



Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Stability of stationary fronts in a non-linear wave equation with spatial inhomogeneity

Christopher J.K. Knight^{a,*}, Gianne Derks^a, Arjen Doelman^b, Hadi Susanto^c

^a Department of Mathematics, University of Surrey, Guildford, Surrey, GU2 7XH, United Kingdom

^b Mathematisch Instituut, Leiden University, P.O. Box 9512, 2300 RA, Leiden, The Netherlands

^c School of Mathematical Sciences, University of Nottingham, University Park, Nottingham, NG7 2RD, United Kingdom

ARTICLE INFO

Article history:

Received 17 April 2012

Available online 27 August 2012

Keywords:

Nonlinear wave equations

Inhomogeneities

Fronts

Stability

ABSTRACT

We consider inhomogeneous non-linear wave equations of the type $u_{tt} = u_{xx} + V'(u, x) - \alpha u_t$ ($\alpha \geq 0$). The spatial real axis is divided in intervals I_i , $i = 0, \dots, N + 1$ and on each individual interval the potential is homogeneous, i.e., $V(u, x) = V_i(u)$ for $x \in I_i$. By varying the lengths of the middle intervals, typically one can obtain large families of stationary front or solitary wave solutions. In these families, the lengths are functions of the energies associated with the potentials V_i . In this paper we show that the existence of an eigenvalue zero of the linearisation operator about such a front or stationary wave is related to zeroes of the determinant of a Jacobian associated to the length functions. Furthermore, the methods by which the result is obtained is fully constructive and can subsequently be used to deduce the stability and instability of stationary fronts or solitary waves, as will be illustrated in examples.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

The existence and stability of stationary or travelling fronts or solitary waves in the non-linear wave equation (or non-linear Klein–Gordon equation)

$$u_{tt} = u_{xx} + V'(u) \quad (1)$$

* Corresponding author.

E-mail addresses: c.j.knight@surrey.ac.uk (C.J.K. Knight), g.derks@surrey.ac.uk (G. Derks), doelman@math.leidenuniv.nl (A. Doelman), hadi.susanto@math.nottingham.ac.uk (H. Susanto).

is well known when the potential V does not depend explicitly on the spatial variable x , see for instance [7,32] and references therein. Indeed, the existence question can be analysed using dynamical systems techniques. Furthermore, Sturm–Liouville arguments give that if a front/solitary wave exists, it is stable when it is monotonic and unstable otherwise. However, if the potential has a spatial inhomogeneity, i.e.

$$u_{tt} = u_{xx} + \frac{\partial V}{\partial u}(u, x), \quad (2)$$

less is known about the existence and stability of fronts or solitary waves.

Various systems are modelled by non-linear wave equations. For instance, taking $V(u) = D(1 - \cos u)$, gives the sine-Gordon equation,

$$u_{tt} = u_{xx} - D \sin u,$$

which models various phenomena including molecular systems, dislocation of crystals and DNA processes [2,3,8,32,33]. It is also a fundamental model for Josephson junctions, two superconductors sandwiching a thin insulator [13]. In the case of Josephson junctions, the factor D represents the Josephson tunnelling critical current. In an ideal uniform Josephson junction, this is a constant. But if there are magnetic variations, e.g. because of non-uniform thickness of the width of the insulator or the insulator being comprised of materials with different magnetic properties next to each other, then the Josephson tunnelling critical current D will vary with the spatial variable x , leading to an inhomogeneous potential $V(u, x) = D(x)(1 - \cos u)$.

Most of the analytical and theoretical work on Josephson junctions with an inhomogeneous critical current consider localised variations in D , hence the inhomogeneity is described by delta-like functions [10,11,18,19,23]. Yet, in real experiments the inhomogeneities vary from a moderate to a large finite length [1,26,27,31]. In [17], the time-dependent dynamics of a travelling front, so-called (Josephson) fluxon, in the presence of a finite size defect is considered within the framework of perturbation theory (D is near 1); while the scattering of a fluxon on a finite length inhomogeneity is studied in [24]. A full analysis of the existence of stationary fluxons in long Josephson junctions with a finite length inhomogeneity is given in [5]. It is shown that often a plethora of solutions exists and a natural question arising from this paper is: “Which solutions are stable, where do changes of stability occur, and what type of changes are these?”

In this paper we generalise this question and study stability and changes of stability of stationary fronts or solitary waves in a general (damped) non-linear wave equation with non-local inhomogeneities in its potential. A front or solitary wave $u(x)$ is a smooth, C^1 , solution with (exponential) decay to a constant value for $x \rightarrow \pm\infty$. Usually a front has different endpoints at $\pm\infty$, while they are the same for a solitary wave. To avoid having to write front/solitary wave, we will use the term “front” to refer both to a front and a solitary wave. We will study inhomogeneous wave equations of the form

$$u_{tt} = u_{xx} + \frac{\partial}{\partial u} V(u, x; I_l, I_{m_1}, \dots, I_{m_n}, I_r) - \alpha u_t, \quad (3)$$

where $\alpha \geq 0$ is a constant damping coefficient and the potential $V(u, x; I_l, I_{m_1}, \dots, I_{m_n}, I_r)$ consists of smooth (C^3) functions $V_i(u)$, defined on a finite number of disjoint open intervals I_{m_i} on the real spatial axis such that $\mathbb{R} = \overline{\bigcup I_i}$. Thus I_l is the uttermost left interval, I_r is the uttermost right interval and $I_{m_1}, \dots, I_{m_n}$ are the middle intervals. For example, if there are four intervals, i.e., $\mathbb{R} = (-\infty, -L_1) \cup (-L_1, 0) \cup (0, L_2) \cup (L_2, \infty)$, then $V(u, x; \{I_i\})$ is given by

$$V(u, x; I_l, I_{m_1}, I_{m_2}, I_r) = \begin{cases} V_l(u), & x \in I_l = (-\infty, -L_1); \\ V_{m_1}(u), & x \in I_{m_1} = (-L_1, 0); \\ V_{m_2}(u), & x \in I_{m_2} = (0, L_2); \\ V_r(u), & x \in I_r = (L_2, \infty). \end{cases} \quad (4)$$

Note that without loss of generality, any potential defined on four intervals can be written as in (4), as it is always possible to fix the endpoint of the second interval to be at zero by translating the x variable and updating the end points of the other intervals accordingly. An example of such a potential, modelling a so-called $0-\pi$ Josephson junction [4] with a defect, is shown in Fig. 1. Here $V_l(u) = V_{m_2}(u) = \cos u + \gamma u$, $V_{m_1}(u) = \gamma u$ and $V_r(u) = -\cos u + \gamma u$ with $\gamma = 0.1$. We will consider this example in more detail in Section 5.2.

