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Principal Eigenvalues and Sturm Comparison via Picone's Identity*

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We characterize the principal eigenvalues and eigenfunctions in $\mathbb{R}^N$, and present comparison results, for higher dimensional p-Laplacian. Our main tool is Picone's identity. In this way we extend several recent results on spectral and comparison properties for differential equations. © 1999 Academic Press

Key Words: Picone's identity; principal eigenvalues; Sturm comparison.

INTRODUCTION

This paper is primarily motivated by the recent articles by Jin [J], Zhu [Z], and Walter [W], on spectral and comparison results for differential equations. Specifically, in [J], there is a discussion of the existence and nonexistence of eigenvalues/solutions for linear Schrödinger equations with indefinite weight functions, while in [Z, W] the authors present Sturmian results for nonlinear ordinary differential inequalities of the p-Laplacian type.

In this article, we extend the results of [J] to the p-Laplacian and present a version of the results in [Z, W] suitable for partial differential equations. We obtain some modest extensions of the latter results even in the one-dimensional case. We do not employ the methods in [J, Z, W], since

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it does not appear that they are amenable to the extensions we seek (division by specific functions thus exploiting linearity in [J], Bernoulli transformations in [Z, W]). Instead we proceed by arguments involving a suitable Picone identity. This approach is of extreme simplicity.

This paper is arranged as follows. In the next section we present some background material. The following two sections deal with principal eigenvalues and with Oscillation and Sturmian theory.

Although we occasionally refer to such topics, we do not consider existence/regularity conditions in this paper. The interested reader will find these subjects discussed in detail in some of the cited references. Here, for the sake of presentational simplicity, we assume that all functions introduced are smooth in their arguments, unless otherwise specified. Solutions to differential inequalities are understood in the usual weak sense and are assumed to be of class $C^{1+\alpha}$ at least. Throughout this paper we will use $\|\cdot\|$ and $\|\cdot\|_r$ to denote the norms in $W^{1,p}$ and L^r $(r>1)$ respectively.

1. BACKGROUND

Let Ω be a domain in $\mathbb{R}^n$, bounded or unbounded, with $N>p>1$. We consider the equation

$$-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = \lambda g\,|u|^{p-2}\,u \qquad \text{in} \quad \Omega \tag{1.1}$$

in two ways: We say that $u \in W^{1,p}_{\mathrm{loc}}$ is a solution of (1.1) if and only if for all $\varphi \in C_0^\infty(\Omega)$ we have

$$L(u, \varphi) := \int_\Omega |\nabla u|^{p-2}\,\nabla u \nabla \varphi = \int_\Omega g\,|u|^{p-2}\,u\varphi := R(u, \varphi), \tag{1.2}$$

while $u \not\equiv 0$ is an eigenfunction of (1.1) if and only if u is a solution of (1.1) and furthermore there exists a sequence $\{\varphi_n\} \subset C_0^\infty(\Omega)$ such that $L(\varphi_n, \varphi_n) \to L(u, u)$ and $R(\varphi_n, \varphi_n) \to R(u, u)$. Note that if Ω is bounded, (1.2) implies that the eigenfunctions satisfy $u=0$ on $\partial\Omega$.

We say u is a supersolution of (1.1), or a solution to the inequality

$$-\Delta_p u \geqslant g\,|u|^{p-2}\,u$$

if and only if for any $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqslant 0$,

$$\int_\Omega |\nabla u|^{p-2}\,\nabla u \nabla \varphi \geqslant \int_\Omega g\,|u|^{p-2}\,u\varphi.$$

Observe that a solution may not be an eigenfunction. If Ω is unbounded, the existence of a positive eigenfunction may be shown under suitable conditions on g, but in general there may not be any. Sufficient conditions can be found in the literature. See, e.g., [AB, AH1, FMST, H].

We assume that the coefficient $g \in L^{N/p}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$. Then by standard regularity results (cf. Tolksdorff [T]), any solution u of (1.1) satisfies that $u \in C^{1+\alpha}(\Omega')$ for any bounded domain $\Omega' \subset \Omega$. Moreover, if u is a supersolution of (1.1), then Theorems 1.2 and 5.1 of [Tr] asserts that, for any ball $B(2r) \subset \Omega$ with radius $2r$, a Harnack inequality

$$\max_{B(r)} u \leqslant C \min_{B(2r)} u \tag{1.3}$$

holds, where C depends on N, p, the radius r and $\|g\|_{\infty,\, B(2r)}$. It then follows that, in particular, if u is a nonnegative solution of (1.1) then either $u > 0$ or $u \equiv 0$.

