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On nontrivial solutions of critical polyharmonic elliptic systems

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ABSTRACT

In the present work, we consider elliptic systems involving polyharmonic operators and critical exponents. We discuss the existence and nonexistence of nontrivial solutions to these systems. Our theorems improve and/or extend the ones established by Bartsch and Guo [T. Bartsch, Y. Guo, Existence and nonexistence results for critical growth polyharmonic elliptic systems, J. Differential Equations 220 (2006) 531–543] in both aspects of spectral interaction and regularity of lower order perturbations.

1. Introduction and main results

In the 80's decade, Brézis and Nirenberg investigated, in the famous paper [7], the existence of nontrivial solutions for the critical equation

$$\begin{cases} -\Delta u = |u|^{\frac{4}{N-2}} u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Since then a lot of attention has been devoted to questions and extensions related to (1). We refer for instance to Chapter 3 of the Struwe's book [22] and references therein for an overview on the so-called Brézis–Nirenberg problem.

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In this work, we consider the system

$$\begin{cases} (-\Delta)^m u_i = f_i(u) + g_i(u) & \text{in } \Omega,\ i = 1, \dots, n, \\ \left(\dfrac{\partial}{\partial \nu}\right)^j u_i = 0 & \text{on } \partial\Omega,\ i = 1, \dots, n,\ j = 0, \dots, m-1. \end{cases} \tag{2}$$

Here $m, n \in \mathbb{N}$, $N > 2m$, Ω is a smooth bounded domain, ν is the exterior normal field on $\partial\Omega$, $u = (u_1, \dots, u_n)$, $f_i(u) = \frac{1}{2^*} D_{u_i} F(u)$ and $g_i(u) = \frac{1}{2} D_{u_i} G(u)$, where $F, G : \mathbb{R}^n \to \mathbb{R}$ are C^1 functions with F positively homogeneous of degree $2^* = 2N/(N-2m)$ and G homogeneous of degree 2, i.e. $F(\alpha t) = \alpha^{2^*} F(t)$ and $G(\alpha t) = \alpha^2 G(t)$ for $\alpha > 0$ and $t \in \mathbb{R}^n$. Such systems are known in the literature as potential type systems (or gradient systems) and F and G are called potential functions. Note that (2) is a natural vector extension of (1) since, when $n = 1$, homogeneity assumption on F and G readily yields $F(t) = |t|^{2^*}$ and $G(t) = \lambda t^2$, modulo multiplication by constants.

The Brézis–Nirenberg problem for the polyharmonic operator, that is $m \geqslant 2$ and $n = 1$, has been discussed in various works, we mention for instance [3–5,9,10,12] for $m = 2$, and [21] for arbitrary $m \geqslant 1$. For $n \geqslant 2$, there are many homogeneous functions of class C^1. Some classical examples are:

(i) $F(t) = |t|_q^{2^*}$, $F(t) = (\sigma_l(t))^{2^*/l}$;
(ii) $G(t) = |t|_q^2$, $G(t) = (\sigma_l(t))^{2/l}$, $G(t) = \langle At, t\rangle = \sum_{i,j=1}^n a_{ij} t_i t_j$,

where $|t|_q := (\sum_{i=1}^n |t_i|^q)^{1/q}$ with $q \geqslant 1$, σ_l is the lth elementary symmetric polynomial, $l = 1, \dots, n$, and $A = (a_{ij})$ is a symmetric $n \times n$ matrix. The problem (2) has been addressed to an extension closely related to the case $n = 1$, precisely $G(t) = \langle At, t\rangle$. In this situation, Amster et al. [1] obtained an existence result for $m = 1$, arbitrary $n \geqslant 1$ and $F(t) = |t|_q^{2^*}$ with $q \geqslant 2$. Recently, Bartsch and Guo [2] established existence and nonexistence results for $m, n \geqslant 1$ and F positively 2^*-homogeneous. In particular, the existence result of [2] in higher dimensions (Theorem 1.3) improves the one of [1]. For other results on elliptic systems with superlinear nonlinearities and $m = 1$, we refer to [8] and [16] for $n = 2$, [19] for arbitrary $n \geqslant 2$ and references therein.

