{\mu_1}{a_2}\int_\Omega v_\varepsilon(1-v_\varepsilon-a_2u_\varepsilon)-\frac{\mu_1}{a_2}\int_\Omega(1-v_\varepsilon-a_2u_\varepsilon)\\
&\le-\mu_1\int_\Omega u_\varepsilon^2-2\mu_1\int_\Omega u_\varepsilon(v_\varepsilon-1)-\frac{\mu_1}{a_2}\int_\Omega(v_\varepsilon-1)^2-d_3\varrho_2\int_\Omega|\nabla w_\varepsilon|^2\\
&\quad-\frac{\mu_1 d_2}{a_2\mu_2}\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}+\frac{\mu_1}{a_2\mu_2}\int_\Omega\frac{\chi_{2\varepsilon}(w_\varepsilon)}{v_\varepsilon}\nabla v_\varepsilon\cdot\nabla w_\varepsilon\\
&\quad-\varrho_2\int_\Omega\frac{(\alpha u_\varepsilon+\beta v_\varepsilon)w_\varepsilon^2}{1+\varepsilon(\alpha u_\varepsilon+\beta v_\varepsilon)w_\varepsilon},
\end{aligned} \tag{3.102}$$

here we have been used the fact that $a_1\ge 1$. Using Young's inequality, we obtain

$$\frac{\mu_1}{a_2\mu_2}\int_\Omega \frac{\chi_{2\varepsilon}(w_\varepsilon)}{v_\varepsilon}\nabla v_\varepsilon\cdot\nabla w_\varepsilon \le \frac{\mu_1 d_2}{2a_2\mu_2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}+\frac{\mu_1\|\chi_{2\varepsilon}\|^2_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}}{2a_2\mu_2 d_2}\int_\Omega |\nabla w_\varepsilon|^2. \tag{3.103}$$

Inserting (3.103) into (3.102), we conclude that

$$\begin{aligned}\frac{d}{dt}E_2(t) \le & -\mu_1\int_\Omega u_\varepsilon^2 - 2\mu_1\int_\Omega u_\varepsilon(v_\varepsilon-1)-\frac{\mu_1}{a_2}\int_\Omega (v_\varepsilon-1)^2-\frac{\mu_1 d_2}{2a_2\mu_2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\\ & -\Big(d_3\varrho_2-\frac{\mu_1\|\chi_{2\varepsilon}\|^2_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}}{2a_2\mu_2 d_2}\Big)\int_\Omega |\nabla w_\varepsilon|^2.\end{aligned} \tag{3.104}$$

Analogous to Lemma 5.5 in [39], there is a constant $\kappa_2>0$ such that

$$-\mu_1\int_\Omega u_\varepsilon^2-2\mu_1\int_\Omega u_\varepsilon(v_\varepsilon-1)-\frac{\mu_1}{a_2}\int_\Omega (v_\varepsilon-1)^2\le -\kappa_2\int_\Omega u_\varepsilon^2-\kappa_2\int_\Omega (v_\varepsilon-1)^2 \tag{3.105}$$

Letting

$$o_2=\varrho_2-\frac{\mu_1\|\chi_{2\varepsilon}\|^2_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}}{2a_2\mu_2 d_2 d_3}$$

and choosing

$$\varrho_2>\frac{\mu_1\|\chi_{2\varepsilon}\|^2_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}}{2a_2\mu_2 d_2 d_3},$$

it is clear to see that $o_2>0$. Taking

$$C_3:=\min\{\frac{\mu_1 d_2}{2a_2\mu_2},\ o_2,\ \kappa_2\},$$

and thus establishes (3.99). (3.100) and (3.101) are direct results on integrating (3.99) in time. This completes the proof. □

Lemma 3.16. *Suppose that the conditions of Lemma 3.15 hold. Then for all $\varsigma>0$ there exists $T(\varsigma)>0$ such that for each $\varepsilon\in(0,1)$,*

$$\|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)}+\|v_\varepsilon(\cdot,t)-1\|_{L^1(\Omega)}+\|w_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\le\varsigma \quad \textit{for all } t>T(\varsigma). \tag{3.106}$$

Proof. The proof is similar to Lemma 3.14, to avoid repetition, we omit giving details on this here. □

Finally, inspired by [50, p.1394], we prove the main theorem.

The Proof of Theorem 3.2. Applying Fubini-Tonelli theorem, Lemma 3.10 provides $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and a null set $\Lambda \subset (0,\infty)$ such that $\varepsilon_j \searrow 0$ as $j \to$ and

$$u_\varepsilon(\cdot,t) \to u(\cdot,t), \quad v_\varepsilon(\cdot,t) \to v(\cdot,t) \text{ and } w_\varepsilon(\cdot,t) \to w(\cdot,t) \quad a.e. \text{ in } \Omega \text{ for all } t \in (0,\infty) \setminus \Lambda,$$

as $\varepsilon = \varepsilon_j \searrow 0$. In accordance with Fatou's lemma and the fact that $L^2(\Omega) \subset L^1(\Omega)$, on one hand, in case of $a_1, a_2 \in (0,1)$, we obtain from Lemma 3.14 that

$$\begin{aligned}
&\|u(\cdot,t) - N_1\|_{L^1(\Omega)} \to 0, \quad \|v(\cdot,t) - N_2\|_{L^1(\Omega)} \to 0 \text{ and} \\
&\|w(\cdot,t)\|_{L^2(\Omega)} \to 0 \text{ as } (0,\infty) \setminus \Lambda \ni t \to \infty,
\end{aligned}$$

where $N_1 = \frac{1-a_1}{1-a_1a_2}$ and $N_2 = \frac{1-a_2}{1-a_1a_2}$. On the other hand, in case of $a_1 \geq 1 > a_2 > 0$, we obtain from Lemma 3.16 that

$$\|u(\cdot,t)\|_{L^1(\Omega)} \to 0, \quad \|v(\cdot,t) - 1\|_{L^1(\Omega)} \to 0 \text{ and } \|w(\cdot,t)\|_{L^2(\Omega)} \to 0 \text{ as } (0,\infty) \setminus \Lambda \ni t \to \infty,$$

which completes the proof of Theorem 3.2. □

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