

Outgoing modal solutions for Galbrun's equation in helioseismology

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Abstract

We construct modal outgoing Green's kernels for the simplified Galbrun's equation under spherical symmetry, in the context of helioseismology. The coefficients of the equation are $\mathcal{C}^2$ functions representing the solar interior model $\mathcal{S}$, complemented with an isothermal atmospheric model. We solve the equation in vectorial spherical harmonics basis to obtain modal equations for the different components of the unknown wave motions. These equations are then decoupled and written in Schrödinger form, whose coefficients are shown to be $\mathcal{C}^2$ apart from at most two regular singular points, and to decay like a Coulomb potential at infinity. These properties allow us to construct an outgoing Green's kernel for each spherical mode. We also compute asymptotic expansions of coefficients up to order r^{-3} as r tends to infinity, and show numerically that their accuracy is improved by including the contribution from the gravity although this term is of order r^{-3} .

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1. Introduction

We study the Galbrun’s equation in the frequency domain, which describes adiabatic wave motion subject to a time-harmonic source $\mathbf{f}$ of angular frequency ω , on top of a time-invariant fluid background,

$$\begin{aligned} -\rho_0(\omega^2 + 2i\omega\Gamma)\check{\xi} - \nabla_{\check{\mathbf{x}}}[\gamma p_0 \nabla_{\check{\mathbf{x}}} \cdot \check{\xi}] + (\nabla_{\check{\mathbf{x}}} p_0)(\nabla_{\check{\mathbf{x}}} \cdot \check{\xi}) - \nabla_{\check{\mathbf{x}}}(\check{\xi} \cdot \nabla_{\check{\mathbf{x}}} p_0) \\ + (\check{\xi} \cdot \nabla_{\check{\mathbf{x}}}) \nabla_{\check{\mathbf{x}}} p_0 + \rho_0(\check{\xi} \cdot \nabla_{\check{\mathbf{x}}}) \nabla_{\check{\mathbf{x}}} \phi_0 = \mathbf{f}. \end{aligned} \quad (1.1)$$

The background is characterized by the pressure p_0 , the density ρ_0 , the adiabatic index γ , the attenuation Γ and the gravity potential ϕ_0 solution to the Poisson equation

$$\Delta_{\check{\mathbf{x}}} \phi_0 = 4\pi G \rho_0, \quad \text{given by } \phi_0(\check{\mathbf{x}}) := -G \int_{\mathbb{R}^3} \frac{\rho_0(\check{\mathbf{y}})}{|\check{\mathbf{x}} - \check{\mathbf{y}}|} d\check{\mathbf{y}}, \quad (1.2)$$

where G is the gravitational constant, $G = 6.67408 \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2}$. In this paper, we consider background quantities to be spherically symmetric, as in the standard model S [1]. The unknown $\check{\xi}(\check{\mathbf{x}}, \omega)$ is the displacement vector describing solar oscillations.

Equation (1.1) is a simplified version of the Galbrun’s equation derived by Lynden-Bell and Ostriker [2] in the context of rotating bodies (stars, galaxies, etc.). Compared to their original equation [2, Eqs. (28–30)], our equation ignores flows, rotation and the perturbation to the gravitational potential by means of the Cowling’s approximation [3]. Equation (1.1) is also [4, Eq. (1.65)] without flow. The original Galbrun’s equation was introduced by Galbrun in [5] and re-derived by Poirée in [6] in the context of aeroacoustics, see [7,4] for a historical discussion. Since the unknown $\check{\xi}$ is a Lagrangian displacement in terms of an Eulerian variable $\check{\mathbf{x}}$, (1.1) is also called *Eulerian–Lagrangian* formulation of the problem.

As a novelty, we study equation (1.1) in $\mathbb{R}^3$, which allows for the presence of a solar atmosphere. Most studies in helioseismology instead use a free-surface boundary condition [8,9]; however this is not appropriate for modeling high-frequency waves, [10]. One approach to include an atmosphere is to describe it as a thin layer of isothermal perfect gas [11]. While this assumption excludes a sharp increase of temperature in the chromosphere, it is reasonable for studying wave propagation in the photosphere (up to 2 Mm above the solar surface). In terms of the coefficients of (1.1), the isothermal assumption (associated with a constant adiabatic exponent) implies constant sound speed, and exponentially decaying density and pressure. These constitute the definition of our atmospheric model AtmoI in Assumption 3. While our model is also ‘isothermal’ as in [11], we do not assume a plane-parallel atmosphere, and the Poisson relation (1.2) (which determines ϕ_0 from ρ_0) is maintained globally, in contrast to [11] which assumes globally hydrostatic equilibrium and imposes ϕ'_0 to be constant in the atmosphere, see further discussion in Remark 1.

The actual working coefficients of equation (1.1) are $\mathcal{C}^2$ representations of the data points given in the model S up to its limit (496 km above the Sun’s surface) and satisfy the assumptions of AtmoI in the atmosphere. The existence and construction of such representations are established by the authors in [12]. The isothermal assumption is also used in the model Atmo for the scalar wave equation employed in [13–15]. The scalar wave equation which is obtained from (1.1) under simplifying assumptions, cf. [10], cannot include gravity forces which are needed

to model the surface gravity waves (f-mode) observed in the Sun, as well as the internal gravity waves (g-mode) recently detected, cf. [16].

The main results of our work, which also serve as novelties, are the following:

- In Section 4, exploiting spherical symmetry, we write (1.1) in basis of vectorial spherical harmonics (VSH), and obtain the modal ODEs whose unknowns are the coefficients of ξ in this basis. The equations for the different coefficients can be decoupled to obtain modal ODEs for the radial displacement ξ_r which are then rewritten in the form of Schrödinger equations $-\partial_r^2 + V_\ell(r, \omega, \Gamma)$, with ℓ the spherical harmonic degree. The VSH basis is also a classical tool to solve analytically, for example, the Maxwell equation, cf. [17, Section 6.5], [18], and the elastic equation, cf. [19, Section 3.17], under spherical symmetry. After preliminary versions in our report [20], we have chosen in this paper the explicit forms of the modal equations (given in Propositions 3 and 5) that are most conducive to numerical simulations and mathematical analysis and which in particular highlight properties such as regularity and decay rate at infinity. In a different approach, [21,9,11] also study the vector equation in the interior of the Sun and under spherical symmetry, by first rewriting the equation to obtain a system with only scalar unknowns and then working in the scalar spherical harmonic basis, see Remarks 2 and 3.
- In Sections 5 and 6, we study the coefficients of the modal operator in terms of local regularity and decay rate at infinity and show that they have long-range behavior at infinity and contain a finite set of regular singular points (in the sense of [22]). This singular set consists solely of the origin with attenuation, while it has an additional point without attenuation. Only the singularity at the origin was considered in [21,9,11] in a less rigorous study and under more restrictive hypothesis (on the background coefficients) for the vector equation either only in the interior of the Sun or with a different (isothermal) atmospheric model, see Remarks 2 and 3.
- Using the techniques of Agmon–Klein [23], we construct in Section 7 a Green’s kernel which is regular at $r = 0$ and whose behavior as $r \rightarrow \infty$ is characterized by an oscillatory term $e^{i\psi(r, k_a)} \sim e^{ik_a r}$ with a complex wavenumber k_a obtained as part of the asymptotic analysis. We refer to Remark 8 to explain our choice of [23] among the vast literature of scattering theory for Schrödinger equation.

We also uncover two interesting properties of the modal operators, which distinguish the vectorial equation from the scalar one. Our modal operators have an extra growth in $[\ell(\ell + 1)]^2$ compared to those of the scalar wave operator. This does not prevent the construction of solutions for each fixed ℓ , however complicates the task of obtaining uniform results in ℓ (not in the scope of the current paper) as was done for the scalar operator in [23], see Remark 10. By numerical experiments, we also retrieve the effect of the gravity, absent in the scalar case, which appears dominantly as a term of order r^{-3} in the potential, and greatly improves the accuracy of the approximation.

2. Physical and scaled parameters

In this section, we introduce the physical solar parameters and their *scaled* representations, motivated by the convenience of a unitless wavenumber and consistency with [15,20,12]. The same analysis goes through if we work with unscaled parameters and coordinates.

We denote by $R_\odot$ the Sun's radius with $R_\odot = 6.96 \times 10^{10}$ cm. We have denoted by $\check{\mathbf{x}} \in \mathbb{R}^3$ the coordinate system in $\mathbb{R}^3$ with the origin placed at the center of the Sun and the set $\{|\check{\mathbf{x}}| = R_\odot\}$ representing its surface. The scaled coordinates $\mathbf{x}$ and radius $r = |\mathbf{x}|$, are defined by

$$\mathbf{x} = \check{\mathbf{x}} / R_\odot, \quad r = R / R_\odot. \quad (2.1)$$

We note three special values of the scaled radius with $r = 1$ corresponding to the surface of the Sun, r_s the end of the model S, and r_a the beginning of the model AtmOI, with

$$r_s = 1.0007126, \quad r_a = 1.00073. \quad (2.2)$$

The original physical background quantities mentioned in the introduction, ρ_0 (in g cm^{-3}), γ (unitless) and p_0 (in $\text{g s}^{-2} \text{cm}^{-1}$) are functions of the unscaled radius R . We define corresponding functions in terms of the scaled radius r by

$$r \mapsto \mathbf{f}(r) = f(R_\odot r), \quad \text{given function } f : R \mapsto f(R). \quad (2.3)$$

The density $r \mapsto \rho_0(r)$ (in g cm^{-3}) and the adiabatic index $r \mapsto \gamma(r)$ are thus defined from the original quantities $R \mapsto \rho(R)$ and $R \mapsto \gamma(R)$ by (2.3), while the scaled pressure field $p_0(r)$ (in $\text{g s}^{-2} \text{cm}^{-3}$) is given by

$$r \mapsto p_0(r) = \frac{p_0(R_\odot r)}{R_\odot^2}. \quad (2.4)$$

The *scaled* adiabatic background sound speed (given in s^{-1}) is defined by

$$c_0^2(r) := \frac{\gamma(r) p_0(r)}{\rho_0(r)} = \left(\frac{c_0(R_\odot r)}{R_\odot} \right)^2, \quad \text{with } c_0 \text{ in } \text{cm s}^{-1}. \quad (2.5)$$

The scaled background gravity potential $\Phi_0(r)$ (in s^{-2}) is related to $\phi_0(R)$ by

$$\mathbf{x} \mapsto \Phi_0(\mathbf{x}) = \frac{\phi_0(R_\odot \mathbf{x})}{R_\odot^2}. \quad (2.6)$$

When ρ_0 is radial, it is the solution of, cf. [20, Appendix G.3],

$$\frac{1}{r^2} \left(r^2 \Phi_0' \right)' = 4\pi G \rho_0, \quad \text{with} \quad \Phi_0'(r) = \frac{4\pi G}{r^2} \int_0^r \rho_0(s) s^2 ds, \quad (2.7)$$

with $'$ denoting the derivative with respect to the radial coordinate r .

We introduce the *inverse scale height* function associated with a $\mathcal{C}^1$ function $g(r)$,

$$\alpha_g(r) := -\frac{g'(r)}{g(r)}, \quad (2.8)$$

and the quantity E_{he} (unitless) which measures the deviation from the hydrostatic equilibrium,

$$E_{\text{he}}(r) := \frac{\Phi_0'(r)}{c_0^2(r)} - \frac{\alpha_{p_0}(r)}{\gamma(r)}. \quad (2.9)$$

With the square root $\sqrt{\cdot}$ taking argument in $[0, \pi)$, the *complex-frequency* σ (in s^{-1}) and wavenumber k_0 (unitless) are defined in terms of the angular frequency ω and attenuation $\Gamma(\omega, r)$,

$$\sigma(r, \omega) = \sqrt{\omega^2 + 2i\omega\Gamma(\omega, r)}, \quad k_0(r) = \frac{\sigma(r)}{c_0(r)}. \quad (2.10)$$

3. Equations of motion for the model AtmoI

In this section, we introduce our working equation (from (1.1)) in the scaled coordinate with the notations of Section 2. With $\xi \cdot \nabla_{\mathbf{x}}$ denoting the directional derivative of a vector, cf. [9, Eq. (4.10) p. 59], we introduce the differential operators $\mathcal{P}$ and $\mathcal{G}$ in the scaled variable $\mathbf{x}$,

$$\mathcal{G}\xi = \rho_0 (\xi \cdot \nabla_{\mathbf{x}}) \nabla_{\mathbf{x}} \Phi_0, \quad (3.1a)$$

$$\mathcal{P}\xi = -\nabla_{\mathbf{x}}[\gamma p_0 \nabla_{\mathbf{x}} \cdot \xi] + (\nabla_{\mathbf{x}} p_0)(\nabla_{\mathbf{x}} \cdot \xi) - \nabla_{\mathbf{x}}[(\xi \cdot \nabla_{\mathbf{x}}) p_0] + (\xi \cdot \nabla_{\mathbf{x}}) \nabla_{\mathbf{x}} p_0. \quad (3.1b)$$

By using $\nabla_{\check{\mathbf{x}}} = R_{\odot}^{-1} \nabla_{\mathbf{x}}$ and $\nabla_{\check{\mathbf{x}}} = R_{\odot}^{-1} \nabla_{\mathbf{x}}$, equation (1.1) in variable $\mathbf{x}$ is

$$-\rho_0 \sigma^2 \xi + \mathcal{P}\xi + \mathcal{G}\xi = \mathbf{f}, \quad \text{in } \mathbb{R}^3, \quad (3.2)$$

with its unknown $\xi(\mathbf{x})$ and right-hand side $\mathbf{f}(\mathbf{x})$ related to $\check{\xi}(\check{\mathbf{x}})$ and $\check{\mathbf{f}}(\check{\mathbf{x}})$ of (1.1) by (2.3).

Definition 1 (Hydrostatic equilibrium). Under hydrostatic equilibrium, the pressure, density and gravity potential satisfy

$$p_0' = -\rho_0 \Phi_0'. \quad (3.3)$$

Definition 2 (Model AtmoI). In the isothermal atmospheric model AtmoI , the attenuation Γ , the adiabatic coefficient γ and the sound speed c_0 are constant, and ρ_0 is exponentially decreasing. In other words, for positive constants $\gamma_a, \alpha_a, c_a > 0$ and $\Gamma_a(\omega) \geq 0$,

$$\Gamma(\omega, r) = \Gamma_a(\omega), \quad \gamma(r) = \gamma_a, \quad c_0(r) = c_a, \quad \rho_0(r) = \rho_0(r_a) e^{-\alpha_a(r-r_a)}. \quad (3.4)$$

We now list our working assumptions regarding the coefficients where, by a *decreasing function* f , we mean $f(a) \leq f(b)$ when $a > b$. In particular, these assumptions are verified by the solar models we use, with $p_0(r)$, $\rho_0(r)$ and $\gamma(r)$ given by the $\mathcal{C}^2$ representation constructed in [12], from the data points of model S in the interior of the Sun ($r \leq r_s$) and properties of the isothermal atmosphere for $r \geq r_a$.

Assumption 1. The functions γ , p_0 , ρ_0 and $\Gamma(\omega, \cdot)$ are radial and satisfy

$$\Gamma(\omega, r) \geq 0, \quad 1 < \gamma(r) < 2, \quad \rho_0(r) > 0, \quad p_0(r) > 0; \quad (3.5a)$$

$$\rho_0, \gamma \in \mathcal{C}^2[0, \infty), \quad p_0 \in \mathcal{C}^3[0, \infty), \quad r \mapsto \Gamma(\omega, r) \in \mathcal{C}^2([0, \infty)); \quad (3.5b)$$

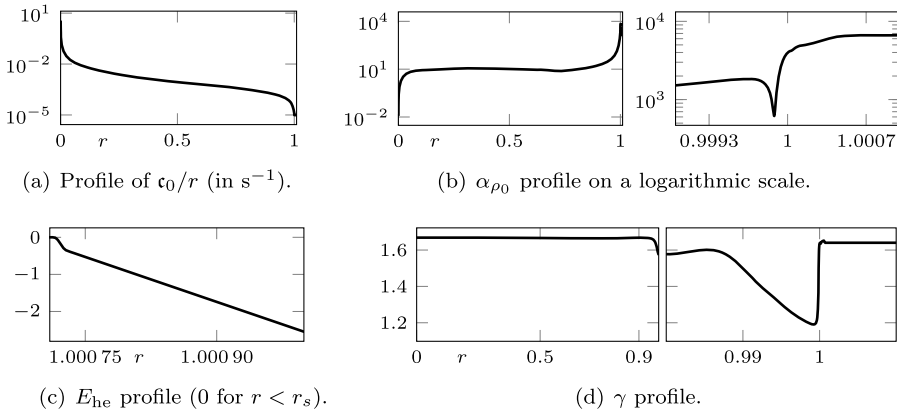


Fig. 1. The working quantities defined from the solar and atmosphere model S-AtmOI.

$$r c_0^{-1} \text{ strictly increasing on } [0, \infty); \quad (3.5c)$$

$$E_{he} \leq 0, \quad E_{he} \text{ decreasing on } [0, \infty). \quad (3.5d)$$

Assumption 2. The hydrostatic equilibrium (3.3) is verified for $r \in [0, r_s]$.

