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On the regularity of the maximal function of a BV function

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Abstract

We show that the non-centered maximal function of a BV function is quasicontinuous. We also show that *if* the non-centered maximal functions of an SBV function is a BV function, then it is in fact a Sobolev function. Using a recent result of Weigt [12], we are in particular able to show that the non-centered maximal function of a set of finite perimeter is a Sobolev function.

MSC: 42B25; 26B30

Keywords: Function of bounded variation; Sobolev function; Non-centered maximal function; Quasicontinuity; Absolute continuity

1. Introduction

An open problem that has attracted significant attention in the past two decades is the so-called $W^{1,1}$-problem: is the Hardy-Littlewood maximal function of a Sobolev function $u \in W^{1,1}(\mathbb{R}^d)$ also (locally) in the $W^{1,1}$-class? Typically also a bound $\|\nabla Mu\|_{L^1(\mathbb{R}^d)} \le C\|\nabla u\|_{L^1(\mathbb{R}^d)}$ is expected to hold. In the case $1 < p < \infty$ the analogous result is known to hold, as first shown by Kinnunen [6]. The same is true for the non-centered maximal function, defined by

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$$Mu(x) := \sup_{x \in B(z,r)} \fint_{B(z,r)} |u|\,dy, \quad x \in \mathbb{R}^d.$$

We will work with this non-centered version that tends to have better regularity than the ordinary Hardy-Littlewood maximal function, in which only balls centered at x are considered. Tanaka [11] gave a positive answer to the $W^{1,1}$-problem in the case $d = 1$. Generalizing this to higher dimensions has received significant attention, but results have been achieved only in very special cases. Luiro [10] gave a positive answer to the problem in the case of radial functions, whereas Aldaz and Pérez Lázaro [2] did the same for *block-decreasing* functions.

In fact, in [2] the authors considered functions $u \in \mathrm{BV}(\mathbb{R}^d)$ rather than just $u \in W^{1,1}(\mathbb{R}^d)$. And apart from the Sobolev regularity, one can consider other continuity properties of the maximal function, but these are also not well understood; see [1] for some positive results when $d = 1$ as well as counterexamples, and [2] for continuity results for block-decreasing BV functions in general dimensions.

In the current paper we show that in general dimensions and for general $u \in \mathrm{BV}(\mathbb{R}^d)$, a very natural continuity property, namely quasicontinuity, can be proven for the maximal function. All definitions will be given in Section 2. Most of the time we will consider the maximal function $M_\Omega u$ where one considers balls contained in an open set $\Omega \subset \mathbb{R}^d$. After proving some preliminary results in Section 3, we prove the following quasicontinuity result in Section 4.

Theorem 1.1. *Let $u \in \mathrm{BV}(\Omega)$. Then $M_\Omega u$ is 1-quasicontinuous.*

Utilizing this result, in Section 5 we study continuity properties of $M_\Omega u$ on lines parallel to coordinate axes. Then in Section 6 we examine absolute continuity on lines and membership in the Sobolev class, proving the following theorem. We say that a BV function is a special function of bounded variation, or SBV function, if the variation measure has no Cantor part.

Theorem 1.2. *Let $u \in \mathrm{SBV}(\mathbb{R}^d)$. If $Mu \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$, then $Mu \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$.*

In a potential breakthrough toward a solution to the $W^{1,1}$-problem, Weigt [12] has shown very recently that for a set of finite perimeter $E \subset \Omega$, we have $M_\Omega \mathbb{1}_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ such that $|DM_\Omega \mathbb{1}_E|(\Omega)$ is at most a constant times $|D\mathbb{1}_E|(\Omega)$. We can utilize this result and go a step further to the desired Sobolev regularity at least in the global case $\Omega = \mathbb{R}^d$, as follows.

Theorem 1.3. *Let $E \subset \mathbb{R}^d$ be a set of finite perimeter. Then $M\mathbb{1}_E \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ with $\|\nabla M\mathbb{1}_E\|_{L^1(\mathbb{R}^d)} \le C_d |D\mathbb{1}_E|(\mathbb{R}^d)$, where C_d only depends on the dimension d.*

Finally, in Sections 7 and 8 we study formulas for the gradient of the maximal function, as well as some further properties in the case $d = 1$.

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2. Notation and definitions

2.1. Basic notation

We will always work in the Euclidean space $\mathbb{R}^d$, $d \geq 1$. We denote the d-dimensional Lebesgue measure by $\mathcal{L}^d$ and the s-dimensional Hausdorff measure by $\mathcal{H}^s$, $s \geq 0$. We denote the characteristic function of a set $E \subset \mathbb{R}^d$ by $\mathbb{1}_E$.

We write $B(x,r)$ for an open ball in $\mathbb{R}^d$ with center x and radius r, that is, $\{y \in \mathbb{R}^d : |y-x| < r\}$, and we write $\mathbb{S}^{d-1}$ for the unit sphere in $\mathbb{R}^d$, that is, $\{y \in \mathbb{R}^d : |y| = 1\}$. We always work with the Euclidean norm $|\cdot|$. When we consider closed balls, we always specify this by the bar $\overline{B}(x,r)$.

If a function u is integrable on some measurable set $D \subset \mathbb{R}^d$ of positive and finite Lebesgue measure, we write

$$\fint_D u(y)\,dy := \frac{1}{\mathcal{L}^d(D)} \int_D u(y)\,dy$$

for its mean value on D.

We will always denote by $\Omega \subset \mathbb{R}^d$ a nonempty open set. The Sobolev space $W^{1,1}(\Omega)$ consists of functions $u \in L^1(\Omega)$ whose first weak partial derivatives $D_k u$, $k = 1, \ldots, d$, belong to $L^1(\Omega)$.

The Sobolev 1-capacity of a set $A \subset \mathbb{R}^d$ is defined by

$$\mathrm{Cap}_1(A) := \inf \int_{\mathbb{R}^n} \big(|u| + |Du|\big)\,dy,$$

where the infimum is taken over Sobolev functions $u \in W^{1,1}(\mathbb{R}^d)$ satisfying $u \geq 1$ in a neighborhood of A. The Sobolev 1-capacity is countably subadditive. Using a cutoff function we find that for every ball $B(x,r)$ with $0 < r \leq 1$, we have

$$\mathrm{Cap}_1(B(x,r)) \leq C_0 r^{d-1} \tag{2.1}$$

for a constant C_0 depending only on d.

We say that a function v on Ω (generally we understand functions to take values in $[-\infty,\infty]$) is 1-quasicontinuous if for every $\varepsilon > 0$ there exists an open set $G \subset \Omega$ such that $\mathrm{Cap}_1(G) < \varepsilon$ and $v|_{\Omega\setminus G}$ is finite and continuous.

By e.g. [5, Theorem 4.3, Theorem 5.1] we know that for any $A \subset \mathbb{R}^d$,

$$\mathrm{Cap}_1(A) = 0 \quad \text{if and only if} \quad \mathcal{H}^{d-1}(A) = 0. \tag{2.2}$$

The non-centered maximal function of a measurable function u on Ω is defined by

$$M_\Omega u(x) := \sup_{x \in B(z,r) \subset \Omega} \fint_{B(z,r)} |u|\,dy, \quad x \in \Omega.$$

Sometimes we do not mention Ω and then it is understood that $\Omega = \mathbb{R}^d$, so that $Mu := M_{\mathbb{R}^d} u$.

For $\ell \in \mathbb{N}$, we denote by $\mathcal{M}(\Omega;\mathbb{R}^\ell)$ the Banach space of vector-valued Radon measures μ, equipped with the *total variation norm* $|\mu|(\Omega) < \infty$, which is defined relative to the Euclidean norm on $\mathbb{R}^\ell$. By the Riesz representation theorem, $\mathcal{M}(\Omega;\mathbb{R}^\ell)$ can be identified with the dual space of $C_0(\Omega;\mathbb{R}^\ell)$ through the duality pairing $\langle \phi, \mu\rangle := \int_\Omega \phi \cdot d\mu := \sum_{j=1}^{\ell} \int_\Omega \phi_j \, d\mu_j$. Thus weak* convergence $\mu_i \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^\ell)$ means $\langle \phi, \mu_i\rangle \to \langle \phi, \mu\rangle$ for all $\phi \in C_0(\Omega;\mathbb{R}^\ell)$.

We denote the restriction of a Radon measure μ to a Borel set $A \subset \mathbb{R}^d$ by $\mu \llcorner A$, that is,

$$\mu \llcorner A(H) := \mu(A \cap H), \quad H \subset \mathbb{R}^d \ \ \mu\text{-measurable}.$$

For a vector-valued Radon measure $\gamma \in \mathcal{M}(\Omega;\mathbb{R}^\ell)$ and a positive Radon measure μ, we can write the Radon-Nikodym decomposition

$$\gamma = \gamma^a + \gamma^s = \frac{d\gamma}{d\mu}\, d\mu + \gamma^s$$

of γ with respect to μ, where $\frac{d\gamma}{d\mu} \in L^1(\Omega, \mu; \mathbb{R}^\ell)$.

2.2. *Functions of bounded variation*

The theory of BV functions presented here can be found e.g. in [3], and we give precise references only for a few key facts. A function $u \in L^1(\Omega)$ is a function of bounded variation, denoted $u \in \mathrm{BV}(\Omega)$, if its distributional derivative is an $\mathbb{R}^d$-valued Radon measure with finite total variation. This means that there exists a (unique) Radon measure $Du \in \mathcal{M}(\Omega;\mathbb{R}^d)$ such that for all $\varphi \in C_c^1(\Omega)$, the integration-by-parts formula

$$\int_\Omega u \frac{\partial \varphi}{\partial y_j}\, dy = -\int_\Omega \varphi \, d(Du)_j, \quad j = 1, \ldots, d$$

holds.