The existence of stationary fronts is governed by the ODE reduction

$$0 = u_{xx} + \frac{\partial}{\partial u} V(u, x; I_l, I_{m_1}, \dots, I_{m_n}, I_r). \quad (5)$$

Due to the discontinuity of V as a function of x , solutions of (5) can be maximally C^1 smooth. These C^1 solutions can be constructed by using a phase plane analysis of the various Hamiltonian ODEs

$$u_{xx} + V'_i(u) = 0, \quad x \in I_i,$$

plus boundary conditions to “match” the solutions and their derivatives at the end points of the intervals and get decay to fixed points of the uttermost left potential V_l for $x \rightarrow -\infty$ and uttermost right potential V_r for $x \rightarrow \infty$. For the existence of a stationary front it is necessary that the outer potentials V_l and V_r have a local maximum. Then the front in the unbounded uttermost left interval I_l is part of the unstable manifold of the fixed point in V_l , while in the unbounded uttermost right interval I_r , it is part of the stable manifold of the fixed point in V_r . Within each of the bounded middle intervals $I_{m_1}, \dots, I_{m_n}$, any solution can be associated with a particular energy or Hamiltonian (h_i). Generically it can be shown that if there exists a stationary front for the middle intervals ($I_{m_1}, \dots, I_{m_n}$), then there are nearby intervals for which stationary fronts also exist [5,29]. The lengths of the n middle intervals and the associated fronts can be parametrised by the values of the n Hamiltonians $\{h_i\}$. These length functions are denoted by $L_i(h_1, \dots, h_n)$; more details can be found in Section 2. An illustration is given in Fig. 2 in the case of two middle intervals and the potential V as sketched in Fig. 1.

Once the families of stationary fronts have been constructed, the next question is their stability. To determine the stability of a front $u(x; \{h_i\})$, we consider linear stability first. A solution of the wave equation (3) is written as $u(x, t) = u(x; \{h_i\}) + \widehat{\psi}(x, t)$; linearising about $u(x; \{h_i\})$ and using the spectral Ansatz $\widehat{\psi}(x, t) = e^{\lambda t} \psi(x)$ gives the eigenvalue problem

$$\mathcal{L}\psi = \Lambda\psi, \quad (6)$$

where $\Lambda := \lambda(\lambda + \alpha)$ and

$$\mathcal{L}(x; \{h_i\}) := D_{xx} + \frac{\partial^2}{\partial u^2} V(u(x; \{h_i\}), x; \{h_i\}), \quad x \in \bigcup I_i. \quad (7)$$

Here we are abusing notation in the potential V and use that the intervals I_{m_i} can be parametrised by their lengths and hence by the parameters $\{h_i\}$ (see Section 2). The natural domain for the linear operator $\mathcal{L}$ is $H^2(\mathbb{R})$, so we define $\mathcal{L}$ to have an eigenvalue Λ if there exist some eigenfunction $\psi \in H^2(\mathbb{R})$ such that (6) holds. The Sobolev Embedding Theorem implies that $H^2(\mathbb{R}) \subset C^1(\mathbb{R})$, so the eigenfunctions will be continuously differentiable. The operator $\mathcal{L}$ is self-adjoint, hence all eigenvalues Λ will be real. Furthermore, $\mathcal{L}$ is a Sturm–Liouville operator, thus Sturm’s Theorem [30,20] can be

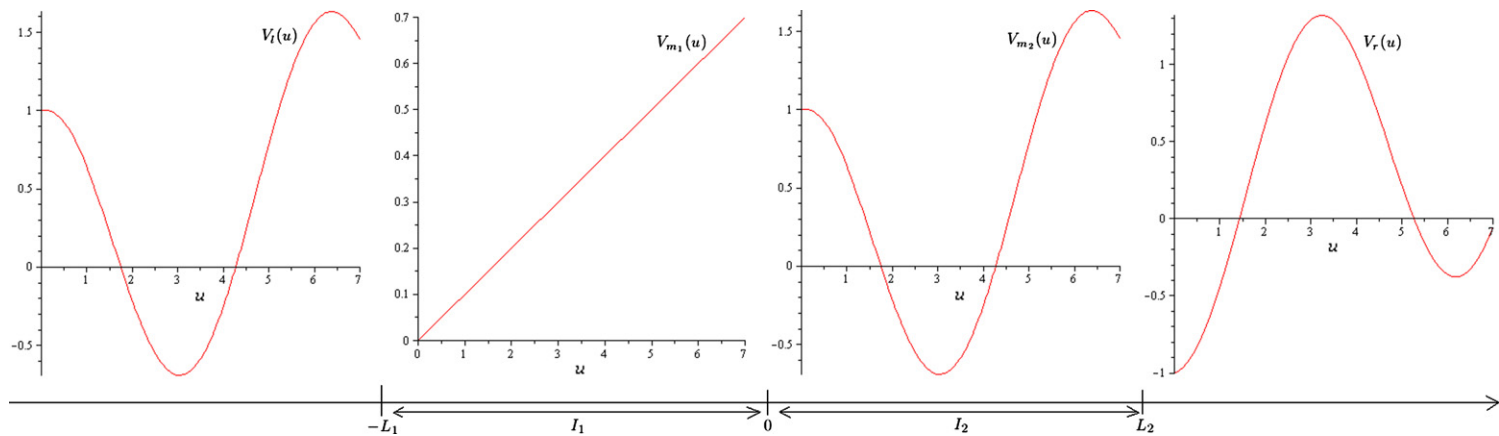


Fig. 1. An example of a non-homogeneous potential on four spatial intervals. This potential models a $0-\pi$ Josephson junction with a defect.