Assumption 3. For $r \geq r_a$, the coefficients follow the model AtmOI of Definition 2.

The global regularity of the coefficients is given by Assumption 1 while the hydrostatic equilibrium prevails in the interior (for $r \leq r_s$, Assumption 2) and the atmospheric model AtmOI for $r \geq r_a$ (Assumption 3). Therefore, a transition region $[r_s, r_a]$ is needed to ensure global regularity, cf. [12].

In the solar case, assumptions (3.5a) and (3.5c) on γ and c_0 come from the profiles given by the model S, as illustrated in Figs. 1(a) and 1(d). Assumption 2 for the interior comes from the fact that the background quantities of model S are solutions of Euler's equation for adiabatic state at equilibrium, with no flow and with only gravity exterior force.¹

In the atmosphere region $r \geq r_a$, Assumption 3 also implies that the density scale height α_{ρ_0} (2.8) and wavenumber k_0 (2.10) are constant and that the fluid pressure p_0 also decays exponentially at the same rate as ρ_0 ,

¹ The Euler system comprises of conservation of mass equation, adiabatic energy equation and the equation of motion, cf. [4, Eqs. (1.14)–(1.17) p. 22],

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \mathbf{v}_0) = 0, \quad \mathbf{v}_0 \cdot \nabla p_0 = c_0^2 \mathbf{v}_0 \cdot \nabla \rho_0; \quad (3.6a)$$

$$\partial_t (\rho_0 \mathbf{v}_0) + \nabla \cdot (\rho_0 \mathbf{v}_0 \otimes \mathbf{v}_0 + p_0 \mathbf{Id}) = -\rho_0 \nabla \Phi_0. \quad (3.6b)$$

At equilibrium, and assuming no background flow ($\mathbf{v}_0 = 0$), the system (3.6) reduces to the equation of motion (3.6b), which takes the form, $\nabla p_0 = -\rho_0 \nabla \Phi_0$. Under radial symmetry, this latter equation becomes $p'_0 = -\rho_0 \Phi'_0$.

$$k_0(r) = k_a := \frac{\sqrt{\omega^2 + 2i\omega\Gamma_a(\omega)}}{c_a}, \quad (3.7)$$

$$\alpha_{\rho_0}(r) = \alpha_{p_0}(r) = \alpha_{\gamma p_0}(r) = \alpha_a > 0, \quad r \geq r_a.$$

In addition to the positive real constants in (3.4), above, we introduce $k_a \in \mathbb{C}$. Under Assumptions 1 and 2, the quantity E_{he} has the following property,

$$E_{\text{he}} = 0, \text{ for } r \in [0, r_s], \quad E_{\text{he}} \leq 0 \text{ and is decreasing for } r \in [r_s, \infty). \quad (3.8)$$

We provide a plot for the parameters in Fig. 1 with r_s and r_a given in (2.2). In Figs. 1(b) and 1(d), we enlarge the area near the solar surface to visualize the drastic variation of α_{ρ_0} and γ is this region. These parameters are constructed in [12].

Remark 1. The hydrostatic equilibrium Assumption 2 is not compatible with the assumption that ρ_0 is exponentially decreasing in the atmosphere. In the model AtmoI , we choose to keep the latter at the expense of the first property in the atmosphere. Another option is to impose Assumption 2 for all $r \geq 0$, at the expense of losing constant adiabatic index in the atmosphere. Note that these two options maintain the Poisson equation relation (2.7) in the atmosphere, which is not the case in the plane-parallel isothermal atmosphere of [21,11] where Φ'_0 is considered as constant in the atmosphere. $\triangle$

4. Galbrun equation in spherical symmetry

Exploiting the spherical symmetry assumption, Subsection 4.2 presents our working equation in the vector spherical harmonics basis $\{\mathbf{P}_\ell^m, \mathbf{B}_\ell^m, \mathbf{C}_\ell^m\}$ whose properties are summarized in Subsection 4.1. The unknowns are now the coefficients $\{a_\ell^m, b_\ell^m, c_\ell^m\}$ of ξ in the aforementioned basis which solve the modal system of ODEs obtained in Proposition 1. The systems of equations solved by $\{a_\ell^m, b_\ell^m\}$ are decoupled from the equation of c_ℓ^m . The remaining works are spent on computing the ODE satisfied by the radial unknown a_ℓ^m , called the decoupled radial ODE given in Proposition 2. We also rewrite this equation in Schrödinger form, in Proposition 4, called the conjugated radial equation. The coefficients of these two equations are explicitly given in Proposition 5.

4.1. Vectorial spherical harmonics

The spherical basis is denoted $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$, with $0 \leq \phi < 2\pi$ and $0 \leq \theta < \pi$. We recall below the definition of the surface gradient ∇_{S^2} acting on a scalar function f , and the divergence operator $\nabla_{S^2} \cdot$ acting on a vector $\mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_\phi \mathbf{e}_\phi$,

$$\nabla_{S^2} f := \partial_\theta f \mathbf{e}_\theta + \frac{\partial_\phi f}{\sin \theta} \mathbf{e}_\phi; \quad (4.1a) \quad \nabla_{S^2} \cdot \mathbf{v} := \frac{\partial_\theta (\sin \theta v_\theta)}{\sin \theta} + \frac{\partial_\phi v_\phi}{\sin \theta}. \quad (4.1b)$$

The vector spherical harmonics are defined from $\mathbf{Y}_\ell^m$ as, cf. [24, Eq. (9.56), Section 9.3.3] or [19, Definition 3.336, p. 107],

$$\mathbf{P}_\ell^m(\hat{\mathbf{x}}) = \mathbf{Y}_\ell^m(\hat{\mathbf{x}}) \mathbf{e}_r, \quad \ell = 0, 1, \dots; \quad (4.2)$$

$$\mathbf{B}_\ell^m(\hat{\mathbf{x}}) = \frac{\nabla_{S^2} \mathbf{Y}_\ell^m}{\sqrt{\ell(\ell+1)}}, \quad \mathbf{C}_\ell^m(\hat{\mathbf{x}}) = -\frac{\mathbf{e}_r \times \nabla_{S^2} \mathbf{Y}_\ell^m}{\sqrt{\ell(\ell+1)}}, \quad \ell = 1, 2, \dots$$

For the definition of the spherical harmonics Y_ℓ^m , we refer to [24, 9.3.1] or [24, Theorem 9.11 p. 238]. The family of vectors in (4.2) forms an orthogonal basis for the vector-valued L^2 functions, while $\mathbf{C}_\ell^m$, $\mathbf{B}_\ell^m$ for $\ell = 1, 2, \dots$ form a complete orthonormal basis for the set of tangent vectors.

For sequence $\{f_0\} \cup \{f_\ell^m\}$, $\ell \in \mathbb{N}^+$, $-\ell \leq m \leq \ell$, we introduce the short-hand notation

$$\sum_{(\ell, m)} f_\ell^m := f_0 + \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} f_\ell^m. \quad (4.3)$$

4.2. System of ODE for level (ℓ, m)

Using the summation notation of (4.3), we write the source $\mathbf{f}$ and solution ξ in the vectorial spherical harmonics basis (4.2),

$$\xi = \xi_r + \xi_h + \xi_h^\times, \quad \mathbf{f} = \mathbf{f}_r + \mathbf{f}_h + \mathbf{f}_h^\times \quad \text{with}, \quad (4.4a)$$

$$\mathbf{f}_r = \sum_{(\ell, m)} f_\ell^m(r) \mathbf{P}_\ell^m(\hat{x}), \quad \xi_r = \sum_{(\ell, m)} a_\ell^m(r) \mathbf{P}_\ell^m(\hat{x}); \quad (4.4b)$$

$$\mathbf{f}_h = \sum_{(\ell, m)} g_\ell^m(r) \mathbf{B}_\ell^m(\hat{x}), \quad \xi_h = \sum_{(\ell, m)} b_\ell^m(r) \mathbf{B}_\ell^m(\hat{x}); \quad (4.4c)$$

$$\mathbf{f}_h^\times = \sum_{(\ell, m)} h_\ell^m(r) \mathbf{C}_\ell^m(\hat{x}), \quad \xi_h^\times = \sum_{(\ell, m)} c_\ell^m(r) \mathbf{C}_\ell^m(\hat{x}). \quad (4.4d)$$

To state the systems satisfied by a_ℓ^m , b_ℓ^m and c_ℓ^m , we define the function-valued matrices B^ℓ and C^ℓ of size 2×2 , in terms of inverse scale height functions $\alpha_\bullet$ (2.8), E_{he} (2.9) and wavenumber k_0 (2.10),

$$B^\ell := \begin{pmatrix} \alpha_{\gamma p_0} - \frac{2}{r} & \frac{\ell(\ell+1)}{r} \\ -\frac{1}{r} & 0 \end{pmatrix}, \quad C^\ell := \begin{pmatrix} C_{11} & C_{12}^\ell \\ C_{21} & C_{22}^\ell \end{pmatrix} \quad \text{with} \quad (4.5)$$

$$\begin{aligned} C_{11} &= -k_0^2 + \frac{\Phi_0''}{c_0^2} + 2\frac{\alpha_{\gamma p_0}}{r} + \frac{2}{r^2} - 2\frac{\alpha_{p_0}}{r\gamma}, & C_{21} &= -\frac{2}{r^2} + \frac{1}{r\gamma} \alpha_{p_0}, \\ C_{12}^\ell &= \ell(\ell+1) \left(\frac{\alpha_{p_0}}{r\gamma} - \frac{\alpha_{\gamma p_0}}{r} - \frac{1}{r^2} \right), & C_{22}^\ell &= -k_0^2 + \frac{\ell(\ell+1)}{r^2} + \frac{E_{\text{he}}}{r}. \end{aligned} \quad (4.6)$$

We note that these quantities C and B are unitless, similarly to the working quantities we have derived from Section 2.

Proposition 1. *If ξ is a solution of (3.2) with right-hand-side $\mathbf{f} \in C^2(\mathbb{R}^3)$ whose expansion is given in (4.4), the coefficients of the harmonic expansion (4.4) of the solution ξ satisfy the following ODEs.*

- For $\ell = 0$, the coefficient a_0 solves the ODE with C_{11} given by (4.6),

$$-\partial_r^2 a_0 + \left(\alpha_{\gamma p_0} - \frac{2}{r} \right) \partial_r a_0 + C_{11} a_0 = \frac{f_0}{\gamma p_0}. \quad (4.7)$$

- For $\ell > 0$, coefficient c_ℓ^m satisfies,

$$c_\ell^m = \frac{h_\ell^m(r)}{c_0^2 \rho_0 (-k_0^2 + r^{-1} E_{\text{he}})}. \quad (4.8)$$

With B^ℓ and C^ℓ given by (4.5) and (4.6), coefficients a_ℓ^m and b_ℓ^m satisfy,

$$\frac{1}{\gamma p_0} \begin{pmatrix} f_\ell^m \\ g_\ell^m \\ \sqrt{\ell(\ell+1)} \end{pmatrix} = \left(\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \partial_r^2 + B^\ell \partial_r + C^\ell \right) \begin{pmatrix} a_\ell^m \\ b_\ell^m \\ \sqrt{\ell(\ell+1)} \end{pmatrix}. \quad (4.9)$$

Proof. Step 1: We first rewrite the action of $\mathcal{P}$ and $\mathcal{G}$ in spherical coordinates. We need the following identities. Since

$$\nabla_{\mathbb{S}^2} \cdot \text{curl}_{\mathbb{S}^2} f = 0 \quad \Rightarrow \quad \nabla_{\mathbb{S}^2} \cdot \mathbf{C}_\ell^m = 0, \quad (4.10a)$$

$$\nabla_{\mathbb{S}^2} \cdot \mathbf{B}_\ell^m = \frac{\nabla_{\mathbb{S}^2} \cdot \nabla_{\mathbb{S}^2} Y_\ell^m}{\sqrt{\ell(\ell+1)}} = \frac{\Delta_{\mathbb{S}^2} Y_\ell^m}{\sqrt{\ell(\ell+1)}} = -\sqrt{\ell(\ell+1)} Y_\ell^m, \quad (4.10b)$$

$$\Rightarrow \nabla \cdot \boldsymbol{\xi} = \frac{1}{r^2} \partial_r (r^2 \boldsymbol{\xi}_r) + \frac{1}{r} \nabla_{\mathbb{S}^2} \cdot \boldsymbol{\xi}_h = \sum_{(\ell,m)} \left[\frac{(r^2 a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)} \frac{b_\ell^m}{r} \right] Y_\ell^m. \quad (4.11)$$

For a function p depending only on r , the material derivative of its gradient along $\mathbf{v}$ simplifies to

$$(\mathbf{v} \cdot \nabla) \nabla p = (\mathbf{v} \cdot \nabla) p'(r) \mathbf{e}_r = p''(r) \mathbf{v}_r + \frac{p'(r)}{r} \mathbf{v}_h. \quad (4.12)$$

Using (4.11) and (4.12), we obtain

$$\mathcal{G} \boldsymbol{\xi} = \rho_0 \Phi_0'' \boldsymbol{\xi}_r + \rho_0 \frac{\Phi_0'}{r} \boldsymbol{\xi}_h \quad \text{and} \quad \mathcal{P} \boldsymbol{\xi} = (\mathcal{P} \boldsymbol{\xi})_r + (\mathcal{P} \boldsymbol{\xi})_h, \quad \text{where} \quad (4.13)$$

$$(\mathcal{P} \boldsymbol{\xi})_r = - \sum_{(\ell,m)} \partial_r \left[\gamma p_0 \left(\frac{(r^2 a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)} \frac{b_\ell^m}{r} \right) \right] Y_\ell^m \mathbf{e}_r \quad (4.14)$$

$$+ \sum_{(\ell,m)} p_0' \left[\frac{(r^2 a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)} \frac{b_\ell^m}{r} \right] Y_\ell^m \mathbf{e}_r + p_0'' \boldsymbol{\xi}_r - \sum_{(\ell,m)} (p_0' a_\ell^m)' Y_\ell^m \mathbf{e}_r, \\ (\mathcal{P} \boldsymbol{\xi})_h = - \sum_{(\ell,m)} \frac{\gamma p_0}{r} \left(\frac{(r^2 a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)} \frac{b_\ell^m}{r} \right) \nabla_{\mathbb{S}^2} Y_\ell^m \\ + \frac{p_0'}{r} \boldsymbol{\xi}_h - \sum_{(\ell,m)} \frac{p_0' a_\ell^m}{r} \nabla_{\mathbb{S}^2} Y_\ell^m. \quad (4.15)$$

Step 2: Putting together results of (4.13) for $\mathcal{P} \boldsymbol{\xi}$ and $\mathcal{G} \boldsymbol{\xi}$, we obtain the ODE in the radial direction as coefficients of $Y_\ell^m \mathbf{e}_r$,

$$\begin{aligned}
 & (-\sigma^2 \rho_0 + \rho_0 \Phi_0'') a_\ell^m - \partial_r \left[\gamma p_0 \left(\frac{(r^2 a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)} \frac{b_\ell^m}{r} \right) \right] \\
 & + p_0' \left[\frac{(r^2 a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)} \frac{b_\ell^m}{r} \right] - p_0' \partial_r a_\ell^m = f_\ell^m.
 \end{aligned} \tag{4.16}$$

The ODE in the tangential direction obtained as coefficients of $\nabla_{\mathbb{S}^2} Y_\ell^m$ is,

$$\begin{aligned}
 & -\sigma^2 \rho_0 \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} - \frac{\gamma p_0}{r} \left(\frac{(r^2 a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)} \frac{b_\ell^m}{r} \right) \\
 & + \frac{p_0' + \rho_0 \Phi_0'}{r} \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} - \frac{p_0' a_\ell^m}{r} = \frac{g_\ell^m}{\sqrt{\ell(\ell+1)}}.
 \end{aligned} \tag{4.17}$$

The ODE in the tangential direction obtained as coefficients of $-\mathbf{e}_r \times \nabla_{\mathbb{S}^2} Y_\ell^m$ is,

$$\left(-\sigma^2 \rho_0 + \frac{p_0' + \rho_0 \Phi_0'}{r} \right) \frac{c_\ell^m}{\sqrt{\ell(\ell+1)}} = \frac{h_\ell^m}{\sqrt{\ell(\ell+1)}}. \tag{4.18}$$

Step 3a: For $\ell = 0$, (4.16) is only in terms of a_0 ,

$$(-\sigma^2 \rho_0 + \rho_0 \Phi_0'') a_0 - \partial_r \left(\frac{\gamma p_0}{r^2} \partial_r (r^2 a_0) \right) + \frac{p_0'}{r^2} \partial_r (r^2 a_0) - p_0' \partial_r a_0 = f_0. \tag{4.19}$$

After simplification, we arrive at (4.7). We next consider the case for $\ell > 0$.