We finally recall the following Picone's identity (cf. [AH2, D]):

Picone's Identity. Suppose $v > 0$ and $u \geqslant 0$ are differentiable. Let

$$\begin{aligned} L_1(u, v) &= |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u\, |\nabla v|^{p-2}\nabla v, \\ R_1(u, v) &= |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v. \end{aligned} \tag{1.4}$$

Then $L_1(u, v) = R_1(u, v)$, $L_1(u, v) \geqslant 0$, and $L_1(u, v) = 0$ a.e. Ω if and only if $\nabla(u/v) = 0$ a.e. Ω, i.e., $u = kv$ for some constant k in each component of Ω.

2. PRINCIPAL EIGENVALUES AND POSITIVE SOLUTIONS

We observe the following consequence of (1.4).

THEOREM 2.1. *Suppose $-\Delta_p u \geqslant g_1 |u|^{p-2} u$ has a positive solution w in Ω. If $g \leqslant g_1$, then so does $-\Delta_p u = g\, |u|^{p-2} u$.*

Proof. Exhaust Ω by a family of bounded domains $\bar{\Omega}_k \subset \Omega_{k+1}$ with $\Omega = \bigcup_k \Omega_k$. For $\varphi \in C_0^{\infty}(\Omega_k)$, let $\varphi^{\pm} = \max\{\pm\varphi, 0\}$. We have, from Picone's identity,

$$\begin{aligned} 0 \leqslant \int_{\Omega_k} L_1(\varphi^{\pm}, w) &= \int_{\Omega_k} |\nabla\varphi^{\pm}|^p - \int_{\Omega_k} \frac{(\varphi^{\pm})^p}{w^{p-1}}(-\Delta_p w) \\ &\leqslant \int_{\Omega_k} (|\nabla\varphi^{\pm}|^p - g_1(\varphi^{\pm})^p), \end{aligned} \tag{2.1}$$

and the equalities hold only if $\varphi^{\pm} \equiv 0$ or $\varphi^{\pm} = cw$, the latter is impossible. It then follows that

$$0 \leqslant \int_{\Omega_k} (|\nabla\varphi|^p - g_1 |\varphi|^p) \leqslant \int_{\Omega_k} (|\nabla\varphi|^p - g\,|\varphi|^p) \tag{2.2}$$

for $\varphi \in C_0^\infty(\Omega_k)$.

We claim that, for any k, there exists $c_k > 0$ such that

$$\int_{\Omega_k} (|\nabla\varphi|^p - g\,|\varphi|^p) \geqslant c_k \int_{\Omega_k} (|\nabla\varphi|^p + |\varphi|^p) \tag{2.3}$$

for $\varphi \in C_0^\infty(\Omega_k)$. Suppose the contrary. Then for some $k > 0$, there exists $\varphi_n \in C_0^\infty(\Omega_k)$ with $\|\varphi_n\| = 1$ such that

$$\int_{\Omega_k} (|\nabla\varphi_n|^p - g\,|\varphi_n|^p) \leqslant \frac{1}{n}.$$

By (2.2) we have

$$0 \leqslant \int_{\Omega_k} (|\nabla\varphi_n|^p - g_1 |\varphi_n|^p) \leqslant \frac{1}{n}.$$

Without loss of generality we can assume that $\varphi_n \to \varphi_0$ in $L^r(\Omega_k)$ for some $\varphi_0 \in W_0^{1,\,p}(\Omega_k)$, where $1 < r < Np/(N-p)$. It then follows from (2.2) and the convergence that

$$0 \leqslant \int_{\Omega_k} (|\nabla\varphi_0|^p - g\,|\varphi_0|^p) = 0.$$

This implies from (2.1) and (2.2) that $\varphi_0 \equiv 0$, contradicting to the fact that $\|\varphi_n\| = 1$. Thus (2.3) must hold.

