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On the number of zeros of Abelian integrals for a polynomial Hamiltonian irregular at infinity☆

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Abstract

Up to now, most of the results on the tangential Hilbert 16th problem have been concerned with the Hamiltonian regular at infinity, i.e., its principal homogeneous part is a product of the pairwise different linear forms. In this paper, we study a polynomial Hamiltonian which is not regular at infinity. It is shown that the space of Abelian integral for this Hamiltonian is finitely generated as a $\mathbb{R}[h]$ module by several basic integrals which satisfy the Picard–Fuchs system of linear differential equations. Applying the bound meandering principle, an upper bound for the number of complex isolated zeros of Abelian integrals is obtained on a positive distance from critical locus. This result is a partial solution of tangential Hilbert 16th problem for this Hamiltonian. As a consequence, we get an upper bound of the number of limit cycles produced by the period annulus of the non-Hamiltonian integrable quadratic systems whose almost all orbits are algebraic curves of degree $k+n$, under polynomial perturbation of arbitrary degree.

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1. Introduction

In this paper, we study the number of zeros of Abelian integral for a polynomial Hamiltonian which is irregular at infinity.

1.1. The tangential Hilbert 16th problem

Let $H(x, y),\ f(x, y),\ g(x, y)$ be polynomials in two-real variables and Γ_h the closed connected component of level set $\{(x, y)|H(x, y) = h\}$. Suppose that

$$\omega = f(x, y)\,dx + g(x, y)\,dy \tag{1.1}$$

is a real polynomial 1-form with degree $d = \max\{\deg f(x, y), \deg g(x, y)\}$. The Abelian integral is defined by

$$I(h) = I(h, H, \omega) = \oint_{\Gamma_h} \omega. \tag{1.2}$$

The tangential Hilbert 16th problem, or the weakened Hilbert 16th problem, posed by Arnold [A1,A2], is to place an upper bound $Z(\deg H, d)$ of the number of zeros of $I(h)$ on the maximal connected interval of existence of Γ_h, in terms of $\deg H$ and d.

The general result of solving the tangential Hilbert 16th problem was achieved by Varchenko [V] and Khovanskii [K], who proved independently the existence of $Z(\deg H, d)$, but no explicit expression of $Z(\deg H, d)$ has been obtained. Many authors have contributed to estimate or to give an explicit upper bound of the number of zeros of $I(h)$ for the cubic and quartic elliptic Hamiltonians $H = y^2 + p(x)$, see for instance Petrov [P1,P2,P3], Rousseau and Zoladek [RZ], Zhao and Zhang [ZZz], Liu [Lc] etc. In the paper [HI2], Horozov and Iliev gave a linear upper bound $Z(3, d) \leqslant 15d + 15$ for general cubic Hamiltonians. The authors of the paper [NY3] constructed a linear differential equation satisfied by $I(h)$ and obtained using the tools from [IY] an asymptotical exponential bound for the number of zeros of $I(h)$. More results of this problem will be recalled in Sections 1.2–1.4.

1.2. Abelian integrals and limit cycles

We briefly recall the connection between the tangential Hilbert 16th problem and the number of limit cycles of planar vector fields.

1.2.1. The polynomial perturbations of Hamiltonian systems

Consider the perturbed system

$$dH(x, y) + \varepsilon\omega = 0, \tag*{$(1.3)_\varepsilon$}$$

where ε is a small parameter. Then the displacement function is

$$d(h,\varepsilon)=\rho(h,\varepsilon)-h=\varepsilon I(h)+\varepsilon^2 M_2(h)+\cdots\varepsilon^k M_k(h)+\cdots. \tag{1.4}$$

Here $\rho(h,\varepsilon)$ is the first return mapping of $(1.3)_\varepsilon$ in terms of h and ε. Let $M_1(h)=I(h)$. It is well known that the number of zeros of the first non-vanishing Melnikov function $M_k(h)$, $k=1,2,\ldots,$ gives an upper bound of the number of limit cycles in $(1.3)_\varepsilon$ which are born out from the period annulus Γ_h surrounding the center of $(1.3)_0$.

For the quadratic perturbations of quadratic Hamiltonian systems, i.e., $\deg H=3$, $d=2$, it has been proved in [GH] for perturbations of *generic* quadratic Hamiltonian, that, if $I(h)\equiv 0$, then $(1.3)_\varepsilon$ is a Hamiltonian system. It has been shown $Z(3,2)=2$ by the works of Horozov, Iliev [HI1], Gavrilov [G3], Li [LZ], etc. If $I(h)\equiv 0$ for *non-generic* quadratic Hamiltonians, then the higher-order Melnikov function $M_k(h)$, $k\geqslant 2$, must be considered. In the paper [I3], Iliev gave the formula of higher-order Melnikov function for quadratic perturbations of non-generic quadratic integrable system. By the study of the number of zeros of higher Melnikov function, we know that the cyclicity of period annulus of non-generic quadratic Hamiltonian systems under quadratic perturbations is 3 for the Hamiltonian triangle case, and 2 for other cases (see [CLY,GI,I1,ZLL,ZZh2]).

In order to obtain more limit cycles of planar systems and various configuration patterns of their relative disposition, which is a part of Hilbert 16th problem, Li et al. study the tangential Hilbert 16th problem for the symmetric planar polynomial systems. For example, he proved that the exact upper bound of the number of limit cycles (*Hilbert number*) for cubic system is at least 11 [LjH]. More results about the number of zeros of higher-order Melnikov function and limit cycles can be found in [F,G2,GI,I4,Lj,ZZh1] and reference therein.

1.2.2. The polynomial perturbations of non-Hamiltonian integrable systems

Consider generalized system

$$\begin{cases}\dot{x}=\dfrac{H_y(x,y)}{M(x,y)}+\varepsilon P(x,y),\\ \dot{y}=-\dfrac{H_x(x,y)}{M(x,y)}+\varepsilon Q(x,y),\end{cases} \tag*{$(1.5)_\varepsilon$}$$

where H_y/M, H_x/M, $P(x,y)$, $Q(x,y)$ are polynomials, $H(x,y)=h$ is a first integral of system $(1.5)_0$ with integrating factor $M(x,y)$. Suppose that $(1.5)_0$ has at least one center. If $M(x,y)$ is not a constant, then $(1.5)_0$ is called a non-Hamiltonian integrable system. The Abelian integrals, associated with system $(1.5)_\varepsilon$, are defined as

$$\widetilde{I}(h)=\oint_{\Gamma_h} M(x,y)(-P(x,y)\,dy+Q(x,y)\,dx). \tag{1.6}$$

Since the integrating factor $M(x,y)$ is no longer a constant, the study of Abelian integrals for non-Hamiltonian integrable systems is more difficult than the one in the Hamiltonian cases.

In the papers [LLLZ,LZLZ,ZLLZ], the authors study quadratic non-Hamiltonian integrable systems whose almost all orbits are conic, cubic and quartic curves, respectively, where the phrase “almost all" means “all except at most a finite number of”. They give a linear estimate of the number of zeros of Abelian integrals $\widetilde{I}(h)$ for these systems. A series papers are concerned with the quadratic perturbations of quadratic non-Hamiltonian integrable systems, see [DLZ,GLLZ,I2,Zo], etc.

1.3. The space of Abelian integrals and Gavrilov theorems

The study of tangential Hilbert 16th problem requires a very basic information concerning the space of Abelian integrals. This problem can be resolved if the Hamiltonian is sufficiently regular at infinity.

Definition 1.1 (*Novikov and Yakovenko [NY5]*). A polynomial $H(x, y) \in \mathbb{C}[x, y]$ of degree n is said to be regular at infinity, if one of the three equivalent conditions holds:

(1) its principle homogeneous part $\hat{H}$,a homogeneous polynomial of degree n, is a product of n pairwise different linear forms;
(2) $\hat{H}$ has an isolated critical point (necessarily of multiplicity $(n-1)^2$) at the origin $(0, 0)$;
(3) the level curve $\{\hat{H} = 1\} \subset \mathbb{C}^2$ is non-singular.

Definition 1.2. A polynomial $H(x, y) \in \mathbb{C}[x, y]$ of degree n is said to be irregular at infinity, if it is not regular at infinity.

In [G1,G2], Gavrilov proved that for polynomial Hamiltonian $H(x, y)$ regular at infinity, the space of Abelian integrals is finitely generated as a $\mathbb{C}[h]$-module by the basic integrals. However, it seems that there is no general result about the space of Abelian integrals for the polynomial Hamiltonian which is irregular at infinity.

1.4. Meandering principle and Picard–Fuchs system

Consider a polynomial vector field in $\mathbb{R}^n$ or $\mathbb{C}^n$, defined by a system of n first-order polynomial ordinary differential equations, whose degrees and the magnitude of coefficients are explicitly bounded. Then the number of isolated intersection points between a integral trajectory of this polynomial vector field and any affine hyperplane in the ambient space can be explicitly characterized in terms of the size of this integral trajectory and the magnitude of the coefficients of vector fields, see [NY1,NY2,NY4,Y1] for details. Using the bounded meandering principle, the authors of the paper [NY1] proved that the number of zeros of Abelian integrals for the elliptic Hamiltonian $H = y^2 + p(x)$ is characterized by a certain tower function depending only on the degree $\deg p(x)$ and d.

Almost all approaches of the solution of the tangential Hilbert 16th problem so far was based on using the system which is called *Picar–Fuchs system*, or *Gauss–Manin connection*. The system, satisfied by the monomial integrals $V = (I_0, I_1, \ldots, I_l)$, has the form

$$\dot{V} = \Upsilon(h)V,$$

with a rational matrix function$\Upsilon(h)$, where $I_0, I_1, \ldots, I_l$ generate the space of Abelian integrals as a $\mathbb{R}[h]$-module or $\mathbb{C}[h]$-module . One can obtain from Picard–Fuchs system more information concerning Abelian integrals.

To investigate the tangential Hilbert 16th problem for the *balanced Hamiltonian*, an explicit system of the monomial integrals is derived in the paper [NY5]. A peculiar feature is that the dimension of this system is approximately two times greater than that one of the standard Picard–Fuchs system, and so it is called *Redundant Picard–Fuchs system*. The above result allow to apply the bounded meandering principle for the balanced Hamiltonians and then one gets an explicit upper bound for the number of zeros of Abelian integral away from the critical locus. The paper [Y2] deals with the bounded decomposition in Brieskorn lattice and Picard–Fuchs system corresponding to semiquasi-homogeneous Hamiltonian.

1.5. The main results of this paper

It seems that most of results on the tangential Hilbert problem so far have been concerned with the Hamiltonian regular at infinity. In this paper, we consider a polynomial Hamiltonian which is irregular at infinity. More precisely, let

$$H(x, y) = x^k(\tfrac{1}{2}y^2 + p(x)) = h, \quad k \in \mathbb{Z}^+ = \{1, 2, 3, \ldots, \}, \tag{1.7}$$

where $p(x)$ is a monic polynomial of degree n,

$$p(x) = \sum_{l=0}^{n} p_l x^l, \quad p_n = 1, \tag{1.8}$$

$H(x, y) \in \mathbb{R}[x, y]$ or $\mathbb{C}[x, y]$. The homogeneous part of $H(x, y)$ has a zero at $x = 0$ with multiplicity $k + n$, which means that $H(x, y)$, defined in (1.7), is irregular at infinity. On the other hand, the Hamiltonian (1.7) is a first integral of non-Hamiltonian integrable system $(1.5)_0$ with the integrating factor $M(x, y) = x^{k-1}$. Of course, it is also a first integral of Hamiltonian system $dH(x, y) = 0$, i.e., the Hamiltonian system $(1.3)_0$.

We investigate in this paper the tangential Hilbert 16th problem for Hamiltonian system (1.7). It is shown in Section 2 that the space of Abelian integrals is finitely generated as a $\mathbb{R}[h]$-module by $n + k$ basic integrals $J_{-1}(h), J_0(h), \ldots, J_{n+k-2}(h)$, which is a counterpart of Gavrilov theorem (Corollary 2.2). The properties of Abelian integrals are given in Section 2.3, provided $p(x) = \pm(x - c)^n$, $c \in \mathbb{R}$. Following the arguments used in [NY5], we derive an explicit system of Picard–Fuchs equations of the form

$$(h\mathbf{E} - \mathbf{A})\dot{\mathbf{J}} = \mathbf{B}\mathbf{J}, \quad \mathbf{A}, \mathbf{B} \in Mat_{(n+k)\times(n+k)}(\mathbb{C}),$$

satisfied by the vector $\mathbf{J} = (J_{-1}, J_0, \ldots, J_{n+k-2})$. The algorithm for derivation of Picard–Fuchs system allow us to give a complete description and obtain an explicit bound on the norms for the matrices of **A** and **B**, see Proposition 3.1, Sections 3.3 and 3.4 for more details. The above information on Picard–Fuchs system and the space of Abelian integrals already suffices to apply the bounded meandering principle and get an explicit upper bound for the number of zeros of $I(h)$ away from the critical locus of the Hamiltonian (1.7) (Theorem 4.2).