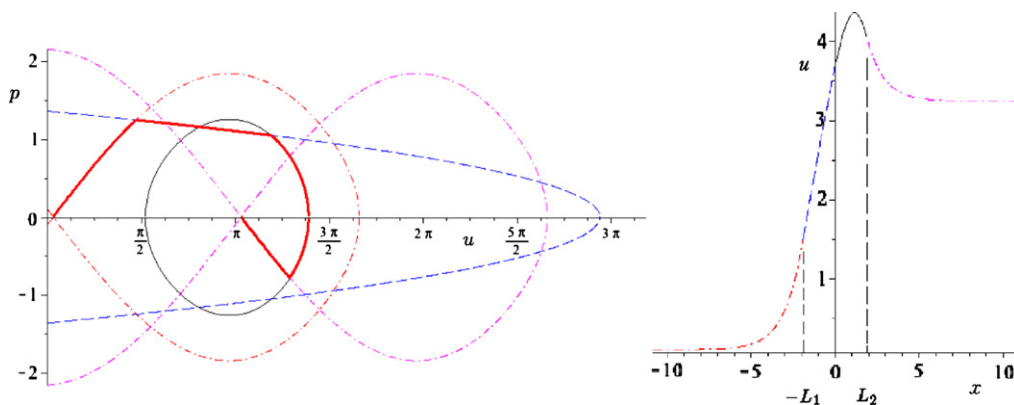


Fig. 2. The bold solid red line in the left phase portrait is an example of a stationary front. The dashed–dotted red line is the unstable manifold to $u_{-\infty} = \arcsin \gamma$ and the dashed–dotted magenta line is the unstable manifold to $u_{\infty} = \arcsin \gamma + 2\pi$. The dashed blue curve is the level set of the Hamiltonian $\frac{1}{2}u_x^2 + V_1(u) = h_1$ and the solid black closed orbit is the level set of the Hamiltonian $\frac{1}{2}u_x^2 + V_2(u) = h_2$. By varying these level sets, a full family with varying lengths can be obtained. On the right, the front is plotted in the x – u representation. The colours correspond to the colours used in the phase portrait. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

applied, leading to the fact that the eigenvalues of $\mathcal{L}$ are simple and bounded from above. And, if v_1 is an eigenfunction of $\mathcal{L}$ with eigenvalue Λ_1 and v_2 is an eigenfunction of $\mathcal{L}$ with eigenvalue Λ_2 with $\Lambda_1 > \Lambda_2$, then there is at least one zero of v_2 between any pair of zeroes of v_1 (including the zeroes at $\pm\infty$). Hence, the eigenfunction v_1 has a fixed sign (no zeroes) if and only if Λ_1 is the largest eigenvalue of $\mathcal{L}$. The continuous spectrum of $\mathcal{L}$ is determined by the system at $\pm\infty$. A short calculation shows that the continuous spectrum is the interval $(-\infty, -\sqrt{\min(-V''(u_{-\infty}), -V''(u_{+\infty}))})$, where $u_{\pm\infty}$ are the asymptotic states at $\pm\infty$ of the stationary front $u(x; \{h_i\})$. It is assumed that both asymptotic states are maxima of the potential, hence $V''(u_{\pm\infty}) < 0$.

Note that “the derivative of the wave” $\frac{\partial}{\partial x}u(x; \{h_i\})$ is in general only in $C^0(\mathbb{R})$. Thus – unlike the standard case of spatially homogeneous potentials – $\frac{\partial}{\partial x}u(x; \{h_i\})$ is not an eigenfunction and $\Lambda = 0$ is not the “translational” eigenvalue (there is no translational invariance). Another consequence of this is that non-monotonic waves (see for example Fig. 2) may be stable, again unlike in the homogeneous case.

If the largest eigenvalue Λ of $\mathcal{L}$ is not positive or if $\mathcal{L}$ does not have any eigenvalues, then the pinned fluxon is linearly stable, otherwise it is linearly unstable. This follows immediately from analysing the quadratic $\Lambda = \lambda(\lambda + \alpha)$ and recalling that $\alpha \geq 0$. Indeed, if $\Lambda > 0$ then there exists a $\lambda > 0$ such that $\Lambda = \lambda(\lambda + \alpha)$ and if $\Lambda < 0$ then $\Re(\lambda) < 0$ for all λ that satisfy $\Lambda = \lambda(\lambda + \alpha)$. Finally, if $\Lambda = 0$ the largest of the two λ ’s that satisfy $\Lambda = \lambda(\lambda + \alpha)$, is $\lambda = 0$. Furthermore, the λ -values associated with the continuous spectrum also have non-positive real part as the continuous spectrum of $\mathcal{L}$ is on the negative real axis. So we conclude that if the largest eigenvalue Λ_0 of $\mathcal{L}$ is positive then the front $u(x; \{h_i\})$ is linearly unstable; if $\Lambda_0 \leq 0$ then the front is linearly stable. Thus the eigenvalue $\Lambda_0 = 0$ in (6) is of particular importance, as this is the point where a change of stability can occur.

The linear stability can be used to show non-linear stability, see also [4,5]. The non-linear wave equation without dissipation is Hamiltonian. Indeed, define $P = u_t$, $U = (u, P)$, then formally Eq. (3) can be written as a Hamiltonian dynamical system with dissipation for x -dependent vector functions U in an infinite dimensional vector space, which is equivalent to $(H^1(\mathbb{R}) \cap L^1(\mathbb{R})) \times L^2(\mathbb{R})$:

$$\frac{d}{dt}U = J\delta\mathcal{H}(U) - \alpha\mathbf{D}U, \quad \text{with } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{aligned}\mathcal{H}(U) = & \frac{1}{2} \int_{-\infty}^0 [P^2 + u_x^2 + 2(V(u, x; \{I_i\}) - V_l(u_{-\infty}))] dx \\ & + \frac{1}{2} \int_0^{\infty} [P^2 + u_x^2 + 2(V(u, x; \{I_i\}) - V_r(u_{+\infty}))] dx.\end{aligned}\quad (8)$$

For any solution $u(x, t)$ of (3), we have

$$\frac{d}{dt} \mathcal{H}(U) = -\alpha \int_{-\infty}^{\infty} P^2 dx \leq 0. \quad (9)$$

As the front $u(x; \{h_i\})$ is a stationary solution, we have $D\mathcal{H}(u(x; \{h_i\}), 0) = 0$ and the Hessian of $\mathcal{H}$ about the front is

$$D^2\mathcal{H}(u(x; \{h_i\}), 0) = \begin{pmatrix} -\mathcal{L} & 0 \\ 0 & I \end{pmatrix}.$$

If $\mathcal{L}$ has only strictly negative eigenvalues, then it follows immediately that $(u(x; \{h_i\}), 0)$ is a minimum of the Hamiltonian and (9) gives that all solutions nearby the stationary front $u(x; \{h_i\})$ will stay nearby for all time.