In this paper, we focus the general case $m, n \geqslant 1$, F positively homogeneous of degree 2^* and G homogeneous of degree 2. Our theorems extend and/or improve the ones of [2] into two directions:

(a) targeting necessary and sufficient conditions to existence of nontrivial solutions of (2), we find a "sharp" interaction between G and the first eigenvalue λ_1 of $(-\Delta)^m$ with homogeneous Dirichlet boundary conditions;
(b) our study is developed for a broad class of 1-homogeneous nonlinearities $g_i(u)$ since we require only C^1 regularity on G, which is natural from the vector view point.

Let $H_0^m(\Omega)$ be the mth order Sobolev space defined as the completion of $C_0^\infty(\Omega)$ with respect to the scalar product

$$\langle u, v\rangle := \begin{cases} \int_\Omega \Delta^{m/2} u \cdot \Delta^{m/2} v\, dx, & \text{if } m \text{ even}, \\ \int_\Omega \nabla(\Delta^{(m-1)/2} u) \cdot \nabla(\Delta^{(m-1)/2} v)\, dx, & \text{if } m \text{ odd}, \end{cases}$$

and associated norm $\|u\|$. Denote by E the vector Sobolev space $H_0^m(\Omega, \mathbb{R}^n) := H_0^m(\Omega) \times \cdots \times H_0^m(\Omega)$ endowed with the scalar product

$$\langle u, v\rangle_E := \sum_{i=1}^n \langle u_i, v_i\rangle$$

for $u, v \in E$ and also write $\|u\|_E$ for the corresponding norm. We denote the norm on the Lebesgue space $L^q(\Omega)$ by

$$|u|_q := \left(\int_{\Omega} |u|^q \, dx \right)^{1/q}$$

and on the corresponding vector Lebesgue space $L^q(\Omega, \mathbb{R}^n)$ by

$$|u|_q := \left(\sum_{i=1}^{n} |u_i|_q^q \right)^{1/q}.$$

A first necessary condition for the existence of nontrivial solutions of (2) arises naturally from a Pohozaev identity due to Pucci and Serrin [20] and Guedda and Véron [15] applied to star-shaped domains.

Theorem 1.1. *Let $F, G : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions with F positively homogeneous of degree 2^* and G homogeneous of degree 2. If G is negative on $\mathbb{R}^n \setminus \{0\}$ and Ω is star-shaped, then (2) has no nontrivial solution.*

Assume now that G is positive somewhere. Set

$$M_G := \max_{|t|_2 = 1} G(t).$$

In this case, we have $M_G > 0$. A second necessary condition is given in

Theorem 1.2. *Let $F, G : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions with F positively homogeneous of degree 2^* and G even and homogeneous of degree 2. Suppose moreover that Ω is a ball if $m > 1$, $D_{u_i} F$ is positive on $\mathbb{R}^n_+$, $D_{u_i} G$ is positive on $\overline{\mathbb{R}^n_+} \setminus \{0\}$ and non-decreasing on $\mathbb{R}^n$. If $M_G \geqslant \lambda_1$, then (2) has no nonnegative nontrivial solution.*

Theorems 1.1 and 1.2 extend, respectively, Theorems 1.2 and 1.1 of [2] to a rather broad class of 2-homogeneous functions G (see remark below). We emphasize that the idea of the proof of Theorem 1.1 of [2] does not work here, since in general the lower order term $g_i(u)$ is nonlinear. Our proof is based on an alternative argument which relies on Hopf's lemma and strong maximum principle to polyharmonic operators.

Remark 1.1. Theorem 1.1 of [2] states a nonexistence result of positive solution for (2) in the case $m \geqslant 1$ and $G(u) = \langle At, t \rangle$ under some additional assumptions on the matrix A. However, its proof has a gap for $m > 1$, since it relies on the positivity of an eigenfunction $\varphi_1 \in H_0^m(\Omega)$ associated to the first eigenvalue λ_1 of $(-\Delta)^m$ and, by an example of [17], one finds for $m > 1$ eigenfunctions that change sign in strictly convex, arbitrarily smooth domains, except in some specific cases as the ball. So, their result is certainly true on general smooth domains for $m = 1$ and on balls for $m > 1$.

