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## Semi-classical limits of ground states of a nonlinear Dirac equation

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## ABSTRACT

We study the semi-classical states of the following nonlinear Dirac equation

$$-i\hbar \sum_{k=1}^3 \alpha_k \partial_k w + a\beta w + V(x)w = W(x)g(|w|)w$$

for  $x \in \mathbb{R}^3$  where the nonlinearity is of superlinear and subcritical growth as  $|w| \rightarrow \infty$ . The Dirac operator is unbounded from below and above so the associate energy functional is strongly indefinite. We develop an argument to establish the existence of least energy solutions for  $\hbar$  small. We also describe the concentration phenomena of the solutions as  $\hbar \rightarrow 0$ .

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## 1. Introduction and main results

In the literature there have been large amounts of works on existence and concentration phenomenon of semi-classical states of nonlinear Schrödinger equations arising in the *non-relativistic* quantum mechanics, see, for example, [3,4,8,9,11,18,20,21,23,24,26,32] and their references. It is quite natural to ask if certain similar results can be forwarded to nonlinear Dirac equations arising in the *relativistic* quantum mechanics. Mathematically, the problem is difficult and very interesting because the Dirac equation is strongly indefinite in the sense that, firstly, both the negative and positive parts of the spectrum of Dirac operator are unbounded and contain essential spectrum respectively, and secondly, the relative energy functional does not satisfy the Palais–Smale condition.

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This paper is concerned with the existence and concentration phenomena of semi-classical ground states to the following stationary Dirac equation:

$$-i\hbar \sum_{k=1}^3 \alpha_k \partial_k w + a\beta w + V(x)w = W(x)g(|w|)w \quad (1)$$

for  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$  where  $g(|w|)w$  is of superlinear and subcritical growth as  $|w| \rightarrow \infty$ . Here,  $\hbar$  denotes Plank's constant,  $\partial_k = \partial/\partial x_k$ ,  $a > 0$  is a constant,  $\alpha_1, \alpha_2, \alpha_3$  and  $\beta$  are  $4 \times 4$  Pauli-Dirac matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and  $V, W : \mathbb{R}^3 \rightarrow \mathbb{R}$ .

The equation or the more general one

$$-i\hbar \sum_{k=1}^3 \alpha_k \partial_k w + a\beta w + M(x)w = F_w(x, w) \quad (2)$$

arises when one seeks for the standing wave solutions of the nonlinear Dirac equation

$$-i\hbar \partial_t \psi = i\hbar \sum_{k=1}^3 \alpha_k \partial_k \psi - mc^2 \beta \psi - V(x)\psi + G_\psi(x, \psi). \quad (3)$$

Assuming that  $G(x, e^{i\theta}\psi) = G(x, \psi)$  for all  $\theta \in [0, 2\pi]$ , a standing wave solution of (3) is a solution of the form  $\psi(t, x) = e^{\frac{i\mu t}{\hbar}} w(x)$ . It is clear that  $\psi(t, x)$  solves (3) if and only if  $w(x)$  solves (2) with  $a = mc$ ,  $M(x) = V(x)/c + \mu I_4$  and  $F(x, w) = G(x, w)/c$ .

There are many works devoted to the study on the existence of solutions of (2) under various hypotheses on the potential and the nonlinearity (see [5,7,14,15,17,16,22] and the references therein). We note that these papers concerned mainly the existence without involving the concentration phenomenon of semi-classical states.

For small  $\hbar$ , the standing waves are referred to as semi-classical states. To describe the transition from quantum to classical mechanics, the existence of solutions  $w_\hbar$ ,  $\hbar$  small, possesses an important physical interest (see [27,30]). Only very recently, the paper [13] studied the existence of a family of ground states of the problem (1) with  $V(x) \equiv 0$  and the special nonlinear  $|w|^{p-1}w$ ,  $p \in (2, 3)$ , for all  $\hbar$  small, and showed that the family concentrates around the maxima of  $W(x)$  as  $\hbar \rightarrow 0$ .

Let us now describe the results of the present paper. For notational convenience, writing  $\varepsilon = \hbar$ ,  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$  and  $\alpha \cdot \nabla = \sum_{k=1}^3 \alpha_k \partial_k$ , we reread Eq. (1) as

$$-i\varepsilon \alpha \cdot \nabla w + a\beta w + V(x)w = W(x)g(|w|)w. \quad (4)$$

Set

$$\begin{aligned}\tau &:= \min V, & \mathcal{V} &:= \{x \in \mathbb{R}^3: V(x) = \tau\}, \\ \tau_\infty &:= \liminf_{|x| \rightarrow \infty} V(x), \\ \pi &:= \max W, & \mathcal{W} &:= \{x \in \mathbb{R}^3: W(x) = \pi\}, \\ \pi_\infty &:= \limsup_{|x| \rightarrow \infty} W(x).\end{aligned}$$

On the linear fields we will use the following hypotheses:

- ( $P_0$ )  $V, W \in C^1(\mathbb{R}^3, \mathbb{R})$ ,  $\max |V| < a$ , and  $\inf W > 0$ .  
 ( $P_1$ )  $\tau < \tau_\infty$ , and there is  $x_v \in \mathcal{V}$  such that  $W(x_v) \geq W(x)$  for all  $|x| \geq R$ , some large  $R > 0$ .  
 ( $P_2$ )  $\pi > \pi_\infty$ , and there is  $x_w \in \mathcal{W}$  such that  $V(x_w) \leq V(x)$  for all  $|x| \geq R$ , some large  $R > 0$ .

On nonlinear potential field, writing  $G(|w|) := \int_0^{|w|} g(s) ds$ , we consider the following hypotheses:

- ( $g_1$ )  $g(0) = 0$ ,  $g \in C^1((0, \infty))$ ,  $g'(s) > 0$  for  $s > 0$ , and there exist  $p \in (2, 3)$ ,  $c_1 > 0$  such that  $g(s) \leq c_1(1 + s^{p-2})$  for  $s \geq 0$ ;  
 ( $g_2$ ) there is  $\theta > 2$  such that  $0 < \theta G(|w|) \leq g(|w|)|w|^2$  if  $w \neq 0$ .

Clearly, the power function  $g(|w|) = |w|^{p-2}$  satisfies these assumptions.

Observe that, in case ( $P_1$ ) we can assume  $W(x_v) = \max_{x \in \mathcal{V}} W(x)$  and set

$$\mathcal{A}_V := \{x \in \mathcal{V}: W(x) = W(x_v)\} \cup \{x \notin \mathcal{V}: W(x) > W(x_v)\};$$

and in case ( $P_2$ ) we can assume  $V(x_w) = \min_{x \in \mathcal{W}} V(x)$  and set

$$\mathcal{A}_W := \{x \in \mathcal{W}: V(x) = V(x_w)\} \cup \{x \notin \mathcal{W}: V(x) < V(x_w)\}.$$

Obviously,  $\mathcal{A}_V$  and  $\mathcal{A}_W$  are bounded. Moreover, if  $\mathcal{V} \cap \mathcal{W} \neq \emptyset$  then  $\mathcal{A}_V = \mathcal{A}_W = \mathcal{V} \cap \mathcal{W}$ .

**Theorem 1.1.** Let ( $g_1$ )–( $g_2$ ) and ( $P_0$ ) be satisfied.

(A) Suppose that ( $P_1$ ) holds. Then, for sufficiently small  $\varepsilon > 0$ , there exists a least energy solutions  $w_\varepsilon$  of (4) with  $w_\varepsilon \in \bigcap_{s \geq 2} W^{1,s}$ . If additionally  $\nabla V$  and  $\nabla W$  are bounded, then  $w_\varepsilon$  satisfies:

- ( $a_1$ ) There exists a maximum point  $x_\varepsilon$  of  $|w_\varepsilon|$  with  $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \mathcal{A}_V) = 0$ , such that, for some  $c, C > 0$

$$|w_\varepsilon(x)| \leq C \exp\left(-\frac{c}{\varepsilon}|x - x_\varepsilon|\right).$$

- ( $a_2$ ) Setting  $v_\varepsilon(x) := w_\varepsilon(\varepsilon x + x_\varepsilon)$ , for any sequence  $x_\varepsilon \rightarrow x_0$  as  $\varepsilon \rightarrow 0$ ,  $v_\varepsilon$  converges in  $H^1$  to a least energy solution of

$$-i\alpha \cdot \nabla v + a\beta v + V(x_0)v = W(x_0)g(|v|)v. \quad (5)$$

If particularly  $\mathcal{V} \cap \mathcal{W} \neq \emptyset$  then  $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \mathcal{V} \cap \mathcal{W}) = 0$  and  $v_\varepsilon$  converges in  $H^1$  (up to subsequences) to a least energy solution of

$$-i\alpha \cdot \nabla v + a\beta v + \tau v = \pi g(|v|)v. \quad (6)$$

(B) Suppose that ( $P_2$ ) holds. Then all the conclusions of (A) (with  $\mathcal{A}_V$  replaced by  $\mathcal{A}_W$ ) remain true.

Theorem 1.1 applies in particular to the following equations, respectively:

$$-i\varepsilon\alpha \cdot \nabla w + a\beta w + V(x)w = \nu g(|w|)w \quad (7)$$

(i.e.,  $W(x) \equiv \nu$ , a positive constant) with

$$(V) \quad |V(x)| < a \text{ and } \tau < \tau_\infty;$$

and

$$-i\varepsilon\alpha \cdot \nabla w + a\beta w + \mu w = W(x)g(|w|)w \quad (8)$$

(i.e.,  $V(x) \equiv \mu$ , a constant in  $(-a, a)$ , see [13]) with

$$(W) \quad \inf W > 0 \text{ and } \pi > \pi_\infty.$$

We have the following consequence.

**Corollary 1.2.** *Let  $(g_1)$ – $(g_2)$  and  $(V)$  (resp.  $(W)$ ) be satisfied. Then, for sufficiently small  $\varepsilon > 0$ , there exists a least energy solutions  $w_\varepsilon$  of (7) (resp. (8)) with  $w_\varepsilon \in \bigcap_{s \geq 2} W^{1,s}$ . If additionally  $\nabla V$  (resp.  $\nabla W$ ) is bounded, then  $w_\varepsilon$  satisfies:*

(a<sub>1</sub>) *There exists a maximum point  $x_\varepsilon$  of  $|w_\varepsilon|$  with  $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \mathcal{V}) = 0$  (resp.  $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \mathcal{W}) = 0$ ), such that, for some  $c, C > 0$*

$$|w_\varepsilon(x)| \leq C \exp\left(-\frac{c}{\varepsilon}|x - x_\varepsilon|\right).$$

(a<sub>2</sub>) *Setting  $v_\varepsilon(x) := w_\varepsilon(\varepsilon x + x_\varepsilon)$ , for any sequence of such  $x_\varepsilon$ ,  $v_\varepsilon$  converges in  $H^1$  to a least energy solution of*

$$\begin{aligned} & -i\alpha \cdot \nabla v + a\beta v + \tau v = \nu g(|v|)v \\ & \text{(resp. } -i\alpha \cdot \nabla v + a\beta v + \mu v = \pi g(|v|)v). \end{aligned}$$

**Remark 1.3.** The sets  $\mathcal{A}_V$  and  $\mathcal{A}_W$  can be replaced by their subsets. Given  $y \in \mathbb{R}^3$ , let  $\gamma_{V(y)W(y)}$  denote the least energy of the equation

$$-i\alpha \cdot \nabla v + a\beta v + V(y)v = W(y)g(|v|)v.$$

There is  $y_v \in \mathcal{A}_V$  such that

$$\gamma_{V(y_v)W(y_v)} = \min_{y \in \mathcal{A}_V} \gamma_{V(y)W(y)}.$$

Set  $\mu = V(y_v)$ ,  $\nu = W(y_v)$  and

$$\Omega_v = \{y \in \mathcal{A}_V: \gamma_{V(y)W(y)} = \gamma_{V(y_v)W(y_v)}\}.$$

Then  $\text{dist}(x_\varepsilon, \Omega_v) = 0$  and  $v_\varepsilon$  converges in  $H^1$  to a least energy solution of

$$-i\alpha \cdot \nabla v + a\beta v + \mu v = v g(|v|)v \quad (9)$$

(see Remark 3.6). Similarly, there is  $y_w \in \mathcal{A}_w$  satisfying  $\gamma_{V(y_w)W(y_w)} = \min_{y \in \mathcal{A}_w} \gamma_{V(y)W(y)}$ . Set  $\Omega_w = \{y \in \mathcal{A}_w: \gamma_{V(y)W(y)} = \gamma_{V(y_w)W(y_w)}\}$ . Then  $\text{dist}(x_\varepsilon, \Omega_w) = 0$  and  $v_\varepsilon$  converges in  $H^1$  to a least energy solution of (9) with  $\mu = V(y_w)$  and  $v = W(y_w)$ .

