

Variational first-order partial differential equations [☆]

Alžběta Haková and Olga Krupková

*Mathematical Institute of the Silesian University in Opava, Bezručovo nám. 13, 746 01 Opava,
Czech Republic*

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Abstract

Geometrical and variational properties of systems of first-order partial differential equations (PDE) on fibered manifolds are studied. Existence of Lagrangians is shown to be equivalent with the existence of a closed form which is global and unique; an explicit construction of this form is given. A bijective map between a set of dynamical forms on J^1Y , representing first-order PDE, and forms on the total space Y is found, providing a geometric description of the equations by means of a (not necessarily closed) ideal generated by a system of n -forms on Y (n = dimension of the base manifold). Conditions for this ideal to be closed are studied. Relations with Hamiltonian structures and with multisymplectic forms are discussed.

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1. Introduction

The aim of this paper is to analyze variability of systems of first-order partial differential equations (PDE) on smooth manifolds by methods of differential geometry. It is known that variability, i.e. existence of a Lagrangian for a system

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E-mail addresses: alhak@centrum.cz (A. Haková), olga.krupkova@math.slu.cz (O. Krupková).

of PDE (of any order) is closely connected with the possibility to represent the equations by means of a *closed form* [11]. However, such a form generally is non-global and non-unique. Also, the known theorem does not answer the question about an *explicit construction* of a corresponding closed form and of its *order*.

On the other hand, for the case of *ordinary* differential equations (of any order) on fibered manifolds it is known that variationality is equivalent with the existence of a certain closed 2-form which is global and unique, and explicit formulas for constructing this form are available (see [7,14,15]).

In this paper we prove that also for a system of *first-order PDE* on a fibered manifold variationality is equivalent with the existence of a *global and unique* closed $(n+1)$ -form (where n is the dimension of the base manifold), and provide an *explicit construction* of such a form (Theorem 4.1 and its proof). As a consequence we obtain an *explicit characterization* of general systems of first-order PDE and their Lagrangians. Our results are a generalization of [6] where *quasilinear* first-order PDE are studied. However, the methods we use are completely different. While in [6] variationality properties are studied by tools of the *theory of formal integrability of PDE* (as e.g. in [3]), the proof is rather complicated, and the main result is local and obtained for a system of C^ω equations, our approach is based on a geometric setting to the calculus of variations in fibered spaces [8,11], leading in a simple and straightforward way based on the Poincaré Lemma to *more complete results*: for first-order PDE in general and valid for the C^∞ case. The obtained results are rather surprising, since it turns out that

- for first-order PDE be variational, *polynomiality* in the first derivatives is a *necessary* property,
- the corresponding closed form is not the exterior derivative of the famous Poincaré–Cartan form θ_λ (as it is in the case of ODE or the quasilinear first-order PDE), but rather a more general form, arising as an exterior derivative of the so-called Krupka form, considered sometimes in the calculus of variations as a possible alternative to the Poincaré–Cartan form (cf. [2,9]).

The last section of this paper concerns a geometric setting for systems of first-order PDE on manifolds, based on the existence of a bijective correspondence between such equations and $(n+1)$ -forms (Proposition 5.2). Due to this correspondence we can assign to a system of first-order PDE (represented by a dynamical form) an exterior differential system generated by n -forms. Such a setting is suitable for applying geometric methods to study local and global properties of solutions of differential equations. We propose a concept of *regularity* of a system of PDE, and clarify its meaning for the corresponding ideal of differential forms (Theorem 5.1). Finally, we discuss the case of first-order *variational equations*, the related Hamiltonian structures, and relations of the corresponding closed $(n+1)$ -forms with *multisymplectic forms*. It should be noticed that our approach covers also an important class of global variational problems which are representable only by local Lagrangians (i.e., such that no global Lagrangian can be found).

For some other closely related results and applications the reader is referred also to [13,17].

2. Fibered manifolds and their jet prolongations

In what follows, all manifolds and mappings are smooth, and summation over repeated indices is understood. We consider a fibered manifold $\pi: Y \rightarrow X$, $\dim X = n$, $\dim Y = m + n$. We denote J^1 the 1-jet prolongation functor, and $\pi_1: J^1 Y \rightarrow X$, $\pi_{1,0}: J^1 Y \rightarrow Y$ the jet projections. A vector field ξ on Y is said to be π -vertical if $T\pi.\xi = 0$. Similarly, a vector field ξ on $J^1 Y$ is called π_1 -vertical (resp. $\pi_{1,0}$ -vertical), if $T\pi_1.\xi = 0$ (resp. $T\pi_{1,0}.\xi = 0$). A q -form η on $J^1 Y$ is called π_1 -horizontal (resp. $\pi_{1,0}$ -horizontal), if $i_\xi \eta = 0$ for every π_1 -vertical (resp. $\pi_{1,0}$ -vertical) vector field ξ on $J^1 Y$. η is called *contact*, if $J^1 \gamma^* \eta = 0$ for every section γ of π . A contact $\pi_{1,0}$ -horizontal q -form η is called *1-contact*, if for every π_1 -vertical vector field ξ on $J^1 Y$, the form $i_\xi \eta$ is π_1 -horizontal; η is called *k-contact*, $2 \leq k \leq q$, if $i_\xi \eta$ is $(k-1)$ -contact. For every q -form on Y there is a unique decomposition

$$\pi_{1,0}^* \eta = \eta_0 + \eta_1 + \cdots + \eta_q, \quad (2.1)$$

where η_0 is a π_1 -horizontal form, and η_i , $1 \leq i \leq q$, is a i -contact form on $J^1 Y$; we set $h\eta = \eta_0$, $p_i \eta = \eta_i$, and call it the *horizontal* and *i-contact part* of η , respectively.

We denote by (x^i, y^σ) (resp. $(x^i, y^\sigma, y_j^\sigma)$) local fibered coordinates on Y (resp. the associated coordinates on $J^1 Y$), and set

$$\begin{aligned} \omega_0 &= dx^1 \wedge dx^2 \cdots \wedge dx^n, \quad \omega^\sigma = dy^\sigma - y_k^\sigma dx^k, \\ \omega_j &= i_{\partial/\partial x^j} \omega_0, \quad \omega_{j_1 j_2} = i_{\partial/\partial x^{j_2}} \omega_{j_1}, \dots, \omega_{j_1 j_2 \dots j_n} = i_{\partial/\partial x^{j_n}} \omega_{j_1 \dots j_{n-1}} = 1. \end{aligned} \quad (2.2)$$

If f is a function on $V \subset Y$, we denote by dif the formal derivative of f , i.e., the component at dx^i of the 1-form hdf . It holds

$$dif = \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^\sigma} y_i^\sigma. \quad (2.3)$$

Throughout the paper we shall use a modification of the Poincaré Lemma, adapted to the contact structure on $J^1 Y$. It is based on the following *contact homotopy operator* A [11]. Let $x \in J^1 Y$ be a point. Consider the mapping

$$\chi: [0, 1] \times U \ni (u, (x^i, y^\sigma, y_j^\sigma)) \rightarrow (x^i, uy^\sigma, uy_j^\sigma) \in U, \quad (2.4)$$

where U is an appropriate neighborhood of x . Then

$$\chi^* \omega^\sigma = \chi^* (dy^\sigma - y_j^\sigma dx^j) = d(uy^\sigma) - uy_j^\sigma dx^j = y^\sigma du + u\omega^\sigma. \quad (2.5)$$

If μ is a $\pi_{1,0}$ -projectable p -form on U , we get a splitting $\chi^* \mu = \mu_0 \wedge du + \mu_1$, where neither μ_0 nor μ_1 contain du . Set

$$A\mu = \int_0^1 \mu_0 \wedge du, \quad (2.6)$$

where integration over the parameter u is understood. $A\mu$ is a $(p-1)$ -form on U . Moreover, if μ is k -contact then $A\mu$ is $(k-1)$ -contact. Thus, the operator A is well-adapted to the canonical decomposition (2.1) of a form into its contact components. Since it holds

$$\mu = A d\mu + dA\mu, \quad (2.7)$$

one finally obtains

Proposition (Contact Poincaré Lemma [11, Krupka]). *Let μ be a closed k -contact p -form on Y , $x \in Y$ a point. There exists a neighborhood U of x and a $(k-1)$ -contact $(p-1)$ -form ρ on U such that $d\rho = \mu$.*

3. Y -pertinent first-order Lagrangians

By a *first-order Lagrangian* we mean a horizontal n -form λ on $J^1 Y$. In fibered coordinates, $\lambda = L\omega_0$, where $L = L(x^i, y^v, y_k^v)$.