If we do not know a priori that a function $u \in L^1_{\mathrm{loc}}(\Omega)$ is a BV function, we consider

$$\mathrm{Var}(u,\Omega) := \sup \left\{ \int_\Omega u \operatorname{div} \varphi \, dy, \ \varphi \in C_c^1(\Omega), \ |\varphi| \le 1 \right\}. \tag{2.3}$$

If $\mathrm{Var}(u,\Omega) < \infty$, then the Radon measure Du exists and $\mathrm{Var}(u,\Omega) = |Du|(\Omega)$ by the Riesz representation theorem, and $u \in \mathrm{BV}(\Omega)$ provided that $u \in L^1(\Omega)$. If $E \subset \mathbb{R}^d$ with $\mathrm{Var}(\mathbb{1}_E, \mathbb{R}^d) < \infty$, we say that E is a set of finite perimeter.

A fact that we will use many times is that if $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, then also $|u| \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ with $|D|u||(\Omega) \le |Du|(\Omega)$.

Let u be a function on Ω. We say that $x \in \Omega$ is a Lebesgue point of u if

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y) - \widetilde{u}(x)|\, dy = 0 \tag{2.4}$$

for some $\widetilde{u}(x) \in \mathbb{R}$. We denote by $S_u \subset \Omega$ the set where this condition fails and call it the *approximate discontinuity set*.

Given $\nu \in \mathbb{S}^{d-1}$, we define the half-balls

$$B_\nu^+(x,r) := \{y \in B(x,r) \colon \langle y-x, \nu\rangle > 0\},$$
$$B_\nu^-(x,r) := \{y \in B(x,r) \colon \langle y-x, \nu\rangle < 0\}.$$

We say that $x \in \Omega$ is an approximate jump point of u if there exist $\nu \in \mathbb{S}^{d-1}$ and distinct numbers $u^+(x), u^-(x) \in \mathbb{R}$ such that

$$\lim_{r\to 0} \fint_{B_\nu^+(x,r)} |u(y) - u^+(x)|\, dy = 0 \tag{2.5}$$

and

$$\lim_{r\to 0} \fint_{B_\nu^-(x,r)} |u(y) - u^-(x)|\, dy = 0.$$

The set of all approximate jump points is denoted by J_u. We have that $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$, see [3, Theorem 3.78].

The lower and upper approximate limits of a function u are defined respectively by

$$u^\wedge(x) := \sup\left\{t \in \mathbb{R} \colon \lim_{r\to 0} \frac{\mathcal{L}^d(B(x,r)\cap\{u<t\})}{\mathcal{L}^d(B(x,r))} = 0\right\}$$

and

$$u^\vee(x) := \inf\left\{t \in \mathbb{R} \colon \lim_{r\to 0} \frac{\mathcal{L}^d(B(x,r)\cap\{u>t\})}{\mathcal{L}^d(B(x,r))} = 0\right\}.$$

Note that for all $x \in \Omega \setminus S_u$, we have $\widetilde{u}(x) = u^\wedge(x) = u^\vee(x)$. Also, for all $x \in J_u$, we have $u^\wedge(x) = \min\{u^-(x), u^+(x)\}$ and $u^\vee(x) = \max\{u^-(x), u^+(x)\}$.

We write the Radon-Nikodym decomposition of the variation measure of u into the absolutely continuous and singular parts as $Du = D^a u + D^s u$. Furthermore, we define the Cantor and jump parts of Du by

$$D^c u := D^s u \llcorner (\Omega \setminus S_u), \qquad D^j u := D^s u \llcorner J_u. \tag{2.6}$$

Since $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$ and $|Du|$ vanishes on $\mathcal{H}^{d-1}$-negligible sets, we get the decomposition

$$Du = D^a u + D^c u + D^j u.$$

We say that $u \in \mathrm{BV}(\Omega)$ is a special function of bounded variation, and denote $u \in \mathrm{SBV}(\Omega)$, if $|D^c u|(\Omega) = 0$.

2.3. One-dimensional sections of BV *functions*

For basic results in the one-dimensional case $d = 1$ (with slightly different notation from ours), see [3, Section 3.2]. In this setting, given an open set $\Omega \subset \mathbb{R}$ and $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, we have $J_u = S_u$, J_u is at most countable, and $Du(\{x\}) = 0$ for every $x \in \Omega \setminus J_u$. For every $x, \widetilde{x} \in \Omega$ in a connected component of Ω, we have

$$|u^\vee(\widetilde{x}) - u^\vee(x)| \le |Du|([x, \widetilde{x}]). \tag{2.7}$$

Thus at every point outside S_u, the pointwise representative $u^\wedge = u^\vee = \widetilde{u}$ is continuous. Moreover, $u^\vee$ is upper semicontinuous.

In $\mathbb{R}^d$, denote by $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ the orthogonal projection onto $\mathbb{R}^{d-1}$: for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$,

$$\pi((x_1, \dots, x_d)) := (x_1, \dots, x_{d-1}).$$

Denote the standard basis vectors by e_k, $k = 1, \dots, d$. For an open set $\Omega \subset \mathbb{R}^d$, and $u \in \mathrm{BV}(\Omega)$, denote $D_k u := \langle Du, e_k \rangle$. For any fixed $k \in \{1, \dots, d\}$ — for simplicity we can assume $k = d$ — for every $z \in \pi(\Omega)$ we denote the slices of Ω at $(z, 0)$ in e_d-direction by

$$\Omega_z := \{t \in \mathbb{R} : (z, te_d) \in \Omega\}.$$

We also denote $u_z(t) := u(z, t)$ for $z \in \pi(\Omega)$ and $t \in \Omega_z$. We know that for $\mathcal{L}^{d-1}$-almost every $z \in \pi(\Omega)$, we have $u_z \in \mathrm{BV}(\Omega_z)$ (see [3, Theorem 3.103]) and also, if $u \in \mathrm{SBV}(\Omega)$, then $u_z \in \mathrm{SBV}(\Omega_z)$ (see [3, Eq. (3.108)]). On the other hand, if u_z is absolutely continuous for almost every $z \in \pi(\Omega)$, and similarly in the other coordinate directions, then $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Finally, for $\mathcal{L}^{d-1}$-almost every $z \in \pi(\Omega)$ it holds that

$$S_{u_z} = (S_u)_z \quad \text{and} \quad (\widetilde{u})_z(t) = \widetilde{u_z}(t) \text{ for every } t \in \mathbb{R} \setminus S_{u_z}, \tag{2.8}$$

see [3, Theorem 3.108].

3. Preliminary results

In this section we record and prove some preliminary results. Let $\Omega \subset \mathbb{R}^d$ always denote an arbitrary nonempty open set.

The following fact is generally well known and used e.g. in [10]. It simply says that in the definition of the non-centered maximal function, the supremum can be taken over balls whose *closure* contains the point x. For the maximal function $M_\Omega u$, this fact is not as trivial as it is for the global version $Mu = M_{\mathbb{R}^d} u$, so we give a short proof.

Lemma 3.1. *Let u be a measurable function on Ω. Then we have*

$$M_\Omega u(x) = \sup_{x \in \overline{B}(z,r),\, B(z,r) \subset \Omega} \fint_{B(z,r)} |u|\, dy, \quad x \in \Omega.$$

Proof. Consider a ball $B(z,r) \subset \Omega$ and a point $x \in \Omega \cap \partial B(z,r)$. For some $\delta > 0$, we have $B(x,\delta) \subset \Omega$. Let $0 < \varepsilon < 1/3$. The ball $B(z + \varepsilon(x - z), (1 - \varepsilon)r)$ is contained in $B(z,r)$. Then clearly for sufficiently small $t \in (0, \varepsilon)$, the ball

$$B(z + (\varepsilon + t)(x - z), (1 - \varepsilon)r)$$

is contained in $B(z,r) \cup B(x,\delta) \subset \Omega$, and contains x and contains $B(z, (1 - 3\varepsilon)r)$. Thus (some of the integrals below could be $+\infty$)

$$M_\Omega u(x) \ge \fint_{B(z+(\varepsilon+t)(x-z),(1-\varepsilon)r)} |u|\,dy \ge \frac{1}{\mathcal{L}^d(B(z,r))} \int_{B(z,(1-3\varepsilon)r)} |u|\,dy.$$

Letting $\varepsilon \to 0$, we obtain

$$M_\Omega u(x) \ge \fint_{B(z,r)} |u|\,dy. \quad \square$$

The following simple property of the non-centered maximal function is crucial for proving the 1-quasicontinuity of $M_\Omega u$.

Proposition 3.2. *Let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. Then $M_\Omega u(x) \ge u^\vee(x)$ for every $x \in \Omega \setminus (S_u \setminus J_u)$, that is, for $\mathcal{H}^{d-1}$-almost every $x \in \Omega$.*

Proof. We obviously have $M_\Omega u(x) \ge \widetilde{u}(x) = u^\vee(x)$ for every $x \in \Omega \setminus S_u$, that is, for Lebesgue points x (recall (2.4)). Assume then that $x \in J_u$. Now $u^\vee(x) = \max\{u^-(x), u^+(x)\}$ (recall (2.5)). Supposing $u^\vee(x) = u^+(x)$, we obtain using Lemma 3.1,

$$M_\Omega u(x) \ge \limsup_{r \to 0} \fint_{B(x+(r/2)\nu,r/2)} u\,dy = u^\vee(x).$$

The case $u^\vee(x) = u^-(x)$ is similar. The proof is completed by recalling that $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$. $\square$

As noted by Aldaz and Pérez Lázaro [1], a BV function need not have any upper semicontinuous representative when $d \ge 2$, which causes difficulties since usually such a representative is used in proving the continuity of $M_\Omega u$. However, the following quasi-semicontinuity result given in [4, Theorem 2.5] is a useful substitute. Alternatively, see [9, Theorem 1.1] and [8, Corollary 4.2] for a proof of this result in more general metric spaces.

Theorem 3.3. *Let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $\varepsilon > 0$. Then there exists an open set $G \subset \Omega$ such that $\mathrm{Cap}_1(G) < \varepsilon$ and $u^\wedge|_{\Omega\setminus G}$ is finite and lower semicontinuous, and $u^\vee|_{\Omega\setminus G}$ is finite and upper semicontinuous.*

The following is a direct consequence of a well-known weak type estimate, whose proof can be found e.g. in [7, Lemma 4.3]. Note that in this reference the estimate is formulated using a capacity that is defined slightly differently from Cap_1, but the capacities are easily seen to have the same null sets.