As a continuation of the work in [LLLZ,LZLZ,ZLLZ], we study the number of zeros of the Abelian integral $\widetilde{I}(h)$, defined in (1.6), for the non-Hamiltonian integrable quadratic system $(1.5)_0$ which has a first integral either (1.7) with $n = 1, 2,\ k \geqslant 3$, or

$$H(x, y) = \widetilde{H}(x, y) = x^{-k-2}(\tfrac{1}{2}y^2 + p_0x^2 + p_1x + p_2) = h,\quad k \geqslant 3. \tag{1.9}$$

It is proved that in Section 2.5 that $\widetilde{I}(h)$, related to $(1.5)_\varepsilon$ and (1.9), can be expressed as a combination with polynomial coefficients of $k + 2$ Abelian integrals $J_{-1},\ J_0, \ldots, J_k$, associated with the system $(1.5)_\varepsilon$ and the Hamiltonian $(1.7)|_{n=2},\ k \geqslant 3$. Therefore, the same Picard–Fuchs system can be used in the study of Abelian integrals for these two different quadratic non-Hamiltonian integrable systems. More information on the Picard–Fuchs system for the Hamiltonian (1.7) with $n = 1, 2$ can be found in Sections 3.5, 3.6.1 and 3.6.2. Since the degenerate Hamiltonian $(1.7)_{n=2}$ has a atypical critical value, we derive the Picard–Fuchs system satisfied by $\dot{J}_l,\ l = -1, 0, 1, \ldots, k-1$, in Sections 3.6.3 and 3.6.4. Finally, we get an upper bound for the number of limit cycles of polynomial perturbations of quadratic non-Hamiltonian system $(1.5)_0$ with the first integrals $(1.7)_{n=2}$ or (1.9) under the assumption $\widetilde{I}(h) \not\equiv 0$. The accurate formulation is given in Section 4.3.

1.6. Conventions

Let $(x, y) \in \mathbb{R}^2$ and $f(x, y),\ g(x, y),\ H(x, y) \in \mathbb{R}[x, y]$ if the planar vector fields and the limit cycles are concerned with. We always suppose that ω is a real or complex 1-form and $H(x, y) \in \mathbb{R}[x, y]$ (or $\mathbb{C}[x, y]$) is defined as (1.7) unless the opposite is claimed.

2. The relative cohomology decomposition of polynomial 1-form

2.1. Notations and conventions

Let $\Lambda^m,\ m = 0, 1, 2$ be the space of polynomial m-forms on $\mathbb{R}^2$ or $\mathbb{C}^2$. The multiplication $\gamma(h) \cdot \omega = \gamma(H)\omega$ holds over the ring of polynomial $\mathbb{C}[h]$. An equivalent relation $\sim$ is defined between two 1-form ω and $\widetilde{\omega}$ as follows: $\omega \sim \widetilde{\omega}$ if and only if $\omega - \widetilde{\omega} = d\xi(x, y, H) + \eta(x, y, H)\, dH$, where $\xi(x, y, H)$ and $\eta(x, y, H)$ are polynomials of $x,\ y$ and H.

In the sequel, $*$ denotes constant. Let

$$\omega_{ij} = x^i y^j\, dx, \quad I_{ij} = I_{ij}(h) = \oint_{\Gamma_h} \omega_{ij} \tag{2.1}$$

and

$$\omega_i = x^i y\,dx, \quad J_i = J_i(h) = \oint_{\Gamma_h} \omega_i, \quad i = \cdots - 1, 0, 1, \cdots. \tag{2.2}$$

We put $deg\,\omega = \max\{\deg f(x, y), \deg g(x, y)\}$, where $\omega \in \Lambda^1$ is defined in (1.1). The symbol $\alpha_l(H)$, $\alpha_{ij}(H)$, $\beta_l(H)$, etc. always means the polynomials of H.

2.2. The relative cohomology decomposition of polynomial 1-form for Hamiltonian (1.7)

2.2.1. The main result

Theorem 2.1. *For every complex polynomial 1-form $\omega \in \Lambda^1$, $\deg \omega = d$, there exists polynomials $\delta(H)$, $\alpha_l(H)$, $l = -1, 0, 1, \ldots, n+k-2$, such that*

$$\omega = \sum_{l=-1}^{n+k-2} \alpha_l(H)\omega_l + \delta(H)\,\frac{dx}{x} + d\xi(x, y, H) + \eta(x, y, H)\,dH. \tag{2.3}$$

Here $\xi(x, y, H)$ and $\eta(x, y, H)$ are polynomials of x, y, H, and

(i) *if $n \geqslant 3$, then $\deg \alpha_l(H) \leqslant \left[(\widetilde{d} - l)/(n+k)\right]$ for $\widetilde{d} \geqslant l+1$ and $\deg \alpha_l(H) = 0$ for $\widetilde{d} \leqslant l$, respectively, where $\widetilde{d} = [(d-1)/2]n + ((-1)^d + 1)/2$ and $[s]$ denotes entire part of s;*

(ii) *if $n = 1, 2$, then $\deg \alpha_l(H) \leqslant [(d-1-l)/(n+k)]$ for $d \geqslant l+1$ and $\deg \alpha_l(H) = 0$ for $d \leqslant l$, respectively;*

(iii) *If $d \leqslant k$, then $\delta(H) \equiv 0$; If $d \geqslant k+1$, then $\deg \delta(H) \leqslant [d/k]$ for $n = 1$ and $\deg \delta(H) \leqslant [((d-2)n+2)/(2k)]$ for $n \geqslant 2$, respectively.*

Corollary 2.2. *Let ω be a real polynomial* 1*-form and $deg\,\omega = d$, $H(x, y) \in \mathbb{R}[x, y]$. Then the Abelian integrals $I(h)$, associated with Hamiltonian* (1.7)*, can be expressed as*

$$I(h) = \oint_{\Gamma_h} \omega = \sum_{l=-1}^{n+k-2} \alpha_l(h) J_l, \tag{2.4}$$

where $\alpha_l(h)$ is defined as in Theorem 2.1.

Corollary 2.2 shows that the space of all Abelian integrals is finitely generated as a free $\mathbb{R}[h]$-module by $n+k$ integrals $J_{-1}(h), J_0(h), \ldots, J_{n+k-2}(h)$.

2.2.2. Proof of Theorem 2.1

The proof consists of a long straightforward calculation. Let $i + j \leqslant d = \max\{\deg f(x, y), \deg g(x, y)\}$, where $f(x, y)$, $g(x, y)$ are defined in (1.1). We have

$$x^i y^j\,dy = \frac{1}{j+1}\,d(x^i y^{j+1}) - \frac{i}{j+1}\omega_{i-1, j+1},$$

so we only need to consider the 1-form ω_{ij}, $i+j\leqslant d$. We split the proof into several steps.

Step 1: In the first step, we will express ω_{ij} as a linear combination of dx/x, $\omega_{kj'/2-1,j'}$ and ω_l, $l=-1,0,1,\ldots,$ with polynomial coefficients, modulo 1-form $d\xi+\eta\,dH$.

(1) *If k and j are odd, then*

$$\omega_{ij}=\sum_{l=i}^{i+(j-1)n/2} *\omega_l+d\xi_{ij}(x,y)+\eta_{ij}(x,y)\,dH, \tag{2.5}$$

where

$$\begin{aligned}\xi_{ij}(x,y)&=x^{i+1}\Bigg(*y^j+\sum_{l_1=0}^{n}*x^{l_1}y^{j-2}+\cdots\\&\quad+\sum_{l_1=0}^{n}\sum_{l_2=0}^{n}\cdots\sum_{l_{(j-3)/2}=0}^{n}*x^{l_1+l_2+\cdots+l_{(j-3)/2}}y^3\Bigg),\\ \eta_{ij}(x,y)&=x^{i-k+1}\Bigg(*y^{j-2}+\sum_{l_1=0}^{n}*x^{l_1}y^{j-4}+\cdots\\&\quad+\sum_{l_1=0}^{n}\sum_{l_2=0}^{n}\cdots\sum_{l_{(j-3)/2}=0}^{n}*x^{l_1+l_2+\cdots+l_{(j-3)/2}}y\Bigg).\end{aligned}$$

It follows from (1.7) that

$$dH=x^ky\,dy+x^{k-1}(\tfrac{1}{2}ky^2+kp(x)+xp'(x))\,dx. \tag{2.6}$$

Multiplying both sides of (2.6) by $x^{i-k+1}y^{j-2}$, we get

$$x^{i-k+1}y^{j-2}\,dH=x^{i+1}y^{j-1}\,dy+\frac{1}{2}kx^iy^j\,dx+x^i(kp(x)+xp'(x))y^{j-2}\,dx. \tag{2.7}$$

On the other hand,

$$x^{i+1}y^{j-1}\,dy=\frac{1}{j}d(x^{i+1}y^j)-\frac{i+1}{j}\omega_{ij}. \tag{2.8}$$

Taking (2.8) into (2.7), we have

$$\frac{kj-2(i+1)}{2j}\omega_{ij} = -\sum_{l=0}^{n}(k+l)p_l\omega_{i+l,j-2} - d\left(\frac{x^{i+1}y^j}{j}\right) + x^{i-k+1}y^{j-2}\,dH. \quad (2.9)$$

Please note that (2.6)–(2.9) hold for $\forall k \in \mathbb{Z}^+$. If k and j are odd, then $kj-2(i+1) \neq 0$, which implies

$$\omega_{ij} = \frac{2j}{kj-2(i+1)}\left(-\sum_{l=0}^{n}(k+l)p_l\omega_{i+l,j-2} - d\left(\frac{x^{i+1}y^j}{j}\right) + x^{i-k+1}y^{j-2}\,dH\right). \quad (2.10)$$

It follows from (2.10) that

$$\begin{aligned}\omega_{ij} &= \sum_{l_1=0}^{n} *\omega_{i+l_1,j-2} + d(*x^{i+1}y^j) + *x^{i-k+1}y^{j-2}\,dH\\ &= \sum_{l_1=0}^{n}\sum_{l_2=0}^{n} *\omega_{i+l_1+l_2,j-4} + d(*x^{i+1}y^j + \sum_{l_1=0}^{n} *x^{i+l_1+1}y^{j-2})\\ &\quad +(*x^{i-k+1}y^{j-2} + \sum_{l_1=0}^{n} *x^{i+l_1-k+1}y^{j-4})\,dH\\ &= \cdots\\ &= \sum_{l_1=0}^{n}\sum_{l_2=0}^{n}\cdots\sum_{l_{(j-1)/2}=0}^{n} *\omega_{i+l_1+\cdots+l_{(j-1)/2}} + d\xi_{ij}(x,y) + \eta_{ij}(x,y)\,dH.\end{aligned}$$

Here we use the inequality $k(j-2m)-2(i+l_1+l_2+\cdots+l_m+1) \neq 0$, $m = 1,2,\ldots,(j-1)/2$, provided j and k are odd.
(2) *Let* $kj-2(i+1)=0$, *i.e.,* $i = kj/2-1$. *If* j *is odd and* k *is even, then*

$$\omega_{kj/2-1,j} = \sum_{l'=0}^{(j-1)/2}\sum_{m=0}^{((j-1)/2-l')n} *H^{l'}\omega_{(j/2-l')k+m-1}. \quad (2.11)$$

If j *is even, then*

$$\begin{aligned}\omega_{kj/2-1,j} &= *H^{j/2}\frac{dx}{x} + d\left(\sum_{l',m} *H^{l'}x^{(j/2-l')k+m}\right)\\ &\quad + \left(\sum_{l',m} *H^{l'-1}x^{(j/2-l')k+m}\right)dH, \quad (2.12)\end{aligned}$$

where $(l',m) \neq (j/2,0)$, $0 \leqslant l' \leqslant j/2$, $0 \leqslant m \leqslant (j/2-l')n$.