There is an important class of first-order Lagrangians, arising from *forms of order zero*: Let ρ be an n -form on Y . Then $\lambda = h\rho$ is a first-order Lagrangian with the function L polynomial of degree $\leq n$ in the first-order derivatives. Indeed, writing

$$\rho = A\omega_0 + \sum_{k=1}^n \frac{1}{k!} A_{\sigma_1 \dots \sigma_k}^{j_1 \dots j_k} dy^{\sigma_1} \wedge \dots \wedge dy^{\sigma_k} \wedge \omega_{j_1 \dots j_k} \quad (3.1)$$

with $A_{\sigma_1 \dots \sigma_k}^{j_1 \dots j_k}$, $1 \leq k \leq n$, completely antisymmetric in both the upper and lower indices, we can see that $\lambda = h\rho = L\omega_0$, where

$$L = A + \sum_{k=1}^n A_{\sigma_1 \dots \sigma_k}^{j_1 \dots j_k} y_{j_1}^{\sigma_1} \dots y_{j_k}^{\sigma_k},$$

$$A_{\sigma_1 \dots \sigma_r \dots \sigma_s \dots \sigma_k}^{j_1 \dots j_r \dots j_s \dots j_k} = -A_{\sigma_1 \dots \sigma_s \dots \sigma_r \dots \sigma_k}^{j_1 \dots j_r \dots j_s \dots j_k} = -A_{\sigma_1 \dots \sigma_r \dots \sigma_s \dots \sigma_k}^{j_1 \dots j_s \dots j_r \dots j_k}, \quad 1 \leq r, s \leq k \leq n. \quad (3.2)$$

Lagrangians of this kind will be called *Y -pertinent*. Moreover, as first found in [9],

$$\pi_{1,0}^* \rho = L\omega_0 + \sum_{k=1}^n \frac{1}{(k!)^2} \frac{\partial^k L}{\partial y_{j_1}^{\sigma_1} \dots \partial y_{j_k}^{\sigma_k}} \omega^{\sigma_1} \wedge \dots \wedge \omega^{\sigma_k} \wedge \omega_{j_1 \dots j_k}, \quad (3.3)$$

which means that every n -form ρ on Y is completely determined by its horizontal part—the Lagrangian of ρ . Thus, there is a *one-to-one correspondence* between Y -pertinent Lagrangians on $J^1 Y$ and n -forms on Y , realized via the mapping h . The inverse mapping, defined by formula (3.3), is denoted by \mathfrak{Lep}_1 and called *Lepage mapping of the first kind*. We set $\rho_\lambda = \mathfrak{Lep}_1(\lambda)$, and call it the *Krupka form* of λ . The *at most 1-contact part* of ρ_λ , i.e., the n -form

$$\theta_\lambda = L\omega_0 + \frac{\partial L}{\partial y_j^\sigma} \omega^\sigma \wedge \omega_j, \quad (3.4)$$

is the well-known *Poincaré–Cartan form* of λ . Note that for Lagrangians affine in the y_k^v 's, θ_λ is projectable onto Y and equals to ρ_λ . On the other hand, for all Y -pertinent Lagrangians at least quadratic in the y_k^v 's, θ_λ is non-trivially of order one, while ρ_λ is projectable onto Y . The $(n+1)$ -form

$$E_\lambda = p_1 d\rho_\lambda = p_1 d\theta_\lambda \quad (3.5)$$

is called the *Euler–Lagrange form* of λ [8]. Computing its chart expression one easily obtains

$$E_\lambda = E_\sigma(L) \omega^\sigma \wedge \omega_0, \quad (3.6)$$

where

$$E_\sigma(L) = \frac{\partial L}{\partial y^\sigma} - d_j \frac{\partial L}{\partial y_j^\sigma} \quad (3.7)$$

are the *Euler–Lagrange expressions*. In view of $\pi_{1,0}$ -projectability of ρ_λ , the Euler–Lagrange form for Y -pertinent Lagrangians is defined on $J^1 Y$. This can be seen also from (3.7), since for L given by (3.2) one has

$$\frac{\partial E_\sigma}{\partial y_{ik}^v} = \frac{\partial^2 L}{\partial y_i^\sigma \partial y_k^v} + \frac{\partial^2 L}{\partial y_k^\sigma \partial y_i^v} = 0. \quad (3.8)$$

Local sections γ of π satisfying the *Euler–Lagrange equations*, i.e.,

$$E_\lambda \circ J^1 \gamma = 0, \quad (3.9)$$

are called *extremals* of λ (note that in Eq. (3.9), E_λ is considered as a section of the bundle $\Lambda^{n+1}(J^1 Y) \rightarrow J^1 Y$). In fibered coordinates Eq. (3.9) means that the Euler–Lagrange expressions (3.7) vanish along the prolongation $J^1 \gamma$ of γ . Thus, for Y -pertinent Lagrangians the Euler–Lagrange equations become a system of m first-order PDE.

Remark 3.1. We have seen above that Y -pertinent Lagrangians are polynomials of degree $\leq n$ in the variables y_k^v , and their Euler–Lagrange equations are of order one. In general, however, a Lagrangian polynomial in the y_k^v 's comes from an n -form on $J^1 Y$ and its Euler–Lagrange equations are of the second order. Indeed, the Krupka form $\rho_\lambda = \mathfrak{L}ep_1(\lambda)$ generally is not projectable onto Y , and it is not a unique form on $J^1 Y$ such that $h\rho_\lambda = \lambda$. Actually, ρ_λ is $\pi_{1,0}$ -projectable (i.e. λ is Y -pertinent) if and only if λ is of the form (3.2) (see [13] for more details).

4. Variational properties of systems of first-order PDE: local and global aspects

In the sequel, we will be interested to study properties of systems of first-order PDE on manifolds. Therefore we start with introducing an appropriate *global* object for such a system of equations.

An $(n+1)$ -form E on $J^1 Y$ is called a *dynamical form* if it is 1-contact and $\pi_{1,0}$ -horizontal. Thus, a form E on $J^1 Y$ is a dynamical form if and only if in every fibered chart,

$$E = E_\sigma \omega^\sigma \wedge \omega_0, \quad (4.1)$$

where E_σ , $1 \leq \sigma \leq m$, are functions of the variables (x^i, y^v, y_k^v) . A (local) section γ of π is called a *path* of E , if

$$E \circ J^1 \gamma = 0. \quad (4.2)$$

In fibered coordinates, Eq. (4.2) becomes the following system of m first-order PDE:

$$E_\sigma \left(x^i, y^v \circ \gamma, \frac{\partial(y^v \circ \gamma)}{\partial x^j} \right) = 0, \quad 1 \leq \sigma \leq m. \quad (4.3)$$

A dynamical form E on $J^1 Y$ is called *locally variational*, if for every point $x \in J^1 Y$ there exists a neighborhood U and Lagrangian $\lambda = L\omega_0$ defined on U such that $E|_U = E_\lambda$. This means that on U , the components E_σ of E coincide with the *Euler–Lagrange expressions* (3.7) of L . For locally variational forms, equations for paths (4.2) are the Euler–Lagrange equations.

From the formula for $E_\sigma(L)$ one can see immediately that if for a first-order locally variational form E a Lagrangian L is a polynomial of degree p ($1 \leq p \leq n$) in the y_k^v 's, then the Euler–Lagrange expressions are polynomials of degree p in the y_k^v 's as well. Now, we shall show that for *any* system of first-order PDE to be variational, polynomiality in the first-order derivatives is a *necessary* property.

Proposition 4.1. *Let E be a dynamical form on $J^1 Y$, $E = E_\sigma \omega^\sigma \wedge \omega_0$. If E is locally variational, then the E_σ are polynomials of degree $\leq n$ in the y_j^v 's.*

Proof. If E is variational then in a neighborhood of every point in $J^1 Y$ we have a Lagrangian L such that

$$E_\sigma = \frac{\partial L}{\partial y^\sigma} - d_j \frac{\partial L}{\partial y_j^\sigma}, \quad (4.4)$$

and (3.8) holds. This means that

$$\frac{\partial E_\sigma}{\partial y_k^v} = \frac{\partial^2 L}{\partial y^\sigma \partial y_k^v} - d_j \frac{\partial^2 L}{\partial y_j^\sigma \partial y_k^v} - \frac{\partial^2 L}{\partial y^v \partial y_k^\sigma}, \quad (4.5)$$

hence

$$\frac{\partial E_\sigma}{\partial y_k^v} + \frac{\partial E_v}{\partial y_k^\sigma} = 0. \quad (4.6)$$