In Section 2, we will give an overview of the results for existence of families of stationary fronts, especially the relation between the length of the intervals $\{I_i\}$ and the Hamiltonian parameters $\{h_i\}$. We will focus mainly on the case of one or two middle intervals, hence one or two length parameters, as more intervals can be dealt with in a similar way. After this section, the main part of this paper follows and it determines the relation between potential changes of linear stability (i.e., the existence of an eigenvalue zero of the linearisation operator $\mathcal{L}$) and (constrained) critical points of the interval length functions parametrised by the Hamiltonians $\{h_i\}$. First we will derive a necessary and sufficient condition for the existence of an eigenvalue zero. This condition is derived by constructing the eigenfunction related to the eigenvalue zero. However, even in case of only one middle interval, this condition is not very transparent. So we continue and show that the condition can be related to the determinant of the Jacobian of the length functions $L_i(h_1, \dots, h_n)$.

Since the construction of the eigenfunction associated to the zero eigenvalue is completely explicit, the number of zeroes of the eigenfunction can be determined. Hence it follows directly whether or not the eigenvalue $\Lambda = 0$ is the largest one or not. In other words, this explicit approach not only determines whether an eigenvalue Λ crosses the through zero, it also yields direct information about the stability type of the underlying wave.

In constructing the eigenfunction, it is found that there is a qualitative difference if there are non-simple zeroes in the spatial derivative of the front. This distinction comes from the fact that if the spatial derivative of the front has a non-simple zero in the interval I_j then the front $u(x; \{h_i\})$ is a constant function in the interval I_j and a fixed point of the dynamics in V_j (see Section 2 for more details). In the generic case that there is a family of stationary fronts with none of them having a non-simple zero in its spatial derivative, the relation between the existence of an eigenvalue zero of the linearisation operator $\mathcal{L}$ and the lengths of the middle intervals parametrised by the Hamiltonians can be summarised as:

The linearisation about a stationary front in a system with spatial inhomogeneities has an eigenvalue zero if and only if the determinant of the Jacobian of the lengths functions $L_i(h_1, \dots, h_n)$ vanishes.

The detailed formulation of this result and results for the non-generic situation can be found in Theorems 4.5, 4.7, 5.1, 5.3 and 6.1. In Section 4, a full analysis is given for one middle interval ($n = 1$); the case of two middle intervals ($n = 2$) is considered in Section 5; and the general case is in Section 6.

After obtaining the results about the existence of an eigenvalue zero of the linearisation operator and (constrained) critical points of the length functions, we illustrate their use by deriving the stability and instability of stationary fronts in two examples related to long Josephson junctions. The first example is a long Josephson junction with a microresistor defect. This system is modelled by a damped inhomogeneous wave equation with a potential with one middle interval ($n = 1$). The existence of stationary fronts for such systems is analysed in [5] and it is shown that a plethora of stationary fronts exist. As promised in [5], in this paper we will show that for each length for which there exist stationary fronts, there is exactly one stable one, which might be non-monotonic. This is different from the homogeneous case where all non-monotonic fronts are unstable.

The second example is a $0-\pi$ Josephson junction with a microresistor defect, the potential for such Josephson junction has two middle intervals ($n = 2$) and a typical example was shown in Fig. 1. In [4], the $0-\pi$ Josephson junction without defects is studied; it is shown that there exist both monotonic and non-monotonic fronts and only monotonic fronts are stable. In this paper we will show that a defect can stabilise a non-monotonic front-kink in the $0-\pi$ Josephson junction.

2. Existence of fronts

Before proceeding with the stability results, this section considers the families of stationary front solutions of (3) in more detail. As indicated in the Introduction, stationary fronts are solutions of a non-autonomous Hamiltonian ODE which in the limit for $x \rightarrow -\infty$ converge to a maximum of V_l and for $x \rightarrow +\infty$ converge to a maximum of V_r . Thus a stationary front is a solution to the following boundary value problem:

$$0 = u_{xx} + \frac{\partial V}{\partial u}(u, x; \{I_i\}) \quad \text{and} \quad \lim_{x \rightarrow \pm\infty} u(x) = u_{\pm\infty}. \quad (10)$$

Here $u_{-\infty}$ is a local maximum of $V_l(u)$ and $u_{+\infty}$ is a local maximum of $V_r(u)$.

We focus on the case where there are two middle intervals ($n = 2$), as it provides a good template for all other general cases. For $n = 2$, the potential is given by (4). From (10), it follows that the only explicit x -dependence in V comes from the positioning of the various potentials $V_i(u)$. Thus within each interval I_i , Eq. (10) is spatially homogeneous and the fronts solve a Hamiltonian ODE in each interval. Multiplying through by u_x in (10), writing $p = u_x$ and integrating with respect to x gives the following Hamiltonian description for the stationary fronts (recall that $\mathbb{R} = I_l \cup I_1 \cup I_2 \cup I_r$):

$$\begin{aligned} g &= \frac{1}{2}p^2 + V_{m_1}(u), \quad x \in I_l; & V_- &:= V_l(u_{-\infty}) = \frac{1}{2}p^2 + V_l(u), \quad x \in I_1; \\ h &= \frac{1}{2}p^2 + V_{m_2}(u), \quad x \in I_2; & V_+ &:= V_r(u_{+\infty}) = \frac{1}{2}p^2 + V_r(u), \quad x \in I_r. \end{aligned}$$