If j is odd and k is even, then it follows from (1.7) that

$$\begin{aligned}\omega_{kj/2-1,j} &= x^{k/2-1}(x^k y^2)^{(j-1)/2} y\,dx = 2^{(j-1)/2} x^{k/2-1} y(H - x^k p(x))^{(j-1)/2}\,dx \\ &= x^{k/2-1} y \sum_{l'=0}^{(j-1)/2} *H^{l'}(x^k p(x))^{(j-1)/2-l'}\,dx \\ &= \sum_{l'=0}^{(j-1)/2} \sum_{m=0}^{((j-1)/2-l')n} *H^{l'} x^{(j/2-l')k+m-1} y\,dx,\end{aligned}$$

which implies (2.11). If j is even, the by the same arguments as above, we have

$$\omega_{kj/2-1,j} = x^{-1}(x^k y^2)^{j/2}\,dx = \sum_{l'=0}^{j/2} \sum_{m=0}^{(j/2-l')n} *H^{l'} x^{(j/2-l')k+m-1}\,dx.$$

Since

$$H^{l'} x^{(j/2-l')k+m-1}\,dx = d(*H^{l'} x^{(j/2-l')k+m}) - *x^{(j/2-l')k+m} H^{l'-1}\,dH$$

for $(l', m) \neq (j/2, 0)$, $l' \neq 0$, the decomposition (2.12) follows.
(3) *Let* $kj - 2(i+1) \neq 0$.
If k *is even and* j *is odd, then*

$$\omega_{ij} = \sum_{l=i}^{i+(j-1)n/2} *\omega_l + \sum_{j' \leqslant j-2} *\omega_{kj'/2-1,j'} + d\widetilde{\xi}_{ij}(x, y) + \widetilde{\eta}_{ij}(x, y)\,dH, \qquad (2.13)$$

where $\widetilde{\xi}_{ij}(x, y)$ *and* $\widetilde{\eta}_{ij}(x, y)$ *are two variables polynomials,* j' *and* $i' = kj'/2 - 1$ *have the forms* $j' = j - 2m$, $i' = i + l_1 + l_2 + \cdots + l_m$, $m \in \mathbb{Z}^+, 1 \leqslant m \leqslant (j-3)/2, 0 \leqslant l_q \leqslant n$, $q = 1, 2, \ldots, m$. *So* $i' + (j'-1)n/2 \leqslant i + (j-1)n/2$.
If j *are even, then*

$$\omega_{ij} = \sum_{j' \leqslant j-2} *H^{j'/2} \frac{dx}{x} + d\xi_{ij}(x, y, H) + \eta_{ij}(x, y, H)\,dH, \qquad (2.14)$$

where $\xi_{ij}(x, y, H)$, $\eta_{ij}(x, y, H)$ *are polynomials of* x, y *and* H. $j' = j - 2m$ *is defined as follows: there exists* $i' = i + l_1 + l_2 + \cdots + l_m$, $1 \leqslant m \leqslant (j-2)/2$, $0 \leqslant l_q \leqslant n$, $q = 1, 2, \ldots, m$, *such that* $kj' - 2(i'+1) = 0$.

We get (2.13) from (2.10) by the same arguments as (1) and (2). The expression (2.14) follows from (2.10) and (2.12), we omit the details.

Step 2: We prove in this step that ω_i can be expressed as a linear combination of ω_l, $l = -1, 0, 1, \ldots, n+k-2$, with polynomial coefficients, modulo 1-form $d\xi + \eta\, dH$.

(4) *For* $i \geqslant n+k-1$, *we have*

$$(n+2i+2)\omega_i = (2(i-n+1)-3k)H\omega_{i-k-n} + \sum_{l=0}^{n-1}(2n-3l-2i-2)p_l\omega_{l+i-n} - d(x^{i-n+1}y^3) + 3x^{i-k-n+1}y\,dH. \tag{2.15}$$

Multiplying (1.7) by $x^{i-k-n}y\,dx$, we get

$$H\omega_{i-k-n} = \frac{1}{2}\omega_{i-n,3} + \omega_i + \sum_{l=0}^{n-1} p_l\omega_{i+l-n}. \tag{2.16}$$

By (2.9), we have

$$\frac{3k-2(i-n+1)}{6}\omega_{i-n,3} = -\sum_{l=0}^{n}(k+l)p_l\omega_{i+l-n} - d\left(\frac{x^{i-n+1}y^3}{3}\right) + x^{i-k-n+1}y\,dH. \tag{2.17}$$

If $i = 3k/2+n-1$, i.e., $3k-2(i-n+1) = 0$, then (2.17) implies

$$(k+n)\omega_{3k/2+n-1} = -\sum_{l=0}^{n-1}(k+l)p_l\omega_{3k/2+l-1} - d\left(\frac{x^{3k/2}y^3}{3}\right) + x^{k/2}y\,dH,$$

which is (2.15) with $i = 3k/2+n-1$. If $i \neq 3k/2+n-1$, then (2.15) follows from (2.16) and (2.17).

(5) $\omega_i, i \geqslant n+k-1$, *can be represented as*

$$\omega_i = \sum_{l=-1}^{n+k-2}\alpha_{il}(H)\omega_l + d\xi_i(x,y) + \eta_i(x,y)\,dH, \tag{2.18}$$

where $\xi_i(x,y), \eta_i(x,y)$ *are polynomials of* x *and* y, $\deg\alpha_{il}(H) \leqslant [(i-l)/(n+k)]$, $l = -1, 0, \ldots, n+k-2$.

We prove (2.18) by induction for i. For $i = n+k-1$, it follows from (2.15) that $(3n+2k)\omega_{n+k-1} \sim -kH\omega_{-1} + \sum_{l=0}^{n-1}(-3l-2k)p_l\omega_{l+k-1}$. Suppose (2.18) hold for $n+k-1 \leqslant m \leqslant i-1$, $m \in \mathbb{Z}^+$, then we get by using (2.15) that

$$\begin{aligned}\omega_i &\sim *H\omega_{i-k-n} + \sum_{l'=0}^{n-1} *\omega_{l'+i-n}\\ &\sim H\sum_{l=-1}^{n+k-2}\alpha_{i-k-n,l}(H)\omega_l + \sum_{l'=0}^{n-1} * \sum_{l=-1}^{n+k-2}\alpha_{l'+i-n,l}(H)\omega_l.\end{aligned}$$

Let $\alpha_{il}(H) = *H\alpha_{i-k-n,l}(H) + \sum_{l'=0}^{n-1} *\alpha_{l'+i-n,l}(H)$. Then

$$\begin{aligned}\deg\alpha_{il}(H) &\leqslant \max\{1+\deg\alpha_{i-k-n,l}(H),\ \deg\alpha_{i-1,l}(H),\\ &\quad \deg\alpha_{i-2,l}(H),\ \ldots,\deg\alpha_{i-n,l}(H)\}\\ &\leqslant \max\left\{1+\left[\frac{i-k-n-l}{n+k}\right], \left[\frac{i-1-l}{n+k}\right], \left[\frac{i-2-l}{n+k}\right],\right.\\ &\quad \left.\ldots, \left[\frac{i-n-l}{n+k}\right]\right\}\\ &= \left[\frac{i-l}{n+k}\right].\end{aligned}$$

Step 3: In the final step, we prove (2.3). First of all, we will show that *the following expression holds, provided j is odd*:

$$\omega_{ij} \sim \sum_{l=-1}^{n+k-2}\alpha_{ijl}(H)\omega_l, \tag{2.19}$$

where $\deg\alpha_{ijl}(H) \leqslant [(i+(j-1)n/2-l)/(n+k)]$ *for* $i+(j-1)n/2 \geqslant l+1$ *and* $\deg\alpha_{ijl}(H) = 0$ *for* $i+(j-1)n/2 \leqslant l$, *respectively*.

If k is odd, then (2.19) follows from (2.5) and (2.18). Now we consider $\omega_{kj/2-1,j}$, where k is even. Using (2.18) again, we have in (2.11),

$$H^{l'}\omega_{(j/2-l')k+m-1} \sim \sum_{l=-1}^{n+k-2} H^{l'}\alpha_{(j/2-l')k+m-1,l}(H)\omega_l.$$

The degree of coefficient of ω_l is explicitly bounded:

$$\begin{aligned}\deg H^{l'}\alpha_{(j/2-l')k+m-1,l}(H) &\leqslant l' + \left[\frac{(j/2-l')k+m-1-l}{n+k}\right]\\ &\leqslant \left[\frac{jk/2+l'n+m-1-l}{n+k}\right]\\ &\leqslant \left[\frac{jk/2-1+(j-1)n/2-l}{n+k}\right]\\ &= \left[\frac{i+(j-1)n/2-l}{n+k}\right],\end{aligned}$$

where $i = kj/2-1$, $l=-1,0,1,\ldots,n+k-2$. In the above proof we use the inequality $0\leqslant m\leqslant ((j-1)/2-l')n$. It follows from (2.11) and the estimate for $\deg H^{l'}\alpha_{(j/2-l')k+m-1,l}(H)$ that (2.19) holds for $\omega_{kj/2-1,j}$. Therefore, it follows from (2.13) and (2.18) that (2.19) holds if k is even.

Since ω is a linear combination of ω_{ij} with constant coefficients, we get (2.3) from (2.19), (2.12) and (2.14).

Please note that dx/x just appears in the decomposition of $\omega_{kj/2-1,j}$, provided j is even. If $i+j=d$, $i=kj/2-1$, then $j/2=(d+1)/(k+2)$, which implies $\delta(H)\equiv 0$ for $d\leqslant k$. In what follows we consider (2.14). Let $i+j=d\geqslant k+1, n\geqslant 2$. Then $j'/2$, defined in (2.14), is explicitly bounded:

$$\begin{aligned}\frac{j'}{2} &= \frac{i'+1}{k} = \frac{i+l_1+l_2+\cdots l_m+1}{k} \leqslant \frac{i+mn+1}{k}\\ &\leqslant \frac{1}{k}\left(i+\frac{j-2}{2}n+1\right) = \frac{1}{k}\left(d-j+\frac{j-2}{2}n+1\right)\\ &= \frac{1}{2k}(2d-2n+2+(n-2)j)\\ &\leqslant \frac{1}{2k}(2d-2n+2+(n-2)d) = \frac{1}{2k}((d-2)n+2),\end{aligned}$$

which yields $\deg\delta(H)\leqslant \max\{[((d-2)n+2)/(2k)], [(d+1)/(k+2)]\} = [((d-2)n+2)/(2k)]$ for $d\geqslant k+1$, $n\geqslant 2$. The estimation for $\deg\alpha_l(H)$ follows from (2.19). $\square$

2.3. Properties of Abelian integral J_l provided $p(x)=\pm(x-c)^n$, $c\in\mathbb{R}$, $c\neq 0$, $n\geqslant 2$

Recall that $p(x)$ is monic. If $p(x)$ has only one real critical point at c with multiplicity n, then $p(x)=(x-c)^n$, which means $H(x,y)=x^k(y^2/2+(x-c)^n)=h$. The corresponding integrable system is

$$\begin{cases}\dot{x} = xy = \frac{\partial H}{\partial y}/x^{k-1},\\ \dot{y} = -ky^2/2-(x-c)^{n-1}((k+n)x-kc) = -\frac{\partial H}{\partial x}/x^{k-1}.\end{cases}$$

The type of critical points can be determined by using the theorems in [ZDHD]. If n is even, then the above system has a center at $(c, 0)$ and a saddle at $(kc/(k+n), 0)$. If n is odd, then $(c, 0)$ is a degenerate non-center critical point and $(kc/(k+n), 0)$ is a center (resp., saddle) for $c > 0$ (resp., $c < 0$). Let $\Gamma_h \subset \{(x, y) \in \mathbb{R}^2 | H(x, y) = h\}$ be the periodic orbit around the center.

Proposition 2.3. *Suppose that the monic polynomial $p(x)$ has only one real critical point at $x = c$, i.e. $p(x) = (x-c)^n$.*

(i) *If n is odd and $c > 0$, then*

$$\oint_{\Gamma_h} (kc - (k+n)x)\, x^{k-1}(c-x)^{n/2-1} y\, dx = 0.$$

(ii) *If $n/2$ is even, then*

$$\oint_{\Gamma_h} ((k+n)x - kc)\, x^{k-1}(x-c)^{n/2-1} y\, dx = 0. \tag{2.20}$$

This implies that $J_{k-1}, J_k, \ldots, J_{n/2+k-1}$ are linearly dependent.