In higher dimensions, we have the first main existence theorem.

Theorem 1.3. *Let $N \geqslant 4m$, $F, G : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions with F positively homogeneous of degree 2^* and G homogeneous of degree 2. If $M_G < \lambda_1$ and $G(t_0) > 0$ for some maximum point t_0 of F on $\mathbb{S}^{n-1}$, then (2) has a nontrivial solution.*

Remark 1.2. The conditions $M_G < \lambda_1$ and $G(t_0) > 0$ assumed above correspond, in the case $m = 1$ and $n = 1$, to $\lambda < \lambda_1$ and $\lambda > 0$, respectively. In other words, Theorem 1.3 extends completely the famous existence result of [7] for (1) in higher dimensions.

For some kinds of nonlinearities F and G and domains Ω, the condition $M_G < \lambda_1$ is necessary and sufficient for the existence of nontrivial solutions of (2) as easily follows from the above theorems.

Corollary 1.1. *Let $N \geqslant 4m$, $F, G : \mathbb{R}^n \to \mathbb{R}$ be positive C^1 functions with F and G homogeneous of degrees 2^* and 2, respectively, and G even. Suppose moreover that $D_{u_i}F$ is positive on $\mathbb{R}^n_+$, $D_{u_i}G$ is positive on $\overline{\mathbb{R}^n_+} \setminus \{0\}$ and non-decreasing on $\mathbb{R}^n$. Then, $M_G < \lambda_1$ if, and only if, (2) has a nontrivial solution on arbitrary smooth bounded domains for $m = 1$, or on balls for $m > 1$.*

As an illustration, we apply this corollary to the functions

$$F(t) = |t|_p^{2^*}, \qquad G(t) = \lambda |t|_q^2$$

with $p, q \geqslant 1$ and $\lambda > 0$. One easily checks that a nontrivial solution of (2) exists on those respective domains if, and only if,

$$\lambda < \begin{cases} n^{1-2/q}\lambda_1, & 1 \leqslant q < 2, \\ \lambda_1, & q \geqslant 2. \end{cases}$$

An interesting fact that deserves mention in this example is the following. When we look for solutions of (2) of the form $u = (v, \dots, v)$, modulo a scale factor, the problem is equivalent to solve

$$\begin{cases} -\Delta v = |v|^{\frac{4}{N-2}} v + n^{2/q-1}\lambda v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, as it is well known, the above problem admits a nontrivial solution if, and only if, $0 < \lambda < n^{1-2/q}\lambda_1$. Note however that for $q > 2$, nontrivial solutions of (2) always exist in the range $0 < \lambda < \lambda_1$, which contains strictly the interval $0 < \lambda < n^{1-2/q}\lambda_1$. The immediate conclusion is that in general solutions of (2) cannot be obtained from solutions of (1), even in canonical examples.

In critical dimensions $2m < N < 4m$, we need the Sobolev best constant

$$K := \sup\bigl\{|u|_{2^*}:\ u \in H_0^m(\Omega),\ \|u\| = 1\bigr\}$$

for the embedding of $H_0^m(\Omega)$ in $L^{2^*}(\Omega)$.

Theorem 1.4. *Let $2m < N < 4m$, $F, G : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions with F positively homogeneous of degree 2^* and G homogeneous of degree 2. Let $\varphi_1 \in H_0^m(\Omega)$ be an eigenfunction of $(-\Delta)^m$ associated to λ_1, normalized by $|\varphi_1|_{2^*} = 1$, and set*

$$\bar{\lambda} = \lambda_1 - \left(K^2 \int_\Omega \varphi_1^2\, dx\right)^{-1}.$$

If $M_G < \lambda_1$ and $G(t_0) > \bar{\lambda}$ for some maximum point t_0 of F on $\mathbb{S}^{n-1}$, then (2) has a nontrivial solution.