Observe that, setting  $u(x) = w(\varepsilon x)$ , Eq. (4) is equivalent to the following one:

$$-i\alpha \cdot \nabla u + a\beta u + V_\varepsilon(x)u = W_\varepsilon(x)g(|u|)u \quad (10)$$

where  $V_\varepsilon(x) = V(\varepsilon x)$  and  $W_\varepsilon(x) = W(\varepsilon x)$ . We will in the sequel focus on this equivalent problem.

Our proofs are variational: the semi-classical solutions are obtained as critical points of the energy functional  $\Phi_\varepsilon$  associated to (10). A linking-type argument yields a minimax value  $c_\varepsilon$  for  $\Phi_\varepsilon$ . Comparing with [13], since the solutions depend not only on the linear potential but also on the nonlinear one, the present argument seems to be more delicate. One new ingredient is a comparison of the least energy of a class of limit problems (Lemma 2.11). Another is an analysis on  $c_\varepsilon$  via certain auxiliary functionals (Lemma 2.12). And the third is describing the tendency of  $c_\varepsilon$  as  $\varepsilon \rightarrow 0$  (Lemma 3.4).

## 2. Preliminary results

To prove our main results some preliminaries are firstly in order.

### 2.1. The functional-analytic framework

In what follows by  $|\cdot|_q$  we denote the usual  $L^q$ -norm, and  $(\cdot, \cdot)_2$  the usual  $L^2$ -inner product. Let  $H_0 = -i\alpha \cdot \nabla + a\beta$  denote the selfadjoint operator on  $L^2(\mathbb{R}^3, \mathbb{C}^4)$  with domain  $\mathcal{D}(H_0) = H^1(\mathbb{R}^3, \mathbb{C}^4)$ . A Fourier analysis shows that  $\sigma(H_0) = \sigma_c(H_0) = \mathbb{R} \setminus (-a, a)$  where  $\sigma(\cdot)$  and  $\sigma_c(\cdot)$  denote the spectrum and continuous spectrum. Thus the space  $L^2$  possesses the orthogonal decomposition:

$$L^2 = L^- \oplus L^+, \quad u = u^- + u^+$$

so that  $H_0$  is negative definite (resp. positive definite) in  $L^-$  (resp.  $L^+$ ). Let  $E := \mathcal{D}(|H_0|^{1/2}) = H^{1/2}$  be equipped with the inner product

$$(u, v) = \Re(|H_0|^{1/2}u, |H_0|^{1/2}v)_2$$

and the induced norm  $\|u\| = (u, u)^{1/2}$ , where  $|H_0|$  and  $|H_0|^{1/2}$  denote respectively the absolute value of  $H_0$  and the square root of  $|H_0|$ . Since  $\sigma(H_0) \subset \mathbb{R} \setminus (-a, a)$ , one has

$$a|u|_2^2 \leq \|u\|^2 \quad \text{for all } u \in E. \quad (11)$$

Note that this norm is equivalent to the usual  $H^{1/2}$ -norm, hence  $E$  embeds continuously into  $L^q$  for all  $q \in [2, 3]$  and compactly into  $L_{loc}^q$  for all  $q \in [1, 3)$ . It is clear that  $E$  possesses the following decomposition

$$E = E^- \oplus E^+ \quad \text{with} \quad E^\pm = E \cap L^\pm,$$

orthogonal with respect to both  $(\cdot, \cdot)_2$  and  $(\cdot, \cdot)$  inner products. This decomposition induces also a natural decomposition of  $L^q$ , hence there is  $\pi_q > 0$  such that

$$\pi_q |u^\pm|_q^q \leq |u|_q^q \quad \text{for all } u \in E. \quad (12)$$

In virtue of the assumptions  $(g_1)$ – $(g_2)$ , for any  $\delta > 0$  with  $\delta < (a - \tau)/4$ , there exist  $r_\delta > 0$ ,  $c_\delta > 0$  and  $c'_\delta > 0$  such that

$$\begin{cases} g(s) < \delta & \text{for all } 0 \leq s \leq r_\delta; \\ G(|u|) \geq c_\delta |u|^\theta - \delta |u|^2 & \text{for all } u \in \mathbb{C}^4; \\ G(|u|) \leq \delta |u|^2 + c'_\delta |u|^p & \text{for all } u \in \mathbb{C}^4. \end{cases} \quad (13)$$

On  $E$  we define the functional

$$\Phi_\varepsilon(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^3} V_\varepsilon(x) |u|^2 - \int_{\mathbb{R}^3} W_\varepsilon(x) G(|u|)$$

for  $u = u^- + u^+$ . Denoting  $a(u, v) := \int_{\mathbb{R}^3} \langle H_0 u, v \rangle$  and setting  $a(u) = a(u, u)$ , one has

$$\begin{aligned} \Phi_\varepsilon(u) &= \frac{1}{2} \Re a(u) + \frac{1}{2} \int_{\mathbb{R}^3} V_\varepsilon(x) |u|^2 - \int_{\mathbb{R}^3} W_\varepsilon(x) G(|u|) \\ &= \frac{1}{2} a(u) + \frac{1}{2} \int_{\mathbb{R}^3} V_\varepsilon(x) |u|^2 - \int_{\mathbb{R}^3} W_\varepsilon(x) G(|u|). \end{aligned}$$

Plainly,  $\Phi_\varepsilon \in C^2(E, \mathbb{R})$ .

**Lemma 2.1.** *Critical points of  $\Phi_\varepsilon$  are solutions of (10).*

**Proof.** Observe that, for any  $u, v \in E$ ,

$$\begin{aligned} \left. \frac{d}{ds} \Phi_\varepsilon(u + sv) \right|_{s=0} &= \Re a(u, v) + \Re \int_{\mathbb{R}^3} V_\varepsilon(x) \langle u, v \rangle - \Re \int_{\mathbb{R}^3} W_\varepsilon(x) g(|u|) \langle u, v \rangle \\ &= (u^+ - u^-, v) + \Re \int_{\mathbb{R}^3} (V_\varepsilon(x) - W_\varepsilon(x) g(|u|)) \langle u, v \rangle. \end{aligned}$$

Let  $u \in E$  be a critical point of  $\Phi_\varepsilon$ . For any real vector  $v \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^4)$  one has formally

$$\begin{aligned} 0 &= (u^+ - u^-, v) + \Re \int_{\mathbb{R}^3} (V_\varepsilon(x) - W_\varepsilon(x) g(|u|)) \langle u, v \rangle \\ &= (H_0(\Re u) + V_\varepsilon(\Re u) - W_\varepsilon g(|u|)(\Re u), v)_2 \end{aligned}$$

and

$$\begin{aligned} 0 &= (u^+ - u^-, iv) + \Re \int_{\mathbb{R}^3} (V_\varepsilon(x) - W_\varepsilon(x) g(|u|)) \langle u, iv \rangle \\ &= (H_0(\Im u) + V_\varepsilon(\Im u) - W_\varepsilon g(|u|)(\Im u), v)_2. \end{aligned}$$

Hence, for any general  $v \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ , there holds

$$0 = (H_0 u + V_\varepsilon u - W_\varepsilon g(|u|)u, v)_2$$

which implies that  $u$  is a weak solution of (10). Now a standard regular argument shows that  $u$  is in fact a solution of (10).  $\square$

Denote

$$\Psi_\varepsilon(u) := \int_{\mathbb{R}^3} W_\varepsilon(x) G(|u|).$$

**Lemma 2.2.**  $\Psi_\varepsilon$  is weakly sequentially lower semicontinuous and  $\Phi'_\varepsilon$  is weakly sequentially continuous.

**Proof.** The lemma follows easily because  $E$  embeds continuously into  $L^q$  for  $q \in [2, 3]$  and compactly into  $L^q_{loc}$  for  $q \in [1, 3)$  (see [12]).  $\square$

For further convenience we introduce the following notations:

$$\begin{aligned} B_r^+ &= \{u \in E^+ : \|u\| \leq r\}, & S_r^+ &= \{u \in E^+ : \|u\| = r\}, \\ E_e &:= E^- \oplus \mathbb{R}^+ e \quad (e \in E^+, \mathbb{R}^+ = [0, \infty)) \\ \lambda_1 &:= |V|_\infty, & \lambda_2 &:= \inf W. \end{aligned}$$

Note that  $\lambda_1 < a$  and  $\lambda_2 > 0$  by  $(P_1)$ .

**Lemma 2.3.**  $\Phi_\varepsilon$  possesses the linking structure:

- 1) There exist  $r > 0$  and  $\rho > 0$  both independent of  $\varepsilon$  such that  $\Phi_\varepsilon|_{B_r^+}(u) \geq 0$  and  $\Phi_\varepsilon|_{S_r^+} \geq \rho$ .
- 2) For any  $e \in E^+ \setminus \{0\}$ , there exist  $R_e > 0$  and  $C = C_e > 0$  both independent of  $\varepsilon$  such that  $\Phi_\varepsilon(u) < 0$  for all  $u \in E_e \setminus B_{R_e}$  and  $\max \Phi_\varepsilon(E_e) \leq C$ .

**Proof.** 1) follows easily because, by (11) and (13) with  $\delta\pi < a - \lambda_1$ , for  $u \in E^+$ ,

$$\begin{aligned} \Phi_\varepsilon(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda_1|u|_2^2 - \pi\delta|u|_2^2 - \pi c'_\delta|u|_p^p \\ &\geq \frac{1}{2}\left(1 - \frac{\lambda_1 + \pi\delta}{a}\right)\|u\|^2 - c''_\delta\|u\|^p \end{aligned}$$

and  $p > 2$ .