(iii) *If n is even but $n/2$ is odd, then*

$$\oint_{\Gamma_h} ((k+n)x - kc)\, x^{k-1}(x-c)^{n/2-1} y\, dx = 2\sqrt{2}\pi h. \tag{2.21}$$

Proof. Denote by $(x_i(h), 0)$, $i = 1, 2$, the intersection point of closed orbits Γ_h and x-axis, which implies $H(x_i(h), 0) = x_i^k(h)(x_i(h) - c)^n = h$. By direct computation, we have

$$H(c, 0) = 0, \quad \widetilde{h} = H\left(\frac{kc}{k+n}, 0\right) = (-1)^n c^{n+k} \left(\frac{k}{k+n}\right)^k \left(\frac{n}{k+n}\right)^n.$$

(i) If n is odd and $c > 0$, then $(kc/(k+n), 0)$ is a center and $kc/(k+n) < c$. Since $x = 0$ is an invariant line, the periodic orbit around the center $(kc/(k+n), 0)$ does not intersect $x = 0$, which implies $c > x$ for $\forall (x, y) \in \Gamma_h$, $h \in (\widetilde{h}, 0)$. Therefore,

$$\begin{aligned}
&\oint_{\Gamma_h} (kc - (k+n)x)\, x^{k-1}(c-x)^{n/2-1} y\, dx \\
&\quad = 2\int_{x_1(h)}^{x_2(h)} (kc - (k+n)x)\, x^{k-1}(c-x)^{n/2-1}\sqrt{2hx^{-k} + 2(c-x)^n}\, dx \\
&\quad = 2\sqrt{2}\int_{x_1(h)}^{x_2(h)} (kc - (k+n)x)\, x^{k-1}(c-x)^{n/2-1}(c-x)^{n/2}\sqrt{\frac{h}{u} + 1}\, dx
\end{aligned}$$

$$= 2\sqrt{2}\int_{x_1(h)}^{x_2(h)} \sqrt{\frac{h}{u}+1}\,du$$
$$= 0.$$

Here, we used the following integration formula:

$$\int \sqrt{\frac{h}{u}+1}\,du = \sqrt{h+u}\sqrt{u} + h\ln(\sqrt{u}+\sqrt{h+u}), \quad u = x^k(c-x)^n.$$

(ii) If $n/2$ is even, then $(c,0)$ is a center and $(x-c)^{n/2} > 0$ for $\forall(x,y) \in \Gamma_h$. Therefore,

$$\oint_{\Gamma_h} ((k+n)x - kc)\,x^{k-1}(x-c)^{n/2-1} y\,dx$$
$$= 2\int_{x_1(h)}^{x_2(h)} ((k+n)x - kc)\,x^{k-1}(x-c)^{n/2-1}\sqrt{2hx^{-k} - 2(x-c)^n}\,dx$$
$$= 2\sqrt{2}\int_{x_1(h)}^{x_2(h)} ((k+n)x - kc)\,x^{k-1}(x-c)^{n-1}\sqrt{\frac{h}{v}-1}\,dx$$
$$= 2\sqrt{2}\int_{x_1(h)}^{x_2(h)} \sqrt{\frac{h}{v}-1}\,dv$$
$$= 0.$$

Here we used the following formula:

$$\int \sqrt{\frac{h}{v}-1}\,du = \sqrt{h-v}\sqrt{v} + h\ \arcsin\sqrt{\frac{v}{h}}, \quad v = x^k(x-c)^n.$$

(iii) If n is even but $n/2$ is odd, then $(x-c)^{n/2} < 0$ for $x < c$ and $(x-c)^{n/2} > 0$ for $x > c$, respectively, if $(x,y) \in \Gamma_h$, where Γ_h is a periodic orbit around the center $(c,0)$. Hence,

$$\oint_{\Gamma_h} ((k+n)x - kc)\,x^{k-1}(x-c)^{n/2-1} y\,dx$$
$$= -2\sqrt{2}\int_{x_1(h)}^{c} ((k+n)x - kc)\,x^{k-1}(x-c)^{n/2-1}(x-c)^{n/2}\sqrt{\frac{h}{v}-1}\,dx$$
$$+2\sqrt{2}\int_{c}^{x_2(h)} ((k+n)x - kc)\,x^{k-1}(x-c)^{n/2-1}(x-c)^{n/2}\sqrt{\frac{h}{v}-1}\,dx$$
$$= -2\sqrt{2}\int_{x_1(h)}^{c} \sqrt{\frac{h}{v}-1}\,dv + 2\sqrt{2}\int_{c}^{x_2(h)} \sqrt{\frac{h}{v}-1}\,dv$$
$$= 2\sqrt{2}\pi h. \qquad \square$$

Proposition 2.3(ii) shows that at most $n+k-1$ integrals $J_l(h)$, $l=-1,0,\ldots,n+k-3$, generate the space of all Abelian integrals as a free $\mathbb{R}[h]$-module, provided that $n/2$ is even and $p(x)=(x-c)^n$. Please compare this conclusion with Corollary 2.2.

We always suppose $p_n=1$, i.e., $p(x)$ is monic univariate polynomial in this paper. However, if $p_n=-1$, we have the similar conclusions as Proposition 2.3(i):

Proposition 2.4. *If $p(x)=-(x-c)^n$, then the identity* (2.20) *holds.*

Proof. In this case, the corresponding integrable system has a center at $(kc/(k+n),0)$ if and only if one of the following conditions holds: (i) n is even, (ii) n is odd, $c<0$. The critical point $(c,0)$ is a cuspidal or saddle point. Since $x=0$ is an invariant line, we know that $sgn(x-c)\equiv -1$ (resp., $sgn(x-c)\equiv 1$) for $\forall(x,y)\in\Gamma_h$ if $c>0$ (resp., $c<0$). By the same arguments as in the proof of Proposition 2.3(i), we have

$$\begin{aligned}&\oint_{\Gamma_h}((k+n)x-kc)\,x^{k-1}(x-c)^{n/2-1}y\,dx\\&=2\sqrt{2}sgn((x-c)^{n/2})\int_{x_1(h)}^{x_2(h)}\sqrt{\frac{h}{x^k(x-c)^n}+1}d\left(x^k(x-c)^n\right)\\&=0,\end{aligned}$$

Here $(x_i(h),0), i=1,2,$ is the intersection point of Γ_h and x-axis, $x_1(h)<x_2(h)$, and we use the integration formula as in the proof of Proposition 2.3(i). $\square$

2.4. Normal form and Abelian integrals for non-Hamiltonian quadratic integrable case $n=1,2$ with $k\geqslant 3$

We have given the main results about Abelian integrals for quadratic system $(1.3)_\varepsilon$ and $(1.5)_\varepsilon$ in Corollary 2.2. In this section, we are going to formulate analogs of Section 2.3 for quadratic case $n=2$. Before that we give a normal form of $(1.5)_0$ with at least one center for $n=1,2$. Using the results from appendix of [I3], we get

Proposition 2.5. *If $n=1$ in $(1.5)_0$, then the parameters p_0,p_1,p_2 can be taken as $p_2=0, p_1=1, p_0=-(k+1)/k$. Moreover, system $(1.5)_0$ has a center at $(1,0)$ and two saddles at $(0,\pm\sqrt{2(k+1)/k})$. The closed orbit $\Gamma_h\subset\{(x,y)|H(x,y)=h\}$ is defined for $h\in\Sigma=(-1/k,0)$.*

Proposition 2.6. *If $n=2$ in $(1.5)_0$, then the parameter p_0, p_1, p_2 can be taken as $p_2=1, p_1=-(k+2+kp_0)/(k+1)$. Let*

$$h_1=H(1,0)=\frac{p_0-1}{k+1},\quad h_2=H(\frac{kp_0}{k+2},0)=\frac{-p_0\left(\frac{kp_0}{k+2}\right)^k(k^2p_0-(k+2)^2)}{(k+1)(k+2)^2}.$$

Moreover, we have

(i) *If* $p_0 < 0$, *then system* $(1.5)_0$ *has two center at* $S_1(1,0)$, $S_2(kp_0/(k+2),0)$ *and two saddles at* $(0, \pm\sqrt{-2p_0})$. *The ovals* Γ_h *around* S_1 (*resp.,* S_2) *are defined in* $\Sigma_1 = (h_1, 0)$ (*resp.,* $\Sigma_2 = (h_2, 0)$ *if* k *is even and* $\Sigma_2 = (0, h_2)$ *if* k *is odd, respectively*).

(ii) *If* $p_0 = 0$, *then system* $(1.5)_0$ *has a center at* S_1 *and a degenerate critical point at* $(0,0)$. *The ovals around* S_1 *are defined for* $h \in \Sigma_1 = (-1/(k+1), 0)$.

(iii) *If* $0 < p_0 < (k+2)/k$, *then system* $(1.5)_0$ *has a center at* S_1 *and a saddle at* S_2. *The ovals around* S_1 *are defined for Hamiltonian values* $h \in \Sigma = (h_1, h_2)$.

(iv) *If* $p_0 = (k+2)/k$, *then system* $(1.5)_0$ *has only one degenerate critical point at* $(1,0)$.

(v) *If* $p_0 > (k+2)/k$, *then system* $(1.5)_0$ *has a center at* S_2 *and a saddle at* S_1. *The closed orbits around* S_2 *are defined in* $\Sigma = (h_2, h_1)$.

From now on, we always suppose that p_0, p_1, p_2 are defined as Propositions 2.5 and 2.6 for the cases $n = 1$ and 2. In the next proposition, we consider the Abelian integrals for quadratic case $n = 2$, provided $k \geqslant 1$.

Proposition 2.7. *If* $n = 2$ *and* $p_0 = 1$ (*resp.,* $p_0 = (k+2)^2/k^2$), *then* $J_k = (kJ_{k-1} + 2\sqrt{2}\pi h)/(k+2)$ (*resp.,* $J_k = J_{k-1} + 2\sqrt{2}\pi h/(k+2)$).

Proof. If $p_0 = 1$ (resp., $p_0 = (k+2)^2/k^2$) holds, then $H = x^k(y^2/2 + (x-1)^2) = h$ (resp., $H = x^k(y^2/2 + (x-(k+2)/k)^2) = h$). The results follow from Proposition 2.3(iii). $\square$

2.5. Abelian integrals for system $(1.5)_\varepsilon$ *with the Hamiltonian* (1.9)

Let $d = \max\{\deg P(x,y), \deg Q(x,y)\} - k + 1$ in $(1.5)_\varepsilon$ and $p_2 = 1$ in (1.9). Rewrite the polynomial perturbed system $(1.5)_\varepsilon$ with Hamiltonian (1.9) as the form

$$\begin{cases} \dot{x} = xy + \varepsilon \sum_{i+j \leqslant d-k+1} * x^i y^j, \\ \dot{y} = (k+2)y^2/2 + kp_0 x^2 + (k+1)p_1 x + (k+2) + \varepsilon \sum_{i+j \leqslant d-k+1} * x^i y^j, \end{cases} \tag{2.22$_\varepsilon$}$$

where $k \geqslant 3$, $d \geqslant k-1$. The unperturbed quadratic integrable system $(2.22)_0$ has a first integral (1.9) with integrating factor x^{-k-3}. Using Poincaré transformation

$$x = \frac{1}{z}, \ y = \frac{u}{z}, \ dt = -z\, d\tau,$$

and then taking $z \to x$, $u \to y$, system $(2.22)_\varepsilon$ is reduced to

$$\begin{cases} \dot{x} = xy + \varepsilon \sum_{i+j \leqslant d-k+1} * x^{-(i+j-3)} y^j, \\ \dot{y} = -ky^2/2 - kp_0 - (k+1)p_1 x - (k+2)x^2 + \varepsilon \\ \qquad \times \sum_{i+j \leqslant d-k+1} x^{-(i+j-2)} y^j (* + *y). \end{cases} \tag{2.23$_\varepsilon$}$$

The quadratic integrable system $(2.23)_0$ has a first integral $(1.7)|_{n=2}$ with integrating factor x^{k-1}. To estimate the number of zero of Abelian integral associated system $(2.22)_\varepsilon$, we study the equivalent system $(2.23)_\varepsilon$. Since

$$x^{-(i+j-k-3)}y^j\,dy = \frac{1}{j+1}\,d\left(x^{-(i+j-k-3)}y^{j+1}\right) + \frac{i+j-k-3}{j+1}x^{-(i+j-k-2)}y^{j+1}\,dx,$$

the Abelian integrals, related to system $(2.23)_\varepsilon$, can be represented as

$$I(h) = \oint_{\Gamma_h} x^{k-1}\left(\sum_{i+j\leqslant d-k+1} x^{-(i+j-2)}(*y^j + *y^{j+1})\right) dx. \tag{2.24}$$

Proposition 2.8. (i) *If* $d\geqslant 2k+2$, *then the Abelian integral* (2.24), *related to system* $(2.23)_\varepsilon$, *can be expressed as*

$$I(h) = h^{-[(d-k-2)/k]}\sum_{l=-1}^{k}\beta_l(h)J_l, \tag{2.25}$$

where $\deg\beta_{-1}(h)\leqslant[(d-2)/k]$, $\deg\beta_l(h)\leqslant[(d-k-2)/k]$, $0\leqslant l\leqslant k$.