Consider a quadratic form on $\mathbb{R}^n$, $G(t) = \sum_{i,j=1}^n a_{ij}t_it_j$, being $A = (a_{ij})$ a symmetric $n \times n$ matrix. Obviously, $M_G \leqslant \|A\|$, since

$$\|A\| := \max_{|t|_2 = 1} \bigl|\langle At, t\rangle\bigr| = \max_{|t|_2 = 1} \bigl|G(t)\bigr|.$$

By Perron–Frobenius Theorem, the equality $M_G = \|A\|$ holds whenever $a_{ij} > 0$ for every i, j (see details in [11]). However, in most examples, we find strict inequality $M_G < \|A\|$. In particular, one easily constructs examples of functions F and symmetric matrices A such that $M_G < \lambda_1$, $\|A\| > \lambda_1$ and $G(t_0) > 0$ in higher dimensions, or $G(t_0) > \bar{\lambda}$ in critical dimensions, for some maximum point t_0 of F on $\mathbb{S}^{n-1}$. Therefore, our theorems extend and also improve Theorems 1.3 and 1.4 of [2].

The paper is organized into three sections. In Section 2, we discuss some properties on sharp Sobolev inequalities associated to the space E, characterize the coercivity of the functional

$$\Phi_G(u) := \|u\|_E^2 - \int_\Omega G(u)\,dx$$

in terms of M_G and λ_1, and introduce the variational setting corresponding to the system (2). Sections 3 and 4 are devoted to the proofs of theorems.

2. Extremal maps, coercivity and local compactness

Throughout this section we assume that $F, G : \mathbb{R}^n \to \mathbb{R}$ are C^1 functions with F positively homogeneous of degree 2^* and G homogeneous of degree 2. We also denote $f_i(u) = \frac{1}{2^*} D_{u_i} F(u)$ and $g_i(u) = \frac{1}{2} D_{u_i} G(u)$.

Let $E = H_0^m(\Omega, \mathbb{R}^n)$ as in the introduction. Define

$$K_F(\Omega) := \sup\left\{ \left(\int_\Omega F(u)\,dx \right)^{1/2^*} : \ u \in E, \ \|u\|_E = 1 \right\}.$$

By a standard scaling argument, it follows that $K_F(\Omega) = K_F(\mathbb{R}^N)$. In order to estimate minimal energy levels associated to (2), we need to find maps that realize the best constant $K_F(\mathbb{R}^N)$. The lemma below exhibits these maps.

Lemma 2.1. *We have*:

(a) $K_F(\mathbb{R}^N) = M_F^{1/2^*} K$,
(b) $K_F(\mathbb{R}^N)$ *is achieved exactly by maps of the form* $u_0 = t_0 v_0$, *where* $t_0 \in \mathbb{S}^{n-1}$ *is a maximum point of* F *and* v_0 *is an extremal function for* K.

Recall from Theorem 2.1 of [23] that K is achieved by the functions

$$v_\varepsilon(x) = c_{m,N}\varepsilon^{(N-2m)/2}\left(|x|^2 + \varepsilon^2\right)^{(2m-N)/2}$$

for any $\varepsilon > 0$; here $c_{m,N}$ is normalized so that $|v_\varepsilon|_{2^*} = 1$. Then, the part (b) of Lemma 2.1 provides explicit extremal maps for $K_F(\mathbb{R}^N)$. Note also that the part (a) improves Lemma 3.1 of [2], since it leads to $K_F(\Omega) = M_F^{1/2^*} K$ without assuming additional hypotheses.

Proof of Lemma 2.1. The 2^*-homogeneity of F yields

$$F(t) \leqslant M_F \left(\sum_{i=1}^n t_i^2 \right)^{2^*/2}$$

for all $t \in \mathbb{R}^n$. So, using Minkowski's inequality, one obtains

$$\begin{aligned} \left(\int_{\mathbb{R}^N} F(u)\,dx \right)^{2/2^*} &\leqslant M_F^{2/2^*} \left(\int_{\mathbb{R}^N} \left(\sum_{i=1}^n u_i^2 \right)^{2^*/2} dx \right)^{2/2^*} \\ &\leqslant M_F^{2/2^*} \sum_{i=1}^n \left(\int_{\mathbb{R}^N} |u_i|^{2^*}\,dx \right)^{2/2^*} \end{aligned}$$