Step 3b: For $\ell > 0$, we rewrite (4.16) and (4.17) in matrix form,

$$\frac{1}{\gamma p_0} \begin{pmatrix} f_\ell^m \\ g_\ell^m \end{pmatrix} = \left(\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \partial_r^2 + B^\ell \partial_r + C^\ell \right) \begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix},$$

with B^ℓ given by (4.5) and

$$C^\ell = \begin{pmatrix} -\frac{\sigma^2 \rho_0}{\gamma p_0} + \frac{\rho_0 \Phi_0''}{\gamma p_0} - \frac{2(\gamma p_0)'}{r \gamma p_0} + \frac{2}{r^2} + \frac{2p_0'}{r \gamma p_0} & \frac{\ell(\ell+1)}{r} \left(-\frac{1}{\gamma p_0} \frac{p_0'}{p_0} + \frac{(\gamma p_0)'}{\gamma p_0} - \frac{1}{r} \right) \\ -\frac{2}{r^2} - \frac{1}{r \gamma p_0} \frac{p_0'}{p_0} & -\frac{\sigma^2 \rho_0}{\gamma p_0} + \frac{\ell(\ell+1)}{r^2} + \frac{1}{r \gamma p_0} \left(\frac{p_0'}{p_0} + \frac{\rho_0 \Phi_0'}{p_0} \right) \end{pmatrix}.$$

The final form is obtained by using the definition scale height functions introduced in (2.8), and by replacing $\frac{\rho_0}{\gamma p_0}$ by c_0^2 (2.5). $\square$

Remark 2. For $r \leq r_s$, using hydrostatic equilibrium Assumption 2, equation (3.2) can be written with unknowns ξ , Eulerian perturbations of density δ_ρ and pressure δ_p ,

$$-\rho_0 \sigma^2 \xi + \nabla \delta_p + \delta_\rho \nabla \Phi_0 = \mathbf{f}; \quad (4.20a)$$

$$\delta_\rho = -(\nabla \rho_0) \cdot \xi - \rho_0 \nabla \cdot \xi; \quad (4.20b)$$

$$\delta_p = -\xi \cdot \nabla p_0 - \rho_0 c_0^2 \nabla \cdot \xi. \quad (4.20c)$$

Under the assumption that the source $\mathbf{f}$ has no component $\mathbf{f}^\times$ in its decomposition (4.4), i.e., $\mathbf{f} = \mathbf{f}_r + \mathbf{f}_h$, system (4.20) is equivalent to

$$-\sigma^2 \xi_r + \frac{1}{\rho_0} \partial_r \delta_p + \frac{\delta_\rho}{\rho_0} \Phi'_0 = \frac{f_r}{\rho_0}; \quad (4.21a)$$

$$c_0^2 \delta_\rho - \delta_p = \xi_r \cdot \left(p'_0 - c_0^2 \rho'_0 \right); \quad (4.21b)$$

$$\sigma^2 \left(\frac{1}{\rho_0} \delta_\rho + \frac{\rho'_0}{\rho_0} \xi_r + \partial_r \xi_r + \frac{2}{r} \xi_r \right) + \frac{1}{r^2 \rho_0} \Delta_{S^2} \delta_p = \frac{\nabla_{S^2} \cdot \mathbf{f}_h}{r p_0}, \quad (4.21c)$$

with scalar unknown $f_r = \mathbf{f}_r \cdot \mathbf{e}_r$, δ_ρ and δ_p . For details on the derivation, we refer to [20, Section 5.1].

Since the unknowns of (4.21) are scalar, one can work in the scalar harmonic basis Y_ℓ^m and solve for the coefficients of these unknowns in this expansion. This is the approach taken in [21,9] which yields to a decoupled system for the coefficients of δ_p and ξ_r , denoted respectively by e_ℓ^m and a_ℓ^m ,

$$\begin{pmatrix} \frac{1}{\rho_0} & 0 \\ 0 & \sigma^2 \end{pmatrix} \partial_r \begin{pmatrix} e_\ell^m \\ a_\ell^m \end{pmatrix} + \begin{pmatrix} -\frac{p'_0}{\gamma p_0 \rho_0} & N^2 - \sigma^2 \\ \frac{\sigma^2 - S_\ell^2}{\rho_0 c_0^2} & \sigma^2 \left(\frac{p'_0}{\gamma p_0} + \frac{2}{r} \right) \end{pmatrix} \begin{pmatrix} e_\ell^m \\ a_\ell^m \end{pmatrix} = \frac{1}{r \rho_0} \begin{pmatrix} r f_\ell^m \\ [f_h]_\ell^m \end{pmatrix}, \quad (4.22)$$

where f_ℓ^m and $[f_h]_\ell^m$ are the coefficients of the components of the source,

$$f_r = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_\ell^m(r) Y_\ell^m(\theta, \phi), \quad \nabla_{S^2} \cdot \mathbf{f}_h = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [f_h]_\ell^m(r) Y_\ell^m(\theta, \phi).$$

In (4.22), S_ℓ is called the Lamb frequency (cf. [21]) and N the buoyancy or Brunt-Väisälä frequency (cf. [9, Eq. (3.73)]) defined by

$$S_\ell^2 := \ell(\ell+1) \frac{c_0^2}{r^2}, \quad N^2 := \Phi'_0 \left(\alpha_{\rho_0} - \frac{\alpha_{p_0}}{\gamma} \right). \quad (4.23)$$

The above system which assumes hydrostatic equilibrium is only applicable in the solar interior and can be found in [21, Eqs. (4.61, 4.62)], [9, Eqs. (14.2, 14.3)], or [20, Eq. (5.15)]. $\triangle$

4.3. Decoupled equation for the radial coefficients

We now obtain the equation satisfied by the radial coefficients a_ℓ^m . This equation is what we refer to as the modal ODE in the introduction.

Proposition 2. *With the scale height quantities $\alpha_\bullet$ defined in (2.8), the radial coefficient $a(r) = a_\ell^m(r)$ solves the ODE*

$$\left(\hat{q}_\ell(r) \partial_r^2 + q_\ell(r) \partial_r + \tilde{q}_\ell(r)\right) a_\ell^m = \mathfrak{f}_\ell^m(r), \quad (4.24)$$

where the right-hand side is given as

$$\mathfrak{f}_\ell^m = -\frac{C_{12}}{C_{22}} \frac{g_\ell^m}{\gamma p_0 \sqrt{\ell(\ell+1)}} - \frac{\ell(\ell+1)}{r} \partial_r \left(\frac{1}{C_{22} \gamma p_0 \sqrt{\ell(\ell+1)}} \right) + \frac{f_\ell^m}{\gamma p_0}, \quad (4.25)$$

and coefficients,

$$\hat{q}_\ell(r) = -1 + \frac{\ell(\ell+1)}{r^2 C_{22}}; \quad (4.26a)$$

$$\begin{aligned} q_\ell(r) = & \alpha_{\gamma p_0} - \frac{2}{r} + \frac{C_{12}}{r C_{22}} + \frac{\ell(\ell+1)}{r} \left(\frac{1}{r C_{22}} \right)' \\ & + \frac{\ell(\ell+1)}{r} \left(\frac{2}{r} - \frac{\alpha_{p_0}}{\gamma} \right) \frac{1}{r C_{22}}; \end{aligned} \quad (4.26b)$$

$$\tilde{q}_\ell(r) = C_{11} + \left(\frac{2}{r} - \frac{\alpha_{p_0}}{\gamma} \right) \frac{C_{12}}{r C_{22}} + \frac{\ell(\ell+1)}{r} \left[\left(\frac{2}{r} - \frac{\alpha_{p_0}}{\gamma} \right) \frac{1}{r C_{22}} \right]'. \quad (4.26c)$$

The tangential coefficient b_ℓ^m is given by

$$\frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} = \frac{1}{r} \frac{1}{C_{22}} \partial_r a_\ell^m + \left(\frac{2}{r} - \frac{\alpha_{p_0}}{\gamma} \right) \frac{1}{r C_{22}} a_\ell^m + \frac{1}{C_{22} \gamma p_0 \sqrt{\ell(\ell+1)}} g_\ell^m. \quad (4.27)$$

Proof. Using the second equation of (4.9) to write b_ℓ^m in terms of a_ℓ^m yields (4.27) and

$$\begin{aligned} \partial_r \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} = & \left(\frac{1}{r C_{22}} \right)' \partial_r a + \left(\frac{1}{r C_{22}} \right) \partial_r^2 a + \left[\left(\frac{2}{r} - \frac{\alpha_{p_0}}{\gamma} \right) \frac{1}{r C_{22}} \right]' a \\ & + \left(\frac{2}{r} - \frac{\alpha_{p_0}}{\gamma} \right) \frac{1}{r C_{22}} \partial_r a + \partial_r \left(\frac{1}{C_{22} \gamma p_0 \sqrt{\ell(\ell+1)}} g_\ell^m \right). \end{aligned} \quad (4.28)$$

These expressions are then used to eliminate b_ℓ^m from the first equation of (4.9). $\square$

By algebraic manipulation, cf. Appendix A, we obtain the explicit form of the coefficients of (4.24).

Proposition 3. *The coefficients of the ODE (4.24) are given by,*

$$\frac{r^2 C_{22} \hat{q}_\ell(r)}{k_0^2} = r \left(r - \frac{E_{he}}{k_0^2} \right); \quad (4.29)$$

$$\frac{r^2 C_{22} q_\ell(r)}{k_0^2} = (2 - \alpha_{\gamma p_0} r) \left(r - \frac{E_{he}}{k_0^2} \right) + \frac{\ell(\ell+1)}{k_0^2} \frac{\alpha_{k_0^2} r^2 - 2r + \frac{(r E_{he})'}{k_0^2}}{r^2 - \frac{\ell(\ell+1)}{k_0^2} - r \frac{E_{he}}{k_0^2}}; \quad (4.30)$$

$$\begin{aligned} \frac{r^2 C_{22} \tilde{q}_\ell(r)}{k_0^2} &= \left(r^2 - r \frac{E_{he}}{k_0^2} \right) \left(k_0^2 - \frac{\Phi_0''}{c_0^2} \right) \\ &+ 2 \left(1 - \frac{E_{he}}{k_0^2 r} \right) \left(-\alpha_{\gamma p_0} r + \frac{\alpha_{p_0}}{\gamma} r - 1 \right) \\ &- \frac{\ell(\ell+1)}{k_0^2} \left(k_0^2 - \frac{\Phi_0''}{c_0^2} + \left(\frac{\alpha_{p_0}}{\gamma} \right)' + \frac{\alpha_{p_0}}{\gamma} \left(-\alpha_{\gamma p_0} + \frac{\alpha_{p_0}}{\gamma} \right) \right) \\ &+ \frac{\ell(\ell+1)}{k_0^2} \left(2 - \frac{\alpha_{p_0}}{\gamma} r \right) \frac{\alpha_{k_0^2} r - 2 + \frac{r(r E_{he})'}{k_0^2}}{r^2 - \frac{\ell(\ell+1)}{k_0^2} - r \frac{E_{he}}{k_0^2}}. \end{aligned} \quad (4.31)$$

4.4. Equation for radial coefficients in Schrödinger form

To state the result, we introduce the functions,

$$\mathfrak{h}_\ell(r) := -\frac{q_\ell(r)}{\hat{q}_\ell(r)}, \quad \mathfrak{g}_\ell(r) := -\frac{\tilde{q}_\ell(r)}{\hat{q}_\ell(r)}, \quad \mathfrak{I}_\ell(r) := \exp \left(\frac{1}{2} \int_0^r \mathfrak{h}_\ell \right). \quad (4.32)$$

Proposition 4. *Via the Liouville change of unknown*

$$\tilde{a}_\ell^m(r) := \frac{a_\ell^m(r)}{\mathfrak{I}_\ell(r)}, \quad (4.33)$$

the ODE (4.24) (with unknown a_ℓ^m and rhs $\mathfrak{f}_\ell^m$), is equivalent to the conjugated ODE

$$\left(-\partial_r^2 + V_\ell(r) \right) \tilde{a}_\ell^m = -\frac{\mathfrak{f}_\ell^m(r)}{\mathfrak{I}_\ell(r) \hat{q}_\ell(r)}, \quad (4.34a)$$

$$\text{with potential} \quad V_\ell(r) := \frac{1}{4} \mathfrak{h}_\ell^2(r) - \frac{1}{2} \mathfrak{h}_\ell'(r) + \mathfrak{g}_\ell(r). \quad (4.34b)$$

Proof. We first normalize (4.24) by using $\mathfrak{h}_\ell(r)$ and $\mathfrak{g}_\ell(r)$,

$$\left(-\partial_r^2 + \mathfrak{h}_\ell(r) \partial_r + \mathfrak{g}_\ell(r) \right) a_\ell^m = -\frac{\mathfrak{f}_\ell^m(r)}{\hat{q}_\ell(r)}. \quad (4.35)$$

For lightness of notation, we temporarily drop the subscripts ℓ and m from the coefficients. We next substitute a by $\tilde{a}$, by using $a = \tilde{a} e^{\frac{1}{2} \int \mathfrak{h}}$ and the identities,

$$\partial_r a = \partial_r \left(\tilde{a} e^{\frac{1}{2} \int \mathfrak{h}(s) ds} \right) = e^{\frac{1}{2} \int \mathfrak{h}(s) ds} \left(\partial_r \tilde{a} + \frac{\mathfrak{h}(r)}{2} \tilde{a} \right), \quad (4.36a)$$

$$-\partial_r^2 a = e^{\frac{1}{2} \int \mathfrak{h}(s) ds} \left(-\partial_r^2 \tilde{a} - \mathfrak{h}(r) \partial_r \tilde{a} - \frac{\mathfrak{h}^2}{4} \tilde{a} - \frac{\partial_r \mathfrak{h}}{2} \tilde{a} \right), \quad (4.36b)$$

into the ODE (4.35). This yields the conjugated ODE satisfied by $\tilde{a}$

$$e^{\frac{1}{2} \int \mathfrak{h}} \left(-\partial_r^2 \tilde{a} - \mathfrak{h}(r) \partial_r \tilde{a} - \left[\frac{1}{2} \partial_r \mathfrak{h} + \frac{1}{4} \mathfrak{h}^2 \right] \tilde{a} + \mathfrak{h} \partial_r \tilde{a} + \frac{1}{2} \mathfrak{h}^2 \tilde{a} + \mathfrak{g} \tilde{a} \right) = -\frac{\mathfrak{f}_\ell^m(r)}{\tilde{q}_\ell(r)}.$$

Dividing both sides by $e^{\frac{1}{2} \int \mathfrak{h}}$ gives,

$$-\partial_r^2 \tilde{a} + \left(\frac{1}{4} \mathfrak{h}^2 - \frac{1}{2} \partial_r \mathfrak{h} + \mathfrak{g} \right) \tilde{a} = -e^{-\frac{1}{2} \int_0^r \mathfrak{h}} \frac{\mathfrak{f}_\ell^m(r)}{\tilde{q}_\ell(r)}. \quad \square$$

Remark 3. The Schrödinger form of the wave equation given in Proposition 4 is derived in [25] for radial modes ($\ell = 0$). A Schrödinger equation can also be derived for the Eulerian perturbation of the pressure δ_p (defined in (4.20c)) under the hypothesis of hydrostatic equilibrium [11, Eq. (5.4.1)] or [20, Eq. (5.44)]. It can be obtained from the system (4.21) by eliminating ξ_r to obtain a decoupled equation for δ_p that can then be reduced to a Schrödinger form. Schrödinger equations for different unknowns can also be found in [26] but in Cartesian geometry (representing a small patch of the Sun). $\triangle$

We next use Proposition 3 to obtain a more explicit expression for the coefficient of the conjugated equation (4.34a). We define the auxiliary functions,

$$\mathbb{F}_\ell := k_0^2 r^2 - r E_{\text{he}} - \ell(\ell + 1); \quad \mathbb{F}_0 := k_0^2 r^2 - r E_{\text{he}} \quad (4.37)$$

and the following:

$$\mathfrak{t}_\ell := \frac{1}{\mathbb{F}_\ell} \frac{\mathbb{F}_0'}{\mathbb{F}_0}; \quad (4.38a) \quad \mathfrak{t}_\ell^\Pi := \frac{1}{\mathbb{F}_\ell} \frac{(\mathbb{F}_0/r^2)'}{(\mathbb{F}_0/r^2)}; \quad (4.38b) \quad \mathfrak{v}_\ell := \frac{\mathbb{F}_0/r}{\mathbb{F}_\ell}; \quad (4.38c)$$

$$\mathfrak{g}^I := \frac{1}{k_0^2 - \frac{E_{\text{he}}}{r}} \left(k_0^2 - \frac{\Phi_0''}{\mathfrak{c}_0^2} + \left(\frac{\alpha_{\text{p}0}}{\gamma} \right)' + \frac{\alpha_{\text{p}0}}{\gamma} \left(-\alpha_{\gamma \text{p}0} + \frac{\alpha_{\text{p}0}}{\gamma} \right) \right); \quad (4.39a)$$

$$\mathfrak{w}_\ell := \mathfrak{g}^I + \frac{2r}{\ell(\ell + 1)} \mathfrak{v}_\ell + \left(1 - \frac{\eta}{2} r \right) r \mathfrak{t}_\ell^\Pi - \frac{r^2}{2} \mathfrak{t}_\ell'. \quad (4.39b)$$