For checking 2) take  $e \in E^+ \setminus \{0\}$ . In virtue (13) with  $2\pi\delta < a - \lambda_1$  and (12), one gets

$$\begin{aligned} \Phi_\varepsilon(u) &\leq \frac{1}{2}\|se\|^2 - \frac{1}{2}\|v\|^2 + \frac{\lambda_1}{2}|se + v|_2^2 + \delta\pi|se + v|_2^2 - s^\theta c_\delta \lambda_2 \pi_\theta |e|_\theta^\theta \\ &\leq \frac{s^2(a + \lambda_1 + 2\delta\pi)}{2a}\|e\|^2 - \frac{a - (\lambda_1 + 2\delta\pi)}{2a}\|v\|^2 - s^\theta c_\delta \lambda_2 \pi_\theta |e|_\theta^\theta, \end{aligned} \quad (14)$$

hence 2) since  $\theta > 2$ .  $\square$

Define the following minimax value (see [25,29,6])

$$c_\varepsilon := \inf_{e \in E^+ \setminus \{0\}} \max_{u \in E_e} \Phi_\varepsilon(u).$$

As a consequence of Lemma 2.3 we have

**Lemma 2.4.** *There is  $C > 0$  independent of  $\varepsilon$  such that  $\rho \leq c_\varepsilon < C$ .*

**Proof.** By 1) of Lemma 2.3 and the definition of  $c_\varepsilon$  one has  $c_\varepsilon \geq \rho$ . Take  $e \in E^+$  with  $\|e\| = 1$ . It follows from (14) the following

$$c_\varepsilon \leq C \equiv C_e,$$

ending the proof.  $\square$

Recall that a sequence  $\{u_n\} \subset E$  is said to be a  $(PS)_c$ ,  $c \in \mathbb{R}$ , sequence for  $\Phi_\varepsilon$  if  $\Phi_\varepsilon(u_n) \rightarrow c$  and  $\Phi'_\varepsilon(u_n) \rightarrow 0$ , and  $\Phi_\varepsilon$  is said to satisfy the  $(PS)_c$  condition if any  $(PS)_c$  sequence for  $\Phi_\varepsilon$  has a convergent subsequence. With Lemmas 2.2 and 2.3 and by a linking argument it follows that  $\Phi_\varepsilon$  has a  $(PS)_{c_\varepsilon}$  sequence (see e.g. [12,29]). Obviously, if  $\Phi_\varepsilon$  satisfies the  $(PS)_c$  condition then  $c_\varepsilon$  is a critical value. Unfortunately, since there is no compactly embedding from  $H^{1/2}(\mathbb{R}^3)$  into  $L^p(\mathbb{R}^3)$ , the  $(PS)$  condition does not in general hold, we have to go through more analysis.

In order to get more information on  $c_\varepsilon$ , motivated by [1] (see also [15,25,29]), we consider, for a fixed  $u \in E^+$ , the map  $\phi_u : E^- \rightarrow \mathbb{R}$  defined by

$$\phi_u(v) = \Phi_\varepsilon(u + v).$$

Observe that, for any  $v, w \in E^-$ ,

$$\begin{aligned} \phi''_u(v)[w, w] &= -\|w\|^2 + \int_{\mathbb{R}^3} V_\varepsilon(x)|w|^2 - \Psi''_\varepsilon(u + v)[w, w] \\ &\leq -\frac{a - \lambda_1}{a} \|w\|^2 - \Psi''_\varepsilon(u + v)[w, w], \end{aligned}$$

and in addition

$$\phi_u(v) \leq \frac{a + \lambda_1}{2a} \|u\|^2 - \frac{a - \lambda_1}{2a} \|v\|^2.$$

Therefore, there is a unique  $h_\varepsilon(u) \in E^-$  such that

$$\phi_u(h_\varepsilon(u)) = \max_{v \in E^-} \phi_u(v).$$

It is clear that

$$\begin{aligned} 0 &= \phi'_u(h_\varepsilon(u))v \\ &= -(h_\varepsilon(u), v) + \Re \int_{\mathbb{R}^3} V_\varepsilon(x) \langle u + h_\varepsilon(u), v \rangle - \Psi'_\varepsilon(u + h_\varepsilon(u))v \end{aligned}$$



for all  $v \in E^-$ , and

$$v \neq h_\varepsilon(u) \Leftrightarrow \Phi_\varepsilon(u+v) < \Phi_\varepsilon(u+h_\varepsilon(u)).$$

For any  $u \in E^+$  and  $v \in E^-$ , setting  $z = v - h_\varepsilon(u)$  and  $\ell(t) = \phi_u(h_\varepsilon(u) + tz)$ , one has  $\ell(1) = \phi_u(v)$ ,  $\ell(0) = \phi_u(h_\varepsilon(u))$  and  $\ell'(0) = 0$ . Thus  $\ell(1) - \ell(0) = \int_0^1 (1-t)\ell''(t) dt$ . This implies that

$$\begin{aligned} & \phi_u(v) - \phi_u(h_\varepsilon(u)) \\ &= \int_0^1 (1-t)\phi_u''(h_\varepsilon(u) + tz)[z, z] dt \\ &= - \int_0^1 (1-t) \left( \|z\|^2 - \int_{\mathbb{R}^3} V_\varepsilon(x)|z|^2 + \int_{\mathbb{R}^3} W_\varepsilon(x)g(|u+h_\varepsilon(u)+tz|)|z|^2 \right. \\ & \quad \left. + \int_{\mathbb{R}^3} W_\varepsilon(x) \frac{g'(|u+h_\varepsilon(u)+tz|)}{|u+h_\varepsilon(u)+tz|} (\Re(u+h_\varepsilon(u)+tz, z))^2 \right) dt \end{aligned}$$

hence,

$$\begin{aligned} \Phi_\varepsilon(u+h_\varepsilon(u)) - \Phi_\varepsilon(u+v) &= \frac{1}{2} \left( \|z\|^2 - \int_{\mathbb{R}^3} V_\varepsilon(x)|z|^2 \right) \\ & \quad + \int_0^1 \int_{\mathbb{R}^3} (1-t)W_\varepsilon(x) \left( g(|u+h_\varepsilon(u)+tz|)|z|^2 \right. \\ & \quad \left. + \frac{g'(|u+h_\varepsilon(u)+tz|)}{|u+h_\varepsilon(u)+tz|} (\Re(u+h_\varepsilon(u)+tz, z))^2 \right) dt. \end{aligned} \quad (15)$$

Define  $I_\varepsilon : E^+ \rightarrow \mathbb{R}$  by

$$\begin{aligned} I_\varepsilon(u) &= \Phi_\varepsilon(u+h_\varepsilon(u)) \\ &= \frac{1}{2}(\|u\|^2 - \|h_\varepsilon(u)\|^2) + \frac{1}{2} \int_{\mathbb{R}^3} V_\varepsilon(x)|u+h_\varepsilon(u)|^2 - \Psi_\varepsilon(u+h_\varepsilon(u)). \end{aligned}$$

Set

$$\mathcal{N}_\varepsilon := \{u \in E^+ \setminus \{0\} : I'_\varepsilon(u)u = 0\}.$$

**Lemma 2.5.** For any  $u \in E^+ \setminus \{0\}$ , there is a unique  $t = t(u) > 0$  such that  $t(u)u \in \mathcal{N}_\varepsilon$ .

**Proof.** See [1,15].  $\square$

**Lemma 2.6.** *We have*

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u).$$

**Proof.** Indeed, given  $e \in E^+$ , if  $u = v + se \in E_e$  with  $\Phi_\varepsilon(u) = \max_{z \in E_e} \Phi_\varepsilon(z)$  then the restriction  $\Phi_\varepsilon|_{E_e}$  of  $\Phi_\varepsilon$  on  $E_e$  satisfies  $(\Phi_\varepsilon|_{E_e})'(u) = 0$  which implies  $v = h_\varepsilon(se)$  and  $I'_\varepsilon(se)(se) = \Phi'_\varepsilon(u)(se) = 0$ , i.e.  $se \in \mathcal{N}_\varepsilon$ . Thus  $\inf_{u \in E_e} \Phi_\varepsilon(u) \leq c_\varepsilon$ . On the other hand, if  $w \in \mathcal{N}_\varepsilon$  then  $(\Phi_\varepsilon|_{E_w})'(w + h_\varepsilon(w)) = 0$  so  $c_\varepsilon \leq \max_{u \in E_w} \Phi_\varepsilon(u) = I_\varepsilon(w)$ . Thus  $\inf_{u \in E_e} \Phi_\varepsilon(u) \geq c_\varepsilon$ . This proves the desired conclusion.  $\square$

**Lemma 2.7.** *For any  $e \in E^+ \setminus \{0\}$ , there is  $T_e > 0$  independent of  $\varepsilon > 0$  such that  $t_\varepsilon \leq T_e$  for  $t_\varepsilon > 0$  satisfying  $t_\varepsilon e \in \mathcal{N}_\varepsilon$ .*

**Proof.** Since  $I'_\varepsilon(t_\varepsilon e)(t_\varepsilon e) = 0$  one sees that the restriction of  $\Phi_\varepsilon$  satisfies  $(\Phi_\varepsilon|_{E_e})'(t_\varepsilon e + h_\varepsilon(t_\varepsilon e)) = 0$ . Thus

$$\Phi_\varepsilon(t_\varepsilon e + h_\varepsilon(t_\varepsilon e)) = \max_{w \in E_e} \Phi_\varepsilon(w).$$

This, together with Lemma 2.6 and (14), implies the desired conclusion.  $\square$

Let  $\mathcal{K}_\varepsilon := \{u \in E: \Phi'_\varepsilon(u) = 0\}$  be the critical set of  $\Phi_\varepsilon$ . It is easy to see that if  $\mathcal{K}_\varepsilon \setminus \{0\} \neq \emptyset$  then

$$c_\varepsilon = \inf\{\Phi_\varepsilon(u): u \in \mathcal{K}_\varepsilon \setminus \{0\}\}$$

(see an argument of [15]). Using the same iterative argument of [16, Proposition 3.2] one obtains easily the following

**Lemma 2.8.** *If  $u \in \mathcal{K}_\varepsilon$  with  $|\Phi_\varepsilon(u)| \leq C_1$  and  $|u|_2 \leq C_2$ , then, for any  $q \in [2, \infty)$ ,  $u \in W^{1,q}(\mathbb{R}^3)$  with  $\|u\|_{W^{1,q}} \leq A_q$  where  $A_q$  depends only on  $C_1, C_2$  and  $q$ .*

Let  $\mathcal{S}_\varepsilon$  be the set of all least energy solutions of  $\Phi_\varepsilon$ . If  $u \in \mathcal{S}_\varepsilon$  then  $\Phi_\varepsilon(u) = c_\varepsilon$  and a standard argument shows that  $\mathcal{S}_\varepsilon$  is bounded in  $E$ , hence,  $|u|_2 \leq C_2$  for  $u \in \mathcal{S}_\varepsilon$ , some  $C_2 > 0$  independent of  $\varepsilon$ . Therefore, as a consequence of Lemmas 2.6 and 2.8 we see that, for each  $q \in [2, \infty)$ , there is  $C_q > 0$  independent of  $\varepsilon$  such that

$$\|u\|_{W^{1,q}} \leq C_q \quad \text{for all } u \in \mathcal{S}_\varepsilon. \quad (16)$$

This, together with the Sobolev embedding theorem, implies that there is  $C_\infty > 0$  independent of  $\varepsilon$  with

$$|u|_\infty \leq C_\infty \quad \text{for all } u \in \mathcal{S}_\varepsilon. \quad (17)$$

## 2.2. The limit problem

We will make use of the limit equation for proving our main result. To this end we discuss in this section the existence and some properties of the least energy solutions of the limit problem.