Proposition 3.4. *Let $u \in \mathrm{BV}(\mathbb{R}^d)$. Then*

$$\mathrm{Cap}_1(\{x \in \mathbb{R}^d : Mu(x) = \infty\}) = 0.$$

We define auxiliary maximal operators M_Ω^R and $M_{\Omega,R}$, $R > 0$, by

$$M_\Omega^R u(x) := \sup_{x\in B(z,r)\subset\Omega,\, r<R} \fint_{B(z,r)} |u|\,dy, \quad x \in \Omega,$$

and

$$M_{\Omega,R}u(x) := \sup_{x\in B(z,r)\subset\Omega,\, r\geq R} \fint_{B(z,r)} |u|\,dy, \quad x \in \Omega. \tag{3.5}$$

Obviously $M_\Omega u = \max\{M_\Omega^R u, M_{\Omega,R}u\}$. Again if $\Omega = \mathbb{R}^d$, we omit it from the notation.

Proposition 3.6. *Let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. Then* $\mathrm{Cap}_1(\{x \in \Omega : M_\Omega u(x) = \infty\}) = 0$.

Proof. Consider the open sets

$$\Omega_j := \{x \in \Omega : \operatorname{dist}(x, \mathbb{R}^d \setminus \Omega) > 2^{-j}\}, \quad j \in \mathbb{N}.$$

Now $\bigcup_{j=1}^\infty \Omega_j = \Omega$. Choose cutoff functions $\eta_j \in C_c^\infty(\Omega)$ with $0 \leq \eta_j \leq 1$ in $\mathbb{R}^d$ and $\eta_j = 1$ in Ω_j. Now $\eta_j u \in \mathrm{BV}(\mathbb{R}^d)$ and so by Proposition 3.4,

$$\mathrm{Cap}_1(\{x \in \Omega_j : M_\Omega^{2^{-2j}} u(x) = \infty\}) \leq \mathrm{Cap}_1(\{x \in \Omega_j : M(\eta_{j+1}u)(x) = \infty\}) = 0.$$

On the other hand, since $u \in L^1(\Omega)$, clearly $M_{\Omega,2^{-2j}}u(x) < \infty$ for every $x \in \Omega$. In total $M_\Omega u(x) = \max\{M_\Omega^{2^{-2j}}u(x), M_{\Omega,2^{-2j}}u(x)\} < \infty$ for Cap_1-almost every $x \in \Omega_j$. Since $\bigcup_{j=1}^\infty \Omega_j = \Omega$, we obtain the result. □

The following fact is well known; for a proof covering the case $u \in L^1(\mathbb{R}^d)$ see e.g. [10, Proposition 3.2], while the case $u \in L^\infty(\mathbb{R}^d)$ follows by a slight modification. This result does not necessarily hold for $M_\Omega u$ in an open set Ω, which is why we formulate some of the main results of this paper only in the global case $\Omega = \mathbb{R}^d$.

Proposition 3.7. *Let $u \in L^1(\mathbb{R}^d)$ (resp. $u \in L^\infty(\mathbb{R}^d)$), and let $R > 0$. Then $M_R u$ is Lipschitz with constant depending only on d, R, and $\|u\|_{L^1(\mathbb{R}^d)}$ (resp. $\|u\|_{L^\infty(\mathbb{R}^d)}$).*

The following result proven in [8, Lemma 3.5] is our key tool for handling the exceptional set of quasi (semi)continuity.

Lemma 3.8. *Let $G \subset \mathbb{R}^d$ and $\varepsilon > 0$. Then there exists an open set $U \supset G$ with $\mathrm{Cap}_1(U) \le C_2 \mathrm{Cap}_1(G) + \varepsilon$ such that*

$$\frac{\mathcal{L}^d(B(x,r)\cap G)}{\mathcal{L}^d(B(x,r))} \to 0 \quad \textit{as } r \to 0$$

uniformly for $x \in \mathbb{R}^d \setminus U$. Here C_2 depends only on d.

We will need the following version. We write $u_+ := \max\{u, 0\}$ for the positive part of a function (we use this notation only in the following lemma; afterwards it is not to be confused with the jump value u^+).

Lemma 3.9. *Let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, let $G \subset \Omega$, and let $\varepsilon > 0$. Then there exists an open set $U \supset G$ such that $\mathrm{Cap}_1(U) \le C_2 \mathrm{Cap}_1(G) + \varepsilon$ and*

$$\frac{1}{\mathcal{L}^d(B(x,r))} \int_{B(x,r)\cap G} |u|\, dy \to 0 \quad \textit{as } r \to 0$$

locally uniformly for $x \in \Omega \setminus U$. Here C_2 is the same constant as in Lemma 3.8.

Proof. We have $|u| \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, and so we can assume that u is nonnegative. By Lemma 3.8, we find an open set $W \supset G$ such that $\mathrm{Cap}_1(W) \le C_2 \mathrm{Cap}_1(G) + \varepsilon/2$ and

$$\frac{\mathcal{L}^d(B(x,r)\cap G)}{\mathcal{L}^d(B(x,r))} \to 0 \quad \text{as } r \to 0 \tag{3.10}$$

uniformly for $x \in \mathbb{R}^d \setminus W$. Let

$$\Omega_j := \{x \in B(0,j) \colon \operatorname{dist}(x, \mathbb{R}^d \setminus \Omega) > 1/j\}, \quad j \in \mathbb{N},$$

so that $\Omega = \bigcup_{j=1}^\infty \Omega_j$. Note that we have by the Sobolev embedding

$$u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \subset L^{d/(d-1)}_{\mathrm{loc}}(\Omega) \subset L^{d/(d-1)}(\Omega_j) \quad \text{for every } j \in \mathbb{N}.$$

Choose numbers $\alpha_j > 0$, $j = 0, 1, \ldots$, such that $\alpha_{j+1} \ge 2\alpha_j$ and

$$\sum_{j=1}^\infty \left(\int_{\Omega_{j+1}} (u - \alpha_{j-1})_+^{d/(d-1)}\, dy \right)^{(d-1)/d} < \frac{\varepsilon}{5^d C_0}.$$

Next take a sequence $\beta_j \searrow 0$, $\beta_j \le 1$, such that still

$$\sum_{j=1}^\infty \frac{1}{\beta_j} \left(\int_{\Omega_{j+1}} (u - \alpha_{j-1})_+^{d/(d-1)}\, dy \right)^{(d-1)/d} < \frac{\varepsilon}{5^d C_0}. \tag{3.11}$$

Define the sets

$$E_j := \{x \in \Omega \colon u(x) \geq \alpha_j\}.$$

Then define the sets

$$A_j := \left\{ x \in \Omega_j \colon \frac{1}{\mathcal{L}^d(B(x,r))} \int_{B(x,r)\cap E_j} u\,dy > \beta_j \right.$$
$$\left. \text{for some } 0 < r \leq 1/5 \text{ with } B(x,r) \subset \Omega_{j+1} \right\}.$$

Consider $j \in \mathbb{N}$ and $x \in A_j$. For some $0 < r_x \leq 1/5$, we have $B(x, r_x) \subset \Omega_{j+1}$ and

$$\frac{1}{\mathcal{L}^d(B(x,r_x))} \int_{B(x,r_x)} (u - \alpha_{j-1})_+\,dy \geq \frac{1}{2} \frac{1}{\mathcal{L}^d(B(x,r_x))} \int_{B(x,r_x)\cap E_j} u\,dy > \frac{\beta_j}{2},$$

and so by Hölder's inequality

$$\frac{1}{r_x^{d-1}} \left(\int_{B(x,r_x)} (u - \alpha_{j-1})_+^{d/(d-1)}\,dy \right)^{(d-1)/d} > \frac{\beta_j}{2}.$$

Now $\{B(x, r_x)\}_{x\in A_j}$ is a covering of A_j. By the 5-covering theorem, we find a countable collection of pairwise disjoint balls $\{B(x_k, r_k)\}_{k=1}^{\infty}$ such that $A_j \subset \bigcup_{k=1}^{\infty} B(x_k, 5r_k)$. Now we have by (2.1), and by using the triangle inequality for the $L^{q/(q-1)}$-norm,

$$\begin{aligned}
\operatorname{Cap}_1(A_j) &\leq \sum_{k=1}^{\infty} \operatorname{Cap}_1(B(x_k, 5r_k)) \\
&\leq 5^{d-1} C_0 \sum_{k=1}^{\infty} r_k^{d-1} \\
&\leq \frac{2 \times 5^{d-1} C_0}{\beta_j} \sum_{k=1}^{\infty} \left(\int_{B(x_k,r_k)} (u - \alpha_{j-1})_+^{d/(d-1)}\,dy \right)^{(d-1)/d} \\
&\leq \frac{2 \times 5^{d-1} C_0}{\beta_j} \left(\int_{\Omega_{j+1}} (u - \alpha_{j-1})_+^{d/(d-1)}\,dy \right)^{(d-1)/d}.
\end{aligned}$$

Now recalling (3.11), we get

$$\mathrm{Cap}_1\left(W \cup \bigcup_{j=1}^{\infty} A_j\right) \le \mathrm{Cap}_1(W) + \sum_{j=1}^{\infty} \mathrm{Cap}_1(A_j) < C_2\,\mathrm{Cap}_1(G) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Finally, take an open set

$$U \supset W \cup \bigcup_{j=1}^{\infty} A_j$$

with $\mathrm{Cap}_1(U) < C_2\,\mathrm{Cap}_1(G) + \varepsilon$.