We also introduce the wavenumber function $k(r)$ and the parameter $\eta(r)$,

$$-k^2 := \frac{\alpha_{\gamma \text{p}0}^2}{4} - \frac{\alpha'_{\gamma \text{p}0}}{2} - k_0^2, \quad (4.40a) \quad \eta := 2 \frac{\alpha_{\text{p}0}}{\gamma} - \alpha_{\gamma \text{p}0}. \quad (4.40b)$$

Proposition 5. In terms of the functions introduced in (4.38) and (4.40), the coefficients of the ODE (4.34a) and (4.35) are given by

$$\mathfrak{h}_\ell(r) = \alpha_{\gamma p_0} - \frac{2}{r} + \ell(\ell+1) \mathfrak{t}_\ell; \quad (4.41a)$$

$$\begin{aligned} \mathfrak{g}_\ell(r) = & -k_0^2 + \frac{\Phi_0''}{\mathfrak{c}_0^2} + \frac{2}{r^2} + \frac{2(\alpha_{\gamma p_0} - \frac{\alpha_{p_0}}{\gamma})}{r} + \frac{\ell(\ell+1)\mathfrak{g}^I}{r^2} \\ & + \ell(\ell+1) \left(\frac{2}{r} - \frac{\alpha_{p_0}}{\gamma} \right) \mathfrak{t}_\ell; \end{aligned} \quad (4.41b)$$

$$V_\ell(r) = -k^2(r) + \frac{\Phi_0''(r)}{\mathfrak{c}_0^2(r)} - \eta(r) \mathfrak{v}_\ell(r) + \frac{\ell(\ell+1)}{r^2} \mathfrak{w}_\ell(r) + \frac{[\ell(\ell+1)]^2}{4} \mathfrak{t}_\ell^2. \quad (4.41c)$$

With S_ℓ the Lamb frequency (4.23), the function $\mathfrak{I}_\ell$ defined in (4.32) is

$$\mathfrak{I}_\ell(r) = \frac{1}{r \mathfrak{c}_0 \sqrt{\rho_0}} \sqrt{\frac{k_0^2 r^2 - \ell(\ell+1) - r E_{he}}{k_0^2 r^2 - r E_{he}}} = \frac{1}{r \mathfrak{c}_0 \sqrt{\rho_0}} \sqrt{\frac{\sigma^2 - S_\ell^2 - \frac{E_{he}}{r} \mathfrak{c}_0^2}{\sigma^2 - \frac{E_{he}}{r} \mathfrak{c}_0^2}}. \quad (4.42)$$

Proof. The expressions (4.41a) and (4.41b) are obtained by substituting the expressions of $\hat{q}_\ell$, q_ℓ and $\tilde{q}_\ell$ given in Proposition 3 in the definition of $\mathfrak{h}_\ell$ and $\mathfrak{g}_\ell$ (4.34b).

• To obtain the expression for V_ℓ , we first compute

$$\frac{1}{4} \mathfrak{h}_\ell^2 = \frac{\alpha_{\gamma p_0}^2}{4} + \frac{1}{r^2} - \frac{1}{r} \alpha_{\gamma p_0} + \frac{[\ell(\ell+1)]^2}{4} \mathfrak{t}_\ell^2 + \frac{\ell(\ell+1)}{2} \mathfrak{t}_\ell \left(\alpha_{\gamma p_0} - \frac{2}{r} \right); \quad (4.43a)$$

$$-\frac{1}{2} \mathfrak{h}_\ell' = -\frac{\alpha_{\gamma p_0}'}{2} - \frac{1}{r^2} - \frac{\ell(\ell+1)}{2} \mathfrak{t}_\ell'. \quad (4.43b)$$

Using the definition of k and η in (4.40), and after some rearrangement, we obtain

$$\begin{aligned} V_\ell = & -k^2 + \frac{\Phi_0''}{\mathfrak{c}_0^2} + \frac{\ell(\ell+1)}{r^2} \mathfrak{g}^I - \eta \left(\frac{1}{r} + \frac{\ell(\ell+1)}{2} \mathfrak{t}_\ell \right) \\ & + \frac{2}{r} \left(\frac{1}{r} + \frac{\ell(\ell+1)}{2} \mathfrak{t}_\ell \right) + \ell(\ell+1) \left(\frac{\ell(\ell+1)}{4} \mathfrak{t}_\ell^2 - \frac{1}{2} \mathfrak{t}_\ell' \right). \end{aligned} \quad (4.44)$$

We next rewrite the term $(\frac{1}{r} + \frac{\ell(\ell+1)}{2} \mathfrak{t}_\ell)$ in (4.44), by using the identities,

$$\mathfrak{t}_\ell = \mathfrak{t}_\ell^I + \mathfrak{t}_\ell^\Pi \quad \text{and} \quad \frac{1}{r} + \frac{\ell(\ell+1)}{2} \mathfrak{t}_\ell^I = \mathfrak{v}_\ell \quad \text{where} \quad \mathfrak{t}_\ell^I := \frac{2}{r F_\ell}.$$

Expression (4.44) can be further written as,

$$V_\ell = -k^2 + \frac{\Phi_0''}{\mathfrak{c}_0^2} - \eta \mathfrak{v}_\ell + \frac{\ell(\ell+1)\mathfrak{g}^I}{r^2} + \frac{2\mathfrak{v}_\ell}{r} + \ell(\ell+1) \left[\left(\frac{1}{r} - \frac{\eta}{2} \right) \mathfrak{t}_\ell^\Pi + \frac{\ell(\ell+1)\mathfrak{t}_\ell^2}{4} - \frac{\mathfrak{t}_\ell'}{2} \right].$$

The expression (4.41c) for V_ℓ is obtained by using definition of $\mathfrak{w}_\ell$ in (4.39b).

- For the expression of $\mathfrak{J}_\ell$ of (4.32), we first write

$$t_\ell = \frac{(k_0^2 r^2 - r E_{\text{he}})'}{k_0^2 r^2 - \ell(\ell+1) - r E_{\text{he}}} - \frac{(k_0^2 r^2 - r E_{\text{he}})'}{r^2 k_0^2 - r E_{\text{he}}}. \quad (4.45)$$

For $g > 0$, $\alpha_g = -(\ln g)'$, while for complex g , using complex log taking argument in $[0, 2\pi)$, we obtain

$$\begin{aligned} \int_0^r \mathfrak{h}_\ell &= -\ln(\gamma p_0) - 2 \ln r + \log(k_0^2 r^2 - \ell(\ell+1) - r E_{\text{he}}) - \log(k_0^2 r^2 - r E_{\text{he}}) \\ &= \log\left(\frac{k_0^2 r^2 - \ell(\ell+1) - r E_{\text{he}}}{r^2 \gamma p_0 (k_0^2 r^2 - r E_{\text{he}})}\right). \end{aligned} \quad (4.46)$$

The final expression (4.42) is obtained by exponentiation. $\square$

Remark 4. The expressions for $\ell = 0$ can be read of from the above results by setting $\ell = 0$. It can also be obtained directly from (4.7),

$$\mathfrak{h}_0 = \alpha_{\gamma p_0} - \frac{2}{r}; \quad \mathfrak{J}_0(r) = \frac{1}{r c_0(r) \sqrt{\rho_0(r)}}; \quad (4.47a)$$

$$\mathfrak{g}_0(r) = -k_0^2(r) + \frac{\Phi_0''(r)}{c_0^2(r)} + \frac{2}{r^2} + \frac{2}{r} \left(\alpha_{\gamma p_0} - \frac{\alpha_{p_0}}{\gamma} \right); \quad (4.47b)$$

$$V_0 = -k^2(r) + \frac{\Phi_0''(r)}{c_0^2(r)} - \frac{\eta(r)}{r} + \frac{2}{r^2}. \quad (4.47c)$$

The expression of $\mathfrak{J}_0$ is the same as in [11, Eq. (4.8.4)] (the difference by a factor r is due to their definition [11, Eq. (4.1.1)]). $\triangle$

Remark 5. For $r \leq r_s$, by using Assumptions 2 and 3, which yields,

$$\begin{aligned} E_{\text{he}} &\equiv 0, \quad E'_{\text{he}} \equiv 0, \quad 2\alpha_{c_0} = \alpha_{\gamma p_0} - \alpha_{\rho_0}, \quad \frac{\Phi_0'}{c_0^2} = \frac{\alpha_{p_0}}{\gamma}, \\ -\frac{\Phi_0''}{c_0^2} + \left(\frac{\alpha_{p_0}}{\gamma}\right)' &= -E'_{\text{he}} - \frac{\Phi_0'}{c_0^2} 2\alpha_{c_0} = -\frac{\Phi_0'}{c_0^2} (\alpha_{\gamma p_0} - \alpha_{\rho_0}), \end{aligned}$$

we can simplify the expressions in Proposition 5 to,

$$\mathfrak{h}_\ell = \alpha_{\gamma p_0} - \frac{2}{r} - \frac{\ell(\ell+1)}{k_0^2} \frac{\alpha k_0^2 r - 2}{r^3 - \frac{\ell(\ell+1)}{k_0^2} r}, \quad (4.48a)$$

$$\mathfrak{g}_\ell = -k_0^2 + \frac{\Phi_0''}{c_0^2} - \frac{2}{r} \left(\frac{\Phi_0'}{c_0^2} - \alpha_{\gamma p_0} - \frac{1}{r} \right) + \frac{\ell(\ell+1)}{k_0^2 r^2} \left(k_0^2 - \frac{\Phi_0'}{c_0^2} \left(\alpha_{\rho_0} - \frac{\Phi_0'}{c_0^2} \right) \right)$$

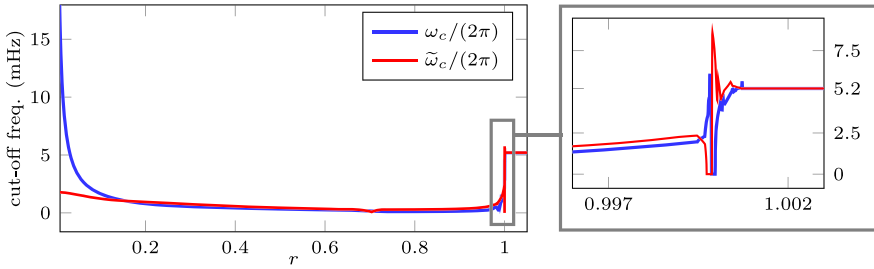


Fig. 2. Comparison of the real parts of the cut-off frequencies for the vector and scalar problems, (4.49) and (4.51) with a zoom near the surface.

$$+ \frac{\ell(\ell+1)}{k_0^2 r^2} \left(-2 + \frac{\Phi_0'}{c_0^2} r \right) \frac{\alpha_{k_0^2 r} - 2}{r^2 - \frac{\ell(\ell+1)}{k_0^2}}. \quad (4.48b)$$

The fourth term of g_ℓ can be written as $\frac{\ell(\ell+1)(\sigma^2 - N^2)}{\sigma^2 r^2}$ where N is the buoyancy or Brunt-Väisälä frequency (4.23). $\triangle$

4.5. Cut-off frequency

We generalize the discussion of cut-off frequency in [13, section 3.1] to our conjugated operator $-\partial_r^2 + V_\ell$. We introduce the local cut-off frequency ω_c ,

$$\frac{\omega_c^2(r)}{c_0^2(r)} := \frac{\alpha_{\gamma p_0}^2(r)}{4} - \frac{\alpha_{\gamma p_0}'(r)}{2} + \frac{\Phi_0''(r)}{c_0^2(r)} - \frac{\eta(r)}{r} + \frac{2}{r^2}, \quad (4.49)$$

and the horizontal wavenumber function k_h ,

$$k_h^2 = 0, \quad \text{for } \ell = 0, \\ k_h^2 = \frac{\ell(\ell+1)}{r^2} \left(g^1 - \frac{r^2 \eta}{2} + r t_\ell - \frac{r^2 t_\ell'}{2} + \frac{\ell(\ell+1)(r t_\ell)^2}{4} \right), \quad \text{for } \ell > 0. \quad (4.50)$$

The second expression comes from expression (4.44) for V_ℓ . These are generalization of the scalar equation for which $k_h = \sqrt{\ell(\ell+1)}/r$, cf. [13, Eq. (7)], and

$$\frac{\tilde{\omega}_c^2(r)}{c_0^2(r)} = \frac{\alpha_\rho^2(r)}{4} + \frac{\alpha_\rho'(r)}{2} + \frac{\alpha_\rho(r)}{r}, \quad \text{scalar cut-off frequency}, \quad (4.51)$$

cf. [13, Eq. (11)]. The cut-off frequency ω_c for the scalar equation does not have the gravity term and adiabatic exponent, and uses α_ρ (instead of $-\alpha_{\gamma p_0}$). Note however that the form of the potential and thus the cut-off frequency depends on the unknown of the problem [26]. Here, our cut-off frequency for the vectorial problem is for ξ_r (even if we were to neglect the gravity in (7.11b)) while the one of the scalar problem is for $\nabla \cdot \xi$. We compare in Fig. 2 the cut-off frequencies of the scalar and vectorial problems using the background solar models S-AtmoI.

We see that they have the same pattern, in particular with the singularity at the origin and the drastic variations near the surface, and we note that they coincide in the atmosphere.

In terms of $\omega_c(r)$ and k_h^2 , the potential V_ℓ can be written in a similar form to [13, Eq. (10)],

$$V_\ell(r) = -\frac{\sigma^2 - \omega_c^2}{c_0^2} + k_h^2. \quad (4.52)$$

The limiting value of ω_c in the atmosphere, called the *constant acoustic cut-off* angular frequency, determines the behavior of waves there, see Section 7. By the asymptotic expansion in (6.3), we have

$$\lim_{r \rightarrow \infty} \omega_c(r) = \omega_t, \quad \text{where } \omega_t := \alpha_a c_a / 2, \quad \alpha_a, c_a \text{ defined in (3.4)}. \quad (4.53)$$

For the solar background configuration, $\omega_t = 5.20$ mHz, cf. Fig. 2.

5. Local regularity

In this section, we determine the local regularity of the coefficients of the radial modal ODE (4.24) as well as the conjugated one (4.34a). Under Assumption 1, their coefficients are made up of C^2 functions and quotients of these. The only concern on compact intervals of $[0, \infty)$ is the loss of regularity at *singular points* where the coefficients might ‘blow up’ due to a division by zero. We first recall the form of the normalized equation (4.24) with zero right-hand-side,

$$(-\partial_r^2 + \mathfrak{h}_\ell \partial_r + \mathfrak{g}_\ell)u = 0. \quad (5.1)$$

In the classical theory of ODE with meromorphic coefficients, cf. [27, Theorem 4 p. 164], $r = r_0$ is a regular singular point of (5.1) if it is a pole of at most first order of $\mathfrak{h}_\ell$ or a pole of at most second order of $\mathfrak{g}_\ell$. However, with only C^2 in our assumptions, we instead appeal to the work of Bôcher [22] to expand this definition.

Definition 3 (*Bôcher regular singular points*). For a fixed ℓ , $r = r_0$ is a regular singular point of (5.1) if there exist constants η_ℓ and $\tilde{\eta}_\ell$, so that

$$\begin{aligned} \text{if } \mathfrak{h}_\ell &= \frac{-\eta_\ell}{r - r_0} + \tilde{\mathfrak{h}}_\ell, \quad \mathfrak{g}_\ell = \frac{-\tilde{\eta}_\ell}{(r - r_0)^2} + \tilde{\mathfrak{g}}_\ell, \text{ then} \\ \text{for fixed } \delta > 0, \quad \lim_{\epsilon \rightarrow 0^+} \int_{r_0 \pm \epsilon}^{r_0 \pm \delta} |\tilde{\mathfrak{h}}_\ell| dr < \infty \text{ and } \lim_{\epsilon \rightarrow 0^+} \int_{r_0 \pm \epsilon}^{r_0 \pm \delta} (r - r_0) |\tilde{\mathfrak{g}}_\ell| dr < \infty. \end{aligned} \quad (5.2)$$

For a regular singular point $r = r_0$, by Definition 3, the *indicial roots* or *characteristics exponents* associated are the roots of the *indicial polynomial*

$$\lambda(\lambda - 1) + \eta_\ell \lambda + \tilde{\eta}_\ell = 0, \quad (5.3a)$$

$$\text{with coefficients } \eta_\ell := -\lim_{r \rightarrow r_0} (r - r_0) \mathfrak{h}_\ell, \quad \tilde{\eta}_\ell := -\lim_{r \rightarrow r_0} (r - r_0)^2 \mathfrak{g}_\ell. \quad (5.3b)$$

These exponents λ describe the dominant behavior of the solution, that is, the solution behaves like r^λ as $r \rightarrow 0$, see Section 7.