For any  $\mu \in (-a, a)$  and  $\nu > 0$ , consider the equation

$$-i\alpha \cdot \nabla u + a\beta u + \mu u = \nu g(|u|)u, \quad u \in H^1(\mathbb{R}^3, \mathbb{C}^4). \quad (18)$$

Its solutions are critical points of the functional

$$\Gamma_{\mu\nu}(u) := \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{\mu}{2} \int_{\mathbb{R}^3} |u|^2 - \nu \int_{\mathbb{R}^3} G(|u|)$$

defined for  $u = u^- + u^+ \in E = E^- \oplus E^+$ . Denote the critical set, the least energy, and the set of least energy solutions of  $\Gamma_{\mu\nu}$  as follows

$$\begin{aligned}\mathcal{L}_{\mu\nu} &:= \{u \in E: \Gamma'_{\mu\nu}(u) = 0\}, \\ \gamma_{\mu\nu} &:= \inf\{\Gamma_{\mu\nu}(u): u \in \mathcal{L}_{\mu\nu} \setminus \{0\}\}, \\ \mathcal{R}_{\mu\nu} &:= \{u \in \mathcal{L}_{\mu\nu}: \Gamma_{\mu\nu}(u) = \gamma_{\mu\nu}, |u(0)| = |u|_\infty\}.\end{aligned}$$

The following lemma is from [15].

**Lemma 2.9.** *There hold the following:*

- i)  $\mathcal{L}_{\mu\nu} \neq \emptyset$ ,  $\gamma_{\mu\nu} > 0$ , and  $\mathcal{L}_{\mu\nu} \subset \bigcap_{q \geq 2} W^{1,q}$ ;
- ii)  $\gamma_{\mu\nu}$  is attained, and  $\mathcal{R}_{\mu\nu}$  is compact in  $H^1(\mathbb{R}^3, \mathbb{C}^4)$ ;
- iii) there exist  $C, c > 0$  such that

$$|u(x)| \leq C \exp(-c|x|) \quad \text{for all } x \in \mathbb{R}^3, u \in \mathcal{R}_{\mu\nu}.$$

As before we introduce the following notations:

$$\begin{aligned}\mathcal{J}_{\mu\nu}: E^+ &\rightarrow E^-: \Gamma_{\mu\nu}(u + \mathcal{J}_{\mu\nu}(u)) = \max_{v \in E^-} \Gamma_{\mu\nu}(u + v); \\ J_{\mu\nu}: E^+ &\rightarrow \mathbb{R}: J_{\mu\nu}(u) = \Gamma_{\mu\nu}(u + \mathcal{J}_{\mu\nu}(u)); \\ \mathcal{M}_{\mu\nu} &:= \{u \in E^+ \setminus \{0\}: J'_{\mu\nu}(u)u = 0\}.\end{aligned}$$

Plainly, critical points of  $J_{\mu\nu}$  and  $\Gamma_{\mu\nu}$  are in one to one correspondence via the injective map  $u \rightarrow u + \mathcal{J}_{\mu\nu}(u)$  from  $E^+$  into  $E$ .

Notice that, similar to (15), for  $u \in E^+$ ,  $v \in E^-$  and  $z = v - \mathcal{J}_{\mu\nu}(u)$ , there holds

$$\begin{aligned}\Gamma_{\mu\nu}(u + \mathcal{J}_{\mu\nu}(u)) - \Gamma_{\mu\nu}(u + v) \\ = \frac{1}{2}(\|z\|^2 - \mu|z|_2^2) + \nu \int_0^1 \int_{\mathbb{R}^3} (1-t) \left( g(|u + \mathcal{J}_{\mu\nu}(u) + tz|) |z|^2 \right. \\ \left. + \frac{g'(|u + \mathcal{J}_{\mu\nu}(u) + tz|)}{|u + \mathcal{J}_{\mu\nu}(u) + tz|} (\Re(u + \mathcal{J}_{\mu\nu}(u) + tz, z))^2 \right) dt. \quad (19)\end{aligned}$$

**Lemma 2.10.** *Let  $u \in \mathcal{M}_{\mu\nu}$  be such that  $J_{\mu\nu}(u) = \gamma_{\mu\nu}$ , and set  $E_u = E^- \oplus \mathbb{R}u$ . Then*

$$\max_{w \in E_u} \Gamma_{\mu\nu}(w) = J_{\mu\nu}(u).$$

**Proof.** Clearly, since  $u + \mathcal{J}_{\mu\nu}(u) \in E_u$ ,

$$J_{\mu\nu}(u) = \Gamma_{\mu\nu}(u + \mathcal{J}_{\mu\nu}(u)) \leq \max_{w \in E_u} \Gamma_{\mu\nu}(w).$$

On the other hand, for any  $w = v + su \in E_u$ ,

$$\begin{aligned} \Gamma_{\mu\nu}(w) &= \frac{1}{2} \|su\|^2 - \frac{1}{2} \|v\|^2 + \frac{\mu}{2} |v + su|_2^2 - v \int_{\mathbb{R}^3} G(|v + su|) \\ &\leq \Gamma_{\mu\nu}(su + \mathcal{J}_{\mu\nu}(su)) = J_{\mu\nu}(su). \end{aligned}$$

Thus, since  $u \in \mathcal{M}_{\mu\nu}$ ,

$$\max_{w \in E_u} \Gamma_{\mu\nu}(w) \leq \max_{s \geq 0} J_{\mu\nu}(su) = J_{\mu\nu}(u),$$

giving the conclusion.  $\square$

**Lemma 2.11.** Let  $\mu_j \in (-a, a)$  and  $v_j > 0$ ,  $j = 1, 2$ , with  $\min\{\mu_2 - \mu_1, v_1 - v_2\} \geq 0$ . Then  $\gamma_{\mu_1 v_1} \leq \gamma_{\mu_2 v_2}$ . If additionally  $\max\{\mu_2 - \mu_1, v_1 - v_2\} > 0$  then  $\gamma_{\mu_1 v_1} < \gamma_{\mu_2 v_2}$ . In particular,  $\gamma_{\mu_1 v_j} < \gamma_{\mu_2 v_j}$  if  $\mu_1 < \mu_2$ , and  $\gamma_{\mu_j v_1} > \gamma_{\mu_j v_2}$  if  $v_1 < v_2$ .

**Proof.** Let  $u \in \mathcal{L}_{\mu_2 v_2}$  with  $\Gamma_{\mu_2 v_2}(u) = \gamma_{\mu_2 v_2}$  and set  $e = u^+$ . Then

$$\gamma_{\mu_2 v_2} = \Gamma_{\mu_2 v_2}(u) = \max_{w \in E_e} \Gamma_{\mu_2 v_2}(w).$$

Let  $u_0 \in E_e$  be such that  $\Gamma_{\mu_1 v_1}(u_0) = \max_{w \in E_e} \Gamma_{\mu_1 v_1}(w)$ . One has

$$\begin{aligned} \gamma_{\mu_2 v_2} &= \Gamma_{\mu_2 v_2}(u) \geq \Gamma_{\mu_2 v_2}(u_0) \\ &= \Gamma_{\mu_1 v_1}(u_0) + \frac{1}{2}(\mu_2 - \mu_1)|u_0|_2^2 + (v_1 - v_2) \int_{\mathbb{R}^3} G(|u_0|) \\ &\geq \gamma_{\mu_1 v_1} + \frac{1}{2}(\mu_2 - \mu_1)|u_0|_2^2 + (v_1 - v_2) \int_{\mathbb{R}^3} G(|u_0|) \end{aligned}$$

as claimed.  $\square$

### 2.3. Auxiliary functionals

Assume that the sequence of functions  $\hat{V}_\varepsilon$  and  $\hat{W}_\varepsilon \in C \cap L^\infty(\mathbb{R}^3, \mathbb{R})$ ,  $0 < \varepsilon \leq 1$ , satisfy

( $\star$ )  $\varpi := \sup_{\varepsilon, x} |\hat{V}_\varepsilon(x)| < a$ ,  $\inf_{\varepsilon, x} \hat{W}_\varepsilon(x) > 0$ ;  $\hat{V}_\varepsilon(x) \rightarrow \mu$  and  $\hat{W}_\varepsilon(x) \rightarrow \nu$  uniformly on bounded sets of  $x$  as  $\varepsilon \rightarrow 0$ .

Consider the equations

$$-i\alpha \cdot \nabla u + a\beta u + \hat{V}_\varepsilon(x)u = \hat{W}_\varepsilon(x)g(|u|)u. \quad (20)$$

Denote

$$\hat{\Phi}_\varepsilon(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^3} \hat{V}_\varepsilon(x)|u|^2 - \int_{\mathbb{R}^3} \hat{W}_\varepsilon(x)G(|u|)$$

and, as before, the associate  $\hat{h}_\varepsilon$ ,  $\hat{I}_\varepsilon$ ,  $\hat{\mathcal{N}}_\varepsilon$ ,  $\hat{c}_\varepsilon$  and so on. Note that, setting  $V_\varepsilon^0(x) = \hat{V}_\varepsilon(x) - \mu$  and  $W_\varepsilon^0(x) = \nu - \hat{W}_\varepsilon(x)$ , we have by definition

$$\hat{\Phi}_\varepsilon(u) = \Gamma_{\mu\nu}(u) + \frac{1}{2} \int_{\mathbb{R}^3} V_\varepsilon^0(x)|u|^2 + \int_{\mathbb{R}^3} W_\varepsilon^0(x)G(|u|). \quad (21)$$

**Lemma 2.12.**  $\limsup_{\varepsilon \rightarrow 0} \hat{c}_\varepsilon \leq \gamma_{\mu\nu}$ .

**Proof.** In virtue of Lemma 2.9 let  $u = u^- + u^+ \in \mathcal{R}_{\mu\nu}$ , a least energy solution of (18) and set  $e = u^+$ . It is clear that  $e \in \mathcal{M}_{\mu\nu}$ ,  $\mathcal{J}_{\mu\nu}(e) = u^-$  and  $J_{\mu\nu}(e) = \gamma_{\mu\nu}$ . There is a unique  $t_\varepsilon > 0$  such that  $t_\varepsilon e \in \hat{\mathcal{N}}_\varepsilon$ . One has

$$\hat{c}_\varepsilon \leq \hat{I}_\varepsilon(t_\varepsilon e). \quad (22)$$

By Lemma 2.7,  $\{t_\varepsilon\}$  is bounded, hence, without loss of generality we can assume  $t_\varepsilon \rightarrow t_0$  as  $\varepsilon \rightarrow 0$ .