Let $f_k \in C_0^1(\Omega_k)$ be such that $f_k \geqslant 0$, $\|f_k\| = 1$, and $\mathrm{supp}(f_k)$ is contained in a neighbourhood of $\partial\Omega_k$. By (2.3) and the maximum principle, the problem

$$\begin{aligned} -\Delta_p u - g\,|u|^{p-2}\,u &= f_k, \qquad x \in \Omega_k, \\ u &= 0, \qquad x \in \partial\Omega_k \end{aligned} \tag{2.4}$$

has a positive solution u_k. We normalize the solution u_k such that $u_k(x_0) = 1$ for some fixed point $x_0 \in \Omega_1$ (note that only f_k would have been modified accordingly). Obviously such u_k is also a supersolution of the modified equation (2.4). Thus Harnack's inequality (1.3) is applicable. We then have, for any fixed Ω_m,

$$\|u_k\|_\infty \leqslant C,$$

by (2.3), (2.4) and the definition of f_k, with C independent of k. Without loss of generality, this implies that $u_k \to u$ in $C^{1+\alpha}$ by [T] for some α on every Ω_m with $u \geqslant 0$ and $u \not\equiv 0$ since $u(x_0) = 1$. It then follows from Harnack's inequality that $u > 0$ on Ω and u solves the equation. This proves the theorem. ∎

Remark 2.2. Note that we do not need a fixed $w > 0$ for all k in Theorem 2.1. We need only assume the existence of a $w_k > 0$ in Ω_k, solution $-\Delta_p u = g_1 |u|^{p-2} u$ in Ω_k. I.e., w can change with k.

Now consider the eigenvalue problem

$$-\Delta_p u = \lambda g\, |u|^{p-2}\, u \tag{2.5}$$

on Ω. Sufficient conditions on g have been given (see, e.g., [AB, AH1, FMST, H, J]) so that (2.5) has positive eigenfunctions. In particular, if $g \in L^{N/p}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, $g^+ \not\equiv 0$ and $g^- \not\equiv 0$, then there exist principal eigenvalues $\lambda^+ > 0 > \lambda^-$ such that (2.5) has positive eigenfuntions associated with λ^+ and λ^- respectively. We note that the principal eigenvalues are characterized by the relation

$$\int_\Omega |\nabla\varphi|^p \geqslant \max\left\{\lambda^+ \int_\Omega g\, |\varphi|^p,\ \lambda^- \int_\Omega g\, |\varphi|^p\right\}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Our next result is related to Theorem 1 of Jin and Theorem 1.1 of Afrouzi and Brown for $p = 2$, which give another characterization of the principal eigenvalues λ^+ and λ^-.

THEOREM 2.3. *Let* $-\Delta_p w_1 = \lambda^+ g w_1^{p-1}$ *and* $-\Delta_p w_2 = \lambda^- g w_2^{p-1}$ *in* Ω *with* w_1 *and* w_2 *positive eigenfunctions,* $\lambda^- < 0 < \lambda^+$. *Then problem* (2.5) *has a positive solution if and only if* $\lambda \in [\lambda^-, \lambda^+]$.

Proof. It follows immediately from Theorem 2.1 that for $\lambda \in [\lambda^-, \lambda^+]$, (2.5) has a positive solution. Indeed, let $\lambda \in (\lambda^-, \lambda^+)$. Observe that for $\varphi \in C_0^\infty(\Omega)$ we have

$$\int (|\nabla\varphi|^p - \lambda g\, |\varphi|^p) \geqslant \max\left\{\lambda^+ \int_\Omega g\, |\varphi|^p,\ \lambda^- \int_\Omega g\, |\varphi|^p\right\} - \lambda \int g\, |\varphi|^p \geqslant 0.$$

Note that equality cannot hold for $\varphi \not\equiv 0$, since there are no eigenvalues in (λ^-, λ^+). We conclude from Theorem 2.1 and (2.2), (2.4) in particular, the existence of a $w \geqslant 0$ such that $-\Delta_p w - \lambda g w^{p-1} = f(x)$ with $0 \leqslant f \in C_0(\Omega)$. Now we apply Theorem 2.1.