Fix $\delta > 0$. Take $j_0 \in \mathbb{N}$ sufficiently large that $j_0 \ge 1/\delta$ and $\beta_{j_0} < \delta/2$. Using (3.10), take $0 < R \le 1/5$ such that

$$\frac{\mathcal{L}^d(G \cap B(x,r))}{\mathcal{L}^d(B(x,r))} < \frac{\delta}{2\alpha_{j_0}} \quad \text{for all } x \in \mathbb{R}^d \setminus U \text{ and } 0 < r \le R.$$

Thus for all

$$x \in \{y \in B(0, 1/\delta) \colon \operatorname{dist}(y, \mathbb{R}^d \setminus \Omega) > \delta\} \setminus U \subset \Omega_{j_0} \setminus U$$

and $0 < r \le \min\{R, \operatorname{dist}(\Omega_{j_0}, \mathbb{R}^d \setminus \Omega_{j_0+1})\}$, we have

$$\begin{aligned} &\frac{1}{\mathcal{L}^d(B(x,r))} \int_{G \cap B(x,r)} u\,dy \\ &\le \frac{1}{\mathcal{L}^d(B(x,r))} \int_{G \cap B(x,r) \setminus E_{j_0}} u\,dy + \frac{1}{\mathcal{L}^d(B(x,r))} \int_{B(x,r) \cap E_{j_0}} u\,dy \\ &< \frac{\delta}{2\alpha_{j_0}} \alpha_{j_0} + \beta_{j_0} \le \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

Since $\delta > 0$ was arbitrary, this proves the local uniform convergence in the set $\Omega \setminus U$. $\square$

4. Quasicontinuity

In this section we prove that the non-centered maximal function of a BV function is 1-quasi-continuous. As before, $\Omega \subset \mathbb{R}^d$ is an arbitrary nonempty open set.

The following theorem is Theorem 1.1 in a slightly more general form.

Theorem 4.1. *Let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$ or $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$. Then $M_\Omega u$ is 1-quasicontinuous.*

Proof. First assume that $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$. Then also $|u| \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$, and so we can assume that u is nonnegative.

Fix $\varepsilon > 0$. By Theorem 3.3 we find an open set $G \subset \Omega$ such that $\mathrm{Cap}_1(G) < \varepsilon/C_2$ and $u^\vee|_{\Omega \setminus G}$ is upper semicontinuous. Since $\mathrm{Cap}_1(S_u \setminus J_u) = \mathcal{H}^{d-1}(S_u \setminus J_u) = 0$ (recall (2.2)) and

$$\mathrm{Cap}_1(\{x \in \Omega \colon M_\Omega u(x) = \infty\}) = 0$$

by Proposition 3.6, we can assume that $G \supset \{x \in \Omega : M_\Omega u(x) = \infty\} \cup (S_u \setminus J_u)$. Then by Lemma 3.9, we can take an open set $U \supset G$ such that $\mathrm{Cap}_1(U) < \varepsilon$ and

$$\frac{1}{\mathcal{L}^d(B(x,r))} \int_{B(x,r)\cap G} u\, dy \to 0 \quad \text{as } r \to 0 \tag{4.2}$$

locally uniformly for $x \in \Omega \setminus U$. Since $M_\Omega u|_{\Omega\setminus U}$ is finite and lower semicontinuous, it is sufficient to prove upper semicontinuity. Fix $x \in \Omega \setminus U$. Take a sequence $x_j \to x$, $x_j \in \Omega \setminus U$, such that

$$\lim_{j\to\infty} M_\Omega u(x_j) = \limsup_{\Omega\setminus U \ni y \to x} M_\Omega u(y).$$

(At this stage we cannot exclude the possibility that the lim sup is ∞.) Now we only need to show that $M_\Omega u(x) \geq \lim_{j\to\infty} M_\Omega u(x_j)$.

We find “almost optimal” balls $B(x_j^*, r_j)$ in the sense that

$$\lim_{j\to\infty} M_\Omega u(x_j) = \lim_{j\to\infty} \fint_{B(x_j^*,r_j)} u\, dy, \tag{4.3}$$

with $x_j \in B(x_j^*, r_j) \subset \Omega$. Since $u \in L^1(\Omega)$, we can assume that the radii r_j are uniformly bounded. Now we consider two cases.

Case 1. Suppose that by passing to a subsequence (not relabeled), we have $r_j \to 0$. Fix $\delta > 0$. By the upper semicontinuity of $u^\vee|_{\Omega\setminus G}$, for some $r > 0$ we have $B(x, r) \subset \Omega$ and

$$u^\vee(x) \geq \sup_{B(x,r)\setminus G} u^\vee - \delta. \tag{4.4}$$

Note also that for sufficiently large $j \in \mathbb{N}$, we have $B(x_j^*, r_j) \subset B(x, r)$. Thus, using Proposition 3.2 (recall that $G \supset S_u \setminus J_u$), we get for large $j \in \mathbb{N}$

$$\begin{aligned}
M_\Omega u(x) &\geq u^\vee(x) \\
&\geq \sup_{B(x,r)\setminus G} u^\vee - \delta \quad \text{by (4.4)} \\
&\geq \frac{1}{\mathcal{L}^d(B(x_j^*, r_j))} \int_{B(x_j^*,r_j)\setminus G} u\, dy - \delta \quad \text{since } B(x_j^*, r_j) \subset B(x, r) \\
&= \frac{1}{\mathcal{L}^d(B(x_j^*, r_j))} \int_{B(x_j^*,r_j)} u\, dy - \frac{1}{\mathcal{L}^d(B(x_j^*, r_j))} \int_{B(x_j^*,r_j)\cap G} u\, dy - \delta \\
&\geq \fint_{B(x_j^*,r_j)} u\, dy - \frac{2^d}{\mathcal{L}^d(B(x_j, 2r_j))} \int_{B(x_j,2r_j)\cap G} u\, dy - \delta.
\end{aligned}$$

Now by (4.2) and (4.3), we get

$$M_\Omega u(x) \geq \lim_{j\to\infty} M_\Omega u(x_j) - \delta,$$

so that letting $\delta \to 0$, we obtain the desired inequality.

Case 2. The other alternative is that passing to a subsequence (not relabeled), we have $r_j \to r \in (0,\infty)$. Passing to a further subsequence (not relabeled), the vectors $x_j^* - x_j$ (since they have length at most r_j) converge to some $v \in \mathbb{R}^d$. Now for $x^* := x + v$ we have $\mathbb{1}_{B(x_j^*,r_j)} \to \mathbb{1}_{B(x^*,r)}$ in $L^1(\mathbb{R}^d)$, with $x \in \overline{B}(x^*,r)$ and $B(x^*,r) \subset \Omega$. In this case we have by Lemma 3.1 that

$$M_\Omega u(x) \geq \fint_{B(x^*,r)} u\,dy = \lim_{j\to\infty} \fint_{B(x_j^*,r_j)} u\,dy = \lim_{j\to\infty} M_\Omega u(x_j).$$

This completes the proof in the case $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$.

Now consider the case $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$. The proof is the same, except that now we need to consider a third case.

Case 3. Suppose that passing to a subsequence (not relabeled), we have $r_j \to \infty$. Then since $B(x_j^*, r_j + 1) \ni x$ for all large $j \in \mathbb{N}$, we get

$$Mu(x) \geq \limsup_{j\to\infty} \fint_{B(x_j^*,r_j+1)} u\,dy \geq \limsup_{j\to\infty} \fint_{B(x_j^*,r_j)} u\,dy = \lim_{j\to\infty} Mu(x_j).$$

This completes the proof. $\square$

Aldaz and Pérez Lázaro [2] showed that the maximal function of a *block-decreasing* BV function is continuous at every point outside a $\mathcal{H}^{d-1}$-negligible set. Such a continuity property is somewhat stronger than 1-quasicontinuity; recall that Cap_1 and $\mathcal{H}^{d-1}$ have the same null sets. The following simple example shows that for general BV functions we cannot have such a stronger continuity property, demonstrating that quasicontinuity seems to be the correct concept to consider.

Example 4.5. Take an enumeration of all the points on the plane with rational coordinates $\{q_j\}_{j=1}^\infty$ and define the "enlarged rationals"

$$E := \bigcup_{j=1}^\infty B(q_j, 2^{-j}).$$

Clearly $\mathcal{L}^2(E) \leq \pi/2$. By lower semicontinuity and subadditivity we have (recall (2.3))

$$\mathrm{Var}(\mathbb{1}_E, \mathbb{R}^2) \leq \sum_{j=1}^\infty \mathrm{Var}(\mathbb{1}_{B(q_j,2^{-j})}, \mathbb{R}^2) = 2\pi \sum_{j=1}^\infty 2^{-j} = 2\pi.$$

Thus $\mathbb{1}_E \in \mathrm{BV}(\mathbb{R}^2)$. We have $M\mathbb{1}_E(x) < 1$ for every $x \in \mathbb{R}^2$ with $\mathbb{1}_E^\vee(x) = 0$. However, for every such x there is a sequence of points $x_j \to x$ such that $x_j \in E$ for every $j \in \mathbb{N}$. Thus $M\mathbb{1}_E(x_j) = 1$ for every $j \in \mathbb{N}$. Hence $M\mathbb{1}_E$ is discontinuous at every point in the set $\{x \in \mathbb{R}^2 : \mathbb{1}_E^\vee(x) = 0\}$, which has even infinite Lebesgue measure.

5. Continuity and Lusin property on lines

In this section we prove that the non-centered maximal function of an SBV function, when restricted to almost every line parallel to a coordinate axis, is continuous and has the Lusin property. The Lusin property for a function v defined on $V \subset \mathbb{R}$ states that

$$\text{if } N \subset V \text{ with } \mathcal{L}^1(N) = 0, \quad \text{then } \mathcal{L}^1(v(N)) = 0. \tag{5.1}$$

First we prove this kind of property in the following form. Recall that S_u denotes the set of non-Lebesgue points of u, and that $\widetilde{u}$ is the Lebesgue representative of u.