Observe that (21) induces that

$$\begin{aligned} & (\hat{\Phi}_\varepsilon(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) - \hat{\Phi}_\varepsilon(t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e))) \\ & + (\Gamma_{\mu\nu}(t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)) - \Gamma_{\mu\nu}(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e))) \\ & = \frac{1}{2} \int_{\mathbb{R}^3} V_\varepsilon^0(x)(|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|^2 - |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|^2) \\ & + \int_{\mathbb{R}^3} W_\varepsilon^0(x)(G(|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|) - G(|t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|)). \end{aligned} \quad (23)$$

Since, denoting  $z_\varepsilon = \mathcal{J}_{\mu\nu}(t_\varepsilon e) - \hat{h}_\varepsilon(t_\varepsilon e)$ ,

$$|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|^2 - |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|^2 = |z_\varepsilon|^2 - 2\Re\langle t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e), z_\varepsilon \rangle$$

and

$$\begin{aligned} & G(|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|) - G(|t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|) \\ & = G(|t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e) - z_\varepsilon|) - G(|t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|) \\ & = -g(|t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|)\Re\langle t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e), z_\varepsilon \rangle + K_\varepsilon(x) \end{aligned}$$

with

$$K_\varepsilon(x) := \int_0^1 (1-s) \left( g(|t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e) - s z_\varepsilon|) |z_\varepsilon|^2 + \frac{g'(|t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e) - s z_\varepsilon|)}{|t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e) - s z_\varepsilon|} (\Re(t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e) - s z_\varepsilon), -z_\varepsilon) \right) ds$$

we get from (23) (remark that  $W_\varepsilon^0(x) \leq \nu$ )

$$\begin{aligned} & (\hat{\Phi}_\varepsilon(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) - \hat{\Phi}_\varepsilon(t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e))) \\ & + (\Gamma_{\mu\nu}(t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)) - \Gamma_{\mu\nu}(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e))) \\ & \leq \frac{1}{2} \int_{\mathbb{R}^3} V_\varepsilon^0(x) |z_\varepsilon|^2 - \Re \int_{\mathbb{R}^3} V_\varepsilon^0(x) \langle t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e), z_\varepsilon \rangle \\ & - \Re \int_{\mathbb{R}^3} W_\varepsilon^0(x) g(|t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|) \langle t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e), z_\varepsilon \rangle + \nu \int_{\mathbb{R}^3} K_\varepsilon(x). \end{aligned} \quad (24)$$

Remark that one has, by (19) (with  $z$  replaced by  $z_\varepsilon$ ),

$$\Gamma_{\mu\nu}(t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)) - \Gamma_{\mu\nu}(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) = \frac{1}{2} (\|z_\varepsilon\|^2 - \mu |z_\varepsilon|_2^2) + \nu \int_{\mathbb{R}^3} K_\varepsilon(x)$$

and, by the representation (15) with  $\Phi_\varepsilon$  replaced by  $\hat{\Phi}_\varepsilon$ ,

$$\hat{\Phi}_\varepsilon(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) - \hat{\Phi}_\varepsilon(t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)) \geq \frac{1}{2} \left( \|z_\varepsilon\|^2 - \int_{\mathbb{R}^3} \hat{V}_\varepsilon(x) |z_\varepsilon|^2 \right).$$

Thus (24) (jointly with  $(g_1)$ ) implies

$$\begin{aligned} \|z_\varepsilon\|^2 - \varpi |z_\varepsilon|_2^2 & \leq -\Re \int_{\mathbb{R}^3} V_\varepsilon^0(x) \langle t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e), z_\varepsilon \rangle \\ & - \Re \int_{\mathbb{R}^3} W_\varepsilon^0(x) g(|t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|) \langle t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e), z_\varepsilon \rangle \\ & \leq \int_{\mathbb{R}^3} |V_\varepsilon^0(x)| |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)| |z_\varepsilon| + c_1 \int_{\mathbb{R}^3} |W_\varepsilon^0(x)| |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)| |z_\varepsilon| \\ & + c_1 \int_{\mathbb{R}^3} |W_\varepsilon^0(x)| |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|^{p-1} |z_\varepsilon| \\ & \leq c_2 \left( \int_{\mathbb{R}^3} (|V_\varepsilon^0(x)| + |W_\varepsilon^0(x)|)^2 |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|^2 \right)^{1/2} |z_\varepsilon|_2 \\ & + c_1 \left( \int_{\mathbb{R}^3} |W_\varepsilon^0(x)|^{p/(p-1)} |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|^p \right)^{(p-1)/p} |z_\varepsilon|_p. \end{aligned} \quad (25)$$

Since  $t_\varepsilon \rightarrow t_0$  and  $e$  is exponentially decay, we have for  $q = 2, p$ ,

$$\limsup_{R \rightarrow \infty} \int_{|x| \geq R} |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|^q = 0$$

which implies that

$$\begin{aligned} & \int_{\mathbb{R}^3} (|V_\varepsilon^0(x)| + |W_\varepsilon^0(x)|)^2 |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|^2 \\ &= \left( \int_{|x| \leq R} + \int_{|x| > R} \right) (|V_\varepsilon^0(x)| + |W_\varepsilon^0(x)|)^2 |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|^2 \\ &\leq \int_{|x| \leq R} (|V_\varepsilon^0(x)| + |W_\varepsilon^0(x)|)^2 |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|^2 + c_3 \int_{|x| > R} |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|^2 \\ &= o(1) \end{aligned}$$

as  $\varepsilon \rightarrow 0$ , and similarly

$$\int_{\mathbb{R}^3} |W_\varepsilon^0(x)|^{p/(p-1)} |t_\varepsilon e + \mathcal{J}_{\mu\nu}(t_\varepsilon e)|^p = o(1)$$

as  $\varepsilon \rightarrow 0$ . Thus, since  $\varpi < a$  by the assumption  $(*)$ , it follows from (25) that  $\|z_\varepsilon\| = \|\hat{h}_\varepsilon(t_\varepsilon e) - \mathcal{J}_{\mu\nu}(t_\varepsilon e)\| \rightarrow 0$ , that is,  $\hat{h}_\varepsilon(t_\varepsilon e) \rightarrow \mathcal{J}_{\mu\nu}(t_0 e)$ . Consequently,

$$\int_{\mathbb{R}^3} V_\varepsilon^0(x) |t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|^2 \rightarrow 0 \quad \text{and} \quad \int_{\mathbb{R}^3} W_\varepsilon^0(x) G(|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|) \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ . This, jointly with (21), implies

$$\hat{\Phi}_\varepsilon(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) = \Gamma_{\mu\nu}(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) + o(1) = \Gamma_{\mu\nu}(t_0 e + \mathcal{J}_{\mu\nu}(t_0 e)) + o(1),$$

that is,

$$\hat{I}_\varepsilon(t_\varepsilon e) = J_{\mu\nu}(t_0 e) + o(1)$$

as  $\varepsilon \rightarrow 0$ . Recalling that by Lemma 2.10

$$J_{\mu\nu}(t_0 e) \leq \max_{v \in E_e} \Gamma_{\mu\nu}(v) = J_{\mu\nu}(e) = \gamma_{\mu\nu},$$

we obtain, jointly with (22),

$$\lim_{\varepsilon \rightarrow 0} \hat{c}_\varepsilon \leq \lim_{\varepsilon \rightarrow 0} \hat{I}_\varepsilon(t_\varepsilon e) = J_{\mu\nu}(t_0 e) \leq \gamma_{\mu\nu}$$

as claimed.  $\square$

Now, for any  $\tau \leq \mu \leq \liminf_{|x| \rightarrow \infty} V(x)$  and  $\pi \geq v \geq \limsup_{|x| \rightarrow \infty} W(x)$ , we set

$$V^\mu(x) := \max\{\mu, V(x)\},$$

$$W^v(x) := \min\{v, W(x)\},$$

and let  $V_\varepsilon^\mu(x) = V^\mu(\varepsilon x)$ ,  $W_\varepsilon^v(x) = W^v(\varepsilon x)$ . Consider the functional

$$\Phi_\varepsilon^{\mu v}(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^3} V_\varepsilon^\mu(x) |u|^2 - \int_{\mathbb{R}^3} W_\varepsilon^v(x) G(|u|)$$

with  $\mathcal{N}_\varepsilon^{\mu v}$ ,  $c_\varepsilon^{\mu v}$  and so on as before. By definition and Lemma 2.11,

$$\gamma_\tau \pi \leq \gamma_{V(0)W(0)} \leq \gamma_{V^\mu(0)W^v(0)}. \quad (26)$$

Moreover, observe that

$$\Phi_\varepsilon^{\mu v}(u) = \Gamma_{\mu v}(u) + \frac{1}{2} \int_{\mathbb{R}^3} (V_\varepsilon^\mu(x) - \mu) |u|^2 + \int_{\mathbb{R}^3} (v - W_\varepsilon^v(x)) G(|u|).$$

This, together with Lemma 2.12, shows

$$\gamma_{\mu v} \leq c_\varepsilon^{\mu v} \quad \text{and} \quad \limsup_{\varepsilon \rightarrow 0} c_\varepsilon^{\mu v} \leq \gamma_{V^\mu(0)W^v(0)}. \quad (27)$$

In particular,

$$\lim_{\varepsilon \rightarrow 0} c_\varepsilon^{\mu v} = \gamma_{\mu v} \quad (28)$$

if  $V(0) \leq \mu$  and  $W(0) \geq v$ .

### 3. Proof of the main result

We are now give the proofs of the main results.

#### 3.1. The case with $(P_0)$ and $(P_1)$

In this section we consider firstly the situation that  $(P_0)$  and  $(P_1)$  are satisfied. We start with observing that, for any  $x_0 \in \mathcal{V}$ , setting  $\tilde{V}(x) = V(x + x_0)$  and  $\tilde{W}(x) = W(x + x_0)$ , if  $\tilde{w}(x)$  is a solution of

$$-i\varepsilon\alpha \cdot \nabla \tilde{w} + a\beta \tilde{w} + \tilde{V}(x)\tilde{w} = \tilde{W}(x)g(|\tilde{w}|)\tilde{w}, \quad (29)$$

then  $w(x) := \tilde{w}(x - x_0)$  solves (4). Thus, without loss of generality, we can assume that  $W(x_v) = \max_{x \in \mathcal{V}} W(x)$  and  $x_v = 0 \in \mathcal{V}$  ( $x_v = 0 \in \mathcal{V} \cap \mathcal{W}$  if  $\mathcal{V} \cap \mathcal{W} \neq \emptyset$ ). Then  $\tau = V(0)$  and  $\kappa := W(0) \geq W(x)$  for all  $|x| \geq R$ .

Consider the equivalent equation (10). The key for the proof is that  $\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq \gamma_{\tau\kappa}$ . Then we argue by contradiction to show the existence of semi-classical solutions. In order to show the concentration phenomena it is sufficient to verify that, for any sequence  $\varepsilon_j \rightarrow 0$  with  $u_j \in \mathcal{S}_{\varepsilon_j}$ , there is a subsequence which converges, up to a shift of  $x$ -variable, to a least energy solution of the limit



problem, and such a subsequence is uniformly small at the infinity with the help of the sub-solution estimate. Lastly, using a comparison principle we complete the proof.

**Lemma 3.1.**  $\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq \gamma_{\tau\kappa}$ . In particular,  $\lim_{\varepsilon \rightarrow 0} c_\varepsilon = \gamma_{\tau\pi}$  if additionally  $\mathcal{V} \cap \mathcal{W} \neq \emptyset$ .