We consider the situations $\ell = 0$ and $\ell > 0$ separately. The existence and uniqueness of the roots are investigated in Proposition 6. The results giving their exponents in both situations are listed in Proposition 8.

5.1. At level $\ell = 0$

Due to its simplicity, we study the equation at $\ell = 0$ separately. Recall from Remark 4,

$$-\partial_r^2 a_0 + \left(\alpha_{\gamma p_0} - \frac{2}{r} \right) \partial_r a_0 + \left(-\frac{\sigma^2}{c_0^2} + \frac{\Phi_0''}{c_0^2} + 2\frac{\alpha_{\gamma p_0}}{r} + \frac{2}{r^2} - 2\frac{\alpha_{p_0}}{r\gamma} \right) a_0 = 0. \quad (5.4)$$

It follows that $r = 0$ is the only singular point on $[0, \infty)$, with indicial equation

$$s(s-1) + 2s - 2 = 0 \quad \text{and exponents} \quad \lambda_0^- = -2, \lambda_0^+ = 1. \quad (5.5)$$

5.2. For $\ell > 0$

From the expressions of the coefficients given in Proposition 5, potential singularities occur at $r = 0$ and additionally, when $\Gamma = 0$, at the zeros of

$$G(r) = 0 \text{ on } (r_s, \infty) \quad \text{with} \quad G(r) := \omega^2 \frac{r}{c_0^2(r)} - E_{\text{he}}(r), \quad (5.6a)$$

$$\text{and } F(r) = 0 \quad \text{with} \quad F(r) := \omega^2 \frac{r^2}{c_0(r)^2} - \ell(\ell+1) - r E_{\text{he}}(r). \quad (5.6b)$$

These functions were first introduced in (4.37). We first investigate the existence of zeros for (5.6a) and (5.6b) and their multiplicity. We define

$$\ell_\omega^* := -\frac{1}{2} + \sqrt{\frac{\omega^2}{c_0(r_s)^2} r_s^2 + \frac{1}{4}}. \quad (5.7)$$

Proposition 6. *We consider Assumption 1–Assumption 3. For $\omega > 0$, the only root of $G = 0$ is $r = 0$. Equation (5.6b) has a unique zero, when $\Gamma = 0$, $\omega > 0$, on $(0, \infty)$, denoted by $r_{\omega, \ell}^*$. If $\ell_\omega^* \in \mathbb{N}$, for $\ell = \ell_\omega^*$, then $r_{\omega, \ell}^* = r_s$; otherwise,*

$$\ell \leq \ell_\omega^* \Rightarrow r_{\omega, \ell}^* \in [0, r_s] \quad \text{and} \quad \ell > \ell_\omega^* \Rightarrow r_{\omega, \ell}^* > r_s. \quad (5.8)$$

Proof. Statement 1 The result is due to Assumption 1 which gives $E_{\text{he}} \geq 0$ and $c_0 > 0$ thus $|G| > \frac{\omega^2}{c_0^2} r > 0$ for $\omega > 0$ and $r > 0$.

Statement 2 Since $r \mapsto -E_{\text{he}} \geq 0$ is increasing by (3.5d), $r \mapsto r(-E_{\text{he}})$ is strictly increasing. By (3.5c), $r \mapsto \omega^2 \frac{r^2}{c_0(r)^2}$ is strictly increasing. This means $r \mapsto F(r)$ is strictly increasing on $[0, \infty)$. Together with the fact that,

$$F(0) = -\ell(\ell + 1) < 0 \quad \text{and} \quad F(r) > 0 \text{ for } r > k_0(r_a)\sqrt{\ell(\ell + 1)}, \quad (5.9)$$

this means that $F(r) = 0$ has a unique zero on $(0, \infty)$, denoted by $r_{\ell, \omega}^*$.

We now consider its position. From (3.8), $E_{\text{he}}(r_s) = 0$, we thus have the equivalence,

$$F(r_s) \geq 0 \quad \Leftrightarrow \quad \ell \leq \ell_{\omega}^*. \quad (5.10)$$

With the strict increase of F and its values in (5.9), equivalence (5.10) means that, if $\ell \leq \ell_{\omega}^*$ then $F(r_s) \geq 0$ and $r_{\ell, \omega}^* \in (0, r_s]$, while if $\ell > \ell_{\omega}^*$, $F(r_s) < 0$ thus $r_{\ell, \omega}^* \in (r_s, \infty)$. If $\ell_{\omega}^* \in \mathbb{N}$, since $F(r_s) = 0$ for $\ell = \ell_{\omega}^*$, r_s is the unique zero on $(0, \infty)$ for this level. $\square$

We next calculate the limits of $\mathfrak{h}_{\ell}$ and $\mathfrak{g}_{\ell}$ at 0 and $r_{\ell, \omega}^*$. These computations are needed to compute the indicial exponents of the singular points.

Proposition 7. For $\omega > 0$ and $\ell > 0$, we have

$$\lim_{r \rightarrow 0} r \mathfrak{h}_{\ell} = -4 \quad ; \quad \lim_{r \rightarrow 0} r^2 \mathfrak{g}_{\ell} = -2 + \ell(\ell + 1). \quad (5.11)$$

When $\Gamma = 0$ also with $\omega > 0$ and $\ell > 0$, then

$$\lim_{r \rightarrow r_{\omega, \ell}^*} (r - r_{\omega, \ell}^*) \mathfrak{h}_{\ell} = 1 \quad ; \quad \lim_{r \rightarrow r_{\omega, \ell}^*} (r - r_{\omega, \ell}^*)^2 \mathfrak{g}_{\ell} = 0. \quad (5.12)$$

Proof. Statement 1 For the limit at $r = 0$, we work with the expression of $\mathfrak{h}_{\ell}$ and $\mathfrak{g}_{\ell}$ in the interior in (4.48a) and (4.48b) and obtain $\lim_{r \rightarrow 0} r \mathfrak{h}_{\ell} = -4$, and

$$\lim_{r \rightarrow 0} r^2 \mathfrak{g}_{\ell} = -2 + \ell(\ell + 1) - \lim_{r \rightarrow 0} \frac{\ell(\ell + 1)}{k_0^2} \frac{\Phi_0'}{\epsilon_0^2} \left(\alpha_{\rho_0} - \frac{\Phi_0'}{\epsilon_0^2} \right). \quad (5.13)$$

It remains to calculate $\Phi_0'(0)$. For this, using (2.7) and L'Hopital's rule yields

$$\Phi_0'(0) = 4\pi G \lim_{r \rightarrow 0} \frac{(\int_0^r s^2 \rho_0(s) ds)'}{2r} = 4\pi G \lim_{r \rightarrow 0} \frac{r^2 \rho_0(r)}{2r} = 0. \quad (5.14)$$

Statement 2 We will simply write $r_{\star} = r_{\omega, \ell}^*$. Since it is a simple root, we obtain immediately the result with $\mathfrak{g}_{\ell}$. For the limit with $\mathfrak{h}_{\ell}$, we first use the assumption of $r_{\star}$ in (5.6b) to obtain the limits,

$$\lim_{r \rightarrow r_{\star}} \frac{\ell(\ell + 1)}{k_0^2} \frac{1}{r^2 - r \frac{E_{\text{he}}}{k_0^2}} = 1; \quad (5.15a) \quad \lim_{r \rightarrow r_{\star}} \frac{\ell(\ell + 1) + r E_{\text{he}}}{k_0^2} = r_{\star}^2. \quad (5.15b)$$

We next separate $(r - r_{\star})\mathfrak{h}_{\ell}$ into regular and singular factors,

$$- \lim_{r \rightarrow r_{\star}} (r - r_{\star}) \mathfrak{h}_{\ell} = \frac{L_1}{L_2}, \quad \text{where} \quad (5.16)$$

$$L_1 = \lim_{r \rightarrow r_\star} \frac{\ell(\ell+1)}{k_0^2} \frac{\alpha_{k_0^2} r^2 - 2r + \frac{1}{k_0^2} (r E_{\text{he}})'}{\left(r^2 - r \frac{E_{\text{he}}}{k_0^2}\right)} \stackrel{(5.15a)}{=} \lim_{r \rightarrow r_\star} \alpha_{k_0^2} r^2 - 2r + \frac{(r E_{\text{he}})'}{k_0^2},$$

$$L_2 = \lim_{r \rightarrow r_\star} \frac{\left(r^2 - \frac{\ell(\ell+1)}{k_0^2} - r \frac{E_{\text{he}}}{k_0^2}\right)}{r - r_\star} \stackrel{\text{L'Hopital}}{=} \lim_{r \rightarrow r_\star} \left(r^2 - \frac{\ell(\ell+1)}{k_0^2} - r \frac{E_{\text{he}}}{k_0^2}\right)'.$$

We next show that $L_1 = -L_2$. Since $(\frac{1}{k_0^2})' = \frac{\alpha_{k_0^2}}{k_0^2}$,

$$L_2 = \lim_{r \rightarrow r_\star} 2r - \frac{\ell(\ell+1) + r E_{\text{he}}}{k_0^2} \alpha_{k_0^2} - \frac{(r E_{\text{he}})'}{k_0^2} \stackrel{(5.15b)}{=} \lim_{r \rightarrow r_\star} 2r - r^2 \alpha_{k_0^2} - \frac{(r E_{\text{he}})'}{k_0^2}. \quad \square$$

In the final result of the current section, for $\ell \in \mathbb{N}$, we summarize the singular points and their indicial exponents for both the original equation (5.1) and the conjugated equation (4.34a),

$$(-\partial_r^2 + V_\ell) \tilde{u} = 0. \quad (5.18)$$

Proposition 8. *Given Assumptions 1 to 3, when $\Gamma = 0$ with $\ell > 0$, the only singular points of (5.1) and (5.18) are $r = 0$ and $r = r_{\omega, \ell}^\star$ (defined in Proposition 6). When $\Gamma > 0$ or $\Gamma = 0$ with $\ell = 0$, $r = 0$ is their only singular point.*

For equation (5.1), the indicial exponents associated to $r = 0$ denoted by $\lambda_\ell^\pm$ are

$$\lambda_0^+ = 1, \lambda_0^- = -2; \quad \lambda_\ell^+ = \ell - 1, \lambda_\ell^- = -\ell - 2, \text{ for } \ell > 0, \quad (5.19)$$

while those for conjugated ODE (5.18) denoted by $\kappa_\ell^\pm$ are given by

$$\kappa_0^+ = 2, \kappa_0^- = -1; \quad \kappa_\ell^+ = \ell + 1, \kappa_\ell^- = -\ell, \text{ for } \ell > 0. \quad (5.20)$$

Associated to singular point $r_{\omega, \ell}^\star$, the indicial exponents of (5.1) and (5.18) are independent of ℓ . Denoted respectively by $\lambda_\star^\pm$ and $\kappa_\star^\pm$, they take value

$$\lambda_\star^+ = 2, \lambda_\star^- = 0; \quad (5.21a) \quad \kappa_\star^+ = \frac{3}{2}, \kappa_\star^- = -\frac{1}{2}. \quad (5.21b)$$

Proof. For (5.1), it suffices to consider $\ell > 0$. The main ingredients are the results of Proposition 7 and the definition of the indicial polynomial given in (5.3). Using (5.11) yields the indicial quadratic associated to $r = 0$,

$$\lambda(\lambda - 1) + 4\lambda + 2 - \ell(\ell + 1) = 0 \quad \text{with roots } \ell - 1 \text{ and } -\ell - 2. \quad (5.22)$$

The one associated to $r = r_{\omega, \ell}^\star$ is obtained with (5.12),

$$\lambda(\lambda - 1) - \lambda = 0 \quad \text{with roots } 2 \text{ and } 0. \quad (5.23)$$

This gives us the results in (5.19) and (5.21a). We next show (5.20) and (5.21b).

For the rest of the proof, we denote by r_0 to mean 0 or $r_\star$. We also write,

$$\eta_\ell := \lim_{r \rightarrow r_0} (r - r_0) \mathfrak{h}_\ell, \quad \tilde{\eta}_\ell := \lim_{r \rightarrow r_0} (r - r_0)^2 \mathfrak{g}_\ell. \quad (5.24)$$

The indicial exponents of (5.1), simply written as $\lambda^\pm$, are roots of the quadratic equation,

$$s(s - 1) - \eta_\ell s - \tilde{\eta}_\ell = 0. \quad (5.25)$$

Since $\lim_{r \rightarrow r_0} (r - r_0)^2 \mathfrak{h}_\ell^2 = \eta_\ell^2$ and $\lim_{r \rightarrow r_0} (r - r_0)^2 \mathfrak{h}_\ell' = -\eta_\ell$,

$$\lim_{r \rightarrow r_0} (r - r_0)^2 V_\ell = \frac{1}{4} \eta_\ell^2 + \frac{1}{2} \eta_\ell + \tilde{\eta}_\ell. \quad (5.26)$$

The indicial equation of (4.34a) at $r = r_0$ is thus

$$\tilde{s}(\tilde{s} - 1) - \frac{1}{4} \eta_\ell^2 - \frac{1}{2} \eta_\ell - \tilde{\eta}_\ell = 0. \quad (5.27)$$

Denote the roots of (5.27) as $\tilde{\lambda}^\pm$. Since

$$\tilde{s}(\tilde{s} - 1) = (s - \frac{1}{2} \eta_\ell) \left((s - 1) - \frac{1}{2} \eta_\ell \right) = s(s - 1) - \eta_\ell s + \frac{1}{2} \eta_\ell + \frac{1}{4} \eta_\ell^2, \quad (5.28)$$

quadratic (5.25) and (5.27) are equivalent by the change of variable, $\tilde{s} = s - \frac{1}{2} \eta_\ell$. This means that the indicial roots of (4.24) and (4.34a) are related by

$$\tilde{\lambda}^\pm = \lambda^\pm - \frac{1}{2} \eta_\ell. \quad (5.29)$$

The proof is finished by recalling from Proposition 7 that $\eta_\ell = -4$ for $\ell > 0$ and $\eta_\ell = -2$ for $\ell = 0$ for singular point $r = 0$, while for $r = r_\star$, $\eta_\ell = 1$, $\ell > 0$. $\square$

5.3. Numerical illustration

In Fig. 3, we compute the values of $r_{\omega, \ell}^\star$ and $\ell_\omega^\star$ for the physical parameters corresponding to the Sun, and for frequencies (up to 10 mHz) and modes (up to 2000) commonly used in solar applications. We observe that the singularity mainly lies near the solar radius $r = 1$, except for relatively low modes. On the other hand, this singularity does not exist when we include attenuation, which is the case in practical applications.

In Fig. 4, we show the value of the potential $V_\ell(r)$ as a function of frequencies ω and r , for different modes, with and without attenuation. We observe the ‘blow-up’ of the potential V_ℓ due to the singularity at the origin in both cases, and, at the additional singularity $r_{\omega, \ell}^\star$ without attenuation. The position of the latter varies with the mode and frequency, as also observed in Fig. 3. In the case with attenuation, we observe that the potential changes sign instead of blowing-up, see Fig. 4(a).

Intuitively, the sign of the potential indicates whether waves are propagative, when $V_\ell < 0$, or exponentially decaying when $V_\ell > 0$. We distinguish two propagative regions.

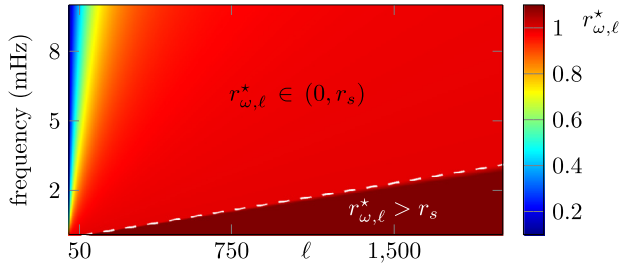
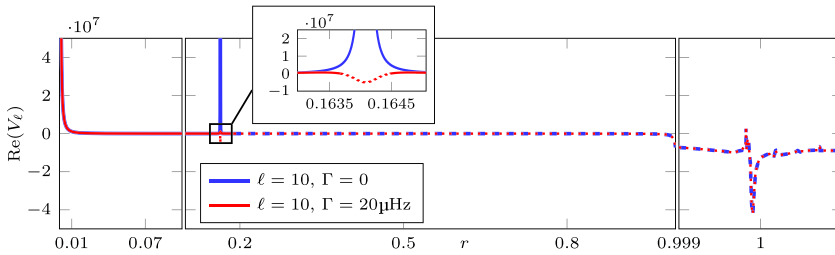


Fig. 3. Value of $r_{\omega, \ell}^*$ in (5.6b) in terms of ω, ℓ . The white dashed line represents ℓ_{ω}^* , that separates the region where the singularity is in the interior (above the line) or in the atmosphere (below it). For visualization, all the positions where $r_{\omega, \ell}^* \geq 1.1$ use the same color scale (dark red). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)



(a) $\text{Re}(V_\ell)$ at 7 mHz with solid and dashed lines respectively for positive and negative values. The potential has a horizontal asymptote (as $r \rightarrow \infty$) at -8.9×10^6 , corresponding to $-\mathbf{k}_a^2$.