(ii) *If* $k-1\leqslant d\leqslant 2k+1$, *then* $I(h)$ *can be expressed as* $(2.4)|_{n=2}$ *with* $\deg\alpha_{-1}(h)\leqslant 1$, $\deg\alpha_l(h) = 0$ *for* $0\leqslant l\leqslant k$.

Proof. Firstly, we point out by symmetry that $\oint_{H=h} x^{-i}y^j\,dx \equiv 0$ if j is even. Therefore, we just consider $I_{-i,j}$ for odd j. The proof is split into several steps.

(1) $I_{-i,j}$, j *is odd and* $j\geqslant 3$, *can be represented as the form*

$$I_{-i,j} = \sum_{l=-i}^{-i+j-1} *J_l + \sum_{kj'/2-1+j'\leqslant -i+j} *I_{kj'/2-1,j'}, \tag{2.26}$$

where $i' = kj'/2-1$ *and* j' *have the forms* $i' = -i+l_1+l_2+\cdots+l_m$, $3\leqslant j' = j-2m\leqslant j-2$, $l_q\in\{0,1,2\}, q=1,2,\ldots,m$, $m\leqslant(j-3)/2$, *and the second term in* (2.26) *vanishes identically if* $-i+j\leqslant 0$.

Since $i\geqslant 0$, $j\geqslant 3$, we have $kj-2(-i+1)\neq 0$. So (2.26) follows from (2.10) by the same arguments as in step 1 of the proof for Theorem 2.1.

(2) $J_{-i}(h)$, $i\geqslant 2$, *can be expressed as*

$$J_{-i}(h) = \sum_{l=-1}^{k}\beta_{-i,l}(h^{-1})J_l, \tag{2.27}$$

where $\beta_{-i,l}(h^{-1})$ *is a polynomial of* h^{-1} *with* $\deg\beta_{-i,l}(h^{-1})\leqslant[(i-2)/k]+1$.

Taking $i \to -i+k+2$ and integrating both sides of (2.15), we get

$$J_{-i} = \frac{1}{(2i+k-2)h}\left(p_0(2i-2k-2)J_{-i+k}\right.$$
$$\left.+p_1(2i-2k-5)J_{-i+k+1}+(2i-2k-8)J_{-i+k+2}\right).$$

Then (2.27) follows by induction.
(3) It follows from (2.24), (2.26) and (2.19) that

$$\begin{aligned} I(h) &= \sum_{i+j\leqslant d-k+1}\left(\sum_{l=-i-j+k+1}^{-i+k+1} *J_l + \sum_{kj'/2-1+j'\leqslant -i+k-2} *I_{kj'/2-1,j'}\right) \\ &= \sum_{l=-d+2k}^{k+1} *J_l + \sum_{kj'/2-1+j'\leqslant k+2} *I_{kj'/2-1,j'} \\ &= \sum_{l=-1}^{k} *J_l + *J_{k+1} + \sum_{l=2}^{d-2k} *J_{-l}, \end{aligned}$$

provided $d \geqslant 2k+2$. We get (2.25) by using (2.27) and (2.18) for $d \geqslant 2k+2$. If $k-1 \leqslant d \leqslant 2k+1$, then $I(h) = \sum_{l=-1}^{k+1} *J_l$, which implies (ii). □

If $p_2 = 0$ in (1.9), then the similar results can be obtained by the sam arguments. □

3. Picard–Fuchs systems

3.1. Gelfand–Leray formula

If a pair of polynomial 1-form ω, θ satisfies the identity $d\omega = dH \wedge \theta$, then for any continuous family of cycle $\Gamma_h \subset \{(x,y)|H(x,y)=h\}$,

$$\frac{d}{dh}\oint_{\Gamma_h} \omega = \oint_{\Gamma_h} \theta, \tag{3.1}$$

which is called Gelfand–Leray formula.

3.2. Derivation of the Picard–Fuchs system for (1.7)

Computations of this section are a modification of a standard derivation of a Picard–Fuchs system for hyperelliptic integrals, see e.g. [NY5,R], etc. Let ω_i be the differential

1-form, defined in (2.2), whose derivative is $d\omega_i = x^i\, dy \wedge dx$. The 2-form $H(x, y)\, d\omega_i$ will be divided by $dH(x, y)$, yielding the identities

$$H(x, y)\, d\omega_i = dH \wedge \lambda_i + \sum_{j=-1}^{n+k-2} a_{ij} d\omega_j$$

with appropriate 1-form λ_i, $i = -1, 0, \ldots, n+k-2$. This implies the Picard–Fuchs equation satisfied by $J_{-1}(h), J_0(h), \ldots, J_{n+k-2}(h)$. More precisely, we have

$$\begin{aligned}
H\, d\omega_i &= x^{i+k}\left(\frac{1}{2}y^2 + p(x)\right) dy \wedge dx \\
&= \frac{1}{2}x^{i+k}y^2\, dy \wedge dx + \left(x^{k-1}(kp(x) + xp'(x))b_i(x) + a_i(x)\right) dy \wedge dx \\
&= \frac{1}{2}x^{i+k}y^2\, dy \wedge dx + \left(H_x - \frac{1}{2}kx^{k-1}y^2\right) b_i(x)\, dy \wedge dx + a_i(x)\, dy \wedge dx \\
&= \frac{1}{2}x^i y H_y\, dy \wedge dx - b_i(x)(dH - H_y\, dy) \wedge dy \\
&\quad -\frac{1}{2}kb_i(x)x^{-1}yH_y\, dy \wedge dx + a_i(x)\, dy \wedge dx \\
&= \left(\frac{1}{2}x^i y - \frac{1}{2}kb_i(x)x^{-1}y\right)(dH - H_x\, dx) \wedge dx - b_i(x)\, dH \wedge dy \\
&\quad +a_i(x)\, dy \wedge dx \\
&= dH \wedge \left(\frac{1}{2}\omega_i + \sum_{j=0}^{i+1}(j - \frac{1}{2}k)b_{ij}\omega_{j-1} - d(yb_i(x))\right) + \sum_{j=0}^{n+k-2} a_{ij}\, d\omega_j,
\end{aligned}$$

where we use the following identities:

(i) *the* 1-*form and the partial differential derivatives of* $H(x, y)$:

$$dH = H_x\, dx + H_y\, dy,\ \ H_x = x^{k-1}\left(\frac{1}{2}ky^2 + kp(x) + xp'(x)\right), \quad H_y = x^k y, \tag{3.2}$$

(ii) *division with remainder*: the polynomials $x^{i+k}p(x)$ of degree $n+k+i$ is divided by $x^{k-1}(kp(x) + xp'(x)) = H_x - (\frac{1}{2})kx^{k-1}y^2$ as

$$\begin{gathered}
x^{i+k}p(x) = x^{k-1}(kp(x) + xp'(x))b_i(x) + a_i(x), \\
\deg a_i(x) \leqslant n+k-2, \ \ \deg b_i(x) \leqslant i+1,
\end{gathered} \tag{3.3}$$

here

$$a_i(x) = \sum_{j=0}^{n+k-2} a_{ij}x^j, \ b_i(x) = \sum_{j=0}^{i+1} b_{ij}x^j, \quad i = -1, 0, 1, \ldots, n+k-2,$$

(iii) the form $b_i(x)\,dy$ is represented as a linear combination

$$b_i(x)\,dy = d(yb_i(x)) - b_i'(x)y\,dx = d(yb_i(x)) - \sum_{j=1}^{i+1} jb_{ij}x^{j-1}y\,dx, \tag{3.4}$$

(iv) the remainder $a_i(x)\,dy \wedge dx$ can be represented as

$$a_i(x)\,dy \wedge dx = \sum_{j=1}^{n+k-2} a_{ij}x^j\,dy \wedge dx = \sum_{j=1}^{n+k-2} a_{ij}\,d\omega_j. \tag{3.5}$$

Integrating over the periodic orbit $\Gamma_h \subset \{(x, y)|H(x, y) = h\}$ (so that exact forms $d(yb_i(x))$ disappear) and using the Gelfand–Leray formula (3.1), one gets

$$h\dot{J}_i - \sum_{j=0}^{n+k-2} a_{ij}\dot{J}_j = \frac{1}{2}J_i + \sum_{j=0}^{i+1}(j - \frac{1}{2}k)b_{ij}J_{j-1}, \tag{3.6}$$

where $\dot{J}_i = dJ_i/dh$. Denote by $\mathbf{J} = col(J_{-1},\ J_0,\ J_1, \ldots, J_{n+k-2})$, $\mathbf{A} = (a_{ij})_{i,j=-1}^{n+k-2}$, $\mathbf{B} = (B_{ij})_{(i,j)=(-1,0)}^{(n+k-2,n+k-1)}$, where we suppose $a_{i,-1} = 0$ and

$$B_{ij} = \begin{cases} (j - k/2)b_{ij}, & j \leqslant i, \\ 1/2 + (i + 1 - k/2)b_{i,i+1}, & j = i + 1, \\ 0, & j \geqslant i + 2. \end{cases} \tag{3.7}$$

The matrix form of (3.6) is

$$(h\mathbf{E} - \mathbf{A})\dot{\mathbf{J}} = \mathbf{BJ}, \quad \mathbf{A},\ \mathbf{B} \in Mat_{(n+k)\times(n+k)}(\mathbb{C}). \tag{3.8}$$

The identities (3.3) imply the following claim, which gives a complete description of the entries of the matrices **A** and **B**.

Proposition 3.1. (i) *Let $p(x)$ be a monic polynomial of degree n, defined as in* (1.8). *If $-1 \leqslant j \leqslant k-2$, then $a_{ij} = 0$; If $k-1 \leqslant j \leqslant n+k-2$, then a_{ij} can be obtained by* *the following recursive formulas*

$$\begin{aligned} a_{-1,j} &= \frac{n-j+k-1}{k+n} p_{j-k+1}, \\ a_{ij} &= a_{i-1,j-1} - \frac{j+1}{k+n} a_{i-1,n+k-2} p_{j-k+1}, \quad i \geqslant 0. \end{aligned} \tag{3.9}$$

(ii) *For* b_{ij}, *we have*

$$b_{i,i+1} = \frac{1}{k+n}, \quad b_{ij} = \frac{1}{k+n} a_{i-j-1,n+k-2}, \quad 0 \leqslant j \leqslant i, \quad i \geqslant -1. \tag{3.10}$$

Proof. The identity (3.3) shows that $a_i(x)$ has a zero at $x = 0$ with multiplicity at least $k-1$, which implies that $a_{ij} = 0$ for $0 \leqslant j \leqslant k-2$. $a_{i,-1} = 0$ follows from our assumption. Recall $p(x)$ is monic, i.e., $p_n = 1$. Using (3.3) again, one gets $b_{-1}(x) \equiv 1/(k+n)$, and

$$\begin{aligned} a_{-1}(x) &= x^{k-1} p(x) - x^{k-1}(kp(x) + xp'(x)) b_{-1}(x) \\ &= x^{k-1} \left(\sum_{l=0}^{n} p_l x^l - \frac{1}{k+n} \sum_{l=0}^{n} (k+l) p_l x^l \right) = \sum_{l=0}^{n} \left(\frac{n-l}{k+n} \right) p_l x^{l+k-1} \\ &= \frac{n}{k+n} p_0 x^{k-1} + \frac{n-1}{k+n} p_1 x^k + \cdots + \frac{1}{k+n} p_{n-1} x^{n+k-2} \\ &= \sum_{j=k-1}^{n+k-2} \left(\frac{n-j+k-1}{k+n} \right) p_{j-k+1} x^j, \end{aligned}$$

which yields the first identity of (3.9). By (3.3), we have

$$x^{i+k-1} p(x) = x^{k-1}(kp(x) + xp'(x)) b_{i-1}(x) + a_{i-1}(x), \tag{3.11}$$

Multiplying both sides of (3.11) by x, we get

$$x^{i+k} p(x) = x^{k-1}(kp(x) + xp'(x))(x b_{i-1}(x)) + x a_{i-1}(x). \tag{3.12}$$

The following identity follows by using division algorithm:

$$\begin{aligned} x a_{i-1}(x) &= x^{k-1}(kp(x) + xp'(x)) \frac{a_{i-1,n+k-2}}{k+n} + a_i(x), \ \deg a_i(x) \\ &\leqslant n+k-2. \end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.12), we obtain

$$b_i(x) = xb_{i-1}(x) + \frac{a_{i-1,n+k-2}}{k+n} \tag{3.14}$$

and

$$\begin{aligned} a_i(x) &= xa_{i-1}(x) - x^{k-1}(kp(x) + xp'(x)) \frac{a_{i-1,n+k-2}}{k+n} \\ &= \sum_{l=k-1}^{n+k-2} a_{i-1,l}x^{l+1} - \frac{a_{i-1,n+k-2}}{k+n} \sum_{l=0}^{n} (k+l)p_l x^{l+k-1} \\ &= \sum_{j=k}^{n+k-1} a_{i-1,j-1}x^j - \frac{a_{i-1,n+k-2}}{k+n} \sum_{j=k-1}^{n+k-1} (j+1)p_{j-k+1}x^j \\ &= \sum_{j=k-1}^{n+k-2} \left(a_{i-1,j-1} - \frac{j+1}{k+n} a_{i-1,n+k-2}p_{j-k+1} \right) x^j, \end{aligned}$$

which implies the second identity of (3.9). Here, we use $p_n = 1$ and $a_{ij} = 0$ for $-1 \leqslant j \leqslant k-2$.