**Proof.** Choose  $\mu = \tau$  and  $\nu = \pi$ . Observe that  $V^\mu(x) = V(x)$ ,  $W^\nu(x) = W(x)$ ,  $\tau = V^\mu(0)$  and  $\kappa = W^\nu(0) \leq \pi$ . Then  $c_\varepsilon = c_\varepsilon^{\mu\nu}$  and the conclusion follows from (27) and (28) directly.  $\square$

**Lemma 3.2.**  $c_\varepsilon$  is attained for all small  $\varepsilon > 0$ .

**Proof.** Given  $\varepsilon > 0$ , let  $u_n \in \mathcal{N}_\varepsilon$  be a minimization sequence:  $I_\varepsilon(u_n) \rightarrow c_\varepsilon$ . By the Ekeland variational principle we can assume that  $u_n$  is, in addition, a  $(PS)_{c_\varepsilon}$  sequence for  $I_\varepsilon$  on  $\mathcal{N}_\varepsilon$ . A standard argument shows that  $u_n$  is in fact a  $(PS)_{c_\varepsilon}$  sequence for  $I_\varepsilon$  on  $E^+$  (see, e.g., [25,31]). Then  $w_n = u_n + h_\varepsilon(u_n)$  is a  $(PS)_{c_\varepsilon}$  sequence for  $\Phi_\varepsilon$  on  $E$ . It is easy to see that  $w_n$  is bounded. We can assume without loss of generality that  $w_n \rightharpoonup w_\varepsilon = z_\varepsilon^+ + z_\varepsilon^- \in \mathcal{K}_\varepsilon$  in  $E$ . If  $w_\varepsilon \neq 0$  then clearly  $\Phi_\varepsilon(w_\varepsilon) = c_\varepsilon$ . So we are going to check that  $w_\varepsilon \neq 0$  for all  $\varepsilon > 0$  small.

Assume by contradiction that there is a sequence  $\varepsilon_j \rightarrow 0$  with  $w_{\varepsilon_j} = 0$ . Then  $w_n = u_n + h_{\varepsilon_j}(u_n) \rightarrow 0$  in  $E$ ,  $u_n \rightarrow 0$  in  $L_{loc}^q$  for  $q \in (1, 3)$ , and  $w_n(x) \rightarrow 0$  a.e. in  $x \in \mathbb{R}^3$ . Choose  $\tau < \mu < \liminf_{|x| \rightarrow \infty} V(x)$ ,  $\nu = \kappa$ , and consider the functional  $\Phi^{\mu\nu}$ . Let  $t_n > 0$  be such that  $t_n u_n \in \mathcal{N}_{\varepsilon_j}^{\mu\nu}$ . We see that  $\{t_n\}$  is bounded and one may assume  $t_n \rightarrow t_0$  as  $n \rightarrow \infty$ . By  $(P_2)$ , the set  $A_\varepsilon := \{x \in \mathbb{R}^3: V_\varepsilon(x) < \mu \text{ or } W_\varepsilon(x) > \nu\}$  is bounded. Additionally,  $\Phi_{\varepsilon_j}(t_n u_n + h_{\varepsilon_j}^{\mu\nu}(t_n u_n)) \leq I_{\varepsilon_j}(u_n)$  by virtue of Lemma 2.10. We obtain

$$\begin{aligned} c_{\varepsilon_j}^{\mu\nu} &\leq I_{\varepsilon_j}^{\mu\nu}(t_n u_n) = \Phi_{\varepsilon_j}^{\mu\nu}(t_n u_n + h_{\varepsilon_j}^{\mu\nu}(t_n u_n)) \\ &= \Phi_{\varepsilon_j}(t_n u_n + h_{\varepsilon_j}^{\mu\nu}(t_n u_n)) + \frac{1}{2} \int_{\mathbb{R}^3} (V_{\varepsilon_j}^\mu(x) - V_{\varepsilon_j}(x)) |t_n u_n + h_{\varepsilon_j}^{\mu\nu}(t_n u_n)|^2 \\ &\quad + \int_{\mathbb{R}^3} (W_{\varepsilon_j}(x) - W_{\varepsilon_j}^\nu(x)) G(|t_n u_n + h_{\varepsilon_j}^{\mu\nu}(t_n u_n)|) \\ &\leq I_{\varepsilon_j}(u_n) + \frac{1}{2} \int_{A_{\varepsilon_j}} (\mu - V_{\varepsilon_j}(x)) |t_n u_n + h_{\varepsilon_j}^{\mu\nu}(t_n u_n)|^2 \\ &\quad + \int_{A_{\varepsilon_j}} (W_{\varepsilon_j}(x) - \nu) G(|t_n u_n + h_{\varepsilon_j}^{\mu\nu}(t_n u_n)|) \\ &= c_{\varepsilon_j} + o(1) \end{aligned}$$

as  $n \rightarrow \infty$ , hence,  $c_{\varepsilon_j}^{\mu\nu} \leq c_{\varepsilon_j}$ . By (27),  $\gamma_{\mu\nu} \leq c_{\varepsilon_j}^{\mu\nu}$ , hence  $\gamma_{\mu\nu} \leq c_{\varepsilon_j}$ . Recalling that  $\nu = \kappa$  and in virtue of Lemma 3.1, letting  $j \rightarrow \infty$  yields

$$\gamma_{\mu\kappa} \leq \gamma_{\tau\kappa},$$

contradicting with  $\gamma_{\tau\kappa} < \gamma_{\mu\kappa}$  (see Lemma 2.11).  $\square$

**Remark 3.3.** It is not difficult to check that  $\mathcal{S}_\varepsilon$  is compact for all small  $\varepsilon > 0$ . Indeed, assume by contradiction that, for some  $\varepsilon_j \rightarrow 0$ ,  $\mathcal{S}_{\varepsilon_j}$  is not compact in  $E$ . Let  $u_n^j \in \mathcal{S}_{\varepsilon_j}$  with  $u_n^j \rightharpoonup 0$  as  $n \rightarrow \infty$ . As done in proving the above Lemma 3.2, one gets a contradiction.

For the later use, letting  $D = -i\alpha \cdot \nabla$ , we write (10) as

$$Du = -a\beta u - V_\varepsilon(x)u + W_\varepsilon(x)g(|u|)u.$$

By Lemma 2.8,  $u \in \bigcap_{q \geq 2} W^{1,q}$  for any  $u \in \mathcal{H}_\varepsilon$ . Acting the operator  $D$  on the two sides of the above representation and noting that  $D^2 = -\Delta$  we get

$$\Delta u = (a^2 - V_\varepsilon^2(x))u + r_\varepsilon(x, |u|)u$$

where

$$\begin{aligned} r_\varepsilon(x, |u|) := & DV_\varepsilon(x) + 2V_\varepsilon(x)W_\varepsilon(x)g(|u|) - W_\varepsilon(x)^2g(|u|)^2 \\ & - \left( DW_\varepsilon(x)g(|u|) + \frac{g'(|u|)}{|u|}W_\varepsilon(x)\Re\langle u, Du \rangle \right). \end{aligned}$$

Letting

$$\operatorname{sgn} u = \begin{cases} \frac{\bar{u}}{|u|} & \text{if } u \neq 0; \\ 0 & \text{if } u = 0, \end{cases}$$

by Kato's inequality [10], there holds

$$\Delta|u| \geq \Re[\Delta u(\operatorname{sgn} u)].$$

Observe that

$$\Re\left[DV_\varepsilon(x)u\frac{\bar{u}}{|u|}\right] = 0$$

and

$$\Re\left[\left(DW_\varepsilon(x)g(|u|) + \frac{g'(|u|)}{|u|}W_\varepsilon(x)\Re\langle u, Du \rangle\right)u\frac{\bar{u}}{|u|}\right] = 0.$$

Hence

$$\Re\left[r_\varepsilon(x, |u|)u\frac{\bar{u}}{|u|}\right] = (2V_\varepsilon(x) - W_\varepsilon(x)g(|u|))W_\varepsilon(x)g(|u|)|u|.$$

We obtain

$$\Delta|u| \geq (a^2 - V_\varepsilon^2(x))|u| + (2V_\varepsilon(x) - W_\varepsilon(x)g(|u|))W_\varepsilon(x)g(|u|)|u|. \quad (30)$$

This, together with (17), implies in particular that there is  $\Lambda > 0$  satisfying

$$\Delta|u| \geq -\Lambda|u|.$$

It then follows from the sub-solution estimate [19,28] that

$$|u(x)| \leq C_0 \int_{B_1(x)} |u(y)| dy \quad (31)$$

with  $C_0$  independent of  $x$  and  $u \in \mathcal{H}_\varepsilon$ ,  $\varepsilon > 0$ , where  $B_1(x) = \{y: |y - x| \leq 1\}$ .

**Lemma 3.4.** Assume additionally that  $\nabla V$  and  $\nabla W$  are bounded. Let  $u_\varepsilon \in \mathcal{S}_\varepsilon$ . There is a maximum point  $y_\varepsilon$  of  $|u_\varepsilon|$  such that  $\lim_{\varepsilon \rightarrow 0} \text{dist}(\varepsilon y_\varepsilon, \mathcal{A}_V) = 0$ , and for any sequence  $\varepsilon y_\varepsilon \rightarrow y_0$ ,  $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$  converges in  $H^1$  to a least energy solution of

$$-i\alpha \cdot \nabla v + a\beta v + V(y_0)v = W(y_0)g(|v|)v. \quad (32)$$

If moreover  $\mathcal{V} \cap \mathcal{W} \neq \emptyset$  then  $\lim_{\varepsilon \rightarrow 0} \text{dist}(\varepsilon y_\varepsilon, \mathcal{V} \cap \mathcal{W}) = 0$ , and, up to subsequences,  $v_\varepsilon$  converges in  $H^1$  to a least energy solution of

$$-i\alpha \cdot \nabla v + a\beta v + \tau v = \pi g(|v|)v. \quad (33)$$

**Proof.** The proof will be carried out in several steps.

*Step 1.* Let  $\varepsilon_j \rightarrow 0$ ,  $u_j \in \mathcal{S}_j$  where  $\mathcal{S}_j = \mathcal{S}_{\varepsilon_j}$ . Then  $\{u_j\}$  is bounded. A concentration argument shows that there exist a sequence  $\{y'_j\} \subset \mathbb{R}^3$  and constants  $r > 0$ ,  $\delta > 0$  such that

$$\liminf_{j \rightarrow \infty} \int_{B_r(y'_j)} |u_j|^2 \geq \delta. \quad (34)$$

Set

$$v_j(x) = u_j(x + y'_j).$$

Then  $v_j$  solves, denoting  $\hat{V}_{\varepsilon_j}(x) = V(\varepsilon_j(x + y'_j))$  and  $\hat{W}_{\varepsilon_j}(x) = W(\varepsilon_j(x + y'_j))$ ,

$$-i\alpha \cdot \nabla v_j + a\beta v_j + \hat{V}_{\varepsilon_j}(x)v_j = \hat{W}_{\varepsilon_j}(x)g(|v_j|)v_j \quad (35)$$

with least energy (using the notations of the previous section)

$$\begin{aligned} \hat{c}_{\varepsilon_j} &= \hat{\Phi}_{\varepsilon_j}(v_j) \\ &:= \frac{1}{2}(\|v_j^+\|^2 - \|v_j^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^3} \hat{V}_{\varepsilon_j}(x)|v_j|^2 - \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x)G(|v_j|) \\ &= \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x)\tilde{G}(|v_j|), \end{aligned}$$

where (and below)

$$\tilde{G}(|u|) := \frac{1}{2}g(|u|)|u|^2 - G(|u|).$$

Plainly,

$$\hat{c}_{\varepsilon_j} = \hat{\Phi}_{\varepsilon_j}(v_j) = \Phi_{\varepsilon_j}(u_j) = c_{\varepsilon_j}.$$

Additionally,  $v_j \rightharpoonup v$  in  $E$  and  $v_j \rightarrow v$  in  $L_{loc}^q$  for  $q \in [1, 3)$ .