$$\leqslant M_F^{2/2^*} K^2 \sum_{i=1}^{n} \|u_i\|^2$$

$$= M_F^{2/2^*} K^2 \|u\|_E^2, \tag{3}$$

so that

$$K_F(\mathbb{R}^N) \leqslant M_F^{1/2^*} K.$$

Consider now the map $u_0 = t_0 v_0 \in E$ with $t_0 \in \mathbb{S}^{n-1}$ such that $M_F = F(t_0)$ and $v_0 \in H_0^m(\Omega)$ an extremal function for K. Then,

$$\begin{aligned}\left(\int_{\mathbb{R}^N} F(u_0)\,dx\right)^{2/2^*} &= M_F^{2/2^*}\left(\int_{\mathbb{R}^n} |v_0|^{2^*}\,dx\right)^{2/2^*}\\ &= M_F^{2/2^*} K^2 \|u_0\|^2\\ &= M_F^{2/2^*} K^2 \|t_0 v_0\|_E^2\\ &= M_F^{2/2^*} K^2 \|u_0\|_E^2,\end{aligned}$$

so that

$$K_F(\mathbb{R}^N) = M_F^{1/2^*} K.$$

Note that $K_F(\mathbb{R}^N)$ is achieved by maps $u_0 = t_0 v_0$ as chosen above. It remain to show that an arbitrary extremal map u for $K_F(\mathbb{R}^N)$ can always be placed in this form. In fact, u satisfies (3) with equality in all steps. Remark also that the second equality corresponds to Minkowski's inequality. So, there exist $t \in \mathbb{S}^{n-1}$ and a function $v \in H_0^m(\Omega)$ such that $u = tv$. Finally, from the first equality, one gets $F(t) = M_F$, and from the third one, one concludes that v is an extremal function for K. $\square$

Assume that G is positive somewhere, so that $M_G > 0$. Define

$$\lambda_{1,G} = \inf_{u \in E_G} \|u\|_E^2,$$

where $E_G := \{u \in E\colon \int_\Omega G(u)\,dx = 1\}$.

Note that $\lambda_{1,G}$ is a positive eigenvalue of the system

$$\begin{cases} (-\Delta)^m u_i = \lambda g_i(u) & \text{in } \Omega,\ i = 1, \dots, n,\\ \left(\dfrac{\partial}{\partial \nu}\right)^j u_i = 0 & \text{on } \partial\Omega,\ i = 1, \dots, n,\ j = 0, \dots, m-1. \end{cases} \tag{4}$$

Moreover, $\lambda_{1,G}$ is the smaller positive eigenvalue of (4) as can easily be checked by multiplying the ith equation by u_i, integrating by parts and, finally, using the relation $\sum_{i=1}^n g_i(u) u_i = G(u)$.

The next lemma expresses the value of $\lambda_{1,G}$ in terms of M_G and λ_1.

Lemma 2.2. *We have*

$$\lambda_{1,G} = \frac{\lambda_1}{M_G}.$$

Proof. First,

$$\begin{aligned}\lambda_{1,G} &= \inf_{u \in E_G} \|u\|_E \geqslant \frac{1}{M_G} \inf_{u \in E_G} \frac{\|u\|_E^2}{|u|_2^2} \\ &= \frac{1}{M_G} \inf_{u \in H_0^m(\Omega)\setminus\{0\}} \frac{\|u\|^2}{|u|_2^2} \\ &= \frac{\lambda_1}{M_G}.\end{aligned}$$

Choose now $t_1 \in \mathbb{S}^{n-1}$ such that $G(t_1) = M_G$ and $\varphi_1 \in H_0^m(\Omega)$ an eigenfunction of $(-\Delta)^m$ associated to λ_1. Then, $u_1 = t_1\varphi_1 \in E$ and

$$\lambda_{1,G} \leqslant \frac{\|u_1\|_E^2}{\int_\Omega G(u_1)\,dx} = \frac{1}{G(t_1)} \frac{\|\varphi_1\|^2}{|\varphi_1|_2^2} = \frac{\lambda_1}{M_G},$$

so that the conclusion of the lemma follows. □

Remark 2.1. From the proof above, it follows that maps of the form $\psi_1 = t_1\varphi_1$, where t_1 is a maximum point of G on $\mathbb{S}^{n-1}$ and $\varphi_1 \in H_0^m(\Omega)$ is a principal eigenfunction of $(-\Delta)^m$, are eigenfunctions of (4) associated to $\lambda_{1,G}$.