Differentiating E_σ and using the above condition we get

$$\begin{aligned}
 & \frac{\partial^{n+1} E_\sigma}{\partial y_{i_1}^{v_1} \cdots \partial y_{i_{k-1}}^{v_{k-1}} \partial y_p^{v_k} \partial y_{i_{k+1}}^{v_{k+1}} \cdots \partial y_{i_{l-1}}^{v_{l-1}} \partial y_p^{v_l} \partial y_{i_{l+1}}^{v_{l+1}} \cdots \partial y_{i_{n+1}}^{v_{n+1}}} \\
 &= - \frac{\partial^{n+1} E_{v_k}}{\partial y_{i_1}^{v_1} \cdots \partial y_{i_{k-1}}^{v_{k-1}} \partial y_p^\sigma \partial y_{i_{k+1}}^{v_{k+1}} \cdots \partial y_{i_{l-1}}^{v_{l-1}} \partial y_p^{v_l} \partial y_{i_{l+1}}^{v_{l+1}} \cdots \partial y_{i_{n+1}}^{v_{n+1}}} \\
 &= - \frac{\partial^{n+1} E_{v_l}}{\partial y_{i_1}^{v_1} \cdots \partial y_{i_{k-1}}^{v_{k-1}} \partial y_p^\sigma \partial y_{i_{k+1}}^{v_{k+1}} \cdots \partial y_{i_{l-1}}^{v_{l-1}} \partial y_p^{v_k} \partial y_{i_{l+1}}^{v_{l+1}} \cdots \partial y_{i_{n+1}}^{v_{n+1}}} \\
 &= - \frac{\partial^{n+1} E_\sigma}{\partial y_{i_1}^{v_1} \cdots \partial y_{i_{k-1}}^{v_{k-1}} \partial y_p^{v_l} \partial y_{i_{k+1}}^{v_{k+1}} \cdots \partial y_{i_{l-1}}^{v_{l-1}} \partial y_p^{v_k} \partial y_{i_{l+1}}^{v_{l+1}} \cdots \partial y_{i_{n+1}}^{v_{n+1}}}, \quad (4.7)
 \end{aligned}$$

since at least two of the indices i_1, \dots, i_{n+1} must take the same value, say, p . Hence, we conclude that

$$\frac{\partial^{n+1} E_\sigma}{\partial y_{i_1}^{v_1} \cdots \partial y_{i_{n+1}}^{v_{n+1}}} = 0. \quad \square \quad (4.8)$$

In view of the above proposition and (4.6), the components E_σ of a locally variational form E on $J^1 Y$ are polynomials of degree at most n in the y_k^v 's with the coefficients completely antisymmetric in both the upper and lower indices. We set

$$\begin{aligned}
 E_\sigma &= B_\sigma + B_{\sigma v_1}^{j_1} y_{j_1}^{v_1} + \cdots + B_{\sigma v_1 \cdots v_n}^{j_1 \cdots j_n} y_{j_1}^{v_1} \cdots y_{j_n}^{v_n}, \\
 B_{\sigma v_1 \cdots j_p \cdots j_q \cdots j_k}^{j_1 \cdots j_p \cdots j_q \cdots j_k} &= B_{\sigma v_1 \cdots v_q \cdots v_p \cdots v_k}^{j_1 \cdots j_q \cdots j_p \cdots j_k}, \quad B_{\sigma v_1 \cdots v_p \cdots v_k}^{j_1 \cdots j_k} = -B_{v_p v_1 \cdots \sigma \cdots v_k}^{j_1 \cdots j_k}, \quad 1 \leq k \leq n. \quad (4.9)
 \end{aligned}$$

Taking into account the proof of Proposition 4.1 we immediately obtain

Corollary 4.1. *Let E be a dynamical form on $J^1 Y$. If its components E_σ satisfy (4.6) then they are of the form (4.9).*

Let us turn to the main result of this section, according to which first-order locally variational forms are *equivalent* to closed $(n+1)$ -forms on Y .

Theorem 4.1. *Let E be a dynamical form on $J^1 Y$. The following conditions are equivalent:*

- (1) *In every fibered chart the components E_σ of E satisfy the following conditions:*

$$\frac{\partial E_\sigma}{\partial y_j^v} + \frac{\partial E_v}{\partial y_j^\sigma} = 0, \quad \frac{\partial E_\sigma}{\partial y^v} - \frac{\partial E_v}{\partial y^\sigma} + d_i \frac{\partial E_v}{\partial y_i^\sigma} = 0, \quad 1 \leq \sigma, v \leq m, \quad 1 \leq j \leq n. \quad (4.10)$$

- (2) *There exists a unique closed $(n+1)$ -form α on Y such that $E = p_1 \alpha$.*
 (3) *E is locally variational.*

Proof. (1) \Rightarrow (2): Let α be a form on Y such that $E = p_1\alpha$. We have $\pi_{1,0}^*\alpha = E + F$ where F is an at least 2-contact form. Put

$$F = \sum_{k=1}^n F_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_k} \wedge \omega_{j_1 \dots j_k}, \quad (4.11)$$

where the components are completely antisymmetric in both the superscripts and the subscripts. The condition $d\alpha = 0$ then gives $p_k d\alpha = 0$, $2 \leq k \leq n+2$. Computing $p_2 d\alpha = 0$ we obtain

$$F_{\sigma v}^j = \frac{1}{2} \frac{\partial E_\sigma}{\partial y_j^v}, \quad \frac{\partial E_v}{\partial y^\sigma} - \frac{\partial E_\sigma}{\partial y^v} + 2d_i F_{\sigma v}^i = 0. \quad (4.12)$$

The first of Eq. (4.12) is a formula for $F_{\sigma v}^j$, and since $F_{\sigma v}^j = -F_{v\sigma}^j$, it gives also the first set of conditions (4.10), which are satisfied by assumption. The second of Eqs. (4.12) is then precisely the second of the identities (4.10).

Proceeding in a similar way with $p_{k+1}d\alpha$ we get

$$\begin{aligned} p_{k+1}d\alpha &= \frac{\partial F_{\sigma v_1 \dots v_{k-1}}^{j_1 \dots j_{k-1}}}{\partial y^{v_k}} \omega^{v_k} \wedge \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_{k-1}} \wedge \omega_{j_1 \dots j_{k-1}} \\ &+ \frac{\partial F_{\sigma v_1 \dots v_{k-1}}^{j_1 \dots j_{k-1}}}{\partial y_{j_k}^{v_k}} \omega_{j_k}^{v_k} \wedge \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_{k-1}} \wedge \omega_{j_1 \dots j_{k-1}} \\ &+ d_i F_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} dx^i \wedge \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_k} \wedge \omega_{j_1 \dots j_k} \\ &- F_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} \omega_i^\sigma \wedge dx^i \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_k} \wedge \omega_{j_1 \dots j_k} \\ &+ F_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} \omega^\sigma \wedge \omega_i^{v_1} \wedge dx^i \wedge \dots \wedge \omega^{v_k} \wedge \omega_{j_1 \dots j_k} + \dots \\ &+ (-1)^{k+1} F_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega_i^{v_k} \wedge dx^i \wedge \omega_{j_1 \dots j_k} = 0. \end{aligned} \quad (4.13)$$

This means that the following equations hold:

$$\frac{\partial F_{\sigma v_1 \dots v_{k-1}}^{j_1 \dots j_{k-1}}}{\partial y^{v_k}} + \frac{\partial F_{\sigma v_1 \dots v_{k-1}}^{j_1 \dots j_{k-1}}}{\partial y^{v_{k-1}}} + \dots + \frac{\partial F_{\sigma v_1 \dots v_k}^{j_1 \dots j_k}}{\partial y^\sigma} - k(k+1)d_{j_k} F_{\sigma v_1 \dots v_{k-1} v_k}^{j_1 \dots j_{k-1} j_k} = 0, \quad (4.14)$$

$$\frac{\partial F_{\sigma v_1 \dots v_{k-1}}^{j_1 \dots j_{k-1}}}{\partial y_{j_k}^{v_k}} - k(k+1)F_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} = 0. \quad (4.15)$$

Now, with help of the formula for $F_{\sigma v}^j$ (4.12), and the first set of conditions (4.10) we can see that for all k , the $\partial F_{\sigma v_1 \dots v_{k-1}}^{j_1 \dots j_{k-1}} / \partial y_{j_k}^{v_k}$ are completely antisymmetric in the indices

$(\sigma_{v_1} \cdots v_k)$ and $(j_1 \cdots j_k)$. Thus (4.15) immediately give

$$\begin{aligned} F_{\sigma_{v_1} \cdots v_k}^{j_1 \cdots j_k} &= \frac{1}{k(k+1)} \frac{\partial F_{\sigma_{v_1} \cdots v_{k-1}}^{j_1 \cdots j_{k-1}}}{\partial y_{j_k}^{v_k}} = \cdots = \frac{2}{k!(k+1)!} \frac{\partial^{k-1} F_{\sigma_{v_1}}^{j_1}}{\partial y_{j_2}^{v_2} \cdots \partial y_{j_k}^{v_k}} \\ &= \frac{1}{k!(k+1)!} \frac{\partial^k E_\sigma}{\partial y_{j_1}^{v_1} \cdots \partial y_{j_k}^{v_k}}. \end{aligned} \quad (4.16)$$

Since the E_σ 's satisfy (4.6), we can apply Corollary 4.1 and assume E_σ of the form (4.9). Then we get

$$\begin{aligned} F_{\sigma_{v_1} \cdots v_k}^{j_1 \cdots j_k} &= \frac{1}{(k+1)!} \left(B_{\sigma_{v_1} \cdots v_k}^{j_1 \cdots j_k} + \binom{k+1}{1} B_{\sigma_{v_1} \cdots v_{k+1}}^{j_1 \cdots j_{k+1}} y_{j_{k+1}}^{v_{k+1}} \right. \\ &\quad + \binom{k+2}{2} B_{\sigma_{v_1} \cdots v_{k+2}}^{j_1 \cdots j_{k+2}} y_{j_{k+1}}^{v_{k+1}} y_{j_{k+2}}^{v_{k+2}} \\ &\quad \left. + \cdots + \binom{n}{n-k} B_{\sigma_{v_1} \cdots v_n}^{j_1 \cdots j_n} y_{j_{k+1}}^{v_{k+1}} \cdots y_{j_n}^{v_n} \right). \end{aligned} \quad (4.17)$$

Finally, Eqs. (4.14), which by (4.15) take the form

$$\frac{\partial F_{\sigma_{v_1} \cdots v_{k-1}}^{j_1 \cdots j_{k-1}}}{\partial y^\rho} + \frac{\partial F_{\rho \sigma_{v_1} \cdots v_{k-2}}^{j_1 \cdots j_{k-1}}}{\partial y^{v_{k-1}}} + \cdots + \frac{\partial F_{v_1 v_2 \cdots v_{k-1}}^{j_1 \cdots j_{k-1}}}{\partial y^\sigma} - d_p \frac{\partial F_{\sigma_{v_1} \cdots v_{k-1}}^{j_1 \cdots j_{k-1}}}{\partial y_p^\rho} = 0, \quad (4.18)$$

are identities. Indeed, they are easily obtained by differentiating relations (4.14) for $k-1$ by y_p^ρ and substituting from (4.16).