Conversely, suppose for some $\lambda_0 > \lambda^+$, (2.5) has a positive solution w. Then, again by Picone's identity, for any $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$,

$$0 \leqslant \int_\Omega L_1(\varphi, w) = \int_\Omega |\nabla\varphi|^p - \int_\Omega \frac{\varphi^p}{w^{p-1}}(-\Delta_p w) \leqslant \int_\Omega (|\nabla\varphi|^p - \lambda_0 g \varphi^p).$$

Letting $\varphi \to w_1$ we obtain

$$0 \leqslant \int_\Omega L_1(w_1, w_1) \leqslant (\lambda^+ - \lambda_0) \int_\Omega g w_1^p < 0,$$

a contradiction. So $\lambda_0 \leqslant \lambda^+$. The fact that $\lambda \geqslant \lambda^-$ can be shown in the same way. This concludes the proof. ∎

3. OSCILLATION AND STURM COMPARISON

In this section we will use Theorems 2.1 and 2.3 to prove and generalize some osillation and comparison results.

Let $\mathscr{L}w = -\Delta_p w - g\,|w|^{p-2}\,w$. As a direct consequence of Theorem 2.3, we have

COROLLARY 3.1. *Let $\tilde{\Omega}$ be a bounded subdomain of Ω. Suppose that $-\Delta_p w - g\,|w|^{p-2}\,w = 0$ in $\tilde{\Omega}$ and $w = 0$ on $\partial\tilde{\Omega}$ has a positive solution. Then $\mathscr{L}w = 0$ has no positive solution in the whole of Ω, i.e., all solutions of $\mathscr{L}w = 0$ in Ω must change sign.*

We note that the subdomain $\tilde{\Omega}$ given in Corollary 3.1 is called a nodal domain.

DEFINITION 3.2. $\mathscr{L}$ is oscillatory if and only if given any neighbourhood $\mathscr{N}$ of infinity, we have a nodal domain M in $\mathscr{N} \cap \Omega$. $\mathscr{L}$ is nonoscillatory otherwise.

COROLLARY 3.3. *$\mathscr{L}$ is nonoscillatory if there exists a neighbourhood $\mathscr{N}$ of infinity such that $\mathscr{L}w = 0$ has a positive solution in $\mathscr{N} \cap \Omega$.*

Next, we generalize Theorem B of [Z], which is proved for radially symmetric function $g(|x|, u)$ and concerns with the existence of positive radial solutions. Consider the problem:

$$\begin{aligned} -\Delta_p u &= g(x, u), && x \in B_K, \\ u &= 0, && x \in \partial B_K, \end{aligned} \tag{3.1}$$

where B_K denotes the ball in $\mathbb{R}^n$ centered at the origin with radius K. Let $M > 0$ and denote $\tilde{M} = (N/M)^{1/p}\,(p/(p-1))^{(p-1)/p}$.

Theorem 3.4. *Suppose $g(x, u)$ satisfies*

$$0 \leqslant g(x, t) \leqslant M t^{p-1}, \qquad t \geqslant 0. \tag{3.2}$$

Then (3.1) *does not have any positive solutions for any $K < \tilde{M}$.*

Proof. Suppose that (3.1) has a positive solution $u_1 > 0$. This shows that the principal eigenvalue to the problem

$$\begin{aligned} -\Delta_p u &= \frac{g(x, u_1)}{u_1^{p-1}}\, u^{p-1}, && x \in B_K, \\ u &= 0, && x \in \partial B_K, \end{aligned} \tag{3.3}$$

is $\lambda^+ = 1$ (the associated positive eigenfunction is u_1). On the other hand, a calculation shows that (cf. the proof of Theorem 4.3 of [Z]), $w(x) = \tilde{M}^{(p-1)/p} - |x|^{p/(p-1)}$ satisfies the inequality $-\Delta_p w \geqslant M w^{p-1}$ on $B_{\tilde{M}}$. Since $g(x, u)/u^{p-1} \leqslant M$ for $u > 0$, Theorem 2.1 implies that

$$-\Delta_p v = \frac{g(x, u_1)}{u_1^{p-1}}\, v^{p-1}$$

has a positive solution in $B_{\tilde{M}}$. In particular $v > 0$ on ∂B_K. We then derive from Theorem 2.3 that the principal eigenvalue λ^+ of (3.3) satisfies $\lambda^+ > 1$, a contradiction. Thus (3.1) cannot have any positive solution. This ends the proof. ▮

As a direct consequence, we have

Corollary 3.5. *Assume $g(x, u)$ satisfies* (3.2). *Then the problem*

$$\begin{aligned} -\Delta_p u &= g(x, u), && x \in \Omega', \\ u &= 0, && x \in \partial \Omega', \end{aligned} \tag{3.1$'$}$$

has no positive solution for any $\Omega' \subset\subset B_{\tilde{M}}$.