Since  $V$  and  $W$  are bounded, we can assume without loss of generality that  $V(\varepsilon_j y'_j) \rightarrow V_0$  and  $W(\varepsilon_j y'_j) \rightarrow W_0$  as  $j \rightarrow \infty$ . Since  $\nabla V$  is bounded:  $|\nabla V(x)| \leq \lambda_3$ , one sees that, given arbitrarily  $r > 0$ , for any  $x \in B_r(0)$ ,

$$|V(\varepsilon_j x + \varepsilon_j y'_j) - V(\varepsilon_j y'_j)| = \left| \int_0^1 \nabla V(\varepsilon_j y'_j + s\varepsilon_j x) \varepsilon_j x ds \right| \leq \varepsilon_j \lambda_3 r.$$

This implies that  $\hat{V}_{\varepsilon_j}(x) \rightarrow V_0$  as  $j \rightarrow \infty$  uniformly on bounded sets of  $x$ . Similarly,  $\hat{W}_{\varepsilon_j}(x) \rightarrow W_0$  as  $j \rightarrow \infty$  uniformly on bounded sets of  $x$ . It then follows from (35) that, for any  $\varphi \in C_0^\infty$ ,

$$\begin{aligned} 0 &= \lim_{j \rightarrow \infty} \int_{\mathbb{R}^3} \langle (-i\alpha \cdot \nabla + a\beta + \hat{V}_{\varepsilon_j}(x))v_j - \hat{W}_{\varepsilon_j}(x)g(|v_j|)v_j, \varphi \rangle \\ &= \int_{\mathbb{R}^3} \langle (-i\alpha \cdot \nabla + a\beta + V_0)v - W_0g(|v|)v, \varphi \rangle, \end{aligned}$$

consequently,  $v$  solves

$$-i\alpha \cdot \nabla v + a\beta v + V_0 v = W_0 g(|v|)v \quad (36)$$

with the energy

$$\begin{aligned} \Gamma_{V_0 W_0}(v) &:= \frac{1}{2} (\|v^+\|^2 - \|v^-\|^2 + V_0 |v|^2) - \int_{\mathbb{R}^3} W_0 G(|v|) \\ &= \int_{\mathbb{R}^3} W_0 \bar{G}(|v|) \geq \gamma_{V_0 W_0} \end{aligned}$$

(since  $\gamma_{V_0 W_0}$  denotes the least energy of (36)). By Fatou's lemma,

$$\int_{\mathbb{R}^3} W_0 \bar{G}(|v|) \leq \lim_{j \rightarrow \infty} \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) \bar{G}(|v_j|)$$

which, jointly with Lemma 2.12, implies

$$\Gamma_{V_0 W_0}(v) \leq \lim_{j \rightarrow \infty} c_{\varepsilon_j} \leq \gamma_{V_0 W_0}.$$

Therefore,

$$\lim_{j \rightarrow \infty} c_{\varepsilon_j} = \Gamma_{V_0 W_0}(v) = \gamma_{V_0 W_0} \quad (37)$$

and

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) \bar{G}(|v_j|) = \int_{\mathbb{R}^3} W_0 \bar{G}(|v|) = \gamma v_0 w_0.$$

Let  $\eta : [0, \infty) \rightarrow [0, 1]$  be a smooth function satisfying  $\eta(s) = 1$  if  $s \leq 1$ ,  $\eta(s) = 0$  if  $s \geq 2$ . Define  $\tilde{v}_j(x) = \eta(2|x|/j)v(x)$ . One has

$$\|v - \tilde{v}_j\| \rightarrow 0 \quad \text{and} \quad |v - \tilde{v}_j|_q \rightarrow 0 \quad \text{as } j \rightarrow \infty \quad (38)$$

for  $q \in [2, 3]$ . Setting  $z_j = v_j - \tilde{v}_j$ , it is not difficult to verify that along a subsequence,

$$\lim_{j \rightarrow \infty} \left| \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) (G(|v_j|) - G(|z_j|) - G(|\tilde{v}_j|)) \right| = 0 \quad (39)$$

and

$$\lim_{j \rightarrow \infty} \left| \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) (g(|v_j|)v_j - g(|z_j|)z_j - g(|\tilde{v}_j|)\tilde{v}_j) \varphi \right| = 0 \quad (40)$$

uniformly in  $\varphi \in E$  with  $\|\varphi\| \leq 1$  (see [2,14,12]). Using the exponentially decay of  $v$ , (38), and the facts that  $\hat{V}_{\varepsilon_j}(x) \rightarrow V_0$ ,  $\hat{W}_{\varepsilon_j}(x) \rightarrow W_0$  as  $j \rightarrow \infty$  uniformly on any bounded set of  $x$ , one checks easily the following

$$\int_{\mathbb{R}^3} \hat{V}_{\varepsilon_j}(x) \langle v_j, \tilde{v}_j \rangle \rightarrow \int_{\mathbb{R}^3} V_0 |v|^2; \quad \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) G(|\tilde{v}_j|) \rightarrow \int_{\mathbb{R}^3} W_0 G(v),$$

consequently,

$$\begin{aligned} \hat{\Phi}_{\varepsilon_j}(z_j) &= \hat{\Phi}_{\varepsilon_j}(v_j) - \Gamma_{V_0 W_0}(v) \\ &\quad + \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) (G(|v_j|) - G(|z_j|) - G(|\tilde{v}_j|)) + o(1) \\ &= o(1) \end{aligned}$$

as  $j \rightarrow \infty$ , which implies that  $\hat{\Phi}_{\varepsilon_j}(z_j) \rightarrow 0$ . Similarly,

$$\begin{aligned} \hat{\Phi}'_{\varepsilon_j}(z_j) \varphi &= \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) (g(|v_j|)v_j - g(|z_j|)z_j - g(|\tilde{v}_j|)\tilde{v}_j) \varphi + o(1) \\ &= o(1) \end{aligned}$$

as  $j \rightarrow \infty$  uniformly in  $\|\varphi\| \leq 1$ , which implies that  $\hat{\Phi}'_{\varepsilon_j}(z_j) \rightarrow 0$ . Therefore,

$$o(1) = \hat{\Phi}_{\varepsilon_j}(z_j) - \frac{1}{2} \hat{\Phi}'_{\varepsilon_j}(z_j) z_j = \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) \bar{G}(|z_j|).$$

This, together with  $(g_2)$ , shows

$$\int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) g(|z_j|) |z_j|^2 \rightarrow 0.$$

Notice that  $\{|z_j|_\infty\}$  is bounded so  $\int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) g(|z_j|) |z_j^+ - z_j^-|^2 \leq C^2$ . As a consequence, we get

$$\begin{aligned} \left(1 - \frac{\lambda_1}{a}\right) \|z_j\|^2 &\leq \|z_j\|^2 + \Re \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) \langle z_j, z_j^+ - z_j^- \rangle \\ &= \Phi'_{\varepsilon_j}(z_j) (z_j^+ - z_j^-) + \Re \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) g(|z_j|) \langle z_j, z_j^+ - z_j^- \rangle \\ &\leq o(1) + C \left( \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) g(|z_j|) |z_j|^2 \right)^{1/2} \\ &= o(1), \end{aligned}$$

that is,  $\|z_j\| \rightarrow 0$  which, together with (38), yields  $v_j \rightarrow v$  in  $E$  as  $j \rightarrow \infty$ .

In order to verify that  $v_j \rightarrow v$  in  $H^1$ , observe that by (35) and (36)

$$H_0 z_j = \hat{W}_{\varepsilon_j}(x) g(|v_j|) v_j - W_0 g(|v|) v - (\hat{V}_{\varepsilon_j}(x) v_j - V_0 v).$$

By the exponential decay of  $v$  and the uniform estimate (17), it is easy to show that  $|H_0 z_j|_2 \rightarrow 0$ . Therefore  $v_j \rightarrow v$  in  $H^1$ .

*Step 2.*  $v_j(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  uniformly in  $j \in \mathbb{N}$ . Assume by contradiction that the conclusion of the lemma does not hold. Then by (31) there exist  $\sigma > 0$  and  $x_j \in \mathbb{R}^3$  with  $|x_j| \rightarrow \infty$  such that  $\sigma \leq |v_j(x_j)| \leq C_0 \int_{B_1(x_j)} |v_j|$ . Since  $v_j \rightarrow v$  in  $H^1$  one gets

$$\begin{aligned} \sigma &\leq C_0 \int_{B_1(x_j)} |v_j| \leq C_0 \int_{B_1(x_j)} |v_j - v| + C_0 \int_{B_1(x_j)} |v| \\ &\leq C' \left( \int_{\mathbb{R}^3} |v_j - v|^2 \right)^{1/2} + C_0 \int_{B_1(x_j)} |v| \rightarrow 0, \end{aligned}$$

a contradiction.

*Step 3.*  $\{\varepsilon_j y'_j\}_j$  is bounded. Assume by contradiction that  $\varepsilon_j |y'_j| \rightarrow \infty$  (along a subsequence). Then  $V_0 > \tau$  and  $W_0 \leq \kappa$ , so  $c_{\varepsilon_j} \rightarrow \gamma_{V_0 W_0} \leq \gamma_{\tau \kappa}$  yielding a contradiction (see Lemma 2.11). Therefore, we can assume  $\varepsilon_j y'_j \rightarrow y_0$ .

$$V_0 = V(y_0), \quad W_0 = W(y_0), \quad (41)$$

and  $v$  is a least energy solution of (32). Now by Step 2 it is easy to see that one may assume that  $y_j = y'_j$  is a maximum point of  $|u_j|$ .

*Step 4.*  $\{\varepsilon y_\varepsilon\}_\varepsilon$  is bounded. Assume by contradiction that there is  $\varepsilon_j \rightarrow 0$  with  $\varepsilon_j |y_j| \rightarrow \infty$  where  $y_j$  is a maximum point of  $|u_j|$  ( $y_j = y_{\varepsilon_j}$ ,  $u_j = u_{\varepsilon_j}$ ). Repeating the above arguments one sees that the associate  $y'_j$  and  $v_j(x) = u_j(x + y'_j)$  satisfies that  $v_j(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  uniformly in  $j \in \mathbb{N}$  and

$\{\varepsilon_j y'_j\}$  is bounded (Steps 2) and 3)). Consequently,  $\varepsilon_j |y_j - y'_j| \geq \varepsilon_j |y_j| - \varepsilon_j |y'_j| \rightarrow \infty$ , particularly,  $|y_j - y'_j| \rightarrow \infty$ . Then,  $\max |u_j| = |u_j(y_j)| = |v_j(y_j - y'_j)| \rightarrow 0$ , a contradiction.