Summarizing, we have obtained the form F completely determined by the components of E . This means, however, that F is globally defined and unique. Moreover, formula (4.17) shows that $\alpha = E + F$ is $\pi_{1,0}$ -projectable: It holds

$$\begin{aligned} \alpha &= E_\sigma \omega^\sigma \wedge \omega_0 + \sum_{k=1}^n \frac{1}{k!(k+1)!} \frac{\partial^k E_\sigma}{\partial y_{j_1}^{v_1} \cdots \partial y_{j_k}^{v_k}} \omega^\sigma \wedge \omega^{v_1} \wedge \cdots \wedge \omega^{v_k} \wedge \omega_{j_1 \cdots j_k} \\ &= B_\sigma dy^\sigma \wedge \omega_0 + \sum_{k=1}^n \frac{1}{(k+1)!} B_{\sigma_{v_1} \cdots v_k}^{j_1 \cdots j_k} dy^\sigma \wedge dy^{v_1} \wedge \cdots \wedge dy^{v_k} \wedge \omega_{j_1 \cdots j_k} \end{aligned} \quad (4.19)$$

(see Appendix for detailed computations). Consequently, α is defined on Y , and is unique.

(2) \Rightarrow (3): Let α be a closed $(n+1)$ -form on Y such that $E = p_1 \alpha$. Then Poincaré Lemma guarantees the existence of a family of local forms ρ on Y such that $d\rho = \alpha$. To construct ρ we shall use the *contact homotopy operator* A (2.6). In this way we get in a neighborhood of every point in $J^1 Y$ an n -form

$$\rho = A(\pi_{1,0}^* \alpha), \quad (4.20)$$

and a Lagrangian

$$\lambda = h\rho = hA(\pi_{1,0}^*\alpha) = Ap_1\alpha = AE. \quad (4.21)$$

Since, by construction of ρ , one has $E = p_1d\rho$ (on the domain of definition of λ), $h\rho = \lambda$ is a local Lagrangian for E , i.e., the dynamical form E is locally variational.

(3) \Rightarrow (1): This is easy to show by a direct computation using (3.7) (cf. Proof of Proposition 4.1). \square

Remark 4.1. Necessary and sufficient conditions for a system of PDE be variational analogous to (4.10), are known for the general case of r th order PDE [1,10]. Compared to known proofs, the above proof is simple and elementary.

Remark 4.2. Computing a Lagrangian for a locally variational form E on J^1Y according to (4.21) one immediately obtains the well-known *Vainberg–Tonti Lagrangian* [19]

$$L = y^\sigma \int_0^1 E_\sigma(x^i, uy^v, uy_k^v) du, \quad (4.22)$$

which due to (4.9) takes the following *polynomial form*:

$$L = L_0 + \sum_{k=1}^n L_{v_1 \dots v_k}^{j_1 \dots j_k} y_{j_1}^{v_1} \dots y_{j_k}^{v_k}, \quad (4.23)$$

where (in the notations of (2.4))

$$L_0 = y^\sigma \int_0^1 (B_{\sigma \circ} \chi) du, \quad L_{v_1 \dots v_k}^{j_1 \dots j_k} = y^\sigma \int_0^1 (B_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} \circ \chi) u^k du, \quad 1 \leq k \leq n. \quad (4.24)$$

As expected, the $L_{v_1 \dots v_k}^{j_1 \dots j_k}$ are completely antisymmetric in the lower and upper indices.

Note that by the above arguments one obtains for a locally variational form E on J^1Y a family of local Vainberg–Tonti Lagrangians λ_U defined on open subsets of J^1Y . Accordingly, there arises a family of local Krupka forms $\rho_{\lambda_U} = \mathfrak{Qep}_1(\lambda_U)$, which are $\pi_{1,0}$ -projectable (cf. (3.1)–(3.3)). The forms $d\rho_{\lambda_U}$ are closed and $p_1d\rho_{\lambda_U} = E$. In view of Theorem 4.1, the local forms $d\rho_{\lambda_U}$ coincide on overlaps of their domains and, in this way, give rise to a unique global form α on Y .

Taking into account the construction and properties of the Krupka form (3.3) and the Poincaré–Cartan form (3.4) related to a Lagrangian λ , we immediately obtain the following assertion.

Corollary 4.2. *A dynamical form E on J^1Y is locally variational if and only if its components are of the polynomial form (4.9), and the $(n+1)$ -form α (4.19) on Y is closed. In this case, every point in Y has a neighborhood U such that $\lambda = L\omega_0$, where L*

is given by (4.23) and (4.24), is a Lagrangian for E defined over U . Moreover,

$$A\alpha = \rho_\lambda, \quad (4.25)$$

(i.e. $\alpha|_U = d\rho_\lambda$) where ρ_λ is the Krupka form of λ .

If, in particular, the E_σ 's are affine functions in y_k^v , then the Poincaré–Cartan form of λ is projectable onto Y and $\theta_\lambda = \rho_\lambda$. Consequently, $\alpha|_U = d\theta_\lambda$.

In the sequel we shall discuss several results which follow directly from Theorem 4.1 and its above corollary.

Remark 4.3. Since locally $\alpha = d\rho_\lambda$, we get just from the expression of α (4.19) that

$$d\rho_\lambda = 0 \Leftrightarrow E_\lambda = 0. \quad (4.26)$$

This interesting and important property of the Krupka form ρ_λ was proved in another way in [2,9]. In this context it is important to notice that in general, the Poincaré–Cartan form θ_λ does not possess this property: $d\theta_\lambda = 0$ obviously implies $p_1 d\theta_\lambda = E_\lambda = 0$, however, vanishing of the Euler–Lagrange form does not mean that $d\theta_\lambda = 0$. From the formula (4.19) we can easily obtain the obstruction for $d\theta_\lambda$ to vanish. If $E_\lambda = 0$ then $\alpha = 0$, and consequently, $p_2\alpha = 0$. However, $p_2\alpha = p_2 d\rho_\lambda = p_2 d\theta_\lambda + p_2 dp_2\rho_\lambda$. Since $d\theta_\lambda$ is at most 2-contact, we obtain the following result:

Suppose that $E_\lambda = 0$. Then $d\theta_\lambda = 0$ if and only if

$$p_2 dp_2\rho_\lambda = 0. \quad (4.27)$$

Computing the above condition in fibered coordinates and using (3.8) we obtain

$$\begin{aligned} p_2 dp_2\rho_\lambda &= \frac{1}{4} p_2 d \left(\frac{\partial^2 L}{\partial y_j^\sigma \partial y_k^v} \omega^\sigma \wedge \omega^v \wedge \omega_{jk} \right) \\ &= \frac{1}{2} d_k \frac{\partial^2 L}{\partial y_j^\sigma \partial y_k^v} \omega^\sigma \wedge \omega^v \wedge \omega_j + \frac{\partial^2 L}{\partial y_j^\sigma \partial y_k^v} \omega_k^\sigma \wedge \omega^v \wedge \omega_j. \end{aligned} \quad (4.28)$$

Thus, $p_2 dp_2\rho_\lambda = 0$ if and only if

$$\frac{\partial^2 L}{\partial y_j^\sigma \partial y_k^v} = 0, \quad (4.29)$$

i.e., L , hence E_σ , are affine functions in the y_k^v 's. (In this case, however, $\theta_\lambda = \rho_\lambda$).

A Lagrangian λ is said to be *trivial* if $E_\lambda = 0$. We have seen above that $E_\lambda = 0 \Leftrightarrow \alpha = d\rho_\lambda = 0$, i.e. in a neighborhood of every point in Y , $\rho_\lambda = dv$. Hence, λ is trivial if and only if in a neighborhood of every point in Y there exists an $(n-1)$ -form v such that $\lambda = hdv$ (cf. [8]).