Observe that a similar result holds for any cylinder with $B_{\tilde{M}}$ as cross section. As an explicit example, note that $w(x_1) = \tilde{M}^{(p-1)/p} - |x_1|^{p/(p-1)}$ satisfies $-\Delta_p w \geqslant M w^{p-1}$ in $(-\tilde{M}, \tilde{M})$, where $\tilde{M} = (1/M)^{1/p} (p/(p-1))^{(p-1)/p}$. Consequently w satisfies the same inequality in the cylinder

$$\mathscr{C} = (-\tilde{M}, \tilde{M}) \times \Pi_{i=2}^{N} (-\infty, \infty).$$

Corollary 3.5 then holds for any $\Omega' \subset\subset \mathscr{C}$. We are not aware of other non-radial results of this type.

Our next results deal with related comparison principles. Consider, for $i = 1, 2$,

$$-\operatorname{div}[\varphi_i(x, u_i) \, |\nabla u_i|^{p-2} \, \nabla u_i] = g_i(x, u_i), \qquad x \in \Omega \tag{3.4$_i$}$$

with

$$\varphi_1(x, u) \geqslant \varphi_2(x, v) > 0, \; \frac{g_1(x, u)}{u^{p-1}} \leqslant \frac{g_2(x, v)}{v^{p-1}}, \qquad \text{for any } 0 \leqslant v \leqslant u \text{ and } x \in \Omega. \tag{3.5}$$

THEOREM 3.6. *Suppose that u_i is a positive solution of* (3.4)$_i$ *for* $i = 1, 2$. *Assume that the Divergence theorem is applicable to u_i on Ω, Ω is bounded and that*

$$\left[\varphi_1(x, u_1) \, |\nabla u_1|^{p-2} \, \frac{\partial u_1}{\partial n} - \varphi_2(x, u_2) \left(\frac{u_1}{u_2} \right)^{p-1} |\nabla u_2|^{p-2} \, \frac{\partial u_2}{\partial n} \right] \leqslant 0 \tag{3.6}$$

on $\partial\Omega$. Then $u_1 \geqslant u_2$ throughout Ω cannot hold, unless $u_1 = c u_2$ for some constant c.

Proof. Suppose, on the contrary, that $u_1 \geqslant u_2$ in Ω. We have u_1^p / u_2^{p-1} is well defined. By Picone's identity, we get

$$0 \leqslant \int_\Omega \varphi_2(x, u_2) \, L_1(u_1, u_2)$$

$$= \int_\Omega \varphi_2(x, u_2) \, |\nabla u_1|^p - \int_\Omega \varphi_2(x, u_2) \, \nabla \left(\frac{u_1^p}{u_2^{p-1}} \right) |\nabla u_2|^{p-2} \, \nabla u_2$$

$$
\begin{aligned}
&= \int_\Omega \left(-\operatorname{div}[\varphi_1(x,u_1)\,|\nabla u_1|^{p-2}\,\nabla u_1]\right) u_1 + \int_\Omega [\varphi_2(x,u_2)-\varphi_1(x,u_1)]\,|\nabla u_1|^p \\
&\quad + \int_\Omega \frac{u_1^p}{u_2^{p-1}} \operatorname{div}[\varphi_2(x,u_2)\,|\nabla u_2|^{p-2}\,\nabla u_2] \\
&\quad + \int_{\partial\Omega} \left(\varphi_1(x,u_1)\,|\nabla u_1|^{p-2}\,u_1 \frac{\partial u_1}{\partial n} - \varphi_2(x,u_2)\,\frac{u_1^p}{u_2^{p-1}}\,|\nabla u_2|^{p-2}\,\frac{\partial u_2}{\partial n}\right) ds \\
&= \int_\Omega \left[\frac{g_1(x,u_1)}{u_1^{p-1}} - \frac{g_2(x,u_2)}{u_2^{p-1}}\right] u_1^p + \int_\Omega [\varphi_2(x,u_2)-\varphi_1(x,u_1)]\,|\nabla u_1|^p \\
&\quad + \int_{\partial\Omega} \left[\varphi_1(x,u_1)\,|\nabla u_1|^{p-2}\,\frac{\partial u_1}{\partial n} - \varphi_2(x,u_2)\left(\frac{u_1}{u_2}\right)^{p-1} |\nabla u_2|^{p-2}\,\frac{\partial u_2}{\partial n}\right] u_1\,ds \\
&\leqslant 0. \qquad (3.
\end{aligned}
$$