Consider the functional on E,

$$\Phi_G(u) = \|u\|_E^2 - \int_\Omega G(u)\,dx.$$

We now characterize the coercivity of Φ_G on E in terms of G and λ_1.

Lemma 2.3. *The functional Φ_G is coercive on E if, and only if, $M_G < \lambda_1$.*

Proof. By Lemma 2.2, it is sufficient to show that coercivity of Φ_G on E is equivalent to $\lambda_{1,G} > 1$. Assume first that Φ_G is coercive on E. Then, there exists a constant $0 < a < 1$ such that

$$\Phi_G(u) = \|u\|_E^2 - 1 \geqslant a\|u\|_E^2$$

for all $u \in E_G$. In other words,

$$\|u\|_E^2 \geqslant \frac{1}{1-a}$$

for all $u \in E_G$, so that $\lambda_{1,G} \geqslant 1/(1-a) > 1$. Let now $\lambda_{1,G} > 1$ and $u \in E$. If $\int_\Omega G(u)\,dx \leqslant 0$, then $\Phi_G(u) \geqslant a\|u\|_E^2$ for any constant $0 < a < 1$. Else, by definition of $\lambda_{1,G}$, we have

$$\frac{\|u\|_E^2}{\int_\Omega G(u)\,dx} \geqslant \lambda_{1,G},$$

so that

$$\Phi_G(u) = \|u\|_E^2 - \int_\Omega G(u)\,dx \geqslant \|u\|_E^2 - \frac{1}{\lambda_{1,G}}\|u\|_E^2 = \frac{\lambda_{1,G} - 1}{\lambda_{1,G}}\|u\|_E^2.$$

This ends the proof. □

As usual, nontrivial solutions of (2) corresponds, up to scaling, to the critical points of $\Phi_G(u)$ constrained to the Nehari manifold $E_F := \{u \in E\colon \int_\Omega F(u)\,dx = 1\}$. In this setting, a classical local compactness result is the following:

Lemma 2.4. *The constrained functional $\Phi_G|_{E_F}$ satisfies the $(PS)_c$-condition for $c < K_F(\mathbb{R}^N)^{-2}$.*

The proof of this lemma is rather standard except that it requires a vector version of the Brézis–Lieb lemma (see [2,6]). We refer the reader to the proof of Lemma 2.3 of [2] combined with a few adaptations.

3. Proof of Theorems 1.3 and 1.4

We now apply the previous lemmas in the proof of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Let

$$c := \inf_{u \in E_F} \Phi_G(u).$$

We first show that $c < K_F(\mathbb{R}^N)^{-2}$. Let $\varphi \in C_0^\infty(\Omega)$ be a cutoff function with $\varphi = 1$ in a neighborhood of 0 and v_ε be the extremal function for K introduced in Section 2. Then, $w_\varepsilon = \varphi v_\varepsilon \in H_0^m(\Omega)$ and

$$\|w_\varepsilon\|^2 = K^{-2} + O\left(\varepsilon^{N-2m}\right),$$
$$|w_\varepsilon|_{2^*}^2 = 1 + O\left(\varepsilon^N\right),$$
$$|w_\varepsilon|_2^2 = \begin{cases} a\varepsilon^{2m} + O(\varepsilon^{N-2m}), & N > 4m, \\ a\varepsilon^{2m}|\log\varepsilon| + O(\varepsilon^{2m}), & N = 4m, \end{cases}$$

for some constant $a > 0$ (cf. [12]). Let now $u_\varepsilon = t_0 w_\varepsilon \in E$, where $t_0 \in \mathbb{S}^{n-1}$, by hypothesis, satisfies $F(t_0) = M_F$ and $G(t_0) > 0$. Consider first the case $N > 4m$, so that