Next, taking into account relation between dynamical forms and PDEs, we obtain an explicit characterization of variational first-order PDE and their Lagrangians:

Theorem 4.2. *A system of C^∞ first-order PDE is variational if and only if for some r , $1 \leq r \leq n$, it is of the form*

$$B_{\sigma_{v_1} \dots v_r}^{j_1 \dots j_r} \frac{\partial y^{v_1}}{\partial x^{j_1}} \dots \frac{\partial y^{v_r}}{\partial x^{j_r}} + \dots + B_{\sigma_{v_1 v_2}}^{ij_2} \frac{\partial y^{v_1}}{\partial x^{j_1}} \frac{\partial y^{v_2}}{\partial x^{j_2}} + B_{\sigma_{v_1}}^{j_1} \frac{\partial y^{v_1}}{\partial x^{j_1}} + B_\sigma = 0, \quad (4.30)$$

where the coefficients are functions of (x^i, y^v) , completely antisymmetric in the upper and lower indices, and the $(n+1)$ -form

$$\begin{aligned} \alpha &= B_\sigma dy^\sigma \wedge \omega_0 + \frac{1}{2!} B_{\sigma_{v_1}}^{j_1} dy^\sigma \wedge dy^{v_1} \wedge \omega_{j_1} + \dots \\ &+ \frac{1}{(r+1)!} B_{\sigma_{v_1} \dots v_r}^{j_1 \dots j_r} dy^\sigma \wedge dy^{v_1} \wedge \dots \wedge dy^{v_r} \wedge \omega_{j_1 \dots j_r} \end{aligned} \quad (4.31)$$

on Y is closed. In this case, α is the exterior derivative of the Krupka form ρ_λ (3.3) associated with the corresponding Vainberg–Tonti Lagrangian L (4.23) and (4.24) (which is a polynomial of degree r in the variables y_j^v).

In keeping with the above results, let us define a mapping \mathfrak{Lep}_2 , associating to every locally variational form on $J^1 Y$ a closed $(n+1)$ -form on Y by the formula

$$\mathfrak{Lep}_2(E) = E_\sigma \omega^\sigma \wedge \omega_0 + \sum_{k=1}^n \frac{1}{k!(k+1)!} \frac{\partial^k E_\sigma}{\partial y_{j_1}^{v_1} \dots \partial y_{j_k}^{v_k}} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_k} \wedge \omega_{j_1 \dots j_k}. \quad (4.32)$$

Corollary 4.3. *The following conditions are equivalent:*

- (1) *A Lagrangian $\lambda = L\omega_0$ on $J^1 Y$ is Y -pertinent, and consequently, a polynomial of degree r in y_j^v , where $1 \leq r \leq n$.*
- (2) *The Euler–Lagrange expressions $E_\sigma(L)$ are polynomials of degree r in y_j^v , where $1 \leq r \leq n$.*
- (3) *The $(n+1)$ -form $\mathfrak{Lep}_2(E_\lambda) = \alpha$ is closed, projectable onto Y , and at most $(r+1)$ -contact.*
- (4) *The n -form $\mathfrak{Lep}_1(\lambda) = \rho_\lambda$ is projectable onto Y and at most r -contact, and $d\rho_\lambda = \alpha$, where $\alpha = \mathfrak{Lep}_2(E_\lambda)$.*

Assertions (3) and (4) above provide a transparent relation between the Euler–Lagrange operator $\varepsilon: \lambda \rightarrow E_\lambda$ and the exterior derivative of differential forms:

Corollary 4.4. *The following diagram is commutative:*

$$\begin{array}{ccc} \lambda & \xrightarrow{\mathfrak{Lep}_1} & \rho_\lambda \\ \varepsilon \downarrow & & \downarrow d \\ E_\lambda & \xrightarrow{\mathfrak{Lep}_2} & d\rho_\lambda. \end{array} \quad (4.33)$$

We note that the fact that the Euler–Lagrange operator is in a sense the exterior derivative operator is the key idea in the construction of the variational sequence: that sequence arises as a quotient sequence of the De Rham sequence [12].

The most important consequence of Theorem 4.1 is a *bijective correspondence between locally variational dynamical forms on $J^1 Y$ and closed $(n+1)$ -forms on Y* :

Theorem 4.3. *The mapping \mathfrak{Lep}_2 of the set of locally variational forms on $J^1 Y$ to the set of closed $(n+1)$ -forms on Y is bijective and inverse to the mapping p_1 .*

Remark 4.4. Theorems 4.1 and 4.3 can be viewed as a generalization to PDE of similar results valid for (higher-order) ODE [15]. It should be noted, however, that in the case of PDE of order ≥ 2 there is no more bijection between locally variational and closed $(n+1)$ -forms. This comes from the fact that a locally variational form of order ≥ 2 has *non-uniquely* determined closed counterparts (see [11,16,17]).

Another consequence of Theorem 4.1 we want to mention concerns *global aspects*. Recall that a locally variational form E on $J^1 Y$ is called *globally variational* if there exists a Lagrangian λ on $J^1 Y$ such that $E = E_\lambda$. Similarly, a trivial Lagrangian λ is called *globally trivial* if $\lambda = h dv$ for some global $(n-1)$ -form v (defined on Y). In this context it is to be stressed that existence of local Lagrangians for a dynamical form in no case implies existence of a global Lagrangian, i.e., it *may not be possible* to choose a family of local Lagrangians over Y which would *coincide on the overlaps of their domains*. Similarly, a trivial Lagrangian need not be globally trivial. Obstructions for global variationality and global triviality are well-known for the general case of PDE of any order, the proof, however, requires tools of the theory of variational bicomplexes or variational sequences (see [1,5,12,18,20,21]). On the other hand, taking into account Theorem 4.1 and Corollary 4.2, the obstructions for global variationality and global triviality in the first-order case are obtained immediately, and one can see at once that they are as follows. As usual, let us denote by $H^k(Y)$ the k th cohomology group of the manifold Y , i.e., the quotient of closed modulo exact k -forms on Y .

Theorem 4.4. *If $H^{n+1}(Y) = \{0\}$ then every locally variational form on $J^1 Y$ is globally variational. If $H^n(Y) = \{0\}$ then every trivial Lagrangian is globally trivial.*

5. The dynamical ideal for first-order PDE

In this section we shall discuss a geometric model for first-order PDE by means of an ideal of differential forms on a fibered manifold. This description turns out to be closely connected with representing PDE as *Hamiltonian systems*, and with *multisymplectic geometry*.

Let E be a dynamical form on $J^1 Y$. By a *Lepage class* of E we shall mean the equivalence class $[\alpha]$ of (possibly local) $(n+1)$ -forms on $J^1 Y$ such that

$$\alpha \in [\alpha] \Leftrightarrow p_1 \alpha = E. \quad (5.1)$$

This means that every element of the class $[\alpha]$ is of the form $\alpha = E + F$ where F is an at least 2-contact form. Lepage classes of dynamical forms are important for a geometric representation of PDEs by means of ideals of differential forms as follows:

Proposition 5.1. *Let E be a dynamical form on $J^1 Y$, $[\alpha]$ its Lepage class. The following conditions are equivalent:*

- (1) *A (local) section γ of the fibered manifold $\pi: Y \rightarrow X$ is a path of E .*
- (2) *For every π -vertical vector field ζ on $J^1 Y$, and every $\alpha \in [\alpha]$,*

$$J^1 \gamma^* i_\zeta \alpha = 0. \quad (5.2)$$

- (3) *For every $\alpha \in [\alpha]$, γ is an integral section of the ideal of differential forms on $J^1 Y$, generated by the system of n -forms $i_\zeta \alpha$, where ζ runs over all π_1 -vertical vector fields on $J^1 Y$.*

By definition, $(n+1)$ -forms belonging to the Lepage class of a first-order dynamical form E are defined on open subsets of $J^1 Y$. We say that E is Y -pertinent if to every point in Y there exists a neighborhood U and a form α_U belonging to the Lepage class of E , projectable onto U . In other words, E is Y -pertinent if it can be represented by a Lepage class defined on Y .

Proposition 5.2. *Let E be a dynamical form on $J^1 Y$.*

The following four conditions are equivalent:

- (1) *E is Y -pertinent.*
- (2) *In every fiber chart, E is of the form $E = E_\sigma dy^\sigma \wedge \omega_0$, where*

$$E_\sigma = B_\sigma + B_{\sigma v_1}^{j_1} y_{j_1}^{v_1} + \cdots + B_{\sigma v_1 \cdots v_n}^{j_1 \cdots j_n} y_{j_1}^{v_1} \cdots y_{j_n}^{v_n},$$

$$B_{\sigma v_1 \cdots v_p \cdots v_q \cdots v_k}^{j_1 \cdots j_p \cdots j_q \cdots j_k} = B_{\sigma v_1 \cdots v_q \cdots v_p \cdots v_k}^{j_1 \cdots j_q \cdots j_p \cdots j_k}, \quad B_{\sigma v_1 \cdots v_p \cdots v_k}^{j_1 \cdots j_k} = -B_{v_p v_1 \cdots \sigma \cdots v_k}^{j_1 \cdots j_k}, \quad 1 \leq k \leq n. \quad (5.3)$$

- (3) *There exists a unique $(n+1)$ -form α on Y such that $E = p_1 \alpha$.*
- (4) *The $(n+1)$ -form*

$$\mathfrak{Lep}_2(E) = E_\sigma \omega^\sigma \wedge \omega_0$$

$$+ \sum_{k=1}^n \frac{1}{k!(k+1)!} \frac{\partial^k E_\sigma}{\partial y_{j_1}^{v_1} \cdots \partial y_{j_k}^{v_k}} \omega^\sigma \wedge \omega^{v_1} \wedge \cdots \wedge \omega^{v_k} \wedge \omega_{j_1 \cdots j_k}, \quad (5.4)$$

is projectable onto Y .