Here we use g and h for the (constant) values of the Hamiltonian in the intervals I_1 respectively I_2 to avoid needless subscripts. Solutions of the set of equations above can be described as the intersection of stable and unstable manifolds with orbits of the dynamics in the middle intervals. Indeed, the unstable manifold of the fixed point $u_{-\infty}$ in the left interval intersects an orbit of Hamiltonian system with potential V_{m_1} , whilst the stable manifold of the fixed point $u_{+\infty}$ in the right interval intersects an orbit of Hamiltonian system with potential V_{m_2} . These orbits in the middle intervals are parametrised by g and h respectively and generically there is a continuum of values of g and h which intersect the unstable manifold of $u_{-\infty}$ and stable manifold of $u_{+\infty}$ respectively. Note that the solutions of the i -th Hamiltonian system can typically lie on a closed orbit in the phase plane. As a consequence, orbits in the middle intervals may have “turning points”, i.e., points at which the u_x -component of the orbit vanishes. These points play a central role in the upcoming analysis.

Next we will show that the parametrisation of the orbits by g and h implies that we can (nearly always) parametrise the solutions in terms of g and h . That is, there is a region in the g - h plane for

which fronts $u(x; g, h)$, and length functions $L_1(g, h)$ and $L_2(g, h)$ can be defined such that $u(x; g, h)$ is a continuously differentiable solution of

$$\begin{aligned} u_x &= p; & g &= \frac{1}{2}p^2 + V_{m_1}(u), & \text{for } x \in I_1 = (-L_1, 0); \\ p_x &= -\frac{\partial V}{\partial u}(u, x; L_1(g, h), L_2(g, h)); & h &= \frac{1}{2}p^2 + V_{m_2}(u), & \text{for } x \in I_2 = (0, L_2); \\ \lim_{x \rightarrow \pm\infty} u(x) &= u_{\pm\infty}. \end{aligned} \quad (11)$$

In (11), L_1 and L_2 are defined implicitly by using that $-L_1$, 0 and L_2 are the points where the dynamics in the various intervals "join" up. At this point, it may be useful to explicitly note that we do not consider the interval lengths in (3) as given and thus fixed. In our approach, we consider the lengths L_i as parameters that may be varied. Of course, our methods in principle do allow us to study front solutions of (3) with prescribed values of the lengths L_i .

In order to see exactly how $L_1(g, h)$ and $L_2(g, h)$ are specified, we define the matching points to be (u_i, p_i) for $i = l, m, r$ (from left to right). That is $u(-L_1) = u_l$, $p(-L_1) = p_l$, $u(0) = u_m$, etc. The Hamiltonian formulation (11) gives that these points can be expressed as functions of the Hamiltonian parameters g and h :

$$\begin{aligned} \frac{1}{2}p_l^2 &= g - V_{m_1}(u_l) = V_- - V_l(u_l); \\ \frac{1}{2}p_m^2 &= g - V_{m_1}(u_m) = h - V_{m_2}(u_m); \\ \frac{1}{2}p_r^2 &= h - V_{m_2}(u_r) = V_+ - V_r(u_r). \end{aligned} \quad (12)$$

So (u_l, p_l) will be functions of g only, (u_r, p_r) will be functions of h only, and (u_m, p_m) will be functions of both g and h . In the first line of (12), the second equality defines a function $g(u_l)$. However the function of interest is $u_l(g)$. If for some $\hat{u}$, $V'_l(\hat{u}) = V'_{m_1}(\hat{u})$ and $V''_l(\hat{u}) \neq V''_{m_1}(\hat{u})$, then $g(u_l)$ has a turning point at $u_l = \hat{u}$ and there will be a bound on the g -values for which a solution exists. From now on we will assume that our potentials satisfy the non-degeneracy condition used above, i.e.,

Assumption 2.1. The potentials V_i , $i = l, 1, \dots, n, r$ are such that if at some point $\hat{u}$ we have $V'_i(\hat{u}) = V'_j(\hat{u})$ then $V''_i(\hat{u}) \neq V''_j(\hat{u})$.

Thus the curve $u_l(g)$ has a bifurcation point at any point $\hat{u}$ with $V'_l(\hat{u}) = V'_{m_1}(\hat{u})$. There will be a bifurcation point at $p_l = 0$ too as p_l is defined in terms of p_l^2 .

Using the same arguments at the other matching points gives the remaining bifurcation points. Thus we define the bifurcation functions $\mathcal{B}_i$, $i = l, m, r$,

$$\begin{aligned} \mathcal{B}_l(g) &= p_l(g)[V'_{m_1}(u_l(g)) - V'_l(u_l(g))], \\ \mathcal{B}_m(g, h) &= p_m(g, h)[V'_{m_2}(u_m(g, h)) - V'_{m_1}(u_m(g, h))], \\ \mathcal{B}_r(h) &= p_r(h)[V'_r(u_r(h)) - V'_{m_2}(u_r(h))]. \end{aligned} \quad (13)$$

If $\hat{g}$ and/or $\hat{h}$ are such that $\mathcal{B}_i(\hat{g}, \hat{h}) = 0$, then there is a degeneracy that must be studied carefully.

Lemma 2.1. Assume that there is a point $(\hat{u}_l, \hat{p}_l)$ such that the first set of equations of (12) are satisfied for some $g = \hat{g}$.