Lemma 5.2. *Let $V \subset \mathbb{R}$ be open and let $u \in \mathrm{BV}_{\mathrm{loc}}(V)$. If $N \subset V \setminus S_u$ with $|Du|(N) = 0$, then*

$$\mathcal{L}^1(\widetilde{u}(N)) = 0.$$

Proof. The claim is equivalent with $\mathcal{L}^1(u^\vee(N)) = 0$; we will work with the everywhere defined representative $u^\vee$ since some of the points that we examine below may be outside N and thus possibly in the set S_u. Fix $\varepsilon > 0$. We can take an open set U with $N \subset U \subset V$ and $|Du|(U) < \varepsilon$. For every $x \in N$, we can choose an arbitrarily short compact interval $I \ni x$ contained in U. Consider the collection of compact intervals (understood to be nondegenerate, i.e. consisting of more than one point)

$$\mathcal{I} := \{I : I \subset U \text{ and } I \cap N \neq \emptyset\}.$$

These form a covering of N. Let

$$H := \{h \in \mathbb{R} : \text{for some } I \in \mathcal{I},\ u^\vee(I) = \{h\}\}.$$

The set H can be at most countable, since the intervals I are nondegenerate. Now the collection of intervals

$$\mathcal{J} := \left\{[\inf u^\vee(I), \sup u^\vee(I)] : I \in \mathcal{I},\ u^\vee \text{ is not constant on } I\right\}$$

is a covering of $u^\vee(N) \setminus H$. It is a fine covering, since every $x \in N \subset V \setminus S_u$ is a point of continuity of $u^\vee$ (recall (2.7)). By Vitali's covering theorem, there exists a countable collection of disjoint intervals $\{J_j\}_{j=1}^\infty$ selected from $\mathcal{J}$ such that

$$\mathcal{L}^1\left(u^\vee(N) \setminus \bigcup_{j=1}^\infty J_j\right) = 0.$$

For every J_j, there exists I_j such that $J_j = [\inf u^\vee(I_j), \sup u^\vee(I_j)]$. Then by (2.7),

$$\mathcal{L}^1(u^\vee(N)) \le \sum_{j=1}^{\infty} \mathcal{L}^1(J_j) \le \sum_{j=1}^{\infty} |Du|(I_j) \le |Du|(U) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

Define

$$H_u := \{x \in \Omega \colon M_\Omega u(x) > |u|^\vee(x)\}.$$

Recall that we denote by $\pi \colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ the orthogonal projection onto $\mathbb{R}^{d-1}$: for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$,

$$\pi((x_1, \dots, x_d)) := (x_1, \dots, x_{d-1}).$$

Proposition 5.3. *Let $A \subset \mathbb{R}^d$. Then*

$$2\mathcal{H}^{d-1}(\pi(A)) \le \mathrm{Cap}_1(A).$$

Proof. Consider $u \in W^{1,1}(\mathbb{R}^d)$ with $u \ge 1$ in a neighborhood of A. Then on almost every line l in the dth coordinate direction intersecting A, we have

$$\int_l \left| \frac{du}{dx_d} \right| ds \ge 2.$$

Integrating over $\mathbb{R}^{d-1}$, we get

$$\int_{\mathbb{R}^d} |\nabla u| \, dx \ge 2\mathcal{H}^{d-1}(\pi(A)).$$

Thus $\|u\|_{W^{1,1}(\mathbb{R}^d)} \ge 2\mathcal{H}^{d-1}(\pi(A))$ and we get the result by taking infimum over all such u. □

Recall from Section 2.2 that a BV function u is in the SBV class if $|D^c u|(\Omega) = 0$.

Theorem 5.4. *Let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$ or $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$. Then $M_\Omega u$ is continuous on almost every line parallel to a coordinate axis.*

If $u \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ or $u \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then Mu also has the Lusin property on almost every line parallel to a coordinate axis.

Proof. Again we can assume that $u \ge 0$. First we prove the continuity on lines. Fix $\varepsilon > 0$. By Theorems 3.3 and 4.1, we can take an open set $G \subset \Omega$ such that $\mathrm{Cap}_1(G) < \varepsilon / C_2$ and $u^\vee|_{\Omega \setminus G}$ is finite and upper semicontinuous, and $M_\Omega u|_{\Omega \setminus G}$ is finite and continuous. Since $\mathrm{Cap}_1(S_u \setminus J_u) = \mathcal{H}^{d-1}(S_u \setminus J_u) = 0$ (recall (2.2)), we can also assume that $G \supset S_u \setminus J_u$. By Lemma 3.9 we can take an open set $U \supset G$ such that $\mathrm{Cap}_1(U) < \varepsilon$ and

$$\frac{1}{\mathcal{L}^d(B(x,r))}\int_{G\cap B(x,r)} u\,dy \to 0 \quad \text{as } r\to 0$$

locally uniformly for $x\in\Omega\setminus U$. We wish to study the behavior of $M_\Omega u$ in $H_u\setminus U$. Consider $x_0\in H_u\setminus U$. We have $\alpha:=M_\Omega u(x_0)\in(0,\infty)$. Let $\delta:=\alpha-u^\vee(x_0)>0$. By upper semicontinuity of $u^\vee|_{\Omega\setminus G}$, for some $R_1>0$ we have

$$u^\vee(x)<\alpha-\frac{3\delta}{4} \tag{5.5}$$

for all $x\in B(x_0,R_1)\setminus G$. By lower semicontinuity of the maximal function, there exists $R_2>0$ such that $M_\Omega u(x)>\alpha-\delta/4$ for all $x\in B(x_0,R_2)$. Moreover, by the choice of the set U, there exists $R_3>0$ such that for all $x\in B(x_0,R_3)\setminus U$ we have

$$\frac{1}{\mathcal{L}^d(B(x,s))}\int_{G\cap B(x,s)} u\,dy\le\frac{\delta}{2^{d+1}}\quad\text{for all } 0<s<R_3. \tag{5.6}$$

Let $R:=\min\{R_1,R_2,R_3\}/4$. Now consider any $x\in B(x_0,R)\setminus U$, and any ball $B(z,r)\ni x$ with $r\in(0,R)$. We have

$$\begin{aligned}
\frac{1}{\mathcal{L}^d(B(z,r))}\int_{B(z,r)} u\,dy &= \frac{1}{\mathcal{L}^d(B(z,r))}\int_{B(z,r)\cap G} u\,dy+\frac{1}{\mathcal{L}^d(B(z,r))}\int_{B(z,r)\setminus G} u\,dy\\
&\le \frac{2^d}{\mathcal{L}^d(B(x,2r))}\int_{B(x,2r)\cap G} u\,dy+\alpha-\frac{3\delta}{4}\quad\text{by (5.5)}\\
&\le\frac{\delta}{2}+\alpha-\frac{3\delta}{4}\quad\text{by (5.6)}\\
&=\alpha-\frac{\delta}{4}.
\end{aligned}$$

On the other hand, we had $M_\Omega u(x)>\alpha-\delta/4$ for all $x\in B(x_0,R)$. Recalling the definition (3.5), we have

$$M_\Omega u(x)=M_{\Omega,R}u(x) \tag{5.7}$$

for every $x\in B(x_0,R)\setminus U$.

Now we examine the behavior of $M_\Omega u$ on lines. Recall the notation and results from Section 2.3. Without loss of generality we can consider lines parallel to the d:th coordinate axis. Recall that π denotes the orthogonal projection onto $\mathbb{R}^{d-1}$. By Proposition 5.3 we know that

$$\mathcal{H}^{d-1}(\pi(U))\le\operatorname{Cap}_1(U)<\varepsilon.$$

Thus it is enough to consider a line not intersecting U. In other words, consider a fixed $z\in\pi(\Omega)\setminus\pi(U)$ and then consider the line (z,t), $t\in\mathbb{R}$. Since we know that $M_\Omega u|_{\Omega\setminus U}$ is continuous, $t\mapsto M_\Omega u(z,t)$ is continuous, proving the first claim.

Now suppose $u \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ or $u \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. The function u_z is in the class $\mathrm{SBV}_{\mathrm{loc}}(\mathbb{R})$ for almost every $z \in \mathbb{R}^{d-1}$, and so we can also assume that $u_z \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R})$. We can further assume that $(J_u)_z$ is at most countable; this follows from the fact that J_u is countably $d-1$-rectifiable, and from the coarea formula, see [3, Theorem 2.93].

Let $N \subset \mathbb{R}$ with zero 1-dimensional Lebesgue measure. We have

$$Mu(z, N) = Mu((\{z\} \times N) \cap H_u) \cup Mu((\{z\} \times N) \setminus H_u).$$

Here the first set has zero one-dimensional Lebesgue measure by the fact that $Mu|_{H_u \setminus U}$ is locally Lipschitz, which is given by (5.7) and Proposition 3.7. For the second set we have, recalling that $U \supset S_u \setminus J_u$,

$$\begin{aligned} Mu((\{z\} \times N) \setminus H_u) &\subset Mu((\{z\} \times N) \setminus (H_u \cup S_u)) \cup Mu((\{z\} \times N) \cap J_u) \\ &= \widetilde{u}((\{z\} \times N) \setminus (H_u \cup S_u)) \cup Mu((\{z\} \times N) \cap J_u) \\ &\subset \widetilde{u_z}(z, N \setminus S_{u_z(\cdot)}) \cup Mu((\{z\} \times N) \cap J_u), \end{aligned}$$

by (2.8) (which we can assume to hold by discarding another $\mathcal{L}^{d-1}$-negligible set). Here the first set has zero measure since $u_z \in \mathrm{SBV}_{\mathrm{loc}}(\Omega_z)$ and so $|Du_z|(N \setminus S_{u_z(\cdot)}) = 0$, and then we can use Lemma 5.2. The second set has zero measure since $(\{z\} \times \mathbb{R}) \cap J_u$ was at most countable. In total, $\mathcal{L}^1(Mu(z, N)) = 0$, and so Mu has the Lusin property on almost every line parallel to a coordinate axis. $\square$

If we could give a positive answer to the following open problem, then we could extend Theorem 5.4 from SBV functions to BV functions.

Open Problem. Let $u \in \mathrm{BV}(\mathbb{R}^d)$. Is it true that $Mu(x) > |u|^\vee(x)$ for $|D^c u|$-almost every $x \in \mathbb{R}^d$?

In one dimension the answer is yes, see Proposition 8.1.

6. Sobolev property

In this section we show that if the non-centered maximal function of an SBV function is locally BV, then it is in fact locally Sobolev.

We rely on the following direction of the Banach-Zarecki Theorem.

Theorem 6.1. *Let $V \subset \mathbb{R}$ be open and let $v \in \mathrm{BV}_{\mathrm{loc}}(V)$ be a continuous function that satisfies the Lusin property. Then v is absolutely continuous on V.*

We say that a function v on $\mathbb{R}^d$ is ACL if it is absolutely continuous on almost every line parallel to a coordinate axis.

The following theorem is Theorem 1.2 in a slightly more general form.