The mapping \mathfrak{Lep}_2 , defined by (5.4) is a bijection between Y -pertinent dynamical forms on $J^1 Y$ and $(n+1)$ -forms on Y . The inverse to \mathfrak{Lep}_2 is the mapping p_1 .

Proof. Suppose (1), and consider a fibered chart (V, ψ) , $\psi = (x^i, y^v)$ on Y . Then E takes the form $E = E_\sigma dy^\sigma \wedge \omega_0$, where E_σ are functions on $\pi_{1,0}^{-1}(V) \subset J^1 Y$. By assumption, there is an $(n+1)$ -form α , defined on $U \subset V$, such that $p_1 \alpha = E$ on U . Let us denote

$$\alpha = B_\sigma dy^\sigma \wedge \omega_0 + \sum_{k=1}^n \frac{1}{(k+1)!} B_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} dy^\sigma \wedge dy^{v_1} \wedge \dots \wedge dy^{v_k} \wedge \omega_{j_1 \dots j_k}, \quad (5.5)$$

where the components $B_{\sigma v_1 \dots v_k}^{j_1 \dots j_k}$, $1 \leq k \leq n$, are completely antisymmetric both in the indices $(j_1 \dots j_k)$ and $(\sigma v_1 \dots v_k)$. Now, computing $E = p_1 \alpha$ we obtain the components E_σ of E in the form (5.3).

Let (2) hold. For every fiber chart (V, ψ) on Y consider a form α_V on V , defined by (5.5). We have to show that the local forms α_V give rise to a global form α on Y , and that this form is the only $(n+1)$ -form on Y such that $p_1 \alpha = E$. First, we show the uniqueness locally. Suppose that on an open set $U \subset V$ one has another form α' such that $E = p_1 \alpha'$. In the chart (V, ψ) ,

$$\alpha' = \alpha_\sigma dy^\sigma \wedge \omega_0 + \sum_{k=1}^n \frac{1}{(k+1)!} \alpha_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} dy^\sigma \wedge dy^{v_1} \wedge \dots \wedge dy^{v_k} \wedge \omega_{j_1 \dots j_k}, \quad (5.6)$$

with the components completely antisymmetric both in the upper and lower indices. The condition $E = p_1 \alpha_V$ gives

$$E_\sigma = \alpha_\sigma + \sum_{k=1}^n \alpha_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} y_{j_1}^{v_1} \dots y_{j_k}^{v_k} = B_\sigma + \sum_{k=1}^n B_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} y_{j_1}^{v_1} \dots y_{j_k}^{v_k}. \quad (5.7)$$

This means that the above polynomials must have equal coefficients, and we obtain $\alpha' = \alpha_V$. Now, the global existence of α easy follows: Indeed, $\alpha_{V_1} \neq \alpha_{V_2}$ on overlap of two charts contradicts with the local uniqueness.

Let us show (3) \Rightarrow (4). In fibered coordinates α (resp. E) is of the form (5.5) (resp. (5.3)). Expressing the form $\mathfrak{L}ep_2(E)$ in the basis (dx^i, dy^σ) we obtain after a straightforward but rather long computation (see Appendix) that $\mathfrak{L}ep_2(E) = \alpha$, proving the $\pi_{1,0}$ -projectability of $\mathfrak{L}ep_2(E)$.

The assertion (4) \Rightarrow (1) is a direct consequence of the definition of Y -pertinent dynamical form.

It remains to show that $\mathfrak{L}ep_2$ realizes a bijective correspondence between Y -pertinent dynamical forms on $J^1 Y$ and $(n+1)$ -forms on Y , inverse to the mapping p_1 . Since for an Y -pertinent form E (5.3), $\mathfrak{L}ep_2(E)$ is the $(n+1)$ -form (5.5), the mapping $\mathfrak{L}ep_2$ is injective. It is also surjective. Indeed, given $\alpha \in \Lambda^{n+1}(Y)$ expressed in fiber coordinates by (5.5), we obtain an Y -pertinent dynamical form E such that $\mathfrak{L}ep_2(E) = \alpha$ by putting (5.3). Finally, the relations $\mathfrak{L}ep_2 \circ p_1 = \text{id}$, $p_1 \circ \mathfrak{L}ep_2 = \text{id}$ become obvious. \square

In view of Proposition 5.2, equations for paths of an Y -pertinent dynamical form E on $J^1 Y$ read

$$\gamma^* i_\xi \alpha_E = 0 \quad \text{for every vertical vector field } \xi \text{ on } Y, \quad (5.8)$$

where α_E is the unique Lepage form on Y , associated to E . In other words, paths of E are integral sections of the ideal of differential forms on Y , generated by the following system of n -forms:

$$\mathcal{D}_{\alpha_E} = \{i_\xi \alpha_E \mid \xi \text{ runs over all vertical vector fields on } Y\}. \quad (5.9)$$

Computing local generators explicitly, we obtain $\mathcal{D}_{\alpha_E} = \text{span}\{\eta_\sigma, 1 \leq \sigma \leq m\}$, where

$$\eta_\sigma = B_\sigma \omega_0 + \sum_{k=1}^n \frac{1}{k!} B_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} dy^{v_1} \wedge \dots \wedge dy^{v_k} \wedge \omega_{j_1 \dots j_k}. \quad (5.10)$$

Definition 5.1. An Y -pertinent dynamical form E on $J^1 Y$ (Eq. (5.8), respectively, an $(n+1)$ -form α on Y) is called *regular* if

$$\text{rank } \mathcal{D}_{\alpha_E} = m. \quad (5.11)$$

Condition (5.11) obviously means that generators (5.10) of \mathcal{D}_{α_E} are linearly independent at each point of Y , or equivalently, that rank of the matrix

$$B = (B_\sigma \quad B_{\sigma v_1}^{j_1} \quad B_{\sigma v_1 v_2}^{j_1 j_2} \quad \dots \quad B_{\sigma v_1 \dots v_n}^{j_1 \dots j_n}), \quad (5.12)$$

where σ labels rows and the other sets of indices label columns, is maximal and equal to $m = \dim Y - \dim X$ at each point of Y .

Denote by $\mathcal{I}(\mathcal{D}_{\alpha_E})$ the ideal generated by the system of n -forms \mathcal{D}_{α_E} .

Theorem 5.1. If E is regular then the ideal $\mathcal{I}(\mathcal{D}_{\alpha_E})$ is closed.

Proof. We shall show that for every $\sigma = 1, 2, \dots, m$, $d\eta_\sigma \in \mathcal{I}(\mathcal{D}_{\alpha_E})$, i.e., there exist 1-forms μ_σ^ρ such that $d\eta_\sigma = \mu_\sigma^\rho \wedge \eta_\rho$. Denote

$$\mu_\sigma^\rho = a_{\sigma j}^\rho dx^j + b_{\sigma v}^\rho dy^v. \quad (5.13)$$

Then

$$\begin{aligned} \mu_\sigma^\rho \wedge \eta_\rho &= (b_{\sigma \kappa}^\rho B_\rho - a_{\sigma j}^\rho B_{\rho \kappa}^j) dy^\kappa \wedge \omega_0 \\ &+ \sum_{k=1}^{n-1} \frac{1}{k!} (b_{\sigma \kappa}^\rho B_{\rho v_1 \dots v_k}^{j_1 \dots j_k} - a_{\sigma i}^\rho B_{\rho \kappa v_1 \dots v_k}^{ij_1 \dots j_k}) dy^\kappa \wedge dy^{v_1} \wedge \dots \wedge dy^{v_k} \wedge \omega_{j_1 \dots j_k} \\ &+ \frac{1}{n!} b_{\sigma \kappa}^\rho B_{\rho v_1 \dots v_n}^{j_1 \dots j_n} dy^\kappa \wedge dy^{v_1} \wedge \dots \wedge dy^{v_n} \wedge \omega_{j_1 \dots j_n}, \end{aligned} \quad (5.14)$$

$$\begin{aligned} d\eta_\sigma &= \left(\frac{\partial B_\sigma}{\partial y^\kappa} - \frac{\partial B_{\sigma \kappa}^i}{\partial x^i} \right) dy^\kappa \wedge \omega_0 \\ &+ \sum_{k=1}^{n-1} \frac{1}{k!} \left(\frac{\partial B_{\sigma v_1 \dots v_k}^{j_1 \dots j_k}}{\partial y^\kappa} - \frac{\partial B_{\sigma \kappa v_1 \dots v_k}^{ij_1 \dots j_k}}{\partial x^i} \right) dy^\kappa \wedge dy^{v_1} \wedge \dots \wedge dy^{v_k} \wedge \omega_{j_1 \dots j_k} \\ &+ \frac{1}{n!} \frac{\partial B_{\sigma v_1 \dots v_n}^{j_1 \dots j_n}}{\partial y^\kappa} dy^\kappa \wedge dy^{v_1} \wedge \dots \wedge dy^{v_n} \wedge \omega_{j_1 \dots j_n} \end{aligned} \quad (5.15)$$