This implies $u_1 = cu_2$ with some c, which contradicts (3.5). Thus the theorem is proved. ▮

COROLLARY 3.7. *If $u_1 = u_2$ on $\partial\Omega$ then $u_1 \geqslant u_2$ cannot hold throughout Ω, unless $u_1 = cu_2$.*

Proof. Indeed if $u_1 = u_2$ on $\partial\Omega$ and $u_1 \geqslant u_2$ in Ω, then (3.6) holds and we obtain a contradiction unless u_1, u_2 are constant multiples. ▮

COROLLARY 3.8. *Suppose g_1, g_2, φ_1 and φ_2 are radially symmetric in the x variable, satisfy* (3.5), *and Ω is a ball centered at the origin. If u_i are positive radially symmetric solutions of* $(3.4)_i$, *$i = 1, 2$, and $u_1 = u_2$ on $\partial\Omega$, then $u_1 \leqslant u_2$ in Ω, provided either $\varphi_1 \equiv \varphi_2$ or $u_2'(r) \leqslant 0$. Consequently, if $u_1(x_0) > u_2(x_0)$ for some $x_0 \in \Omega$, and u_i solves* $(3.4)_i$, *then $u_1 > u_2$ in $\bar{\Omega}$.*

Proof. Suppose the contrary. We can assume that $u_1 > u_2$ on a subdomain Ω' and $u_1 = u_2$ on $\partial\Omega'$. Since u_1 and u_2 are radially symmetric, the Divergence theorem is applicable on Ω'. Thus we can repeat the proof of Theorem 3.6 on Ω'. Observe that $\partial u_1/\partial n \leqslant \partial u_2/\partial n$ on $\partial\Omega'$, and thus the boundary integral in (3.7) is nonpositive. This leads to a contradiction as before. The proof is complete. ▮

Remark 3.9. A scrutiny of the proof of Theorem 3.6 shows that, if we replace (3.4) by

$$
\begin{aligned}
&-\operatorname{div}[\varphi_1(x,u_1)\,|\nabla u_1|^{p-2}\,\nabla u_1] \leqslant g_1(x,u_1) \\
&-\operatorname{div}[\varphi_2(x,u_2)\,|\nabla u_2|^{p-2}\,\nabla u_2] \geqslant g_2(x,u_2),
\end{aligned}
\qquad (3.4)'
$$

the conclusions of Theorem 3.6 and Corollary 3.8 remain valid.

We now apply these arguments to the inequalities considered in Theorem A of [Z] and Theorem 4 of [W]. Consider the inequalities

$$-\operatorname{div}[m_1(x)\,\psi(u)\,|\nabla u|^{p-2}\,\nabla u] \leqslant q_1(x)\,f(u),$$
$$-\operatorname{div}[m_2(x)\,\psi(v)\,|\nabla v|^{p-2}\,\nabla v] \geqslant q_2(x)\,f(v), \tag{3.8}$$

with $m_1 \geqslant m_2$ and $q_1 \leqslant q_2$, $\psi(u)$ is positive for $u > 0$. Observe that if $f(u)/u^{p-1}$ is nonincreasing for $u > 0$, then this is considered in (3.4)′ and Remark 3.9 is directly applicable. Note also that the transformation

$$w = \int_0^u [\psi(\xi)]^\mu \, d\xi := T(u)$$

with $\mu = 1/(p-1)$ changes these into inequalities of the earlier form with $f(u)$ replaced by $f(T^{-1}(w))$. We thus require by (3.5) that $f(T^{-1}(w))/w^{p-1}$, or $f^\mu(T^{-1}(w))/w$ be monotone nonincreasing. Expressing

$$f^\mu(T^{-1}(w)) = \int_0^1 [f^\mu(T^{-1}(tw))]' \, w \, dt,$$

for this it will suffice that

$$[f^\mu(T^{-1}(w))]' = \psi^{-\mu}(u)\,\frac{d}{du}\,f^\mu(u)$$

be nonincreasing. This is the condition given in [Z, W]. Note that, unlike [Z, W], we do not require that f be monotone. Our other condition in (3.6) becomes

$$m_1\,|\nabla u|^{p-2}\,\frac{\partial u}{\partial n} - m_2\left[\frac{\int_0^u \psi^\mu(\xi)}{\int_0^v \psi^\mu(\xi)}\right]^{p-1} \cdot \frac{\psi(v)}{\psi(u)} \cdot |\nabla v|^{p-2}\,\frac{\partial v}{\partial n} \leqslant 0. \tag{3.6$'$}$$