$$\begin{aligned} c &\leqslant \frac{\Phi_G(u_\varepsilon)}{(\int_\Omega F(u_\varepsilon)\,dx)^{2/2^*}} = \frac{\|u_\varepsilon\|_E^2 - \int_\Omega G(u_\varepsilon)\,dx}{(\int_\Omega F(u_\varepsilon)\,dx)^{2/2^*}} \\ &= \frac{\|w_\varepsilon\|^2 - G(t_0)|w_\varepsilon|_2^2}{M_F^{2/2^*}|w_\varepsilon|_{2^*}^2} \\ &= \frac{K^{-2} - aG(t_0)\varepsilon^{2m} + O(\varepsilon^{N-2m})}{M_F^{2/2^*} + O(\varepsilon^N)} \\ &< K^{-2} M_F^{-2/2^*} = K_F\left(\mathbb{R}^N\right)^{-2} \end{aligned}$$

provided $\varepsilon > 0$ is small enough.

In the case $N = 4m$, arguing in a similar manner, one arrives at

$$c \leqslant \frac{K^{-2} - aG(t_0)|\log\varepsilon|\varepsilon^{2m} + O(\varepsilon^{2m})}{M_F^{2/2^*} + O(\varepsilon^{4m})} < K_F\left(\mathbb{R}^N\right)^{-2}$$

for $\varepsilon > 0$ small.

By Lemma 2.4, there exists a minimizer $u \in E_F$ of the functional Φ_G constrained to E_F. Thus, we find a Lagrange multiplier μ such that

$$(-\Delta)^m u_i = g_i(u) + \mu f_i(u) \quad \text{in } \Omega,\ i = 1, \ldots, n.$$

Moreover, $\mu = \Phi_G(u)$, since $\sum_{i=1}^n f_i(u)u_i = F(u)$ and $\sum_{i=1}^n g_i(u)u_i = G(u)$. So, the assumption $M_G < \lambda_1$ and Lemma 2.3 readily yield $\mu > 0$. Now, one easily checks that $\mu^{(N-2m)/(N+2m)}u$ is a nontrivial solution of (2). $\square$

Proof of Theorem 1.4. Let $\varphi_1 \in H_0^m(\Omega)$ be an eigenfunction of $(-\Delta)^m$ corresponding to λ_1, normalized by $|\varphi_1|_{2^*} = 1$. Set $u_1 = t_0\varphi_1 \in E_F$, where $t_0 \in \mathbb{S}^{n-1}$ satisfies $F(t_0) = M_F$ and $G(t_0) > \bar{\lambda}$, as assumed in the statement of theorem. Then,

$$\begin{aligned} c &\leqslant \frac{\Phi_G(u_1)}{(\int_\Omega F(u_1)\,dx)^{2/2^*}} = \frac{\|u_1\|_E^2 - \int_\Omega G(u_1)\,dx}{M_F^{2/2^*}} \\ &= \frac{\|\varphi_1\|^2 - G(t_0)|\varphi_1|_2^2}{M_F^{2/2^*}} \\ &= \frac{(\lambda_1 - G(t_0))|\varphi_1|_2^2}{M_F^{2/2^*}} \\ &< K^{-2}M_F^{-2/2^*} = K_F\big(\mathbb{R}^N\big)^{-2}. \end{aligned}$$

The proof of Theorem 1.4 proceeds now as the one of Theorem 1.3. $\square$

4. Proof of Theorems 1.1 and 1.2

For the proof of Theorem 1.1 we need the following generalized Pohozaev identity due to Pucci and Serrin [20] for classical solutions and, by means of an approximation scheme, to Guedda and Véron [15] for weak solutions in $C^{m,\alpha}(\overline{\Omega}, \mathbb{R}^n)$.

Lemma 4.1. *Let $u \in C^{m,\alpha}(\overline{\Omega}, \mathbb{R}^n)$ be a weak solution of the system*

$$\begin{cases} (-\Delta)^m u_i = h_i(u) & \text{in } \Omega,\ i = 1,\dots,n, \\ \left(\dfrac{\partial}{\partial\nu}\right)^j u_i = 0 & \text{on } \partial\Omega,\ i = 1,\dots,n,\ j = 0,\dots,m-1, \end{cases}$$

where $h_i(u) = D_{u_i}H(u)$ for some $H \in C^1(\mathbb{R}^n)$ with $H(0) = 0$. Then, for m even,

$$\sum_{i=1}^n \int_{\partial\Omega} \big|\Delta^{m/2}u_i\big|^2 x\cdot\nu(x)\,ds = 2N\int_\Omega H(u)\,dx - (N-2m)\sum_{i=1}^n \int_\Omega h_i(u)u_i\,dx$$

and, for m odd,

$$\sum_{i=1}^n \int_{\partial\Omega} \big|\nabla\big(\Delta^{(m-1)/2}u_i\big)\big|^2 x\cdot\nu(x)\,ds = 2N\int_\Omega H(u)\,dx - (N-2m)\sum_{i=1}^n \int_\Omega h_i(u)u_i\,dx.$$