From the requirement $d\mathcal{I}(\mathcal{D}_{\alpha_E}) \subset \mathcal{I}(\mathcal{D}_{\alpha_E})$ we obtain the following system of equations:

$$\begin{aligned} b_{\sigma\kappa}^{\rho} B_{\rho} &= \frac{\partial B_{\sigma}}{\partial y^{\kappa}} - \frac{\partial B_{\sigma\kappa}^i}{\partial x^i} + a_{\sigma j}^{\rho} B_{\rho\kappa}^j, \\ \left(b_{\sigma\kappa}^{\rho} B_{\rho v_1 \dots v_k}^{j_1 \dots j_k} &= \frac{\partial B_{\sigma v_1 \dots v_k}^{j_1 \dots j_k}}{\partial y^{\kappa}} - \frac{\partial B_{\sigma\kappa v_1 \dots v_k}^{ij_1 \dots j_k}}{\partial x^i} + a_{\sigma i}^{\rho} B_{\rho\kappa v_1 \dots v_k}^{ij_1 \dots j_k} \right)_{\text{alt}(\kappa v_1 \dots v_k)}, \quad 1 \leq k \leq n-1, \\ \left(b_{\sigma\kappa}^{\rho} B_{\rho v_1 \dots v_n}^{j_1 \dots j_n} &= \frac{\partial B_{\sigma v_1 \dots v_n}^{j_1 \dots j_n}}{\partial y^{\kappa}} \right)_{\text{alt}(\kappa v_1 \dots v_n)}, \end{aligned} \quad (5.16)$$

where $\text{alt}(\dots)$ means complete antisymmetrization in the indicated indices. These equations are satisfied, in particular, for $a_{\sigma j}^{\rho} = 0$, and $b_{\sigma\kappa}^{\rho}$ satisfying

$$\begin{aligned} b_{\sigma\kappa}^{\rho} B_{\rho} &= \frac{\partial B_{\sigma}}{\partial y^{\kappa}} - \frac{\partial B_{\sigma\kappa}^i}{\partial x^i}, \\ b_{\sigma\kappa}^{\rho} B_{\rho v_1 \dots v_k}^{j_1 \dots j_k} &= \frac{\partial B_{\sigma v_1 \dots v_k}^{j_1 \dots j_k}}{\partial y^{\kappa}} - \frac{\partial B_{\sigma\kappa v_1 \dots v_k}^{ij_1 \dots j_k}}{\partial x^i}, \quad 1 \leq k \leq n-1, \\ b_{\sigma\kappa}^{\rho} B_{\rho v_1 \dots v_n}^{j_1 \dots j_n} &= \frac{\partial B_{\sigma v_1 \dots v_n}^{j_1 \dots j_n}}{\partial y^{\kappa}}. \end{aligned} \quad (5.17)$$

At each point of Y and for every fixed pair of indices σ, κ , Eqs. (5.17) are a system of linear non-homogeneous algebraic equations for m unknowns $b_{\sigma\kappa}^{\rho}$, and the matrix of the system is B^T . By assumption, $\text{rank } B = \text{rank } B^T = m$, i.e., Eqs. (5.17) have a unique solution $b_{\sigma\kappa}^{\rho}$, $1 \leq \rho \leq m$. This completes the proof. \square

The definition of regularity of E provides us with different *sufficient conditions* of regularity. Below we list some of them.

Proposition 5.3. *Let E be an Y -pertinent dynamical form on $J^1 Y$. For E be regular any of the following n conditions is sufficient:*

$$\text{rank} \left(\frac{\partial^k E_{\sigma}}{\partial y_{j_1}^{v_1} \dots \partial y_{j_k}^{v_k}} \right) = m, \quad 1 \leq k \leq n, \quad (5.18)$$

where σ labels rows and the other indices label columns.

Proof. The assertion follows from the fact that the matrix (5.12) is equivalent with the matrix

$$\left(E_{\sigma} \frac{\partial E_{\sigma}}{\partial y_{j_1}^{v_1}} \frac{\partial^2 E_{\sigma}}{\partial y_{j_1}^{v_1} \partial y_{j_2}^{v_2}} \cdots \frac{\partial^n E_{\sigma}}{\partial y_{j_1}^{v_1} \cdots \partial y_{j_n}^{v_n}} \right). \quad \square \quad (5.19)$$

Proposition 5.4. Let α be an $(n+1)$ -form on Y . α is regular if and only if the map

$$V\pi \ni \zeta \rightarrow i_{\zeta}\alpha \in \Lambda^n Y, \quad (5.20)$$

mapping vertical vector fields to n -forms on Y , is injective.

Proof. Denote $\eta_{\sigma} = i_{\partial/\partial y^{\sigma}}\alpha$. Let $\zeta \in V\pi$ be such that $i_{\zeta}\alpha = 0$. Since $\zeta = \zeta^{\sigma} \partial/\partial y^{\sigma}$, we get $i_{\zeta}\alpha = \zeta^{\sigma} \eta_{\sigma} = 0$. However, by the regularity assumption, the n -forms η_{σ} are linearly independent at each point of Y , i.e., the condition $\zeta^{\sigma} \eta_{\sigma} = 0$ implies $\zeta^{\sigma} = 0$ for all σ , meaning that $\zeta = 0$.

Conversely, if $\zeta^{\sigma} \eta_{\sigma} = 0$ implies $\zeta^{\sigma} = 0$ for all σ , then the n -forms η_{σ} are linearly independent at each point of Y , hence $\text{rank } \mathcal{D}_{\alpha_E} = m$. \square

If E is a locally variational form, then α_E (which is closed) is called the zero-order Hamiltonian system associated to E [17]. Due to Theorem 4.1, Lagrangian systems (represented by locally variational forms on $J^1 Y$) are in bijective correspondence with zero-order Hamiltonian systems. The ideal $\mathcal{I}(\mathcal{D}_{\alpha_E})$ is called the Hamiltonian ideal of E , and the equations for integral sections of the Hamiltonian ideal, i.e., (5.8), are said to be Hamilton equations. In this case, integral sections of $\mathcal{I}(\mathcal{D}_{\alpha_E})$ are the extremals of E . Each of the regularity conditions (5.18) can be expressed in terms of a Lagrangian; for example, for $k = 1$ we obtain the following sufficient condition of regularity:

$$\text{rank} \left(\frac{\partial E_{\sigma}}{\partial y_i^v} \right) = \text{rank} \left(\frac{\partial^2 L}{\partial y^{\sigma} \partial y_i^v} - \frac{\partial^2 L}{\partial y_i^{\sigma} \partial y^v} \right) = m. \quad (5.21)$$

Recall that an $(n+1)$ -form α on Y is called multisymplectic if it is closed, and the map $\zeta \rightarrow i_{\zeta}\alpha$, mapping vector fields on Y to n -forms, is injective [4]. In view of this definition, a closed $(n+1)$ -form α on Y is multisymplectic if and only if for every fiber chart (V, ψ) , $\psi = (x^i, y^{\sigma})$ on Y , the n -forms

$$\eta_i = i_{\partial/\partial x^i} \alpha, \quad \eta^{\sigma} = i_{\partial/\partial y^{\sigma}} \alpha, \quad 1 \leq i \leq n, \quad 1 \leq \sigma \leq m, \quad (5.22)$$

are linearly independent at each point of V .

Now, we immediately get the relation between regular closed $(n+1)$ -forms and multisymplectic forms on Y :

Proposition 5.5. Every multisymplectic form on Y is regular.

We stress that the converse is not true. Indeed, the following is an *example of a regular closed $(n+1)$ -form which is not multisymplectic*.