- If $\mathcal{B}_l(\hat{g}) \neq 0$ then there exists a neighbourhood of $\hat{g}$ in which unique smooth curve of left matching points $(u_l(g), p_l(g))$ can be defined which satisfy (12) and $(u_l(\hat{g}), p_l(\hat{g})) = (\hat{u}_l, \hat{p}_l)$.
- If $\mathcal{B}_l(\hat{g}) = 0$ and $(\hat{u}_l, \hat{p}_l) \neq (u_{-\infty}, 0)$ (i.e., $(\hat{u}_l, \hat{p}_l)$ is not a fixed point of the V_l -dynamics), then $\hat{g}$ is an edge of the existence interval for left matching points (u_l, p_l) . At one side of $\hat{g}$ two solutions curves of left matching points emerge from $(\hat{u}_l, \hat{p}_l)$ and there are none at the other side. The two solutions curves form one smooth curve in the (u, u_x) -plane.
- If $\mathcal{B}_l(\hat{g}) = 0$ and $(\hat{u}_l, \hat{p}_l) = (u_{-\infty}, 0)$, then there are two smooth solutions curves of left matching points in the (u, u_x) -plane, both containing the point $(\hat{u}_l, \hat{p}_l)$. If $V'_{m_1}(\hat{u}_l) \neq V'_l(\hat{u}_l)$, then these curves can be smoothly parametrised by g . If $V'_{m_1}(\hat{u}_l) = V'_l(\hat{u}_l)$, then $\hat{g}$ is an edge of the existence interval for the left matching points (u_l, p_l) .

An analogue result can be formulated for the right matching points. The proof and detailed local descriptions of the curves can be found in Appendix A.3 and Lemma A.2. The middle matching points are slightly more complicated as they depend on two variables.

Lemma 2.2. Assume that there is a point $(\hat{u}_m, \hat{p}_m)$ such that the middle set of equations of (12) are satisfied for some $g = \hat{g}$ and $h = \hat{h}$.

- If $\mathcal{B}_m(\hat{g}, \hat{h}) \neq 0$, then there exists a neighbourhood of $(\hat{g}, \hat{h})$ in which unique smooth functions of middle matching points $(u_m(g, h), p_m(g, h))$ can be defined which satisfy (12) and $(u_m(\hat{g}, \hat{h}), p_m(\hat{g}, \hat{h})) = (\hat{u}_m, \hat{p}_m)$.
- If $\mathcal{B}_m(\hat{g}, \hat{h}) = 0$ and $(\hat{u}_m, \hat{p}_m)$ is not a fixed point of the V_{m_1} -dynamics nor of the V_{m_2} -dynamics (i.e., if $\hat{p}_m = 0$, then $V'_{m_1}(\hat{u}_m) \neq 0$ and $V'_{m_2}(\hat{u}_m) \neq 0$), then nearby $\hat{g}$, then there exist unique smooth curves $\tilde{u}_m^h(g)$, $\tilde{p}_m^h(g)$, and $\tilde{h}(g)$ such that the middle equations of (12) are satisfied, $\mathcal{B}_m(g, \tilde{h}(g)) = 0$, and $\tilde{u}_m(\hat{g}) = \hat{u}$, $\tilde{p}_m(\hat{g}) = \hat{p}$, and $\tilde{h}(\hat{g}) = \hat{h}$. Furthermore, the curve $\tilde{h}(g)$ is bijective, hence near $\hat{h}$ there also exist unique smooth curves $\tilde{u}_m^g(h)$, $\tilde{p}_m^g(h)$, and $\tilde{g}(h)$ satisfying the criteria above and $\tilde{g}(\tilde{h}(g)) = g$, $\tilde{u}_m^g(\tilde{h}(g)) = \tilde{u}_m^h(g)$, etc.

Finally, the curve $\tilde{h}(g)$ (or equivalently $\tilde{g}(h)$) forms an edge of the existence region for middle matching points (u_m, p_m) . For each fixed g : if h is at one side of $\tilde{h}(g)$, then two solutions curves of left matching points emerge from $(\hat{u}_m, \hat{p}_m)$ and there are none at the other side. As before, these two solutions curves form one smooth curve in the (u, u_x) -plane.

Again, the proof and detailed local descriptions of the curves can be found in Appendix A.3 and Lemma A.5.

In this paper, we will assume that there exists a connected set in the (g, h) parameter space such that the matching points functions $(u_i(g, h), p_i(g, h))$, for $i = l, m, r$, are well defined for all (g, h) -values in this set. From the lemmas above, it follows that at the (g, h) -values on the boundary of this set, one or more of the bifurcation functions $\mathcal{B}_i$ vanishes. In the (u, u_x) phase space, the front solutions follow the g - and h -orbits (in the V_{m_1} - resp. V_{m_2} -dynamics) to connect the matching points (u_i, p_i) . The length functions $L_1(g, h)$ and $L_2(g, h)$ are determined by the “time of flight” (in x) along these orbits.

In order to find expressions for $L_1(g, h)$ and $L_2(g, h)$, we define functions $p_1(u, g)$ and $p_2(u, h)$ to be such that $p_1(u(x; g, h), g) = u_x(x; g, h)$ for any $x \in I_1$ and $p_2(u(x; g, h), h) = u_x(x; g, h)$ for any $x \in I_2$. Hence

$$\begin{aligned} [p_1(u, g)]^2 &= 2[g - V_{m_1}(u)], \quad \text{for } u \in \{u(x; g, h): x \in \overline{I_1}\}; \\ [p_2(u, h)]^2 &= 2[h - V_{m_2}(u)], \quad \text{for } u \in \{u(x; g, h): x \in \overline{I_2}\}; \end{aligned}$$

and the sign of p_i is determined by the position of $u(x)$ on the orbit: if u is increasing then p_i is defined to be positive and if u is decreasing p_i is defined to be negative. Thus the turning points of u are very important as these are points where the sign in the definition of p_i changes. Also the

expressions for the lengths $L_i(g, h)$ depend on how many turning points the function $u(x; g, h)$ has on the middle intervals I_i . If there are no turning points in either middle interval, then the expressions are straightforward and given by

$$L_1(g, h) = \int_0^{L_1} dx = \int_{u_l(g)}^{u_m(g, h)} \frac{du}{p_1(u, g)} \quad \text{and} \quad L_2(g, h) = \int_{u_m(g, h)}^{u_r(h)} \frac{du}{p_2(u, h)}. \quad (14)$$

However, if the function $u(x; g, h)$ has turning points in one or both middle intervals, the expressions get more complicated.