Theorem 6.2. *Let $u \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ or $u \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. If $Mu \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$, then Mu is ACL and in the class $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$.*

Proof. It is sufficient to consider the d:th coordinate direction. Since $Mu \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$, the function $Mu(z,\cdot)$ is in the class $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ for almost every $z \in \mathbb{R}^{d-1}$. By Theorem 5.4, $Mu(z,\cdot)$ is also continuous and has the Lusin property for almost every $z \in \mathbb{R}^{d-1}$. Now by the Banach-Zarecki theorem, $Mu(z,\cdot)$ is absolutely continuous on $\mathbb{R}$ for almost every $z \in \mathbb{R}^{d-1}$. In conclusion, Mu is ACL. Since we already know that $Mu \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$, it follows that $Mu \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$. □

In Theorem 6.2, we would of course like to prove that $Mu \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$, instead of assuming this. For sets of finite perimeter, such a better result is possible due to a very recent result of Weigt [12]. We restate Theorem 1.3, which is our third main theorem:

Theorem 6.3. *Let $E \subset \mathbb{R}^d$ be a set of finite perimeter. Then $M\mathbb{1}_E \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ with $\|\nabla M\mathbb{1}_E\|_{L^1(\mathbb{R}^d)} \le C_d|D\mathbb{1}_E|(\mathbb{R}^d)$, where C_d only depends on the dimension d.*

Proof. Theorem 1.3 of [12] states that $M\mathbb{1}_E \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ with $|DM\mathbb{1}_E|(\mathbb{R}^d) \le C_d|D\mathbb{1}_E|(\mathbb{R}^d)$. Then by Theorem 6.2, the conclusion follows. □

7. Formula for the gradient

When $u \in \mathrm{BV}(\Omega)$ and $M_\Omega u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, there is a weak gradient $DM_\Omega u \in L^1_{\mathrm{loc}}(\Omega)$. In this section we derive a formula for it, generalizing [10, Lemma 2.2(1)] where the case $u \in W^{1,1}(\mathbb{R}^d)$ was considered. However, as we will see, in the case $u \in \mathrm{BV}(\Omega)$ the formula is, and can be, valid only under certain conditions.

As usual, $\Omega \subset \mathbb{R}^d$ is an arbitrary nonempty open set.

We have the following standard approximation result for BV functions.

Proposition 7.1. *Let $u \in \mathrm{BV}(\Omega)$. Then there exists a sequence of functions $\{v_i\}_{i\in\mathbb{N}}$ in $C^\infty(\Omega)$ such that $v_i \to u$ in $L^1(\Omega)$, $\lim_{i\to\infty}|Dv_i|(\Omega) = |Du|(\Omega)$, and $Dv_i \overset{*}{\rightharpoonup} Du$ and $|Dv_i| \overset{*}{\rightharpoonup} |Du|$ in Ω.*

Proof. By [3, Theorem 3.9] we find a sequence of functions $\{v_i\}_{i\in\mathbb{N}}$ in $C^\infty(\Omega)$ such that $v_i \to u$ in $L^1(\Omega)$ and $\lim_{i\to\infty}|Dv_i|(\Omega) = |Du|(\Omega)$. Then by [3, Proposition 3.13] we have in fact $Dv_i \overset{*}{\rightharpoonup} Du$ in Ω, and by [3, Proposition 1.80] also $|Dv_i| \overset{*}{\rightharpoonup} |Du|$ in Ω. □

We also have the following simple fact concerning the measures of spheres.

Lemma 7.2. *Let ν be a positive Radon measure on Ω with $\nu(\Omega) < \infty$ and $\nu \ll \mathcal{H}^{d-1}$. Then $\nu(\partial B) > 0$ for at most countably many spheres ∂B with $\overline{B} \subset \Omega$.*

Proof. For any distinct $y, z \in \Omega$ with $\overline{B}(y,r) \subset \Omega$ and $\overline{B}(z,R) \subset \Omega$ for some $r, R > 0$, the intersection of the spheres $\partial B(y,r)$ and $\partial B(z,R)$ has zero $\mathcal{H}^{d-1}$-measure and thus zero ν-measure. Thus for every $\alpha > 0$, there can be only finitely many balls $\overline{B} \subset \Omega$ such that $\nu(\partial B) > \alpha$. The result follows. □

Given a ball $B = B(x,r)$ (open, as usual) and $k \in \{1,\dots,d\}$, define the "half-open" balls

$$\overline{B}^{k,+} := B \cup \{y \in \partial B : y_k > x_k\}, \quad \overline{B}^{k,-} := B \cup \{y \in \partial B : y_k < x_k\}.$$

The following definition of "optimal balls" is convenient when studying the non-centered maximal function, and has been used e.g. in [10]. Recall also Lemma 3.1.

Definition 7.3. For a function u on Ω and $x \in \Omega$, let

$$\mathcal{B}_x := \left\{ B(z,r) \subset \Omega \colon x \in \overline{B}(z,r),\ r>0,\ \text{and} \fint_{B(z,r)} |u|\,dy = M_\Omega u(x) \right\}.$$

Note that if $u \in L^1(\Omega)$ and $x \in \Omega$ is a Lebesgue point of u with $\mathcal{B}_x = \emptyset$, then we necessarily have $M_\Omega u(x) = \widetilde{u}(x)$.

The following is our first version of a formula for $DM_\Omega u$. Note that when $M_\Omega u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, at almost every $x \in \Omega$ we can interpret $D_k M_\Omega u(x)$, $k \in \{1,\dots,d\}$, to be either the density of the measure $D_k M_\Omega u$ or the classical partial derivative of $M_\Omega u$ (for the classical partial derivative to make sense, we need the correct pointwise representative of $M_\Omega u$, but the first part of Theorem 5.4 guarantees that $M_\Omega u$ itself is suitable). The same applies to $D_k \widetilde{u}$.

Theorem 7.4. *Let $u \in \mathrm{BV}(\Omega)$ be such that $M_\Omega u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. Then for almost every $x \in \Omega$ and every $k \in \{1,\dots,d\}$, we have*

$$(1)\ \frac{D_k|u|(\overline{B}^{k,+})}{\mathcal{L}^d(B)} \le D_k M_\Omega u(x) \le \frac{D_k|u|(\overline{B}^{k,-})}{\mathcal{L}^d(B)} \quad \text{if } B \in \mathcal{B}_x,\ \overline{B} \subset \Omega,$$

$$(2)\ D_k M_\Omega u(x) = D_k|\widetilde{u}|(x) \quad \text{if } \mathcal{B}_x = \emptyset.$$

Proof. As usual, we can assume that $u \ge 0$. Take $x \in \Omega$ such that all $D_k M_\Omega u(x)$, $k = 1,\dots,d$, exist (both as densities and as classical derivatives).

First suppose that $B(z,r) \in \mathcal{B}_x$ and $\overline{B(z,r)} \subset \Omega$. Fix $k \in \{1,\dots,d\}$. Now $B(z+he_k,r) \subset \Omega$ for $h \in \mathbb{R}$ close to zero.

Consider momentarily $v \in C^\infty(\Omega)$. We get for small $h>0$ that

$$\begin{aligned}
\frac{1}{h}\left(\fint_{B(z+he_k,r)} v\,dy - \fint_{B(z,r)} v\,dy \right) &= \frac{1}{h}\left(\fint_{B(z,r)} v(y+he_k) - v(y)\,dy \right) \\
&= \frac{1}{h}\left(\fint_{B(z,r)} \int_0^h D_k v(y+te_k)\,dt\,dy \right) \qquad (7.5) \\
&= \frac{1}{h}\int_0^h \left(\fint_{B(z+te_k,r)} D_k v\,dy \right) dt.
\end{aligned}$$

By Proposition 7.1 we find a sequence $\{v_i\}_{i\in\mathbb{N}}$ in $C^\infty(\Omega)$ such that $v_i \to u$ in $L^1(\Omega)$, $\lim_{i\to\infty} |Dv_i|(\Omega) = |Du|(\Omega)$, and $Dv_i \overset{*}{\rightharpoonup} Du$ and $|Dv_i| \overset{*}{\rightharpoonup} |Du|$ in Ω. For every ball B with $\overline{B} \subset \Omega$ and $|Du|(\partial B) = 0$, this implies (see [3, Proposition 1.62(b)])

$$Dv_i(B) \to Du(B) \quad \text{so in particular} \quad D_k v_i(B) \to D_k u(B).$$

By Lemma 7.2, there are at most countably many spheres ∂B with $\overline{B} \subset \Omega$ and $|Du|(\partial B) > 0$. In particular, $|Du|(\partial B(z + te_k, r)) = 0$ for almost every $t \in [0, h]$. Writing (7.5) with $v = v_i$ and taking the limit $i \to \infty$, we get by Lebesgue's dominated convergence theorem

$$\frac{1}{h}\left(\fint_{B(z+he_k,r)} u\,dy - \fint_{B(z,r)} u\,dy \right) = \frac{1}{h}\int_0^h \left(\frac{D_k u(B(z+te_k,r))}{\mathcal{L}^d(B(z,r))} \right) dt.$$

Thus

$$\begin{aligned} D_k M_\Omega u(x) &= \lim_{h\to 0^+} \frac{1}{h}\left(M_\Omega u(x+he_k) - M_\Omega u(x)\right) \\ &\geq \lim_{h\to 0^+} \frac{1}{h}\left(\fint_{B(z+he_k,r)} u\,dy - \fint_{B(z,r)} u\,dy \right) \\ &= \lim_{h\to 0^+} \frac{1}{h}\int_0^h \left(\frac{D_k u(B(z+te_k,r))}{\mathcal{L}^d(B(z,r))} \right) dt. \end{aligned}$$