The first formula of (3.10) is obtained by using (3.3). It follows from (3.14) by induction that

$$\begin{aligned} b_i(x) &= x\left(xb_{i-2}(x) + \frac{a_{i-2,n+k-2}}{k+n} \right) + \frac{a_{i-1,n+k-2}}{k+n} \\ &= x^2 b_{i-2}(x) + \frac{a_{i-2,n+k-2}}{k+n}x + \frac{a_{i-1,n+k-2}}{k+n} \\ &= x^2 \left(xb_{i-3}(x) + \frac{a_{i-3,n+k-2}}{k+n} \right) + \frac{a_{i-2,n+k-2}}{k+n}x + \frac{a_{i-1,n+k-2}}{k+n} \\ &= \cdots \\ &= x^i b_0(x) + \frac{a_{0,n+k-2}}{k+n}x^{i-1} + \frac{a_{1,n+k-2}}{k+n}x^{i-2} + \cdots + \frac{a_{i-1,n+k-2}}{k+n} \\ &= \frac{1}{k+n}x^{i+1} + \frac{a_{-1,n+k-2}}{k+n}x^i + \frac{a_{0,n+k-2}}{k+n}x^{i-1} + \frac{a_{1,n+k-2}}{k+n}x^{i-2} \\ &\quad + \cdots + \frac{a_{i-1,n+k-2}}{k+n}, \end{aligned}$$

which yields the second formula of (3.10). The proof is finished. □

3.3. Spectral properties of matrices **A** *and* **B**

The matrices of **A** and **B** can be completely described by the following propositions and corollaries.

Proposition 3.2. (i) *Let* $\tilde{x} \in \mathbb{C}$, $\tilde{x} \neq 0$, *be a critical point of* $x^k p(x)$ *and* $\tilde{h} = \tilde{x}^k p(\tilde{x})$ *the corresponding critical value. Then the column vector* $col(\tilde{x}^{-1}, 1, \tilde{x}, \tilde{x}^2, \dots, \tilde{x}^{n+k-2}) \in \mathbb{C}^{n+k}$ *is the eigenvector of* **A** *with the eigenvalue* $\tilde{h}$.

(ii) *Denote by* V_0 *the subspace spanned by the eigenvectors of matrix* **A** *with the eigenvalue* $h = 0$ *and* $dim\, V_0$ *the dimension of* V_0. *If for any non-zero critical point, the corresponding critical value is not equal to zero, then* $dim\, V_0 = k$.

Proof. If $\tilde{x}$ is a critical point of $x^k p(x)$ and $\tilde{x} \neq 0$, then $\tilde{x}^{k-1}(kp(\tilde{x})) + \tilde{x}p'(\tilde{x}) = 0$. It follows from (3.3) that

$$\tilde{h}\tilde{x}^i = a_i(\tilde{x}) = \sum_{j=-1}^{n+k-2} a_{ij}\tilde{x}^j,$$

which yields (i). Since Proposition 3.1 shows $a_{ij} = 0$ for $-1 \leqslant j \leqslant k-2$, we conclude that $det(h\mathbf{E} - \mathbf{A}) = 0$ has a zero at $h = 0$ with multiplicity k. Let $\delta_j = col(\delta_j^{-1}, \delta_j^0, \delta_j^1, \dots, \delta_j^{n+k-2})$, $j = -1, 0, 1, \dots, n+k-2$, where

$$\delta_j^i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Using $a_{ij} = 0$ for $-1 \leqslant j \leqslant k-2$ again, we have $0 \cdot \delta_j = \mathbf{A}\delta_j$, $-1 \leqslant j \leqslant k-2$, which implies that $\delta_{-1}, \delta_0, \dots, \delta_{k-2}$ are the eigenvectors of **A** with the eigenvalue $h = 0$. $dim\, V_0 = k$ follows from the assumption. $\square$

To convenience, we give the following definition.

Definition 3.3. Function $F(x)$ having neither degenerate critical point nor multiple critical value for $\forall x \in \mathcal{D}$ are said to be a Morse function in $\mathcal{D}$.

Corollary 3.4. *Suppose that* $x^k p(x)$ *is a Morse polynomial in* $\mathcal{D} = \{x | x \neq 0\} \subset \mathbb{C}$, *and for any critical point in* $\mathcal{D}$, *the corresponding critical value is not equal to zero. Then*

(i) **A** *is diagonalizable and its eigenvalues are the critical values of* $H(x, y)$;

(ii) *All finite singular points of Picard–Fuchs equations* (3.8) *are Fuchsian, which, by definition, means the matrix* $(h\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$ *of coefficients of* **J** *has poles of first order. This implies that* $I(h)$, *defined in* (2.4), *is an* (*multiple-valued*) *analytic function in* $\mathbb{C} \backslash \{h | det(h\mathbf{E} - \mathbf{A}) = 0\}$.

Proof. By the assumptions, the matrix **A** has neither degenerate critical point nor multiple critical value for $x \in \mathcal{D}$. The result (i) follows from Proposition 3.2.

Singular points of (3.8) are determined by the equation $det(h\mathbf{E}-\mathbf{A})=0$, which means that $h=\tilde{h}$ is a singular point of (3.8) if and only if it is a eigenvalue of the matrix $\mathbf{A}$. Solving $\dot{J}_i(h)$ from (3.8) by Cramer rule, one can express $\dot{J}_i(h)$ as the form

$$\dot{J}_i(h)=\frac{1}{det(h\mathbf{E}-\mathbf{A})}\sum_{l=-1}^{n+k-2}\gamma_{il}(h)J_l,$$

where $\gamma_{il}(h)$ is a polynomial of h. Since we have shown $a_{ij}=0,\ -1\leqslant j\leqslant k-2$ in Proposition 3.1, $\gamma_{il}(h)$ has a zero at $h=0$ with multiplicity at least $k-1$. Noting that $det(h\mathbf{E}-\mathbf{A})$ has a zero at $h=0$ with multiplicity k, we obtain that the singular point $h=0$ is Fuchsian. It follows from the assumptions that any non-zero critical value of $x^k p(x)$ is simple eigenvalue of the matrix $\mathbf{A}$, so the corresponding singular point of (3.8) is Fuchsian, too. □

It follows from (3.7) and Proposition 3.1 that the diagonal entries of the matrix $\mathbf{B}$ are $B_{i,i+1}=(n+2i+2)/(2(k+n)),\ i=-1,0,1,\dots,n+k-2$, which yields

Proposition 3.5. *The matrix* $\mathbf{B}$ *is triangular. Its spectrum consists of the numbers* $(n+2i+2)/(2(k+n)),\ i=-1,0,1,\dots,n+k-2$.

3.4. Bounds for the matrix norms

For a polynomial $p(x)\in\mathbb{C}[x]$ let $\|p\|$ be the sum of absolute valued of its coefficients, which is called the *norm*, or l^1-*norm* of $p(x)$. The norm of matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$\|\mathbf{A}\|=\max\{\sum_{j=-1}^{n+k-2}|a_{ij}|,\ -1\leqslant i\leqslant n+k-2\},$$

$$\|\mathbf{B}\|=\max\{\sum_{j=0}^{n+k-1}|B_{ij}|,\ -1\leqslant i\leqslant n+k-2\}. \tag{3.15}$$

Theorem 3.6. *The entries of matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are explicitly bounded*:

$$\|\mathbf{A}\|+\|\mathbf{B}\|\leqslant\frac{3}{2}+\frac{n}{k+n}\left(C+C^2+\cdots+C^{n+k}\right),\quad C=\|p\|-1>0. \tag{3.16}$$

Proof. It follows from Proposition 3.1(i) that

$$\|a_{-1}(x)\|=\sum_{j=k-1}^{n+k-2}\left|\frac{n-j+k-1}{k+n}\right||p_{j-k+1}|\leqslant\frac{n}{k+n}(\|p\|-1). \tag{3.17}$$

Using (3.13) again, we get by induction

$$\begin{aligned}
\|a_i(x)\| &= \left\|\sum_{j=k-1}^{n+k-2} a_{i-1,j}x^{j+1} - \left(\sum_{l=0}^{n}\left(\frac{k+l}{k+n}\right)p_l x^{l+k-1}\right)a_{i-1,n+k-2}\right\| \\
&= \left\|\sum_{j=k-1}^{n+k-3} a_{i-1.j}x^{j+1} - \left(\sum_{l=0}^{n-1}\left(\frac{k+l}{k+n}\right)p_l x^{l+k-1}\right)a_{i-1,n+k-2}\right\| \\
&\leqslant \left\|\sum_{j=k-1}^{n+k-3} a_{i-1,j}x^{j+1}\right\| + \left\|\sum_{l=0}^{n-1}\left(\frac{k+l}{k+n}\right)p_l x^{l+k-1}\right\| |a_{i-1,n+k-2}| \\
&\leqslant \|a_{i-1}(x)\| - |a_{i-1,n+k-2}| + (\|p\|-1)|a_{i-1.n+k-2}| \\
&= \|a_{i-1}(x)\| + (\|p\|-2)|a_{i-1,n+k-2}| \\
&\leqslant \|a_{i-2}(x)\| + (\|p\|-2)(|a_{i-2,n+k-2}| + |a_{i-1,n+k-2}|) \\
&\leqslant \cdots \\
&\leqslant \|a_0(x)\| + (\|p\|-2)\sum_{l=0}^{i-1}|a_{l,n+k-2}| \\
&\leqslant \frac{n}{k+n}(\|p\|-1) + (\|p\|-2)\sum_{l=-1}^{i-1}|a_{l,n+k-2}| \\
&\leqslant \frac{n}{k+n}C + (C-1)\sum_{l=-1}^{n+k-3}|a_{l,n+k-2}|.
\end{aligned}$$

Using the same arguments as above, we have

$$\begin{aligned}
\|a_l(x)\| &\leqslant \|a_{l-1}(x)\| + (\|p\|-2)|a_{l-1,n+k-2}| \\
&\leqslant \|a_{l-1}(x)\| + (\|p\|-2)\|a_{l-1}(x)\| = C\|a_{l-1}(x)\|,
\end{aligned}$$

which yields by induction

$$|a_{l,n+k-2}| \leqslant \|a_l(x)\| \leqslant C^{l+1}\|a_{-1}(x)\| \leqslant \frac{n}{k+n}C^{l+2}.$$

The following inequality is obtained from (3.7) and (3.10):

$$\sum_{j=0}^{i+1}|B_{ij}| = \frac{1}{2} + \sum_{j=0}^{i+1}|j-\frac{k}{2}|\cdot|b_{ij}| \leqslant \frac{1}{2} + (n+k-1)\left(\sum_{j=0}^{i+1}|b_{ij}|\right)$$

$$= \frac{1}{2} + \frac{n+k-1}{k+n}\left(1 + \sum_{l=-1}^{i-1} |a_{l,n+k-2}|\right)$$

$$\leqslant \frac{3}{2} + \sum_{l=-1}^{i-1} |a_{l,n+k-2}| \leqslant \frac{3}{2} + \sum_{l=-1}^{n+k-3} |a_{l,n+k-2}|.$$

The above discussions imply

$$\begin{aligned}\|\mathbf{A}\| + \|\mathbf{B}\| &\leqslant \frac{3}{2} + \frac{n}{k+n}C + C\sum_{l=-1}^{n+k-3} |a_{l,n+k-2}| \\ &\leqslant \frac{3}{2} + \frac{n}{k+n}(C + C^2 + \cdots + C^{n+k}). \qquad \square\end{aligned}$$

3.5. *Picard–Fuchs equation for quadratic integrable system with $n = 1$*

We will describe the Picard–Fuchs equation for (1.7) with $n = 1$. Let $p_1 = 1$, $p_0 = -(k+1)/k$ (cf. Proposition 2.5). It is obvious that $\mathbf{A}_1 = \mathbf{A}|_{n=1}$ and $\mathbf{B}_1 = \mathbf{B}|_{n=1}$ are two $(k+1)\times(k+1)$ matrices. The entries of $\mathbf{A}_1$ are

$$a_{i,k-1} = -\frac{1}{k}, \quad a_{ij} = 0, \quad -1 \leqslant j \leqslant k-2, \tag{3.18}$$

and the entries B_{ij} of the matrix $\mathbf{B}$ are defined as (3.7) with

$$b_{i,i+1} = \frac{1}{k+1}, \quad b_{ij} = -\frac{1}{k(k+1)}, \qquad 0 \leqslant j \leqslant i. \tag{3.19}$$

Indeed, it follows from Proposition 3.1 that

$$a_{i,k-1} = -\frac{k}{k+1}a_{i-1,k-1}p_0 = -\frac{k}{k+1}a_{i-1,k-1}\left(-\frac{k+1}{k}\right) = a_{i-1,k-1},$$

which implies $a_{i,k-1} = a_{i-1,k-1} = \cdots = a_{-1,k-1} = p_0/(k+1) = -1/k$. We get (3.19) from (3.10) and (3.18).