Let us assume that $u(x; g, h)$ has ν turning points in interval I_1 , denoted by x_i , for $i = 1, \dots, \nu$ (with $x_i < x_{i+1}$), and μ turning points in interval I_2 , denoted by x_i , for $i = \nu + 2, \dots, \nu + 1 + \mu$ (with $x_i < x_{i+1}$). In the next sections, we will use the notation $x_{\nu+1} = 0$ (recall that I_1 and I_2 join at $x = 0$), hence the “gap” in the counting. The length functions are then given by

$$\begin{aligned} L_1(g, h) &:= \int_{u_l(g)}^{u(x_1; g)} \frac{du}{p_1(u, g)} + \sum_{i=2}^{\nu} \int_{u(x_{i-1}; g)}^{u(x_i; g)} \frac{du}{p_1(u, g)} + \int_{u(x_{\nu}; g)}^{u_m(g, h)} \frac{du}{p_1(u, g)}; \\ L_2(g, h) &:= \int_{u_m(g, h)}^{u(x_{\nu+2}; h)} \frac{du}{p_2(u, h)} + \sum_{i=\nu+3}^{\mu+\nu+1} \int_{u(x_{i-1}; h)}^{u(x_i; h)} \frac{du}{p_2(u, h)} + \int_{u(x_{\mu+\nu+1}; h)}^{u_r(h)} \frac{du}{p_2(u, h)}. \end{aligned} \quad (15)$$

Hence, we see immediately

$$\frac{\partial L_1}{\partial h}(g, h) = \frac{1}{\mathcal{B}_m(g, h)} = \frac{\partial L_2}{\partial g}(g, h). \quad (16)$$

This fact will be used frequently later on.

It is important to realise that all of the integrals in (15) are positive: if $u(x_i) > u(x_{i-1})$ then $p_j(u) > 0$ for $u \in [u(x_{i-1}), u(x_i)]$ so $\int_{u(x_{i-1})}^{u(x_i)} \frac{du}{p_j(u)} > 0$. If however $u(x_i) < u(x_{i-1})$ then $p_j(u) < 0$ for $u \in [u(x_i), u(x_{i-1})]$ meaning $\int_{u(x_{i-1})}^{u(x_i)} \frac{du}{p_j(u)} = - \int_{u(x_i)}^{u(x_{i-1})} \frac{du}{p_j(u)} = \int_{u(x_i)}^{u(x_{i-1})} \frac{du}{-p_j(u)} > 0$. In the arguments above, we have implicitly assumed that (u_m, p_m) is not a fixed point of the V_{m_1} -dynamics or the V_{m_2} -dynamics. If $(\hat{g}, \hat{h})$ is such that $(u_m(\hat{g}, \hat{h}), p_m(\hat{g}, \hat{h}))$ is a fixed point of the V_{m_i} -dynamics, then a front can be constructed for any length L_i . This is a highly degenerate situation and such families will be considered separately.

The condition that (u_m, p_m) is not a fixed point of the V_{m_i} -dynamics, $i = 1, 2$, can be rephrased as the condition that the derivative of the front $u_x(x; g, h)$ does not have a non-simple zero in the middle intervals. Indeed, if $u_x(x)$ has a non-simple zero at $x = \hat{x} \in I_j$, then $u_x(\hat{x}) = 0 = u_{xx}(\hat{x})$ and $u(x)$ satisfies $u_{xx} + V'_j(u) = 0$ for all $x \in I_j$. Thus $V'_j(u(\hat{x})) = 0$ and $\hat{u} = u(\hat{x})$ is a fixed point of the dynamics in I_j . As V_j is smooth, we have uniqueness of solutions to initial value problems and hence $u(x) = \hat{u}$ for all $x \in I_j$. So the condition that (u_m, p_m) is not a fixed point of the V_{m_i} -dynamics, $i = 1, 2$, also implies that all turning points x_i , $i = 1, \dots, \nu, \nu + 2, \dots, \nu + \mu + 1$ correspond to simple roots of u_x .

It follows immediately from the construction above how to extend the results and expressions for the length functions to the case that the potential V in (10) has more than two middle intervals ($n > 2$). If the potential V in (10) has only one middle interval ($n = 1$) then setting $V_m := V_{m_1} = V_{m_2}$, $L := L_1 = L_2$, and $g = h$ gives the relevant relations for the matching points, bifurcation functions, and length functions; more details can be found in Section 4

3. Eigenfunctions in the kernel of the linearisation

In this section we look at the linearisation about a stationary fluxon whose derivative has only simple zeroes and construct the eigenfunction associated with a potential eigenvalue zero. This construction will involve matching conditions at the boundaries of each interval and leads to a compatibility criterion for the existence of an eigenvalue zero. Again, we will give the details in the case where there are two middle intervals as the other cases follow in an analogous way.

For an eigenvalue zero of the linearisation $\mathcal{L}$ to exist, there must be a function $\Psi \in H^2(\mathbb{R})$ such that $\mathcal{L}\Psi = 0$, hence with (7),

$$\Psi_{xx} + \frac{\partial^2 V(u, x; g, h)}{\partial u^2} \Psi = 0, \quad (17)$$

where $u = u(x; g, h)$ is the stationary front whose stability we are analysing. As was already remarked in the Introduction, the linearisation about a smooth stationary front u always has an eigenvalue zero with eigenfunction u_x for a homogeneous wave equation. However, for an inhomogeneous equation generically $u_x \notin H^2(\mathbb{R})$ and thus cannot be an eigenfunction. In spite of this, if zero is an eigenvalue of the linear operator $\mathcal{L}$, the function u_x still plays an important role in the eigenfunction.

To study (17), each interval is considered individually and then the resulting solutions are pieced together. The ODE is second order and Lipschitz continuous in u , hence by uniqueness of solutions there must be two linearly independent solutions in each interval. One of those solutions will be u_x . Note that $u_x \neq 0$ on any interval as it is assumed that u_x has no non-simple zeroes. Furthermore, the end points, $u_{\pm\infty}$, of the solution $u(x; g, h)$ are fixed points of V_i , $i = l$ respectively r , and are local maxima. Thus these points are saddle points of the ODEs $u_{xx} + V'_i(u) = 0$, $i = l$ respectively r , and we can conclude that the far field linearised system has exactly one exponentially decaying solution as $x \rightarrow \pm\infty$; this is the solution u_x .