Here

$$\begin{aligned} &\left| \frac{1}{h}\int_0^h \left(\frac{D_k u(B(z+te_k,r))}{\mathcal{L}^d(B(z,r))} \right) dt - \frac{D_k u(\overline{B}^{k,+}(z,r))}{\mathcal{L}^d(B(z,r))} \right| \\ &\leq \frac{1}{h}\int_0^h \left| \frac{D_k u(B(z+te_k,r))}{\mathcal{L}^d(B(z,r))} - \frac{D_k u(\overline{B}^{k,+}(z,r))}{\mathcal{L}^d(B(z,r))} \right| dt \\ &\leq \frac{1}{\mathcal{L}^d(B(z,r))}\frac{1}{h}\int_0^h |Du|\left(\bigcup_{s\in(0,h)} \left(\overline{B}^{k,+}(z,r)\Delta B(z+se_k,r) \right)\right) dt \\ &= \frac{1}{\mathcal{L}^d(B(z,r))}|Du|\left(\bigcup_{s\in(0,h)} \left(\overline{B}^{k,+}(z,r)\Delta B(z+se_k,r) \right)\right) \\ &\to 0 \quad \text{as } h \to 0 \end{aligned}$$

since $\bigcup_{s\in(0,h)}\left(\overline{B}^{k,+}(z,r)\Delta B(z+se_k,r)\right) \to \emptyset$. Combining the previous inequalities, we get

$$D_k M_\Omega u(x) \geq \frac{D_k u(\overline{B}^{k,+}(z,r))}{\mathcal{L}^d(B(z,r))}.$$

Similarly we get

$$\begin{aligned} D_k M_\Omega u(x) &= \lim_{h\to 0^+} \frac{1}{h}\left(M_\Omega u(x) - M_\Omega u(x-he_k)\right) \\ &\le \lim_{h\to 0^+} \frac{1}{h}\left(\fint_{B(z,r)} u\,dy - \fint_{B(z-he_k,r)} u\,dy \right) \\ &= \frac{D_k u(\overline{B}^{k,-}(z,r))}{\mathcal{L}^d(B(z,r))}. \end{aligned}$$

Then assume that $\mathcal{B}_x = \emptyset$. Discarding another $\mathcal{L}^d$-negligible set, we can assume that x is a Lebesgue point of u, and so necessarily $M_\Omega u(x) = \widetilde{u}(x)$. We can also assume that $D_k\widetilde{u}(x)$, $k = 1, \ldots, d$, exist (again, both as densities and as partial derivatives). Thus

$$\begin{aligned} D_k M_\Omega u(x) &= \lim_{h\to 0^+} \frac{1}{h}\left(M_\Omega u(x+he_k) - M_\Omega u(x)\right) \\ &\ge \lim_{h\to 0^+} \frac{1}{h}\left(\widetilde{u}(x+he_k) - \widetilde{u}(x)\right) \\ &= D_k\widetilde{u}(x). \end{aligned}$$

Similarly,

$$\begin{aligned} D_k M_\Omega u(x) &= \lim_{h\to 0^+} \frac{1}{h}\left(M_\Omega u(x) - M_\Omega u(x-he_k)\right) \\ &\le \lim_{h\to 0^+} \frac{1}{h}\left(\widetilde{u}(x) - \widetilde{u}(x-he_k)\right) \\ &= D_k\widetilde{u}(x). \quad \square \end{aligned}$$

From the viewpoint of having a formula for $DM_\Omega u$, we would of course like to have equality in Theorem 7.4(1), which in particular happens if $|Du|(\partial B) = 0$. With this in mind, we prove the following lemma.

Lemma 7.6. *Let $d \ge 2$ and let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. For $\mathcal{L}^d$-almost every $x \in \Omega$, we have*

$$|Du|(\partial B) = 0$$

for every ball B with $\overline{B} \subset \Omega$ and $x \in \partial B$.

Proof. We can apply Lemma 7.2 with the choice $\nu = |Du|$ to obtain that there are at most countably many spheres ∂B such that $|Du|(\partial B) > 0$. Thus $|Du|(\partial B) > 0$ and $x \in \partial B$ can only be true if x belongs to the countable union of spheres, and such a union of course has Lebesgue measure zero. $\square$

The following fact is easy to prove, see [10, Lemma 2.2(2)].

Lemma 7.7. *Let u be a function on Ω and let $x \in \Omega$. If there is a ball $B \in \mathcal{B}_x$ with $x \in B$, then $DM_\Omega u(x) = 0$ (as a classical derivative).*

Now we can prove the following formula for the gradient of the maximal function.

Theorem 7.8. *Let $u \in \mathrm{BV}(\Omega)$ with $M_\Omega u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. Then for almost every $x \in \Omega$,*

$$(1)\ DM_\Omega u(x) = \frac{D|u|(B)}{\mathcal{L}^d(B)} \quad \text{if } B \in \mathcal{B}_x,\ \overline{B} \subset \Omega,\ DM_\Omega u(x) \neq 0, \text{ and } d \geq 2,$$

$$(2)\ DM_\Omega u(x) = D|\widetilde{u}|(x) \quad \text{if } \mathcal{B}_x = \emptyset.$$

In Examples 7.9 and 7.10 we will show that the assumptions $\overline{B} \subset \Omega$, $DM_\Omega u(x) \neq 0$, and $d \geq 2$ are needed.

Proof. First suppose that $B \in \mathcal{B}_x$, $\overline{B} \subset \Omega$, $DM_\Omega u(x) \neq 0$, and $d \geq 2$. Note that since $DM_\Omega u(x) \neq 0$, we have $x \in \partial B$ by Lemma 7.7. Then by Lemma 7.6 we can assume that $|Du|(\partial B) = 0$ (recall (2.6)). Thus by Theorem 7.4(1), we get (1).

If $\mathcal{B}_x = \emptyset$, Theorem 7.4(2) gives (2). □

Example 7.9. On the real line, take $\Omega = \mathbb{R}$ and

$$u(x) := \begin{cases} x & \text{when } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $u \in \mathrm{BV}(\mathbb{R})$. Now obviously for every $x \in [0, 1]$ we have $\mathcal{B}_x = \{(x, 1)\}$, so that

$$Mu(x) = \fint_{(x,1)} u(t)\, dt = \frac{1}{2}(1 + x).$$

Hence $DMu(x) = 1/2$ for all $x \in (0, 1)$, but

$$\frac{Du((x,1))}{\mathcal{L}^1((x,1))} = \frac{Du([x,1))}{\mathcal{L}^1((x,1))} = 1 \quad \text{and} \quad \frac{Du((x,1])}{\mathcal{L}^1((x,1))} = \frac{Du([x,1])}{\mathcal{L}^1((x,1))} = 1 - \frac{1}{1-x}.$$

Thus Theorem 7.8(1) fails, also if B is replaced by any half-open interval. Hence the assumption $d \geq 2$ is necessary.

A small modification of this example shows that the assumption $\overline{B} \subset \Omega$ is also needed. On the plane, let $\Omega = (0, 1) \times (-2, 2)$ and

$$u(x_1, x_2) := x_1.$$

Now obviously for every $x \in (0, 1) \times (-1, 1)$, we have $\mathcal{B}_x = \{B_x\}$ with

$$B_x := \left\{ B\left(\left(\frac{1 + x_1}{2}, x_2\right), \frac{1 - x_1}{2}\right)\right\},$$

so that

$$M_\Omega u(x) = \fint_{B_x} u\,dy = \frac{1}{2}(1+x_1).$$

Hence $D_1 M_\Omega u(x) = 1/2$ for all $x \in (0,1)\times(-1,1)$, but

$$\frac{D_1 u(B_x)}{\mathcal{L}^2(B_x)} = 1,$$

showing that the assumption $\overline{B} \subset \Omega$ is necessary. Moreover $|Du|(\partial B_x \setminus \partial\Omega) = 0$, so again including any part of the boundary of B_x would not help either.

Finally we give an example showing that the assumption $DM_\Omega u(x) \neq 0$ is also needed in Theorem 7.8.

Example 7.10. On the plane, let $\Omega = \mathbb{R}^2$ and

$$E_0 := B(0,1) \setminus B(0,1-\delta)$$

for a small δ; choose $\delta = 0.01$. Then for every $x \in B(0,\delta)$, we claim that $\mathcal{B}_x^{E_0} = \{B(0,1)\}$; we use the superscript E_0 to specify that we consider the collection of optimal balls with respect to the function $\mathbb{1}_{E_0}$. To see this, fix $x \in B(0,\delta)$ and $B(z,r) \in \mathcal{B}_x^{E_0}$ (such a disk is easily seen to exist). Note first that necessarily $0.49 \le r \le 1$. Now we simply check three cases.

First suppose $0.49 \le r \le 0.63$. Now $B(z,r)$ intersects less than $22/100$ of $\partial B(0,1-\delta)$, and so

$$\begin{aligned} m(B(z,r)) &:= \frac{\mathcal{L}^2(E_0 \cap B(z,r))}{\mathcal{L}^2(B(z,r))} \le \frac{\mathcal{L}^2(E_0 \cap B(z,0.63))}{\mathcal{L}^2(B(z,0.49))} \\ &< \frac{\frac{22}{100} \times 2\pi\delta}{0.49^2 \mathcal{L}^2(B(0,1))} < \frac{0.99 \times 2\pi\delta}{\mathcal{L}^2(B(0,1))} < \frac{\mathcal{L}^2(E_0)}{\mathcal{L}^2(B(0,1))} = m(B(0,1)). \end{aligned}$$

Then suppose $0.63 \le r \le 0.8$. Now $B(z,r)$ intersects less than $1/3$ of $\partial B(0,1-\delta)$, and so

$$m(B(z,r)) < \frac{\frac{1}{3} \times 2\pi\delta}{0.63^2 \mathcal{L}^2(B(0,1))} < m(B(0,1)).$$

Finally suppose $0.8 \le r \le 0.95$. Now $B(z,r)$ intersects less than $1/2$ of $\partial B(0,1-\delta)$, and so

$$m(B(z,r)) < \frac{\frac{1}{2} \times 2\pi\delta}{0.8^2 \mathcal{L}^2(B(0,1))} < m(B(0,1)).$$

Thus necessarily $0.95 \le r \le 1$. But now $B(z,r) \supset \partial B(0,1-\delta)$, because otherwise $B(z,r)$ covers less than $3/4$ of $\partial B(0,1-\delta/2)$, implying that

$$m(B(z,r)) < \frac{\frac{7}{8} \times 2\pi\delta}{0.95^2 \mathcal{L}^2(B(0,1))} < m(B(0,1)).$$

Thus $B(z,r)$ contains $B(0,1-\delta)$, and now clearly the maximum value of $m(B(z,r))$ is obtained by choosing $B(z,r)=B(0,1)$. Thus $\mathcal{B}_x^{E_0}=\{B(0,1)\}$ as desired.