It is easy to get $\|\mathbf{A}_1\| = 1/k \leqslant 1$. Using (3.7) and (3.19), we have

$$\|\mathbf{B}_1\| \leqslant \frac{1}{2} + \left(k - 1 + 1 - \frac{k}{2}\right)\frac{1}{k+1} + \sum_{j=0}^{k-1}\left|j - \frac{k}{2}\right|\frac{1}{k(k+1)}$$

$$\leqslant \frac{1}{2} + \frac{k}{2(k+1)} + \frac{k}{2}\sum_{j=0}^{k-1} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{k}{2(k+1)} + \frac{k}{2(k+1)}$$
$$\leqslant \frac{3}{2},$$

which implies $\|\mathbf{A}_1\| + \|\mathbf{B}_1\| \leqslant 5/2$. The Picard–Fuchs system $(3.8)|_{n=1}$ has two Fuchsian singular points at $h=0$ and $h=-1/k$.

3.6. Picard–Fuchs systems for quadratic integrable systems $n=2$

In this section, we always suppose that $p_2 = 1$, $p_1 = -(k+2+kp_0)/(k+1)$, see Proposition 2.6. The matrix $\mathbf{A}_2 = \mathbf{A}|_{n=2}$ and $\mathbf{B}_2 = \mathbf{B}|_{n=2}$ will be completely described by p_0 and k.

3.6.1. General cases: the description of the entries of $\mathbf{A}_2$ and $\mathbf{B}_2$

Proposition 3.7. *Let $p_2 = 1$, $p_1 = -(k+2+kp_0)/(k+1)$. Then $\mathbf{A}_2 = (a_{ij})$ and $\mathbf{B}_2 = (B_{ij})$ are two $(k+2)\times(k+2)$ matrices, and*

$$a_{i,k-1} = -\frac{p_0-1}{k+1}\left(\sum_{l=1}^{i+1}\left(\frac{kp_0}{k+2}\right)^l\right) + \frac{2p_0}{k+2}\left(\frac{kp_0}{k+2}\right)^{i+1}, \quad i\geqslant 0, \tag{3.20}$$

$$a_{-1,k-1} = \frac{2}{k+2}p_0,\ a_{i,k} = \frac{p_0-1}{k+1} - a_{i,k-1},$$
$$a_{ij} = 0,\ -1\leqslant j\leqslant k-2,\ 0\leqslant i\leqslant k. \tag{3.21}$$

The entries B_{ij} of the matrix $\mathbf{B}_2$ are defined by (3.7) *with* $(3.10)|_{n=2}$, (3.20) *and* (3.21).

Proof. It follows from Proposition 3.1 that

$$a_{i,k-1} + a_{i,k} = -\frac{k}{k+2}a_{i-1,k}p_0 + a_{i-1,k-1} - \frac{k+1}{k+2}a_{i-1,k}p_1 = a_{i-1,k-1} + a_{i-1,k},$$

which implies by induction that

$$a_{i,k-1} + a_{i,k} = a_{i-1,k-1} + a_{i-1,k} = \cdots = a_{-1,k-1} + a_{-1,k} = \frac{p_0-1}{k+1}, \tag{3.22}$$

where we use (3.9) to get $a_{-1,k-1}$ and $a_{-1,k}$. The formulas in (3.21) are obtained by using (3.22) and Proposition 3.1.

We are going to prove (3.20) by induction. For $i=0$, (3.20) holds by direct computations. Suppose that (3.20) holds for $i-1$. Then using (3.22) and Proposition 3.1

again,

$$a_{i,k-1} = -\frac{k}{k+2} a_{i-1,k} p_0 = -\frac{kp_0}{k+2} \left(\frac{p_0 - 1}{k+1} - a_{i-1,k-1} \right),$$

which implies that (3.20) holds for i. $\square$

3.6.2. The case $p_0 = 0$

In this case, it follows from Proposition 3.7 that $a_{i,k} = -1/(k+1)$, $a_{ij} = 0$, $-1 \leqslant j \leqslant k-1$. The matrix $\mathbf{A}_2$ has two eigenvalues at $h = 0$ with multiplicity $k+1$ and $h = -1/(k+1)$ with multiplicity 1, respectively. Using the same arguments as in Proposition 3.2 and Corollary 3.4, we conclude that $\mathbf{A}_2$ is diagonalizable and the two singular points of Picard–Fuchs equation $(3.8)|_{n=2}$ are Fuchsian, too. Therefore, the Abelian integral $I(h)$, defined in (2.4), is an (multiple-valued) analytic function in $\mathbb{C}\backslash\{0, -1/(k+1)\}$. Using the inequality obtained in Section 3.4, we have

$$\sum_{j=-1}^{i+1} |B_{ij}| \leqslant \frac{3}{2} + \sum_{l=-1}^{i-1} |a_{l,k}| = \frac{3}{2} + \frac{i+1}{k+1} \leqslant \frac{3}{2} + \frac{k+1}{k+1} = \frac{5}{2},$$

which means $\|\mathbf{B}_2\| \leqslant 5/2$. The norm of $\mathbf{A}_2$ is $\|\mathbf{A}_2\| = 1/(k+1) \leqslant \frac{1}{2}$.

3.6.3. The degenerate case $p_0 = 1$

We consider the critical values of $x^k p(x)$, $\deg p(x) = 2$. It follows from Proposition 2.6 that for non-zero critical point $x = 1$, the corresponding critical value h_1 is equal to zero. This means Corollary 3.4 does not hold. In what follows we derive a Picard–Fuchs system satisfied by $\dot{\widetilde{\mathbf{J}}} = col(\dot{J}_{-1}, \dot{J}_0, \ldots, \dot{J}_{k-1})$ and show that all singular points are Fuchsian for such system.

Corollary 3.8. *Let $p_0 = 1$. The vector $\widetilde{\mathbf{J}} = col(J_{-1}, J_0, \ldots, J_{k-1})$ satisfies the following Picard–Fuchs system*

$$(h\mathbf{E} - \widetilde{\mathbf{A}}_2)\ddot{\widetilde{\mathbf{J}}} = \widetilde{\mathbf{B}}_2 \dot{\widetilde{\mathbf{J}}}, \tag{3.23}$$

where $\ddot{\widetilde{\mathbf{J}}} = d^2\widetilde{\mathbf{J}}/dh^2$, $\widetilde{\mathbf{A}}_2 = (\widetilde{a}_{ij})_{i,j=-1}^{k-1}$ and $\widetilde{\mathbf{B}}_2 = (\widetilde{B}_{ij})_{(i,j)=(-1,0)}^{(k-1,k)}$ are two $(k+1)\times(k+1)$ matrices with

$$\widetilde{a}_{ij} = 0, \ -1 \leqslant j \leqslant k-2, \ \widetilde{a}_{i,k-1} = \frac{4}{(k+2)^2} \left(\frac{k}{k+2} \right)^{i+1}, \quad -1 \leqslant i \leqslant k-1. \tag{3.24}$$

$\widetilde{\mathbf{B}}_2$ is defined by

$$\widetilde{B}_{i,i+1} = \frac{i-k}{k+2}, \ \widetilde{B}_{ij} = B_{ij} = \left(\frac{k-2j}{(k+2)^2} \right) \left(\frac{k}{k+2} \right)^{i-j}, \quad 0 \leqslant j \leqslant i, \tag{3.25}$$

and $\widetilde{B}_{ij} = 0$ for $i+2 \leqslant j \leqslant k$.

Proof. Differentiating both sides of (3.8), we get

$$(h\mathbf{E}-\mathbf{A})\ddot{\mathbf{J}} = (\mathbf{B}-\mathbf{E})\dot{\mathbf{J}}. \tag{3.26}$$

We have known that the matrix $\mathbf{B}$ is triangular and $B_{ij}=0$ for $j \geqslant i+2$. Proposition 3.5 shows that the diagonal entries are $B_{i,i+1}|_{n=2} = (i+2)/(k+2)$, $i=-1,0,\ldots,k$, which implies that $B_{k,k+1}|_{n=2}=1$. Therefore, the matrix $(\mathbf{B}-\mathbf{E})|_{n=2}$ is triangular with the form

$$(\mathbf{B}-\mathbf{E})\,|_{n=2} = \begin{pmatrix} \widetilde{\mathbf{B}}_2 & 0 \\ \star & 0 \end{pmatrix},$$

where $\star = (B_{k,0}, B_{k,1}, \ldots, B_{k,k})$. On the other hand, it follows from Proposition 2.7 that $\ddot{J}_k = k\ddot{J}_{k-1}/(k+2)$. Substituting it into the right-hand side of $(3.26)|_{n=2}$, we get (3.23).

In fact, (3.23) is the first $k+1$ equations of system $(3.26)|_{n=2}$ with $\ddot{J}_k = (k/(k+2))\ddot{J}_{k-1}$, $p_0=1$. $\square$

Using the same arguments as in Proposition 3.2 and Corollaries 3.4, 3.8 yields that the Picard–Fuchs equation (3.23) has two Fuchsian singular points at $h=0$ and $h=\widetilde{a}_{k-1,k-1}$, which are the eigenvalues of $\widetilde{\mathbf{A}}_2$ with multiplicity k and multiplicity 1, respectively. The norm $\widetilde{\mathbf{A}}_2$ satisfies $\|\widetilde{\mathbf{A}}_2\| \leqslant 1$. By (3.25), we have

$$\begin{aligned}\sum_{j=0}^{i+1} |\widetilde{B}_{ij}| &= \left|\frac{i-k}{k+2}\right| + \sum_{j=0}^{i} \left|\frac{k-2j}{(k+2)^2}\right| \left(\frac{k}{k+2}\right)^{i-j} \leqslant 1 + \frac{k}{(k+2)^2} \sum_{j=0}^{i} \left(\frac{k}{k+2}\right)^{i-j} \\ &= 1 + \frac{k}{2(k+2)}\left(1 - \left(\frac{k}{k+2}\right)^{i+1}\right) < \frac{3}{2},\end{aligned}$$

which shows $\|\widetilde{\mathbf{B}}_2\| \leqslant 3/2$.

3.6.4. The degenerate case $p_0 = (k+2)^2/k^2$

In this case, we know from Proposition 2.6 that for the non-zero critical point $x = (k+2)/k$, the corresponding critical value of $x^k p(x)$, $\deg p(x) = 2$, is equal to zero. So Corollary 3.4 does not hold for this case. Using the same arguments as Section 3.6.3, one gets

Corollary 3.9. *Let $p_0 = (k+2)^2/k^2$. The vector $\widetilde{\mathbf{J}} = col(J_{-1}, J_0, \ldots, J_{k-1})$ satisfies*

$$(h\mathbf{E} - \bar{\mathbf{A}}_2)\ddot{\widetilde{\mathbf{J}}} = \bar{\mathbf{B}}_2\dot{\widetilde{\mathbf{J}}}, \tag{3.27}$$

where $\bar{\mathbf{A}}_2 = (\bar{a}_{ij})$ and $\bar{\mathbf{B}}_2 = (\bar{B}_{ij})$ are two $(k+1)\times(k+1)$ matrices,

$$\bar{a}_{i,k-1} = \frac{p_0-1}{k+1} = \frac{4}{k^2}, \quad \bar{a}_{ij} = 0, \quad -1 \leqslant i \leqslant k-1, \quad -1 \leqslant j \leqslant k-2, \tag{3.28}$$

and the entries $\bar{B}_{ij}$ of the matrix $\bar{\mathbf{B}}$ are defined by

$$\bar{B}_{i,j} = \begin{cases} (k-2j)/(k(k+2)), & 0 \leqslant j \leqslant i, \\ (i-k)/(k+2), & j = i+1, \\ 0, & i+2 \leqslant j \leqslant k. \end{cases} \tag{3.29}$$

Corollary 3.9 shows that (3.27) has two Fuchsian singular points at $h = 0$ and $h = 4/k^2$. It follows from (3.29) that

$$\sum_{j=0}^{i+1} |\bar{B}_{ij}| = \left|\frac{i-k}{k+2}\right| + \frac{1}{k(k+2)} \sum_{j=0}^{i} |k-2j|$$

$$< 1 + \frac{1}{k(k+2)} \sum_{j=0}^{i} k < 2,$$

which implies $\|\bar{\mathbf{B}}_2\| \leqslant 2$. The norm of $\bar{\mathbf{A}}_2$ is $\|\bar{\mathbf{A}}_2\| = 4/k^2 \leqslant 4$.