Proof of Theorem 1.1. Let

$$H(u) = \frac{1}{2^*}F(u) + \frac{1}{2}G(u).$$

Since F and G are homogeneous of degree 2^* and 2, respectively, it follows that $\sum_{i=1}^n h_i(u)u_i = F(u) + G(u)$. Let $u \in E$ be a solution of (2). By regularity results of [18], we have $u \in C^{m,\alpha}(\overline{\Omega}, \mathbb{R}^n)$. Applying Lemma 4.1 to this function H, we obtain for m even,

$$\sum_{i=1}^{n}\int_{\partial\Omega}\left|\Delta^{m/2}u_i\right|^2 x\cdot\nu(x)\,ds=\left[\frac{2N}{2^*}-(N-2m)\right]\int_{\Omega}F(u)\,dx+2m\int_{\Omega}G(u)\,dx$$
$$=2m\int_{\Omega}G(u)\,dx.$$

Since G is negative on $\mathbb{R}^n\setminus\{0\}$, the right-hand side of the above identity is negative unless $u=0$ on Ω. On the other hand, $x\cdot\nu(x)>0$ on $\partial\Omega$ because Ω is star-shaped, so that the unique solution of (2) is the trivial one. A similar procedure works to m odd. $\square$

The proof of Theorem 1.2 requires Hopf's lemma and strong maximum principle both applied to the Laplacian operator on general smooth domains and to polyharmonic operators on balls.

Proof of Theorem 1.2. In this proof, Ω is an arbitrary smooth bounded domain for $m=1$, or a ball for $m>1$. Let $u\in E$ be a nonnegative nontrivial solution of (2). Regularity results of [18] imply that $u\in C^{m,\alpha}(\overline{\Omega},\mathbb{R}^n)$. So, combining the assumptions of theorem, Hopf's lemma and strong maximum principle (cf. [13,14]), one easily checks that $u>0$ in Ω and $(-\frac{\partial}{\partial\nu})^m u>0$ on $\partial\Omega$. Let $\psi_1=(\psi_1^1,\dots,\psi_1^n)\in E$ be a smooth eigenfunction of (4) corresponding to $\lambda_{1,G}$. Since G is even, we can assume that, at least, one component of ψ_1 is positive somewhere. Thus, the set

$$S:=\{s>0:\ u>s\psi_1\ \text{in}\ \Omega\}$$

is upper bounded and nonempty, since $u>0$ in Ω and $(-\frac{\partial}{\partial\nu})^m u>0$ on $\partial\Omega$. Define $s^*:=\sup S>0$. Clearly, $u\geqslant s^*\psi_1$ in Ω. Moreover,

$$\begin{aligned}(-\Delta)^m\bigl(u_i-s^*\psi_1^i\bigr)&=(-\Delta)^m u_i-s^*(-\Delta)^m\psi_1^i\\&=f_i(u)+g_i(u)-s^*\lambda_{1,G}\,g_i(\psi_1)\\&>g_i(u)-g_i(s^*\psi_1)\\&\geqslant 0,\end{aligned}$$

since $\lambda_{1,G}=\lambda_1/M_G\leqslant 1$. Evoking again Hopf's lemma and strong maximum principle, we find $u-s^*\psi_1>0$ in Ω and $(-\frac{\partial}{\partial\nu})^m(u-s^*\psi_1)>0$ on $\partial\Omega$. Therefore, $u-(s^*+\varepsilon)\psi_1>0$ in Ω for $\varepsilon>0$ small enough, and this contradicts the definition of S. So, this concludes the proof of the theorem. $\square$

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