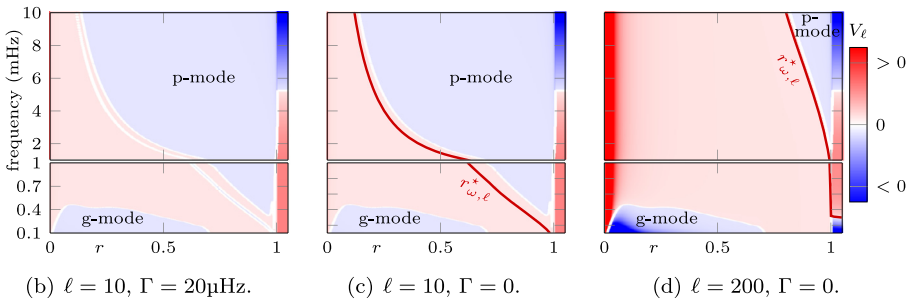


Fig. 4. $\text{Re}(V_\ell)$ in (4.41c) for different frequencies and modes, with and without attenuation. In Fig. (b)–(d), the sign of the potential is indicated using blue for negative, red for positive, and the position where V_ℓ changes sign is indicated in white. In these figures, the low-frequency regime is enlarged for visualization (bottom part). For $\Gamma = 0$, the red line in Fig. (c)–(d) indicates where V_ℓ blows up i.e. at singularity $r_{\omega, \ell}^*$ which corresponds to the zero of (5.6b).

- At low frequencies (below 0.5 mHz) and where $V_\ell < 0$, waves are propagative near the origin. The modes contained in this region are referred to as solar *g-modes* (gravity modes), see Figs. 4(b) to 4(d). These waves are difficult to observe at the solar surface as $V_\ell > 0$ for $r > 0.7$.
- The region of the diagram with higher frequencies and where $V_\ell < 0$ contains the *p-modes* (pressure modes). These are the main modes used to probe the solar interior in helioseismology. This area can be further divided in a region below and one above the cut-off frequency

$\omega_t = 5.20$ mHz, see Subsection 4.5. For $\omega < \omega_t$, the propagative region (for a fixed ω) is located in the interval $[r_{tl}(\omega), r_{tu}(\omega)]$, with r_{tl} lying in the solar interior and r_{tu} just below the solar surface. These positions are referred to as the lower and upper turning points in the solar community [9]. For $\omega > \omega_t$, the propagative region extends into the atmosphere. This means that high-frequency waves propagate into the atmosphere, and thus to simulate these waves, one requires an appropriate boundary condition and atmospheric model, [13].

Upon studying the position where the potential changes sign (in white in Fig. 4) corresponding to the harmonic degree $\ell = 10$ versus that for $\ell = 200$ (see Figs. 4(b) to 4(d)), we observe that the location of the zeros of the potential for g-modes is barely modified with ℓ , while those of p-modes, as well as the singularity $r_{\omega,\ell}^*$ (when $\Gamma = 0$), change drastically, shifting closer to the surface as ℓ increases. High-degree p-modes are thus expected to be confined close to the surface, while the ones of lower order have propagative region reaching deep inside the Sun and can thus provide important information about the solar internal structure.

Remark 6. Interestingly, the position of $r_{\omega,\ell}^*$ is usually located slightly before the lower turning point (r_{tu}) but these two positions do not coincide. On the other hand, in [9, Section 5], which uses a simplified potential, $r_{\omega,\ell}^*$ and r_{tu} coincide. $\triangle$

6. Asymptotic analysis at infinity

In this section, we study the behavior of the coefficients of the radial modal ODE (4.24) as well as the conjugated one (4.34a) as $r \rightarrow \infty$. The main result is given in Proposition 10. We start directly from the form of the coefficients in the atmosphere $r \geq r_a$, on which the functions α_{ρ_0} , c_0 and γ are constant, and ρ_0 is a decaying exponential. These constants were introduced in (3.4) and (3.7), for $r \geq r_a$,

$$c_0(r) = c_a, \quad k_0(r) = k_a, \quad \gamma(r) = \gamma_a, \quad \alpha_{\rho_0}(r) = \alpha_{p_0}(r) = \alpha_{\gamma p_0}(r) = \alpha_a.$$

For $r \geq r_a$, the wavenumber k and parameter η defined in (4.40) become constant with

$$k^2(r) = k_a^2 := k_a^2 - \frac{\alpha_a^2}{4}, \quad \eta(r) = \eta_a := \frac{\alpha_a}{\gamma_a} (2 - \gamma_a). \quad (6.1)$$

We note that $k_a^2 = 0$ at the constant cut-off ω_t introduced in (4.53).

We make a preliminary observation: by substituting the exponential decay of ρ_0 in $r \geq r_a$ into (2.7), we can write Φ'_0 as a sum of a r^{-2} term plus analytic and exponentially decaying terms (a.e.d.t),

$$\Phi'_0 = \frac{G}{r^2} m + \text{a.e.d.t}, \quad \Phi''_0 = -2 \frac{G}{r^3} m + \text{a.e.d.t}, \quad (6.2a)$$

$$\text{with constant } m := 4\pi \int_0^{r_a} s^2 \rho_0(s) ds + 4\pi \rho_0(r_a) \frac{(\alpha_a r_a)^2 + 2r_a \alpha_a + 2}{(\alpha_a)^3}. \quad (6.2b)$$

We have the same type of expressions for E_{he} and its derivatives. These observations together with Proposition 8 immediately give the following result.

Proposition 9. Functions $\mathfrak{h}_\ell$, $\mathfrak{g}_\ell$ and V_ℓ are meromorphic functions in r ; in fact, they are analytic on $r \geq \mathfrak{r}_{\text{analytic}}$ where $\mathfrak{r}_{\text{analytic}} = r_a$ for $(\Gamma > 0, \ell > 0)$ or $\ell = 0$ and $\mathfrak{r}_{\text{analytic}} = r_\ell^*$ for $(\Gamma = 0, \ell > 0)$. They are rational functions in r modulo a.e.d.t on $r \geq r_a$. The acoustic cut-off frequency ω_c (4.49) has asymptotic expansion,

$$\omega_c^2 = \omega_t^2 - \frac{\eta_a \mathfrak{c}_a^2}{r} + \frac{2 \mathfrak{c}_a^2}{r^2} - \frac{2 G \mathfrak{m}}{r^3} + \text{a.e.d.t}, \quad r \geq r_a, \quad (6.3)$$

$$\text{where} \quad \omega_t := \frac{\alpha_a \mathfrak{c}_a}{2}.$$

The asymptotic expansion of $\mathfrak{g}_\ell$, $\mathfrak{h}_\ell$ and V_ℓ as $r \rightarrow \infty$ for any order r^{-n} can be obtained explicitly. In the following proposition, we obtain, for $\Gamma > 0$, the leading terms in their expansion modulo r^{-3} . They are given by $\mathfrak{h}_0$ (4.47a) and the following functions,

$$\mathfrak{g}_\ell^{\text{app}} := -k_a^2 + \frac{2(\alpha_a - \frac{\alpha_a}{\gamma_a})}{r} + \frac{\mu_\ell^2 - \frac{1}{4}}{r^2}, \quad \mathfrak{g}_\ell^G := \mathfrak{g}_\ell^{\text{app}} - \frac{2}{r^3} \frac{G \mathfrak{m}}{\mathfrak{c}_a^2}, \quad (6.4a)$$

$$Q_\ell := k_a^2 + \frac{\eta_a}{r} - \frac{\mu_\ell^2 - \frac{1}{4}}{r^2}, \quad Q_\ell^G := Q_\ell + \frac{2}{r^3} \frac{G \mathfrak{m}}{\mathfrak{c}_a^2}, \quad (6.4b)$$

We have defined the index μ_ℓ as,

$$\mu_\ell^2 - \frac{1}{4} = 2 + \ell(\ell + 1) + \frac{\ell(\ell + 1)}{k_a^2} \frac{\alpha_a}{\gamma_a} \left(\frac{\alpha_a}{\gamma_a} - \alpha_a \right). \quad (6.5)$$

Note that for $\ell = 0$, $\mu_0^2 - 1/4 = 2$. After obtaining the asymptotic with error of r^{-3} , by considering ℓ , ωr^{-1} as large and Γ_a as a small parameter, we also introduce the so-called ‘gravity-enriched’ approximation. This is due to observation by numerical experiments that terms with constant $G \mathfrak{m} / \mathfrak{c}_a^2$ have a significant contribution at level r^{-3} .

For $\ell > 0$, with the ℓ -dependent part of these coefficients coming from the factor F_ℓ^{-1} with F_ℓ defined in (4.37), the main ingredient of Proposition 10 is the following uniform bound for $|F_\ell|^{-1}$ on $r \geq r_a$ and $\ell \in \mathbb{N}^+$. It is stated below for $r > 0$,

$$\begin{aligned} |F_\ell(r)|^2 &= |(\text{Re } k^2) r^2 - r E_{\text{he}} - \ell(\ell + 1)|^2 + |\text{Im } k^2|^2 r^4 \\ \Rightarrow |F_\ell(r)|^2 &\geq \frac{1}{2} \left(|(\text{Re } k^2) r^2 - r E_{\text{he}} - \ell(\ell + 1)| + |\text{Im } k^2| r^2 \right)^2 \\ \Rightarrow \frac{1}{|F_\ell(r)|} &\leq \frac{\sqrt{2}}{|\text{Im } k^2| r^2} = \frac{\mathfrak{c}_0^2(r)}{\sqrt{2} \Gamma_a(r) \omega} r^{-2}, \quad r > 0, \quad \ell \in \mathbb{N}^+. \end{aligned} \quad (6.6)$$

Proposition 10. For $\ell = 0$, we have the following expansion with bounding constant independent of ω ,

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{g}_0^G + \text{a.e.d.t} = \mathfrak{g}_0^{\text{app}} + \mathcal{O}(r^{-3}), & r \geq r_a, \quad r \rightarrow \infty; \\ V_0 &= -Q_0^G + \text{a.e.d.t} = -Q_0 + \mathcal{O}(r^{-3}), & r \geq r_a, \quad r \rightarrow \infty. \end{aligned} \quad (6.7)$$

For $\ell > 0$, assuming $0 < \Gamma_a < 1$ and $\omega > \omega_0 > 0$, as $r \rightarrow \infty$, we have the following expansions uniformly with respect to $\ell > 0$,

$$\mathfrak{h}_\ell = \mathfrak{h}_0 + \frac{\ell(\ell+1)}{\Gamma_a \omega} \mathcal{O}(r^{-3}); \quad (6.8a)$$

$$\mathfrak{g}_\ell = \mathfrak{g}_\ell^G + \frac{\ell(\ell+1)}{\Gamma_a \omega} \mathcal{O}(r^{-3}) = \mathfrak{g}_\ell^{\text{app}} + \frac{\ell(\ell+1)}{\Gamma_a \omega} \mathcal{O}(r^{-3}); \quad (6.8b)$$

$$V_\ell = -Q_\ell^G + \frac{\ell(\ell+1)}{\Gamma_a^2} \mathcal{O}(r^{-3}) + \frac{[\ell(\ell+1)]^2}{(\Gamma_a \omega)^2} \mathcal{O}(r^{-6}) \quad (6.8c)$$

$$= -Q_\ell + \frac{\ell(\ell+1)}{\Gamma_a^2} \mathcal{O}(r^{-3}) + \frac{[\ell(\ell+1)]^2}{(\Gamma_a \omega)^2} \mathcal{O}(r^{-6}). \quad (6.8d)$$

Proof. The statement for $\ell = 0$ is straightforward using Remark 4, and the expansion of Φ_0 in (6.2). For $\ell > 0$, we use

$$\frac{1}{|\mathbf{F}_\ell(r)|} \leq \frac{\sqrt{2}}{|\text{Im } k_a^2| r^2} = \frac{\mathfrak{c}_a^2}{\sqrt{2} \Gamma_a \omega} r^{-2}, \quad r \geq r_a, \quad \ell \in \mathbb{N}^+. \quad (6.9)$$

We will also need the estimate of E_{he} . With $\mathfrak{d}$ dependent of ℓ , we have

$$\frac{E_{\text{he}}}{r} = \mathcal{O}(r^{-1}), \quad \left(\frac{E_{\text{he}}}{r} \right)' = \mathcal{O}(r^{-2}) \quad \Rightarrow \quad \left| \frac{-(E_{\text{he}}/r)'}{k_0^2 - \frac{E_{\text{he}}}{r}} \right| \leq \frac{\mathfrak{d}}{|k_0^2|} r^{-2}. \quad (6.10)$$

The second inequality follows from $E_{\text{he}} < 0$ by assumption (3.5d), thus $|k_a^2 - E_{\text{he}}| > |k_a^2|$.

- The estimate (6.8a) for $\mathfrak{h}_\ell$ follows from that for $\mathfrak{t}_\ell$. Since $(k_0^2)' = 0$, we write

$$\mathfrak{t}_\ell = \frac{1}{\mathbf{F}_\ell} \left(\frac{2}{r} + \frac{-(E_{\text{he}}/r)'}{k_0^2 - E_{\text{he}}} \right) \Rightarrow |\mathfrak{t}_\ell| \leq \frac{\mathfrak{d}}{\Gamma_a \omega} r^{-3}, \quad r \geq r_a, \ell \in \mathbb{N}^+. \quad (6.11)$$

The inequality is obtained by using (6.9) for $\mathbf{F}_\ell$ and (6.10).

- In the defining expression (4.39a) of $\mathfrak{g}^I$, with $(\frac{\alpha_{p0}}{\gamma})' = 0$, if we leave out $\frac{\Phi_0''}{\mathfrak{c}_a^2}$ in the numerator and $\frac{E_{\text{he}}}{r}$ in the denominator, we obtain,

$$\mathfrak{g}^I = 1 + \frac{1}{k_a^2} \frac{\alpha_a}{\gamma_a} \left(\frac{\alpha_a}{\gamma_a} - \alpha_a \right) + \frac{\mathcal{O}(r^{-2})}{|k_a^2|}; \quad (6.12a)$$

$$\Rightarrow \ell(\ell+1) \mathfrak{g}^I + 2 = \mu_\ell^2 - \frac{1}{4} + \frac{\ell(\ell+1)}{|k_a^2|} \mathcal{O}(r^{-2}). \quad (6.12b)$$

Combining with (6.11) for $\mathfrak{t}_\ell$ yields

$$2 + \ell(\ell+1) \left(\mathfrak{g}^I + \left(\frac{2}{r} - \frac{\alpha_a}{\gamma_a} \right) r^2 \mathfrak{t}_\ell \right) = \mu_\ell^2 - \frac{1}{4} + \frac{\ell(\ell+1)}{|k_a^2|} \mathcal{O}(r^{-2}) + \ell(\ell+1) \frac{\mathcal{O}(r^{-1})}{\Gamma_a \omega}.$$

After dividing by r^2 , this gives the bound for the last two terms of $\mathbf{g}_\ell$ in (4.41b). We thus obtain (6.8b) for $\mathbf{g}_\ell$, in noting $|k_a^2| \geq 2\Gamma_a\omega$.