Denote $c:=(0,1)\in\partial B(0,1)$. Next we "perturb" E_0 slightly by removing and adding a small ball:

$$E:=(E_0\setminus B(c,\delta^2))\cup(B(c,\delta^2)\setminus B(0,1)).$$

Since the perturbation is so small, almost the same calculations as above show that for every $x\in B(0,\delta)$, we have $\mathcal{B}_x^E=\{B(0,1)\}$. Thus $DM\mathbb{1}_E=0$ in $B(0,\delta)$, but

$$\frac{D_2\mathbb{1}_E(B(0,1))}{\mathcal{L}^2(B(0,1))}<0,\quad \frac{D_2\mathbb{1}_E(\overline{B}(0,1))}{\mathcal{L}^2(B(0,1))}>0,$$
$$\frac{D_2\mathbb{1}_E(\overline{B}^{2,+}(0,1))}{\mathcal{L}^2(B(0,1))}<0,\quad \frac{D_2\mathbb{1}_E(\overline{B}^{2,-}(0,1))}{\mathcal{L}^2(B(0,1))}>0.$$

Thus none of these equal $DM\mathbb{1}_E$ in $B(0,\delta)$, which is of course a set of nonzero Lebesgue measure.

8. The one-dimensional case

In this section we investigate the properties of the non-centered maximal function in the special case $d=1$. Let $\Omega\subset\mathbb{R}$ be an arbitrary nonempty open set.

Aldaz and Pérez Lázaro [1] proved in one dimension that the non-centered maximal function of a function $u\in\mathrm{BV}(\Omega)$ is in the Sobolev class $W^{1,1}_{\mathrm{loc}}(\Omega)$, with $\|DM_\Omega u\|_{L^1(\Omega)}\le|Du|(\Omega)$. Now we investigate the behavior of $M_\Omega u$ a little further.

Recall that

$$H_u=\{x\in\Omega\colon\, M_\Omega u(x)>|u|^\vee(x)\}.$$

Recall also from the proof of Theorem 5.4 and the Open Problem on Page 69 that it is in some sense desirable that the set H_u be as large as possible. On the real line, we are able to show the following. Recall that S_u is the set of non-Lebesgue points, or (as we are in one dimension) the set of discontinuity points of $u^\vee$ (alternatively some other good pointwise representative). Moreover, denote by ∂^*E the measure-theoretic boundary of a set $E\subset\mathbb{R}$, i.e. the set of points $x\in\mathbb{R}$ for which

$$\limsup_{r\to 0}\frac{\mathcal{L}^1(B(x,r)\cap E)}{\mathcal{L}^1(B(x,r))}>0\quad\text{and}\quad\limsup_{r\to 0}\frac{\mathcal{L}^1(B(x,r)\setminus E)}{\mathcal{L}^1(B(x,r))}>0.$$

Proposition 8.1. *Let $u\in\mathrm{BV}_{\mathrm{loc}}(\Omega)$ with $|Du|(\Omega)<\infty$. Then $|Du|(\Omega\setminus(H_u\cup S_u))=0$.*

Proof. Abbreviate super-level sets by

$$\{u>t\}:=\{x\in\Omega\colon\, u(x)>t\},\quad t\in\mathbb{R}.$$

We have the coarea formula (see e.g. [3, Theorem 3.40])

$$|Du|(A) = \int_{-\infty}^{\infty} |D\mathbb{1}_{\{u>t\}}|(A)\, dt \tag{8.2}$$

for every Borel set $A \subset \Omega$. In particular, for almost every $t \in \mathbb{R}$, the super-level set $\{u > t\}$ has finite perimeter in Ω. Let $N \subset \mathbb{R}$ be the exceptional set. For later purposes, we also include in N the at most countably many $t \in \mathbb{R}$ for which $\mathcal{L}^1(\{u = t\}) > 0$.

Consider $t \in \mathbb{R} \setminus N$. First assume that $t \geq 0$. The fact that $\{u > t\}$ has finite perimeter in Ω means that after redefinition in a set of measure zero, giving a set that we can denote E_t, the relative boundary $\partial E_t \cap \Omega \subset \Omega$ consists of finitely many points; see [3, Proposition 3.52]. It follows that if $x \in \partial E_t \cap \Omega$, then for small enough $r > 0$, necessarily $\mathcal{L}^1((x - r, x) \setminus \{u \leq t\}) = 0$ and $\mathcal{L}^1((x, x + r) \setminus \{u > t\}) = 0$, or vice versa. Supposing without loss of generality the former, we have

$$M_\Omega u(x) \geq \fint_{(x,x+r)} u\, dy > t.$$

If also $x \notin S_u$, then $t = u^\vee(x) = |u|^\vee(x)$, and we get $x \in H_u$. In conclusion, if $x \in \partial^*\{u > t\} \cap \Omega \setminus S_u \subset \partial E_t \cap \Omega \setminus S_u$ for some $t \in \mathbb{R} \setminus N$, then $x \in H_u$.

For $t < 0$, we similarly get that if $x \in \partial E_t \cap \Omega$ for $t \in \mathbb{R} \setminus N$, then necessarily $\mathcal{L}^1((x - r, x) \setminus \{u < t\}) = 0$ and $\mathcal{L}^1((x, x + r) \setminus \{u > t\}) = 0$, or vice versa (recall that $\mathcal{L}^1(\{u = t\}) = 0$). Supposing the former,

$$M_\Omega u(x) \geq \fint_{(x-r,x)} |u|\, dy > |t|.$$

If $x \notin S_u$, then $|t| = |u|^\vee(x)$ and we get $x \in H_u$, as before.

Consider a point x in the Borel set $A := \Omega \setminus (H_u \cup S_u)$. Now $x \notin \partial^*\{u > t\}$ for all $t \in \mathbb{R} \setminus N$. For every $t \in \mathbb{R} \setminus N$ we also have

$$|D\mathbb{1}_{\{u>t\}}|(A) = \mathcal{H}^0(\partial^*\{u > t\} \cap A), \tag{8.3}$$

see e.g. [3, Theorems 3.59 & 3.61]. Thus by (8.2) we get

$$|Du|(A) = \int_{-\infty}^{\infty} \mathcal{H}^0(\partial^*\{u > t\} \cap A)\, dt = \int_N \mathcal{H}^0(\partial^*\{u > t\} \cap A)\, dt = 0. \quad \square$$

Since $M_\Omega u$ is continuous (even absolutely) and $u^\vee$ is upper semicontinuous, H_u is an open set, so it is the union of disjoint open intervals

$$H_u = \bigcup_{j=1}^{\infty} (a_j, b_j) \subset \Omega.$$

Note that for two of these intervals, a_j or b_j may be $\pm\infty$.

Theorem 8.4. *Let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ with $|Du|(\Omega) < \infty$. For each $j \in \mathbb{N}$ there exists a point $c_j \in (a_j, b_j)$ such that we have the representation*

$$|DM_\Omega u|(\Omega) = \sum_{j=1}^{\infty} \big[|M_\Omega u(a_j) - M_\Omega u(c_j)| + |M_\Omega u(c_j) - M_\Omega u(b_j)|\big].$$

If a_j (or b_j) is $\pm\infty$, we interpret $M_\Omega u(a_j)$ as a limit; it exists since $M_\Omega u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ with $\|DM_\Omega u\|_{L^1(\Omega)} \le |Du|(\Omega)$ by [1, Theorem 2.5].

Proof. The set S_u is at most countable and $M_\Omega u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, so we have

$$|DM_\Omega u|(S_u) = 0. \tag{8.5}$$

Consider a point $x \in \Omega \setminus (S_u \cup H_u)$. Now $M_\Omega u(x) = |u|^\vee(x)$. Since both functions $M_\Omega u$ and $|u|^\vee$ are continuous at x, this point can be in $\partial^*\{M_\Omega u - |u| > t\}$ only for $t = 0$. By the coarea formula (8.2) and (8.3), we have

$$\begin{aligned} |D(M_\Omega u - |u|)|(\Omega \setminus (S_u \cup H_u)) &= \int_{-\infty}^{\infty} \mathcal{H}^0(\Omega \cap \partial^*\{M_\Omega u - |u| > t\} \setminus (S_u \cup H_u))\, dt \\ &= 0. \end{aligned}$$

By Proposition 8.1, it follows that

$$|DM_\Omega u|(\Omega \setminus (S_u \cup H_u)) = |D|u||(\Omega \setminus (S_u \cup H_u)) \le |Du|(\Omega \setminus (S_u \cup H_u)) = 0.$$

Thus by (8.5),

$$|DM_\Omega u|(\Omega \setminus H_u) = 0.$$

Thus all of the total variation of $M_\Omega u$ is in the set $H_u = \bigcup_{j=1}^{\infty}(a_j, b_j)$.

Now we follow an argument given in [1]. Suppose that for some $j \in \mathbb{N}$, there exist points d_1, d_2, d_3 with $a_j < d_1 < d_2 < d_3 < b_j$ and $M_\Omega u(d_1) < M_\Omega u(d_2)$ and $M_\Omega u(d_3) < M_\Omega u(d_2)$. We can assume that $M_\Omega u(d_2) = \max\{M_\Omega u(x) : x \in [d_1, d_3]\}$. Then by [1, Lemma 3.6] we have $M_\Omega u(d_2) = |u|^\vee(d_2)$, a contradiction with $d_2 \in H_u$.

It follows that for every $j \in \mathbb{N}$, either $M_\Omega u$ is monotone on (a_j, b_j) or there exists $c_j \in (a_j, b_j)$ such that $M_\Omega u$ is decreasing on $[a_j, c_j]$ and increasing on $[c_j, b_j]$. In the former case, we can just choose an arbitrary $c_j \in (a_j, b_j)$. Now

$$|DM_\Omega u|(\Omega) = |DM_\Omega u|(H_u) = \sum_{j=1}^{\infty} \big[|M_\Omega u(a_j) - M_\Omega u(c_j)| + |M_\Omega u(c_j) - M_\Omega u(b_j)|\big]. \qquad \square$$

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