Example 5.1. On $R^n \times R^m \rightarrow R^n$ consider a form α as follows:

$$\alpha = B_\sigma dy^\sigma \wedge \omega_0 + \frac{1}{2} B_{\sigma\nu}^1 dy^\sigma \wedge dy^\nu \wedge \omega_1, \quad d\alpha = 0, \quad \det(B_{\sigma\nu}^1) \neq 0. \quad (5.23)$$

The corresponding dynamical form is obviously $E = (B_\sigma + B_{\sigma\nu}^1 y_1^\nu) dy^\sigma \wedge \omega_0$, where B_σ , $B_{\sigma\nu}^1$ are functions independent of the variables y_j^ρ , and satisfying the variationality conditions

$$B_{\sigma\nu}^1 = -B_{\nu\sigma}^1, \quad \frac{\partial B_\sigma}{\partial y^\nu} - \frac{\partial B_\nu}{\partial y^\sigma} + \frac{\partial B_{\sigma\nu}^1}{\partial x^1} = 0, \quad \frac{\partial B_{\sigma\rho}^1}{\partial y^\nu} + \frac{\partial B_{\rho\nu}^1}{\partial y^\sigma} + \frac{\partial B_{\nu\sigma}^1}{\partial y^\rho} = 0. \quad (5.24)$$

Since, by assumption, the rank of the $(m \times (m+1))$ -matrix $(B_\sigma, B_{\sigma\nu}^1)$ is maximal (equal to m), the form α is regular. However, computing

$$\begin{aligned} i_\zeta \alpha &= \zeta^\sigma (B_\sigma \omega_0 + B_{\sigma\nu}^1 dy^\nu \wedge \omega_1) - \zeta^i (B_\sigma dy^\sigma \wedge \omega_i - \frac{1}{2} B_{\sigma\nu}^1 dy^\sigma \wedge dy^\nu \wedge \omega_{1i}) \\ &= \zeta^\sigma B_\sigma \omega_0 - (\zeta^\nu B_{\sigma\nu}^1 + \zeta^1 B_\sigma) dy^\sigma \wedge \omega_1 - \sum_{i=2}^n \zeta^i B_\sigma dy^\sigma \wedge \omega_i \\ &\quad + \frac{1}{2} \zeta^i B_{\sigma\nu}^1 dy^\sigma \wedge dy^\nu \wedge \omega_{1i} = 0 \end{aligned} \quad (5.25)$$

for a vector field

$$\zeta = \zeta^i \frac{\partial}{\partial x^i} + \zeta^\sigma \frac{\partial}{\partial y^\sigma}, \quad (5.26)$$

and taking into account that $\omega_{11} = 0$, we obtain

$$\zeta^\sigma = -M^{\sigma\nu} B_\nu \zeta^1, \quad 1 \leq \sigma \leq m, \quad \zeta^2 = \dots = \zeta^n = 0, \quad (5.27)$$

where $(M^{\sigma\nu})$ is the inverse matrix to $(B_{\sigma\nu}^1)$. Hence, the map $\zeta \rightarrow i_\zeta \alpha$ has a one-dimensional kernel spanned by the everywhere non-zero vector field

$$\Gamma = \frac{\partial}{\partial x^1} - M^{\sigma\nu} B_\nu \frac{\partial}{\partial y^\sigma}, \quad (5.28)$$

meaning that the form α is not multisymplectic.

As a consequence of the above we get that *multisymplectic forms do not coincide with regular Hamiltonian systems*.

Appendix

Lemma A.1. *Let α be a $\pi_{1,0}$ -projectable $(n+1)$ -form on J^1Y ,*

$$\alpha = \sum_{k=0}^n \alpha_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_k} \wedge \omega_{j_1 \dots j_k}, \quad (\text{A.1})$$

its expression in a fiber chart. Then $\alpha_\sigma \omega^\sigma \wedge \omega_0 = \alpha_\sigma dy^\sigma \wedge \omega_0$, and for all $k = 1, 2, \dots, n$,

$$\begin{aligned} \alpha_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_k} \wedge \omega_{j_1 \dots j_k} &= \alpha_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} dy^\sigma \wedge dy^{v_1} \wedge \dots \wedge dy^{v_k} \wedge \omega_{j_1 \dots j_k} \\ &+ \sum_{l=1}^{k-1} (-1)^l l! \binom{k+1}{l} \binom{k}{l} \alpha_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} y_{j_{k-l+1}}^{v_{k-l+1}} \dots y_{j_k}^{v_k} dy^\sigma \wedge dy^{v_1} \wedge \dots \wedge dy^{v_{k-l}} \wedge \omega_{j_1 \dots j_{k-l}} \\ &+ (-1)^k (k+1)! \alpha_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} y_{j_1}^{v_1} \dots y_{j_k}^{v_k} dy^\sigma \wedge \omega_0. \end{aligned} \quad (\text{A.2})$$

Proof. We shall proceed by induction on k . For $k = 1$ we have

$$\alpha_{\sigma v}^j \omega^\sigma \wedge \omega^v \wedge \omega_j = \alpha_{\sigma v}^j dy^\sigma \wedge dy^v \wedge \omega_j - 2\alpha_{\sigma v}^j y_j^v dy^\sigma \wedge \omega_0,$$

as desired. For $k+1$ it holds

$$\begin{aligned} \alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_{k+1}} \wedge \omega_{j_1 \dots j_{k+1}} \\ = \alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_{k+1}} \wedge i_{\partial/\partial x^{j_{k+1}}} \omega_{j_1 \dots j_k}, \end{aligned}$$

and since

$$\begin{aligned} i_{\partial/\partial x^{j_{k+1}}} (\alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_{k+1}} \wedge \omega_{j_1 \dots j_k}) \\ = (-1)^{k+2} (k+2) \alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} y_{j_{k+1}}^{v_{k+1}} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_k} \wedge \omega_{j_1 \dots j_k} \\ + (-1)^{k+2} \alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_{k+1}} \wedge \omega_{j_1 \dots j_{k+1}}, \end{aligned}$$

we obtain

$$\begin{aligned} & \alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_{k+1}} \wedge \omega_{j_1 \dots j_{k+1}} \\ &= -(k+1) \alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} y_{j_{k+1}}^{v_{k+1}} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_k} \wedge \omega_{j_1 \dots j_k} \\ &+ (dy^{k+1} - y_l^{k+1} dx^l) \wedge i_{\partial/\partial x^{j_{k+1}}} (\alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_k} \wedge \omega_{j_1 \dots j_k}). \end{aligned}$$

Applying formula (A.2) leads after short calculations to the following result:

$$\begin{aligned} & \alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} \omega^\sigma \wedge \omega^{v_1} \wedge \dots \wedge \omega^{v_{k+1}} \wedge \omega_{j_1 \dots j_{k+1}} \\ &= \alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} dy^\sigma \wedge dy^{v_1} \wedge \dots \wedge dy^{v_{k+1}} \wedge \omega_{j_1 \dots j_{k+1}} \\ &- (k+2)(k+1) \alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} y_{j_{k+1}}^{v_{k+1}} dy^\sigma \wedge dy^{v_1} \wedge \dots \wedge dy^{v_k} \wedge \omega_{j_1 \dots j_k} \\ &+ \sum_{l=2}^k (-1)^l \frac{(k+2)!}{l!(k+2-l)!} \frac{(k+1)!}{(k+1-l)!} \alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} y_{j_{k-l+2}}^{v_{k-l+2}} \dots y_{j_{k+1}}^{v_{k+1}} \\ &\times dy^\sigma \wedge dy^{v_1} \wedge \dots \wedge dy^{v_{k-l+1}} \wedge \omega_{j_1 \dots j_{k-l+1}} \\ &+ (-1)^{k+1} (k+2)! \alpha_{\sigma v_1 \dots v_{k+1}}^{j_1 \dots j_{k+1}} y_{j_1}^{v_1} \dots y_{j_{k+1}}^{v_{k+1}} dy^\sigma \wedge \omega_0, \end{aligned}$$

which is the desired formula for $k+1$. \square

By the above lemma we immediately obtain

Lemma A.2. *Let α be a $\pi_{1,0}$ -projectable $(n+1)$ -form on $J^1 Y$, $\alpha_{\sigma v_1 \dots v_k}^{j_1 \dots j_k}$ and $\beta_{\sigma v_1 \dots v_k}^{j_1 \dots j_k}$, where $0 \leq k \leq n$, its components with respect to the adapted basis $(dx^i, dy^\sigma, \omega^\sigma, dy_j^\sigma)$ and the canonical basis $(dx^i, dy^\sigma, dy_j^\sigma)$, respectively. Then*

$$\begin{aligned} \beta_\sigma &= \alpha_\sigma + \sum_{l=1}^n (-1)^l (l+1)! \alpha_{\sigma v_1 \dots v_l}^{j_1 \dots j_l} y_{j_1}^{v_1} \dots y_{j_l}^{v_l}, \\ \beta_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} &= \alpha_{\sigma v_1 \dots v_k}^{j_1 \dots j_k} + \sum_{l=1}^{n-k} (-1)^l l! N(k, l) \alpha_{\sigma v_1 \dots v_{k+l}}^{j_1 \dots j_{k+l}} y_{j_{k+1}}^{v_{k+1}} \dots y_{j_{k+l}}^{v_{k+l}}, \\ \beta_{\sigma v_1 \dots v_n}^{j_1 \dots j_n} &= \alpha_{\sigma v_1 \dots v_n}^{j_1 \dots j_n}, \end{aligned} \tag{A.3}$$

where

$$N(k, l) = \binom{k+l+1}{l} \binom{k+l}{l}, \quad 1 \leq k \leq n-1.$$

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