- For the statement of V_ℓ , we need to bound $\mathbf{v}_\ell$, $\mathbf{t}_\ell^{\text{II}}$ and $\mathbf{t}'_\ell$. In writing $\mathbf{v}_\ell$ as

$$\mathbf{v}_\ell = \frac{1}{r} \left(1 + \frac{\ell(\ell+1)}{k_0^2 r^2 - \ell(\ell+1) - rE_{\text{he}}} \right), \quad (6.13)$$

we obtain the uniform bound on $r \geq r_a$,

$$\mathbf{v}_\ell = \frac{1}{r} + \frac{\ell(\ell+1)}{\Gamma_a\omega} \mathcal{O}(r^{-3}) \Rightarrow \eta \mathbf{v}_\ell = \frac{\eta_a}{r} + \frac{\ell(\ell+1)}{\Gamma_a\omega} \mathcal{O}(r^{-3}). \quad (6.14)$$

Using estimate (6.10), for $\mathbf{t}_\ell^{\text{II}}$ yields

$$|\mathbf{t}_\ell^{\text{II}}| \leq \frac{\mathfrak{d} \mathbf{c}_a^4}{\sqrt{2} |k_a^2| \Gamma_a \omega} r^{-4}, \quad r \geq r_a, \ell \in \mathbb{N}^+. \quad (6.15)$$

For the bound of $\mathbf{t}'_\ell$, since $(rE_{\text{he}})' = -\frac{\alpha_a}{\gamma_a} - \frac{G\mathbf{m}}{\mathbf{c}_a^2 r^2} + \text{a.e.d.t}$ and

$$\mathbf{t}'_\ell = -\frac{(k_a^2 2r - (rE_{\text{he}})')}{F_\ell^2} \left(\frac{2}{r} + \frac{(k_0^2 - E_{\text{he}}/r)'}{k_0^2 - \frac{E_{\text{he}}}{r}} \right) + \frac{1}{F_\ell} \left(-\frac{2}{r^2} + \left(\frac{(-E_{\text{he}}/r)'}{k_0^2 - \frac{E_{\text{he}}}{r}} \right)' \right), \quad (6.16)$$

with constants $\mathfrak{d}, \tilde{\mathfrak{d}}$ independent of ℓ , for $r \geq r_a$, we have

$$|\mathbf{t}'_\ell| \leq \left(\frac{\mathbf{c}_a^4 |k_a^2|}{2(\Gamma_a\omega)^2} + \frac{\mathbf{c}_a^2}{\sqrt{2}\Gamma_a\omega} \right) \frac{\mathfrak{d}}{r^4} \leq \left(\frac{1}{2\Gamma_a^2} + \frac{\frac{1}{\sqrt{2}} + 1}{\Gamma_a\omega} \right) \frac{\mathfrak{d} \mathbf{c}_a^2}{r^4} = \frac{\tilde{\mathfrak{d}}}{\Gamma_a^2} r^{-4}. \quad (6.17)$$

The second inequality uses $|k_a^2| \leq \mathbf{c}_a^{-2}(\omega^2 + 2\omega\Gamma_a)$. We also note that $\Gamma_a \ll \omega$ i.e. a small parameter, while ω is considered a large parameter.

Put together the bound of $\mathbf{g}^{\text{I}}, \mathbf{v}_\ell, \mathbf{t}'_\ell$ and $\mathbf{t}_\ell^{\text{II}}$, to bound $\ell(\ell+1)\mathbf{w}_\ell$ in expression (4.41c) of V_ℓ , (note that we have multiplied (6.15) and (6.17) by r^2),

$$\begin{aligned} \ell(\ell+1)\mathbf{w}_\ell &\stackrel{(6.15), (6.17)}{=} \ell(\ell+1)\mathbf{g}^{\text{I}} + 2r\mathbf{v}_\ell + \frac{\ell(\ell+1)}{\Gamma_a^2} \mathcal{O}(r^{-2}) \\ &\stackrel{(6.14)}{=} \ell(\ell+1)\mathbf{g}^{\text{I}} + 2 + \frac{\ell(\ell+1)}{\Gamma_a\omega} \mathcal{O}(r^{-2}) + \frac{\ell(\ell+1)}{\Gamma_a^2} \mathcal{O}(r^{-2}) \\ &\stackrel{(6.12b)}{=} \mu_\ell^2 - \frac{1}{4} + \frac{\ell(\ell+1)}{\Gamma_a^2} \mathcal{O}(r^{-2}). \end{aligned} \quad (6.18)$$

The final bound (6.11) for V_ℓ is finally obtained by using (6.11) for $\mathbf{t}_\ell$ to bound the last term on the right-hand-side of (4.41c), $[\ell(\ell+1)]^2 \mathbf{t}_\ell^2$,

$$V_\ell = -k_a^2 + \frac{\Phi_0''}{c_a^2} + \left(-\frac{\eta_a}{r} + \frac{\ell(\ell+1)}{\Gamma_a \omega} O(r^{-3}) \right) + \left(\frac{\mu_\ell^2 - \frac{1}{4}}{r^2} + \frac{\ell(\ell+1)}{\Gamma_a^2} O(r^{-4}) \right) + \frac{[\ell(\ell+1)]^2}{(\Gamma_a \omega)^2} O(r^{-6}). \quad (6.19)$$

The second term is obtained from the bound of $\eta \mathbf{v}_\ell$ given in (6.14).

The proof is complemented by the following observation. The constant G occurs at most at order r^{-3} in the term $\frac{\Phi_0''}{c_a^2}$ in expression (4.41b) for $\mathbf{g}_\ell$ and (4.41c) for V_ℓ , while its appearance in $\mathbf{t}_\ell$, $\mathbf{g}^I$ and $\mathbf{v}_\ell$ is of lower order (at most r^{-4}). To account for the influence of G , we add the leading term of $\frac{\Phi_0''}{c_a^2}$ to $\mathbf{g}_\ell^{\text{app}}$ and $-Q_\ell$ in (6.4). $\square$

Remark 7. When $\Gamma = 0$, we still obtain the same bound with bounding constants independent of ℓ , however for $r \geq \tau_{\omega, \ell, \delta}$ dependent on ℓ . Specifically, for $0 < \delta < 1$, we have to replace the bound (6.9) of F_ℓ^{-1} by

$$\frac{1}{F_\ell} \leq \frac{1}{\delta(k_a^2 r^2 - r E_{\text{he}})} < \frac{1}{\delta k_a^2 r^2}, \quad r \geq \tau_{\omega, \ell, \delta}, \quad \ell \in \mathbb{N}^+, \quad (6.20)$$

where τ is chosen as the unique root to $k_a^2 \tau^2 - \tau E_{\text{he}} = (1 + \frac{\delta}{1-\delta})\ell(\ell+1)$. For $\ell < \ell^*$ (defined in (5.7)), and small enough $\delta > 0$, we can choose $\tau = r_a$. $\triangle$

6.1. Numerical validation

We evaluate the performance of the asymptotic approximations by introducing the relative error ϵ defined, for $\bullet = \{\text{app}, G\}$ (see (6.4)), by

$$\epsilon_{\mathbf{g}_\ell^\bullet} = \frac{|\mathbf{g}_\ell - \mathbf{g}_\ell^\bullet|}{|\mathbf{g}_\ell|}, \quad \epsilon_{V_\ell^\bullet} = \frac{|V_\ell - V_\ell^\bullet|}{|V_\ell|}, \quad \epsilon_{h_0} = \frac{|h_\ell - h_0|}{|h_\ell|}. \quad (6.21)$$

The relative error is pictured in Fig. 5 at 7 mHz, for modes $\ell = 0, 10$ and 750, where we indicate the convergence orders.

As expected, the slope of the error using $\mathbf{g}_\ell^{\text{app}}$, h_0 and V_ℓ^{app} is of order 3 as predicted by (6.8). The relative error when using as approximations $\mathbf{g}_\ell^G$ and V_ℓ^G decreases substantially, with an improvement of 2 to 7 orders of magnitude. This highlights the importance of the gravity term in the approximation. For the mode $\ell = 0$, the relative error for $\mathbf{g}_\ell^G$ and V_ℓ^G is at machine precision, while it increases with higher modes, due to the term $\ell(\ell+1)$ in (6.8).

7. Existence results for the modal radial ODEs

In this section, with the necessary ingredients given in Sections 5 and 6, we can apply the techniques in [23] to construct the outgoing Green function $\mathcal{G}_\ell^+$, a solution to the conjugated equation,

$$\left(-\partial_r^2 + V_\ell \right) \mathcal{G}_\ell(r, s) = \delta(r - s), \quad (7.1)$$

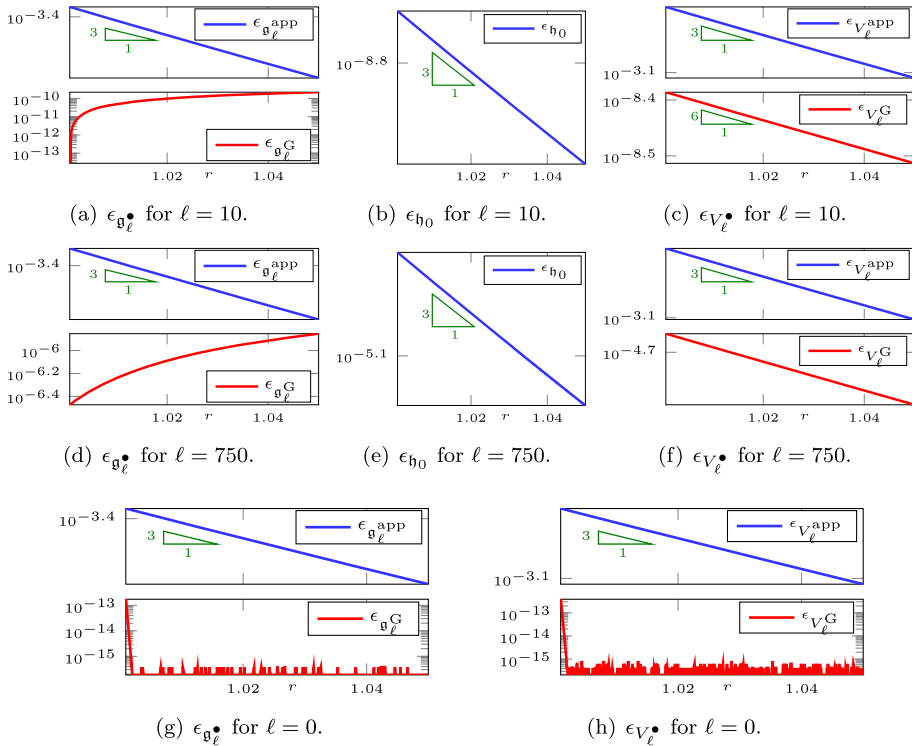


Fig. 5. Relative error $\epsilon_{\bullet}$ (6.21) of the asymptotic functions for the modes $\ell = 0$, $\ell = 10$ and $\ell = 750$ for frequency 7 mHz, with attenuation $\Gamma = 20$ μ Hz.

given by,

$$\mathcal{G}_{\ell}^{+}(r, s) := -\frac{H(s-r)\varphi_{\ell}(r)\varphi_{\ell}^{+}(s) + H(r-s)\varphi_{\ell}(s)\varphi_{\ell}^{+}(r)}{W_{\ell}^{+}}. \quad (7.2)$$

In the above expression, φ_{ℓ} and φ_{ℓ}^{+} are solutions to

$$(-\partial_r^2 + V_{\ell})u = 0, \quad r \in (0, \infty), \quad (7.3)$$

with φ_{ℓ} regular at $r = 0$ and a solution φ_{ℓ}^{+} outgoing at infinity. Their characterization and construction are the heart of this section. We have also denoted by H the Heaviside function and by W_{ℓ}^{+} the Wronskian W of $\varphi_{\ell}(s)$ and $\varphi_{\ell}^{+}(s)$, which for equation (7.1), is independent of r ,

$$W_{\ell}^{+} := W\{\varphi_{\ell}, \varphi_{\ell}^{+}\}. \quad (7.4)$$

The ‘outgoing’ kernel G_{ℓ}^{+} for the original ODE (4.24),

$$\left(\hat{q}(r)\partial_r^2 + q(r)\partial_r + \tilde{q}(r)\right)G_{\ell}(r, s) = \delta(r-s), \quad (7.5)$$

is constructed from $\mathcal{G}_\ell^+$ by relation,

$$G_\ell^+(r, s) := -\frac{\mathfrak{J}_\ell(r)}{\mathfrak{J}_\ell(s)} \frac{1}{\hat{q}_\ell(s)} \mathcal{G}_\ell^+(r, s), \quad (7.6)$$

with $\mathfrak{J}_\ell(r)$ and $\hat{q}_\ell$ given in (4.26a) and (4.42). Under the assumption that

$$\mathbf{W}_\ell^+ \neq 0, \quad (7.7)$$

the radial coefficient a_ℓ^m is given as a $\mathcal{C}^2((0, \infty))$ solution to (4.24) by

$$a_\ell^m(r) := \int_0^\infty G_\ell(r, s) \mathfrak{f}_\ell^m(s) ds, \quad \mathfrak{f}_\ell^m \text{ given by (4.25)}. \quad (7.8)$$

It remains to discuss the construction of φ_ℓ and φ_ℓ^+ . In this work, we employ only part of the extensive results of [23] to obtain the existence and uniqueness of these solutions without indicating the dependence on ℓ of the lower order terms in their expansions, see also Remark 10. To fit completely in the framework of [23], we work under the following assumption,

$$\Gamma > \delta' > 0, \quad 0 < \delta < \omega \neq \omega_t, \quad (7.9)$$

with ω_t defined in (4.53). Under this assumption, the set of singular points consists only of $r = 0$, and we stay away from the ‘energy threshold’ of scattering theory in [23], see also Remark 8.

7.1. Property of V_ℓ

We rewrite the conjugated operator in a form similar to the operator, $-\partial_r^2 - k^2 + V_{A-K} - \ell(\ell + 1)r^{-2}$, considered in [23, Eq. (3.1)], and show that our potential has the same property as theirs, $V_{A-K} - \ell(\ell + 1)r^{-2}$, in terms of regularity at $r = 0$ and decay rate at infinity. In scattering theory, decaying potentials are classified into long and short-range. A potential V is called short-range if $V = \mathcal{O}(r^\delta)$ as $r \rightarrow \infty$, for $\epsilon > 0$, with $\delta = -1 - \epsilon$, $\epsilon > 0$, and long-range if $\delta = -1 + \epsilon'$, $\epsilon' \geq 0$ cf. [23, Eq. (1.2)].

With κ_a^2 defined in (6.1), η in (4.40), for $\ell \in \mathbb{N}$, we rewrite $-\partial_r^2 + V_\ell$ as,

$$-\partial_r^2 + V_\ell = -\partial_r^2 - \kappa_a^2 + \mathcal{V} + \frac{\mathcal{W}_\ell}{r^2}, \quad (7.10)$$

$$\text{where } \mathcal{V} := \kappa_a^2 - k^2 + \frac{\Phi_0''}{\epsilon_0^2} - \frac{\eta(r)}{r+1}.$$

The expression for $\mathcal{W}_\ell = V_\ell + \kappa_a^2 - \mathcal{V}$ is, with the notation (4.37)–(4.40),

$$\mathcal{W}_0 = 2 - \eta \frac{r}{r+1}, \quad (7.11a)$$

$$\ell > 0, \quad \mathcal{W}_\ell = \ell(\ell + 1) \mathfrak{w}_\ell - r^2 \eta \left(\mathfrak{v}_\ell - \frac{1}{r+1} \right) + \frac{[\ell(\ell + 1)]^2}{4} (r \mathfrak{t}_\ell)^2. \quad (7.11b)$$

Under assumption (7.9), the wavenumber k_a is chosen with the same square branch $\sqrt{\cdot}$ in (2.10), thus $k_a \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Im} k_a > 0$. This also means $\mathcal{W}_\ell$ and $\mathcal{V}$ are complex-valued. In addition, potential $\mathcal{V}$ is independent of ℓ and is long-range with dominant behavior at infinity given by the Coulomb-type potential $\eta(r+1)^{-1}$. We recall that $\eta \in C^2([0, \infty))$ and is constant on $r \geq r_a$. From (6.6) and the proof of Proposition 10, there exist constants $\mathfrak{C} > 0$ and $\mathfrak{D} > 0$ independent of σ^2 and ℓ , so that

$$v_\ell \in C^2([0, \infty)), \quad \frac{v_\ell}{\ell(\ell+1)} \leq \mathfrak{C} (1+r^2)^{-1/2}, \quad r \geq 0; \quad (7.12a)$$

$$w_\ell \in C^2([0, \infty)), \quad w_\ell \leq \mathfrak{D}, \quad \lim_{r \rightarrow 0} w_\ell = 1. \quad (7.12b)$$

Potential $\mathcal{W}_\ell/r^2$ is thus short-range and satisfies

$$(\mathcal{W}_\ell(r) - \ell(\ell+1)) r^{-1} \in C^2([0, \infty)). \quad (7.13)$$

With $\Gamma > 0$, the only singularity of V_ℓ comes from the terms $\frac{\mathcal{W}_\ell}{r^2}$ which are regular singular with indicial exponents $\kappa_\ell^\pm$, cf. (5.20) in Proposition 8. We are now ready to apply the procedure in [23] and show the existence and uniqueness of the solutions φ_ℓ and φ_ℓ^+ , respectively in Subsections 7.2 and 7.3.