4. Zeros of Abelian integrals away from the singular locus and limit cycles of vector fields

In this section, we give the main results of this paper.

4.1. Meandering theorem

Consider the system

$$\Delta(h)\dot{X}(h) = \mathcal{A}(h)X(h), \quad \mathcal{A}(h) = \sum_{i=0}^{d} \mathcal{A}_i h^i, \tag{4.1}$$

where $X(h) = col(X_1(h), X_2(h), \ldots, X_m(h))$, $\Delta(h) \in \mathbb{C}[h]$. The right-hand side of (4.1) contains the matrix polynomial $\mathcal{A}(h) \in Mat_{m\times m}(\mathbb{C}[h])$ of degree d and controlled *height* (the maximal absolute value of coefficients of polynomial). Application the bound meandering principle allow to prove

Lemma 4.1 (*Novikov and Yakovenko [NY2, Appendix B]*). *With any linear polynomial system* (4.1) *of degree $\leqslant d$ having at most d Fuchsian singularities in the finite plane, the number of isolated intersection between any trajectory $X(h)$ of system* (4.1) *of height $\leqslant R$ and an any arbitrary linear hyperplane $<\sigma, X> = \sigma_1 X_1 + \sigma_2 X_2 + \cdots \sigma_m X_m = 0$ over any simply connected sub-domain of the set $\{h \in \mathbb{C} \mid |h - h_j| > 1/R,\ |h| < R\}$ is bounded by $(2+R)^N$, where $N = N(m,d) \in \mathbb{N}$ is a primitive recursive function of m, d growing no fast than*

$$N(m,d) \leqslant \exp\exp\exp\exp(4m \ln d + O(1)).$$

4.2. The upper bounds for the number of zeros of Abelian integrals and limit cycles

Corollary 2.2 shows that the linear span of all function $h^{i_l} J_l$, $l = -1, 0, \ldots, n + k - 2$, $0 \leqslant i_l \leqslant \max_{\deg \omega \leqslant d} \{\deg \alpha_l(h)\}$ contains all Abelian integrals of forms of degree $\leqslant d$. To use Lemma 4.1, we should derive the equation satisfied by $h^{i_l} J_l$. In fact, the generator $\{h^i \mathbf{J}\}$ of the space of Abelian integrals satisfy the following system by derivation of (3.8)

$$(h\mathbf{E} - \mathbf{A}) \frac{d}{dh} (h^i \mathbf{J}) = \mathbf{B} h^i \mathbf{J} + i(h\mathbf{E} - \mathbf{A}) h^{i-1} \mathbf{J}. \tag{4.2}$$

Suppose that $x^k p(x)$ is a Morse polynomial in $\mathcal{D} = \{x | x \neq 0\} \in \mathbb{C}$, and for any critical point in $\mathcal{D}$, the corresponding critical value is not equal zero. It follows from Corollary 3.4 and Theorem 3.6 that system (4.2) has the following properties:

(1) all finite singular points are Fuchsian and coincide with the singularities of Picard–Fuchs system (3.8);

(2) system (4.2) can be written in the matrix form as (4.1) and the entries of the matrix will be explicitly bounded by d and $\|p\|$.

Denote by Λ the collection of all critical values of $H(x, y)$, i.e. $\Lambda = \{h | det(h\mathbf{E} - \mathbf{A}) = 0\}$. Let R be a finite positive number and $K_R \subset \mathbb{C} \backslash \Lambda$ the set obtained by cutting the set

$$\{h \in \mathbb{C} : \forall j = 1, 2, \ldots, n+1,\ |h - h_j| > 1/R,\ |h| < R,\ h_j \in \Lambda\}$$

along no more than $n+1$ line segment. K_R is a simply connected compact set "on the distance $1/R$ from both Λ and the infinite critical locus". It follows from Corollary 3.4 that $I(h)$, defined in (2.4), is a single-valued analytic function in K_R.

Please note that $\widetilde{I}(h)$, defined in (1.6) with $\max\{\deg P(x, y), \deg Q(x, y)\} = d - k + 1$, $P(x, y), Q(x, y), H(x, y) \in \mathbb{R}[x, y]$, can be represented as (2.4).

The above discussions and Lemma 4.1 imply the main theorem of this paper:

Theorem 4.2. *Let* $\deg \omega = d$ *and* $\max\{\deg P(x, y), \deg Q(x, y)\} = d - k + 1$. *Suppose that $x^k p(x)$ is a Morse polynomial in $\mathcal{D} = \{x | x \neq 0\}$, and for any critical point in $\mathcal{D}$, the corresponding critical value is not equal zero, then*

(i) *the number of zeros inside K_R of the Abelian integrals $I(h)$, defined in* (2.4), *does not exceed $(2 + R)^N$, where $N = N(k + n, d)$ is a certain elementary function depending only on k, n, and d,*

(ii) *the Abelian integral $\widetilde{I}(h)$, defined in* (1.6) *with* $P(x, y), Q(x, y), H(x, y) \in \mathbb{R}[x, y]$, *has at most $(2 + R)^N$ zeros in K_R.*

Corollary 4.3. *With the assumption of Theorem* 4.2 *and* $I(h) \not\equiv 0$ (*resp.,* $\widetilde{I}(h) \not\equiv 0$), *system* $(1.3)_\varepsilon$ (*resp.,* $(1.5)_\varepsilon$) *has at most $(2 + R)^N$ limit cycles inside the domain $\{(x, y) \in \mathbb{R}^2 | h_c + 1/R < H(x, y) < h_s - 1/R\}$ as $\varepsilon \to 0$, where h_c, h_s are two critical values of $H(x, y)$, and the closed orbit $\Gamma_h \subset \{(x, y) | H(x, y) = h\}$ is defined in (h_c, h_s).*

4.3. Limit cycles for quadratic non-Hamiltonian integrable system

We will list some results for limit cycles of system $(1.5)_\varepsilon$ with $n = 1, 2$ and system $(2.22)_\varepsilon$. It follows from Corollary 4.3 that, if $n = 1$, then system $(1.5)_\varepsilon$ has at most $(2+R)^N$, $N = N(k+1, d)$, limit cycles inside the domain $\{(x, y) \in \mathbb{R}^2 | -1/k+1/R < H(x, y) < -1/R,\ H = x^k(y^2/2)+x-(k+1)/k\}$. Under the assumption $p_0 \neq 0$, $p_0 \neq 1$, $p_0 \neq k^2/(k+2)^2$, $h_1 \neq h_2$, Corollary 4.3 holds for $n = 2$, too.

If $p_0 = 0$, then it follows from Section 3.6.2 that $\widetilde{I}(h)$ is an analytic function in $h \in \mathbb{C}\backslash\{0, -1/(k+1)\}$ and two singular points of system (3.8) are Fuchsian. Using the same arguments as Section 4.2, we can get that if $\widetilde{I}(h) \not\equiv 0$, then system $(1.5)_\varepsilon$ has at most $(2 + R)^{N_2}$ limit cycles in $\{(x, y) \in \mathbb{R}^2 | -1/k + 1/R < H(x, y) < -1/R\}$, where $N_2 = N_2(k + 2, d)$ is a certain elementary function depending only on k and d.

Now we study the Abelian integrals for the degenerate cases. To be more concrete, we only consider the degenerate case $p_0 = 1$ in Lemma 4.4 and Proposition 4.5.

For degenerate cases $p_0 = 1$, Corollary 3.4 does not hold. However, all singular points of Picard–Fuchs system (3.23) are Fuchsian, and the norm of the matrices can be explicitly bounded, see Section 3.6.3. To use these results, our strategy is to reduce the problem of estimating the number of zeros of $I(h)$ to a problem of estimating the number of zeros of certain Abelian integral, which can be expressed as a linear combination of $\dot{J}_l$, $l = -1, 0, 1, \ldots, k - 1$.

Since det $\mathbf{B} \neq 0$, it follows from (3.8) and (2.4) that $I(h)$ and $\dot{I}(h)$ can be represented as

$$I(h) = \sum_{l=-1}^{k} \widetilde{\alpha}_l(h)\dot{J}_l, \quad \dot{I}(h) = \sum_{l=-1}^{k} \widetilde{\beta}_l(h)\dot{J}_l, \tag{4.3}$$

where $\deg \widetilde{\alpha}_l(h) \leqslant \deg \alpha_l(h) + 1$, $\deg \widetilde{\beta}_l(h) \leqslant \deg \alpha_l(h)$. Eliminating $\dot{J}_k$ from the two equations of (4.3), we have

$$\widetilde{\alpha}_k(h)\dot{I}(h) = \widetilde{\beta}_k(h)I(h) + S(h), \quad S(h) = \sum_{l=-1}^{k-1} \gamma_l(h)\dot{J}_l, \tag{4.4}$$

where $\gamma_l(h) = \widetilde{\alpha}_k(h)\widetilde{\beta}_l(h) - \widetilde{\alpha}_l(h)\widetilde{\beta}_k(h)$, $\deg \gamma_l(h) \leqslant \deg \alpha_l(h) + \deg \alpha_k(h) + 1$. The following lemma has been used in many papers, see for instance [HI2,LZLZ,P1,R,ZZz], etc.

Lemma 4.4. *Denote by $\#I(h)$ the number of zeros of $I(h)$. We have*

$$\#I(h) \leqslant \#\widetilde{\alpha}_k(h) + \#S(h) + 1. \tag{4.5}$$

Proof. Suppose that κ_1, κ_2 are two consecutive zeros of $I(h)$, then it follows from (4.4) that $\widetilde{\alpha}_k(\kappa_i)\dot{I}(\kappa_i) = S(\kappa_i)$, $i = 1, 2$, which implies $S(\kappa_1)S(\kappa_2) \leqslant 0$ if $\widetilde{\alpha}_k(h) \neq 0$

in $[\kappa_1, \kappa_2]$. Therefore, between any two consecutive zeros of $I(h)$, there must exist at least one zeros of $S(h)$ or $\widetilde{\alpha}_k(h)$. The result of this lemma follows. □

Section 3.6.3 shows that $\dot{J}_l$, $l = -1, 0, \ldots, k-1$ satisfies Picard–Fuchs equation (3.23). Using Meandering theorem and the same arguments as in Section 4.2, we obtain that $S(h)$ has at most $(2+R)^{\bar{N}}$ zeros inside the domain K_R, where $\bar{N} = \bar{N}(k+2, d)$ is a certain elementary function depending only on k and d. This implies

$$\#I(h) \leqslant (2+R)^{\bar{N}} + \deg \alpha_k(h) + 2, \quad h \in K_R.$$

Since $\widetilde{I}(h)$ can be represented as (2.4), the above estimate holds for $\#\widetilde{I}(h)$, which yields

Proposition 4.5. *Let* $p_0 = 1$, $n = 2$ *and* $\max\{\deg P(x, y), \deg Q(x, y)\} = d - k + 1$, $d \geqslant k - 1$. *If* $\widetilde{I}(h) \not\equiv 0$, *then system* $(1.5)_\varepsilon$ *has at most* $(2+R)^{\bar{N}} + [(d-k-1)/(k+2)] + 2$ (*resp.*, $(2+R)^{\bar{N}} + 2$) *limit cycles for* $d \geqslant 2k+3$ (*resp.*, $k - 1 \leqslant d \leqslant 2k+2$) *inside the domain* $\{(x, y) | 1/R < H(x, y) < \widetilde{a}_{k-1,k-1} - 1/R,\ H(x, y) = x^k(y^2/2 + (x-1)^2)\}$.

For the degenerate case $p_0 = (k+2)^2/k^2$, the same arguments can be used as above.

We can get the similar results on the number of limit cycles and zeros of Abelian integrals for system $(2.22)_\varepsilon$ by using Proposition 2.8. The details are omitted.

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