Thus we have,

Corollary 3.10. *Let u and v be positive solutions of* (3.8). *Assume that $\psi^{-\mu}(u)(d/du)\,f^\mu(u)$ is nonincreasing, where $\mu = 1/(p-1)$, and* (3.6)′ *holds on $\partial\Omega$. Then $u \geqslant v$ cannot hold throughout Ω.*

Observe that on surfaces where $u = v$ enclosing regions with $u > v$, (3.6)′ is satisfied if $\partial v/\partial n \leqslant 0$ there. Indeed if $v' \leqslant 0$ then $u' \leqslant v' \leqslant 0$ and $m_1 \geqslant m_2$ give the result. On the other hand, in keeping with [Z, W], (3.6)′ will also hold if

$$u(0) \geqslant v(0), \qquad u'(0) \geqslant v'(0), \qquad 0 \geqslant v'(0).$$

Note that since $v'(0) \leqslant 0$ then Eq. (3.8) indicates that $v'(r) \leqslant 0$ for all r if $q_2 \geqslant 0$, $f \geqslant 0$. Thus if $u'(0) \geqslant 0 \geqslant v'(0)$ the result follows as (3.6)′ holds at zero and at any point $r_0 > 0$ where $u = v$ and $u(r) > v(r)$ for $r < r_0$.

Suppose now $u(0) \geqslant v(0)$ with $v'(0) \leqslant u'(0) \leqslant 0$. Again (3.6)′ holds at any $r_0 > 0$ with the same properties as in the previous case. But, in this case (3.6)′ becomes an added condition at 0, i.e., we require

$$m_1(0)\,\frac{|u'(0)|^{p-1}\,\psi(u(0))}{\left[\int_0^{u(0)} \psi^\mu\right]^{1/\mu}} \leqslant m_2(0)\,\frac{|v'(0)|^{p-1}\,\psi(v(0))}{\left[\int_0^{v(0)} \psi^\mu\right]^{1/\mu}}. \tag{3.6*}$$

This is similar to condition (F) of [W] and (H_4) of [Z], where $\left[\int_0^{u(0)} \psi^\mu\right]^{1/\mu}$ and $\left[\int_0^{v(0)} \psi^\mu\right]^{1/\mu}$ are replaced by $f(u(0))$ and $f(v(0))$, respectively. Note that our (3.6)* is the same for all f which satisfy the earlier condition.

Furthermore, suppose condition (F) of [W] (or (H_4) of [Z]) holds. In the present case, one has

$$m_1(0)\,\frac{|u'(0)|^{p-1}\,\psi(u(0))}{f(u(0))} \leqslant m_2(0)\,\frac{|v'(0)|^{p-1}\,\psi(v(0))}{f(v(0))};$$

then since $f(T^{-1}(w))/w^{p-1}$ is assumed nonincreasing, then $u(0) \geqslant v(0)$ implies

$$\frac{f(u(0))}{\left[\int_0^{u(0)} \psi^\mu\right]^{1/\mu}} \leqslant \frac{f(v(0))}{\left[\int_0^{v(0)} \psi^\mu\right]^{1/\mu}},$$

and thus our condition (3.6)* holds.

Finally, suppose $v'(0) > 0$. Much as in [W], if $m_1 \equiv m_2$ the result is still true. Indeed (3.6)′ holds as before at any point r_0 where $u = v$ and $u(r) \geqslant v(r)$ for $r < r_0$. We still require (3.6)* at zero, now with $m_1 \equiv m_2$. If we do not require $m_1 \equiv m_2$, then (3.6)′ is a new criterion, however ψ enters in our condition. As for the case of Corollary 3.8, we see that Corollary 3.10 applied to radially symmetric functions and solutions extends Theorem A of [Z] and Theorem 4 of [W].

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