Remark 8. While the literature on long-range scattering of Schrödinger equation is vast, we have chosen the work of [23], since its hypothesis allows for the (weak) singularity at $r = 0$ and the complex-valued V_ℓ . Without these two constraints, one can also appeal to other works for regular real-valued long-range potentials such as [28] in 1D, or [29] for complex regular potentials and [30] for real-valued singular long-range potentials in general dimension. The work of Agmon–Klein [23] is also used for the scalar wave equation in helioseismology in [31,32,15]. See also the introduction of [28] for a discussion on long-range theory for real-valued potentials in 1D. We also note that our construction is away from the threshold energy, i.e. for $k_a \in \mathbb{C} \setminus \{0\}$, which is equivalent with ω away from the acoustic cut-off ω_l . Note that even if our potential is real when $\Gamma = 0$, we cannot use [33] because of the singularity at 0 and $r_\star$. $\triangle$

7.2. Regular solution at $r = 0$

Following the proof of Theorem 3.3 in [23], or Sections 2 of [22], there exists a unique solution φ_ℓ to (7.3) which satisfies

$$\varphi_\ell(r) = r^{\kappa_\ell^+} (1 + o(1)), \quad \text{as } r \rightarrow 0. \quad (7.14)$$

The existence of φ_ℓ is obtained by solving the Volterra integral equation (VIE), following [23, Eqs. (3.18)–(3.20)],

$$\varphi_\ell = r^{\kappa_\ell^+} + \int_0^r K_\ell(r, s) F_\ell(s) \varphi_\ell(s) ds, \quad (7.15)$$

$$\text{with } K_\ell(r, s) = \frac{r^{\kappa_\ell^+} s^{\kappa_\ell^-} - s^{\kappa_\ell^+} r^{\kappa_\ell^-}}{(\kappa_\ell^+ - \kappa_\ell^-) s^{\kappa_\ell^+ + \kappa_\ell^- - 1}} = r^{\kappa_\ell^+} \frac{s^{1-\kappa_\ell^+}}{\kappa_\ell^+ - \kappa_\ell^-} + r^{\kappa_\ell^-} \frac{s^{1-\kappa_\ell^-}}{\kappa_\ell^+ - \kappa_\ell^-},$$

$$\text{and } F_0 = -k_a^2 + \mathcal{V}, \quad F_\ell(s) = -k_a^2 + \mathcal{V} + \frac{\mathcal{W}_\ell - \ell(\ell+1)}{r^2}, \quad \ell > 0.$$

From (7.13), we have $s\mathcal{F}_\ell(s) \in \mathcal{C}^2[0, r]$, and is thus integrable on $[0, r]$. For this reason, (7.15) can be solved by standard procedure of iteration, cf. Section 2 of [22] for a detailed discussion. Note that the term $r^{\kappa_\ell^+}$ serves as first approximation in the iteration.

Uniqueness is obtained by solving (7.15) with $r^{\kappa_\ell^-}$ as the first approximation, which shows the existence of a linearly independent solution $\varphi_\ell^{\text{sing}}$ satisfying

$$\varphi_\ell^{\text{sing}}(r) = r^{\kappa_\ell^-} (1 + o(1)), \quad \text{as } r \rightarrow 0. \quad (7.16)$$

This is mentioned briefly in [23, p. 357] but one can refer to Section 3 of [22] for a detailed exposition.

7.3. Outgoing solution at infinity

Following the same procedure of Theorem 3.1 and the discussion of [23, p. 355] (which used Theorem 2.2 therein), we can construct a global phase ψ as a solution of the eikonal equation (cf. [23, Eq. (1.10)]),

$$|\nabla \psi|^2 = k_a^2 - \mathcal{V}(r), \quad (7.17)$$

$$\text{satisfying } \psi(r, k_a) = r k_a (1 + o(1)), \quad \text{as } r \rightarrow \infty.$$

A choice of the phase function $\psi(r, k_a)$ can be, for $r_0 > 0$,

$$\psi(r, k_a) = \int_{r_0}^r \sqrt{k_a^2 - \mathcal{V}} ds = \int_{r_0}^r \sqrt{k^2(s) - \frac{\Phi_0''(s)}{c_0^2(s)} + \frac{\eta(s)}{s+1}} ds. \quad (7.18)$$

Under assumption (7.9), and for model S-AtmOI, the real part of the quantity in the square root is sign definite (> 0 for $\omega > \omega_c$, and < 0 for $\omega < \omega_c$, both with $r_0 = r_a$).

By Theorem 2.2 of [23], there exists a unique solution φ_ℓ^+ to (7.3) satisfying

$$\varphi_\ell^+(r) = e^{i\psi(r, k_a)} (1 + o(1)), \quad \text{as } r \rightarrow \infty. \quad (7.19)$$

The existence of φ_ℓ^+ is obtained from solving by iteration the VIE, cf. [23, Eqs. (2.20) and (2.21)],

$$\varphi_{\ell}^{+} = T^{+}(r) - \int_r^{\infty} \mathbf{K}(r, s) \mathfrak{F}_{\ell}(s) \varphi_{\ell}^{+}(s) \, ds,$$

$$\text{with } T^{\pm}(r) := \left(\frac{k_a}{\psi'} \right)^{1/2} e^{\pm i\psi}, \quad \mathfrak{F}_{\ell}(s) := \frac{1}{2} \{\psi, s\} + \frac{\mathcal{W}_{\ell}(s)}{s^2}, \quad (7.20)$$

$$\text{and } \mathbf{K}(r, s) := \frac{T^{+}(r) T^{-}(s) - T^{+}(s) T^{-}(r)}{2i k_a}.$$

The Schwarzian derivative $\{\psi, r\}$ is defined as, cf. [23, Eq. 2.19],

$$\{\psi, r\} := \frac{\psi'''}{\psi'} - \frac{3}{2} \left(\frac{\psi''}{\psi'} \right)^2 = -\frac{1}{2} \frac{\mathcal{V}''}{k_a^2 - \mathcal{V}} - \frac{5}{4} \left(\frac{\mathcal{V}'}{k_a^2 - \mathcal{V}} \right)^2. \quad (7.21)$$

The approximate solution $T^{\pm}$ satisfies the comparison equation, $(-\partial_r^2 - k_a^2 + \mathcal{V} - \{\psi, r\})T^{\pm} = 0$. To show that the above VIE can be solved by iteration, cf. [23, Eq. (2.27)],

$$\varphi_{\ell}^{+} = \sum_{j=0}^{\infty} \varphi_{\ell; j}^{+}, \quad \text{where}$$

$$\varphi_{\ell; 0}^{+} = T^{+}, \quad \varphi_{\ell; j+1}^{+} := - \int_r^{\infty} \mathbf{K}(r, s) \mathfrak{F}_{\ell}(s) \varphi_{\ell; j}^{+}(s) \, ds, \quad j \in \mathbb{N},$$

one needs to show the integral defining $\varphi_{\ell; j}^{+}$ and the series defining φ_{ℓ}^{+} converge. These follow firstly from the decay of $\mathcal{V}^{(j)} = \mathcal{O}(r^{-1-j})$ and that of $\mathcal{W}_{\ell}/r^2 = \mathcal{O}(r^{-2})$, which yields

$$|\mathfrak{F}_{\ell}| \leq C(1+r)^{-2}, \quad (7.22)$$

and thus the integrability of $\mathfrak{F}_{\ell}$ on $[r, \infty)$. Secondly, for $s > r$ and $\text{Im } k_a > 0$,

$$|e^{i(\varphi(s) - \varphi(r))}| \leq C, \quad \left(\frac{k_a}{\psi'} \right)^{1/2} = \left(1 - \frac{\mathcal{V}}{k_a^2} \right)^{-1/2} = 1 + o(1), \quad r \rightarrow \infty,$$

$$\implies |\mathbf{K}(r, s)| \leq C \left| \frac{T^{+}(r)}{T^{+}(s)} \right|.$$

By induction, one can show that, cf. [23, Eqs. (2.28) and (2.29)],

$$|\varphi_{\ell; j}^{+}(r)| \leq |T^{+}(r)| \frac{(\mathfrak{L}(r))^j}{j!},$$

$$\text{where } \mathfrak{L}(r) := \int_r^{\infty} C |F(s)| \, ds = o(1), \quad \text{as } r \rightarrow \infty.$$

The uniqueness of the solution of (7.19) is obtained by solving (7.20) with T^- as approximate solution (or first iteration) to obtain a linearly independent solution φ_ℓ^- which (non-unique for now) is characterized by

$$\varphi_\ell^-(r) = e^{-i\psi(r, k_a)} (1 + o(1)), \quad \text{as } r \rightarrow \infty. \quad (7.23)$$

Remark 9. The above construction works with a more general decomposition of V_ℓ . Specifically, one writes $V_\ell = q_L + q_\ell^S$ with short-range $q_\ell^S := V_\ell - q_L$, and potential $q_L \in C^2([0, \infty))$, independent of ℓ and containing the long range part of V_ℓ . In (7.10), we have chosen q_ℓ as $\mathcal{V}$; however, it can simply contain $\frac{\eta_a}{r+1}$, and to this, one can add (or not) terms that are ℓ -independent, $\mathcal{C}([0, \infty))$ and short-range. The phase function ψ is now $\psi = \int_{r_0}^r \sqrt{k_a^2 - q_L} ds$ (assuming q_L small enough). One then solves (7.20) with $\mathfrak{F}_\ell(r) = \frac{1}{2}\{\psi, r\} + q_\ell^S(r)$ and $K, T^\pm$ defined as before with this new phase ψ . $\triangle$

Remark 10. Our construction is obtained for each fixed ℓ . In [23], Agmon and Klein have also obtained uniform results for $r \in [0, \infty)$ as $\ell \rightarrow \infty$, cf. Theorem 5.6 there. Control in ℓ is needed to study the convergence of (4.4) so that a solution ξ can be obtained from the modal solutions. Unfortunately, we cannot apply directly the results of [23], because the dominant term in the potential of [23] is only $\ell(\ell+1)/r^2$ as $\ell \rightarrow \infty$, while ours has an additional term (last term of $\mathcal{W}_\ell$ in (7.11b)) of order $[\ell(\ell+1)]^2$. On a more positive note, since this new term is of order r^{-6} as $r \rightarrow \infty$, and is regular at $r=0$, the same analysis in [23] can probably be modified to obtain uniform results, however with an expected larger growth in ℓ . $\triangle$

8. Conclusion

We have studied the Galbrun's equation under spherical symmetry and obtained outgoing modal solutions which are the coefficients of the vector displacement in the vectorial spherical harmonics basis. The construction of the modal solutions is justified via a global analysis which shows that the modal ODE fits in the theoretical framework of [23]. In particular, the coefficients of the ODE are C^2 except at a finite number of regular singular points, and have long-range growth at infinity. Our analysis also provides an efficient framework to compute the complete Green's kernel from only two simulations using (7.2) and leads way to the efficient numerical resolution of the Galbrun's equation under spherical symmetry.

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Appendix A. Proof of Proposition 3

Proof. It suffices to derive the expression for the coefficient $r^2 C_{22} q_\ell$ and $r^2 C_{22} \tilde{q}_\ell$. We will need the following identities involving C_{22} given in (4.6),

$$C'_{22} = -(k_0^2)' - 2 \frac{\ell(\ell+1)}{r^3} - \frac{E_{\text{he}}}{r^2} + \frac{E'_{\text{he}}}{r}; \quad (\text{A.1a})$$

$$-\frac{C'_{22}}{C_{22}} - \frac{2}{r} = \frac{(k_0^2)' r^2 + 2k_0^2 r - (E_{\text{he}} + rE'_{\text{he}})}{-k_0^2 r^2 + \ell(\ell+1) + rE_{\text{he}}}. \quad (\text{A.1b})$$

Second coefficient. From the expression given in (4.26b), we have

$$\begin{aligned} r^2 C_{22} q_\ell(r) &= C_{22}(\alpha_{\gamma p_0} r^2 - 2r) + rC_{12} - \ell(\ell+1) \frac{C'_{22}}{C_{22}} + \ell(\ell+1) \left(\frac{1}{r} - \frac{\alpha_{p_0}}{\gamma} \right) \\ &= C_{22}(\alpha_{\gamma p_0} r^2 - 2r) - \ell(\ell+1) \alpha_{\gamma p_0} - \ell(\ell+1) \frac{C'_{22}}{C_{22}}. \end{aligned} \quad (\text{A.2})$$

The first two terms of (A.2) are rewritten by replacing C_{22} with (4.6),

$$C_{22}(\alpha_{\gamma p_0} r^2 - 2r) - \ell(\ell+1) \alpha_{\gamma p_0} \quad (\text{A.3a})$$

$$= \left(-k_0^2 \alpha_{\gamma p_0} r^2 + rE_{\text{he}} \alpha_{\gamma p_0} \right) + \left(2k_0^2 r - 2 \frac{\ell(\ell+1)}{r} - 2E_{\text{he}} \right) \quad (\text{A.3b})$$

$$= (\alpha_{\gamma p_0} r - 2) \left(-k_0^2 r + E_{\text{he}} \right) - 2 \frac{\ell(\ell+1)}{r}. \quad (\text{A.3c})$$

We substitute this back into (A.2), and use (A.1b) to get rid of $2 \frac{\ell(\ell+1)}{r}$,

$$r^2 C_{22} q_\ell(r) = (\alpha_{\gamma p_0} r - 2) \left(-k_0^2 r + E_{\text{he}} \right) + \ell(\ell+1) \frac{(k_0^2)' r^2 + 2k_0^2 r - (E_{\text{he}} + rE'_{\text{he}})}{-k_0^2 r^2 + \ell(\ell+1) + rE_{\text{he}}}.$$

After rearrangement, and division by k_0^2 , we obtain (4.30).

Third coefficient. From (4.26c), we write $r^2 C_{22} \tilde{q}_\ell(r) = I_1 + I_2$ with,

$$I_1 := r^2 C_{22} C_{11} + 2C_{12} - \frac{\alpha_{p_0}}{\gamma} r C_{12} \quad (\text{A.4a})$$

$$I_2 := -\ell(\ell+1) \left(\frac{\alpha_{p_0}}{\gamma} \right)' + \ell(\ell+1) \left(-\frac{2}{r^2} - \left(\frac{2}{r} - \frac{\alpha_{p_0}}{\gamma} \right) \left(\frac{1}{r} + \frac{C'_{22}}{C_{22}} \right) \right). \quad (\text{A.4b})$$

The term I_1 is rewritten by substituting in the definition of C_{11} , C_{12} and C_{22} in (4.6),

$$\begin{aligned} I_1 &= \left(-k_0^2 r^2 + \ell(\ell+1) + rE_{\text{he}} \right) \left(-k_0^2 + \frac{\Phi_0''}{c_0^2} \right) \\ &\quad + 2 \left(k_0^2 r^2 - rE_{\text{he}} \right) \left(-\frac{\alpha_{\gamma p_0}}{r} + \frac{\alpha_{p_0}}{r\gamma} - \frac{1}{r^2} \right) - \frac{\ell(\ell+1) \alpha_{p_0}}{\gamma} \left(-\alpha_{\gamma p_0} + \frac{\alpha_{p_0}}{\gamma} - \frac{1}{r} \right). \end{aligned}$$

We next consider I_2 . We rewrite (A.1b) as,

$$-\frac{C'_{22}}{C_{22}} - \frac{1}{r} = \frac{1}{r} + \frac{(k_0^2)' r^2 + 2k_0^2 r - (E_{\text{he}} + rE'_{\text{he}})}{-k_0^2 r^2 + \ell(\ell+1) + rE_{\text{he}}}, \quad (\text{A.5})$$

and thus the second term of I_2 (without the factor $\ell(\ell+1)$) as,

$$\begin{aligned}
& -\frac{2}{r^2} - \left(\frac{2}{r} - \frac{\alpha_{p0}}{\gamma} \right) \left(\frac{1}{r} + \frac{C'_{22}}{C_{22}} \right) \\
& = \left(\frac{2}{r} - \frac{\alpha_{p0}}{\gamma} \right) \frac{(k_0^2)' r^2 + 2k_0^2 r - (E_{\text{he}} + rE'_{\text{he}})}{-k_0^2 r^2 + \ell(\ell+1) + rE_{\text{he}}} - \frac{1}{r} \frac{\alpha_{p0}}{\gamma}.
\end{aligned} \tag{A.6}$$

Put together the above results of I_1 and I_2 , we obtain

$$\begin{aligned}
r^2 C_{22} \tilde{q}_\ell(r) &= \left(k_0^2 r^2 - rE_{\text{he}} \right) \left(k_0^2 - \frac{\Phi_0''}{c_0^2} \right) + 2 \left(k_0^2 r - E_{\text{he}} \right) \left(-\alpha_{\gamma p_0} + \frac{\alpha_{p_0}}{\gamma} - \frac{1}{r} \right) \\
&+ \ell(\ell+1) \left(-k_0^2 + \frac{\Phi_0''}{c_0^2} - \left(\frac{\alpha_{p_0}}{\gamma} \right)' \right) - \ell(\ell+1) \frac{\alpha_{p_0}}{\gamma} \left(-\alpha_{\gamma p_0} + \frac{\alpha_{p_0}}{\gamma} \right) \\
&+ \ell(\ell+1) \left(2 - \frac{\alpha_{p_0}}{\gamma} r \right) \frac{(k_0^2)' r + 2k_0^2 - (\frac{E_{\text{he}}}{r} + E'_{\text{he}})}{-k_0^2 r^2 + \ell(\ell+1) + rE_{\text{he}}}.
\end{aligned}$$

After rearrangement, and division by k_0^2 , we obtain (4.31). $\square$

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