Step 5).  $\lim_{\varepsilon \rightarrow 0} \text{dist}(\varepsilon y_\varepsilon, \mathcal{A}_V) = 0$ . It is sufficient to check that  $y_0 \in \mathcal{A}_V$ . Assume indirectly that  $y_0 \notin \mathcal{A}_V$ . Then it is easy to see that  $\gamma_{V(y_0)W(y_0)} > \gamma_{\tau\kappa}$ , which, together with (37) and Lemma 3.1, implies

$$\lim_{\varepsilon \rightarrow 0} c_\varepsilon = \gamma_{V(y_0)W(y_0)} > \gamma_{\tau\pi} \geq \lim_{\varepsilon \rightarrow 0} c_\varepsilon,$$

a contradiction. Finally, assuming in addition that  $\mathcal{V} \cap \mathcal{W} \neq \emptyset$ , one has  $\mathcal{A}_V = \mathcal{V} \cap \mathcal{W}$ , so  $\lim_{\varepsilon \rightarrow 0} \text{dist}(\varepsilon y_\varepsilon, \mathcal{V} \cap \mathcal{W}) = 0$  and  $v_\varepsilon$  converges in  $H^1$  to a least energy solution of (33). The proof is hereby complete.  $\square$

**Lemma 3.5.** *There exists  $C > 0$  such that for all  $j \in \mathbb{N}$*

$$|u_j(x)| \leq C e^{-\sqrt{\omega/2}|x-y_j|}, \quad \forall j \in \mathbb{N},$$

where  $\omega = a^2 - |V|_\infty^2$ .

**Proof.** By Step 2 of the proof of Lemma 3.4 we may take  $\delta > 0$  and  $r > 0$  such that  $|v_j(x)| \leq \delta$  and

$$\left| \Re \left[ r_{\varepsilon_j}(x, |v_j|) v_j \frac{\bar{v}_j}{|v_j|} \right] \right| \leq \frac{\omega}{2} |v_j|$$

for all  $|x| \geq r$ ,  $j \in \mathbb{N}$ . This, together with (30), implies

$$\Delta |v_j| \geq \frac{\omega}{2} |v_j| \quad \text{for all } |x| \geq r, j \in \mathbb{N}.$$

Let  $\Gamma(y) = \Gamma(y, 0)$  be a fundamental solution to  $-\Delta + \omega/2$  (see, e.g., [28]). Using the uniform boundedness, one may choose  $\Gamma$  so that  $|v_j(y)| \leq \frac{\omega}{2} \Gamma(y)$  holds on  $|y| = r$ , all  $j \in \mathbb{N}$ . Let  $z_j = |v_j| - \frac{\omega}{2} \Gamma$ . Then

$$\begin{aligned} \Delta z_j &= \Delta |v_j| - \frac{\omega}{2} \Delta \Gamma \\ &= \frac{\omega}{2} \left( |v_j| - \frac{\omega}{2} \Gamma \right) = \frac{\omega}{2} z_j. \end{aligned}$$

By the maximum principle we can conclude that  $z_j(y) \leq 0$  on  $|y| \geq r$ . It is well known that there is  $C' > 0$  such that  $\Gamma(y) \leq C' \exp(-\sqrt{\omega/2}|y|)$  on  $|y| \geq 1$ . We see that

$$|v_j(y)| \leq C \exp(-\sqrt{\omega/2}|y|)$$

for all  $y \in \mathbb{R}^3$  and all  $j \in \mathbb{N}$ , that is,

$$|u_j(x)| \leq C \exp(-\sqrt{\omega/2}|x - y_j|)$$

for all  $x \in \mathbb{R}^3$  and all  $j \in \mathbb{N}$ .  $\square$

**Proof of Theorem 1.1(A).** Writing  $\varepsilon = \varepsilon_j$  and going back to Eq. (4) with the variable substitution  $x \mapsto x/\varepsilon$ ,  $w_\varepsilon(x) := u_\varepsilon(x/\varepsilon)$  is a semi-classical solution of (4) with least energy and  $w_\varepsilon \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$

for all  $q \geq 2$  by Lemma 3.2. Finally, it is clear that  $x_\varepsilon := \varepsilon y_\varepsilon$  is a maximum point of  $|w_\varepsilon|$  and the proof of (A) of Theorem 1.1 is completed by Lemmas 3.4 and 3.5.  $\square$

**Remark 3.6.** For any  $y \in \mathcal{A}_c$  set  $\hat{V}(x) = V(x + y)$  and  $\hat{W}(x) = W(x + y)$ . Then  $\hat{V}_\varepsilon(x) = \hat{V}(\varepsilon x) = V(\varepsilon x + y)$  and  $\hat{W}_\varepsilon(x) = W(\varepsilon x + y)$  satisfy the assumption  $(\star)$  (see Section 2.3) with  $\hat{V}_\varepsilon(x) \rightarrow \mu = V(y)$  and  $\hat{W}_\varepsilon(x) \rightarrow v = W(y)$  uniformly on any bounded set of  $x$ . The associate equation (20) possesses the least energy  $\hat{c}_\varepsilon = c_\varepsilon$ . By Lemma 2.9 we have

$$\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq \gamma_{V(y)W(y)}.$$

Therefore, it follows from (37) and (41) that

$$\gamma_{V(y_0)W(y_0)} \leq \gamma_{V(y)W(y)}.$$

This implies that  $\gamma_{V(y_0)W(y_0)} = \gamma_{V(y_v)W(y_v)}$  (see Remark 1.3). This proves that  $\text{dist}(x_\varepsilon, \Omega_v) = 0$  and  $v_\varepsilon$  converges to a least energy solution of (9).

### 3.2. The case with $(P_0)$ and $(P_2)$

As before, setting  $\tilde{V}(x) = V(x + x_w)$  and  $\tilde{W}(x) = W(x + x_w)$ ,  $\tilde{w}(x)$  solves

$$-i\varepsilon\alpha \cdot \nabla \tilde{w} + a\beta \tilde{w} + \tilde{V}(x)\tilde{w} = \tilde{W}(x)g(|\tilde{w}|)\tilde{w},$$

if and only if  $w(x) := \tilde{w}(x - x_w)$  solves (4). So we can assume that  $x_w = 0 \in \mathcal{W}$ . Note that  $\kappa := V(0) \geq \tau$  and  $\pi = W(0)$ .

Consider the problem (10). We have as Lemma 3.1 the following

**Lemma 3.7.**  $\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq \gamma_{\kappa\pi}$ . And  $\lim_{\varepsilon \rightarrow 0} c_\varepsilon = \gamma_{\kappa\pi}$  if additionally  $\mathcal{V} \cap \mathcal{W} \neq \emptyset$ .

Quite similar to Lemma 3.3 one gets the existence result.

**Lemma 3.8.**  $c_\varepsilon$  is attained for all small  $\varepsilon > 0$ .

**Proof.** We are sketchy. Let  $u_n \in \mathcal{N}_\varepsilon$  be a minimization sequence:  $I_\varepsilon(u_n) \rightarrow c_\varepsilon$ , and set  $w_n = u_n + h_\varepsilon(u_n)$ . Then is a  $(PS)_{c_\varepsilon}$  sequence for  $\Phi_\varepsilon$  with  $w_n \rightharpoonup w_\varepsilon = w_\varepsilon^+ + w_\varepsilon^- \in \mathcal{H}_\varepsilon$  in  $E$ . It suffices to show that  $w_\varepsilon \neq 0$  for all  $\varepsilon > 0$  small. Assume by contradiction that there is a sequence  $\varepsilon_j \rightarrow 0$  with  $w_{\varepsilon_j} = 0$ . Then  $w_n \rightarrow 0$  in  $E$ ,  $u_n \rightarrow 0$  in  $L_{loc}^q$  for  $q \in (1, 3)$ , and  $w_n(x) \rightarrow 0$  a.e. in  $x \in \mathbb{R}^3$ . Choose  $\mu = \kappa$  and  $\limsup_{|x| \rightarrow \infty} W(x) < v < \pi$  and consider the functional  $\Phi^{\mu v}$ . Let  $t_n > 0$  be such that  $t_n u_n \in \mathcal{N}_{\varepsilon_j}^{\mu v}$ . One may assume  $t_n \rightarrow t_0$  as  $n \rightarrow \infty$ . By  $(P_3)$  the set  $A_\varepsilon := \{x \in \mathbb{R}^3 : V_\varepsilon(x) < \kappa \text{ or } W_\varepsilon(x) > v\}$  is bounded. Additionally,  $\Phi_{\varepsilon_j}(t_n u_n + h_{\varepsilon_j}^{\mu v}(t_n u_n)) \leq I_{\varepsilon_j}(u_n)$  by virtue of Lemma 2.10. We obtain

$$\begin{aligned} c_{\varepsilon_j}^{\mu v} &\leq I_{\varepsilon_j}^{\mu v}(t_n u_n) \leq I_{\varepsilon_j}(u_n) + \frac{1}{2} \int_{A_{\varepsilon_j}} (\mu - V_{\varepsilon_j}(x)) |t_n u_n + h_{\varepsilon_j}^{\mu v}(t_n u_n)|^2 \\ &\quad + \int_{A_{\varepsilon_j}} (W_{\varepsilon_j}(x) - v) G(|t_n u_n + h_{\varepsilon_j}^{\mu v}(t_n u_n)|) \\ &= c_{\varepsilon_j} + o(1) \end{aligned}$$



as  $n \rightarrow \infty$ , hence,  $c_{\varepsilon_j}^{\mu\nu} \leq c_{\varepsilon_j}$ . This, together with (27), implies  $\gamma_{\mu\nu} \leq c_{\varepsilon_j}^{\mu\nu} \leq c_{\varepsilon_j}$ . In virtue of Lemma 3.7, letting  $j \rightarrow \infty$  yields  $\gamma_{\mu\nu} \leq \gamma_{\mu\pi}$ , contradicting with  $\gamma_{\mu\pi} < \gamma_{\mu\nu}$ .  $\square$

Finally, arguing along the lines carried out in the proofs of Lemmas 3.4 and 3.5 with obvious modifications one gets the following lemma.

**Lemma 3.9.** *We have the following conclusions.*

- (i) *Assume additionally that  $\nabla V$  and  $\nabla W$  are bounded. Let  $u_\varepsilon \in \mathcal{S}_\varepsilon$ . There is a maximum point  $y_\varepsilon$  of  $|u_\varepsilon|$  such that  $\lim_{\varepsilon \rightarrow 0} \text{dist}(\varepsilon y_\varepsilon, \mathcal{A}_W) = 0$ , and for any sequence  $\varepsilon y_\varepsilon \rightarrow y_0$ ,  $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$  converges in  $H^1$  to a least energy solution of*

$$-i\alpha \cdot \nabla v + a\beta v + V(y_0)v = W(y_0)|v|^{p-2}v.$$

*If moreover  $\mathcal{V} \cap \mathcal{W} \neq \emptyset$  then  $\lim_{\varepsilon \rightarrow 0} \text{dist}(\varepsilon x_\varepsilon, \mathcal{V} \cap \mathcal{W}) = 0$ , and, up to subsequences,  $v_\varepsilon$  converges in  $H^1$  to a least energy solution of*

$$-i\alpha \cdot \nabla v + a\beta v + \tau v = \pi|v|^{p-2}v.$$

- (ii) *There exists  $C > 0$  such that*

$$|u_\varepsilon(x)| \leq Ce^{-\sqrt{\omega/2}|x-y_\varepsilon|}$$

*for all  $x \in \mathbb{R}^3$  where  $\omega = a^2 - |V|_\infty^2$ .*

**Proof of Theorem 1.1(B).** Going back to Eq. (4) with the variable substitution  $x \mapsto x/\varepsilon$ ,  $w_\varepsilon(x) := u_\varepsilon(x/\varepsilon)$ , one obtains Theorem 1.1(B).  $\square$

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