In the middle interval(s), we also need a second solution. With the method of variation of constants it can be seen that the function

$$x \mapsto u_x(x) \int \frac{d\xi}{u_x^2(\xi)} \quad (18)$$

is a solution of (17) in each interval, which is linearly independent of u_x . Using the principle of superposition in each interval, the general eigenfunction can be constructed as a linear combination of u_x and the function in (18). However the integral expression in (18) is only defined on intervals in which u has no turning points. Generally this is not the case for an entire interval, so we split each interval in subintervals which don't contain zeroes of u_x . As in Section 2, we assume that there are ν turning points of the front u in interval I_1 , denoted by x_i for $i = 1, \dots, \nu$; and μ turning points in interval I_2 , denoted by x_i for $i = \nu + 2 \dots \nu + 1 + \mu$ with $x_i < x_{i+1}$. We also define $x_0 = -L_1$, $x_{\nu+1} = 0$, $x_{\nu+\mu+2} = L_2$ and intermediate points $M_i = \frac{x_i + x_{i+1}}{2}$, $i = 0, \dots, \nu + \mu + 1$. Note that $u_x(M_i) \neq 0$.

Thus if $\mathcal{L}$ has an eigenvalue zero, its eigenfunction $\Psi(x)$ is given by

$$\Psi(x) = \begin{cases} u_x(x), & x < -L_1 := x_0; \\ A_i u_x(x) + B_i u_x(x) \int_{M_i}^x \frac{d\xi}{u_x^2(\xi)}, & x_i < x < x_{i+1} \text{ for } i = 0, \dots, \nu; \\ A_i u_x(x) + B_i u_x(x) \int_{M_i}^x \frac{d\xi}{u_x^2(\xi)}, & x_i < x < x_{i+1} \text{ for } i = \nu + 1, \dots, \nu + \mu + 1; \\ \hat{k} u_x(x), & x > L_2 := x_{\nu+\mu+2}. \end{cases} \quad (19)$$

The parameters A_i , B_i , $i = 0, \dots, \nu + \mu + 1$ and $\hat{k}$ have to be such that $\Psi \in H^2$, hence also in C^1 . The condition of continuous differentiability at all points x_i , $i = 0, \dots, \nu + \mu + 2$ leads to $2(\nu + \mu + 3)$ conditions on the $2(\nu + \mu + 2) + 1$ constants A_i , B_i and $\hat{k}$. This difference between the number of

parameters and the number of constants will lead to a compatibility condition for the existence of the eigenvalue zero.

The extension of these observations to the general case of n middle intervals with $n > 2$ and the reduction to the case of one middle interval goes along similar lines. Details will follow in the later sections where the compatibility condition will be derived and linked to derivatives of the length function for the various cases.

In order to simplify notation in the following sections, functions $\tilde{G}$ and G_i are introduced. These functions are related to a regularisation of $\int \frac{d\xi}{u_x^2(\xi)}$ when the front u has a turning point (hence u_x vanishes) at one of the end points of integration or in the inside of the interval of integration respectively.

- For $x_{i-1} < a < x_{i+1}$ and $a \neq x_i$, define

$$\tilde{G}(a, x_i) := \int_a^{x_i} \left[\frac{1}{u_x^2(\xi)} - \frac{1}{u_{xx}^2(x_i)(\xi - x_i)^2} \right] d\xi - \frac{1}{(x_i - a)u_{xx}^2(x_i)}. \quad (20)$$

- For $x_{i-1} < a < b < x_{i+1}$ and $a \neq x_i \neq b$, define

$$G_i(a, b) := \tilde{G}(a, x_i) - \tilde{G}(b, x_i). \quad (21)$$

Note that by the assumption that u_x has only simple zeroes it follows that $u_{xx}(x_i) \neq 0$ if $u_x(x_i) = 0$. This is used to show the existence of the integral in the definition of $\tilde{G}$. For details showing these functions are well defined, see Lemma A.1.

The following identities hold for G_i and $\tilde{G}$ if a and b are such that there is no zero of u_x between them and follow by direct calculation.

Lemma 3.1. *If $x_{i-1} < a < b < x_i$ or $x_i < a < b < x_{i+1}$, then*

- $G_i(a, b) = \int_a^b \frac{d\xi}{u_x^2(\xi)}$;
- $\tilde{G}(a, x_i) = \int_a^{x_i} \frac{d\xi}{u_x^2(\xi)} + \tilde{G}(b, x_i)$.

4. Stability of three-interval stationary fronts

In this section we focus on the wave equation with one middle interval ($n = 1$) using the notation $L := L_1 = L_2$ and $V_m := V_{m_1} = V_{m_2}$, thus the potential satisfies $V(x) = V_m(x)$ for $x \in I_1 = (-L, L)$. Note that now $x = 0$ is the midpoint of the middle interval and there is no endpoint of an interval at $x = 0$ anymore. Initially we will look at a stationary front whose turning points are associated with simple zeroes in the derivative u_x . Let $u(x; g)$ be such a front with ν zeroes $x_1(g), \dots, x_\nu(g)$ in the middle interval. The expression for the eigenfunction associated with a potential eigenvalue zero of the linearisation about the stationary front as given in (19), reduces to

$$\psi(x) = \begin{cases} u_x(x), & x < -L; \\ A_i u_x(x) + B_i u_x(x) \int_{M_i}^x \frac{d\xi}{u_x^2(\xi)}, & x_i < x < x_{i+1} \text{ for } i = 0, \dots, \nu; \\ \hat{k} u_x(x), & x > L; \end{cases} \quad (22)$$

where $x_0 = -L$, $x_{\nu+1} = L$.

First we will derive the compatibility condition for the existence of an eigenvalue zero of the linearisation. This condition is not very transparent, so we will show that it can be expressed as a product of the derivative of the length curve and the bifurcation functions. After having analysed the

stationary fronts with simple zeroes in their derivative, we will look at stationary fronts with non-simple zeroes, i.e., stationary fronts for which one or both matching points are fixed points of one or more of the potentials V_l , V_r or V_m . We will finish this section with an example of a Josephson junction with an inhomogeneity to illustrate how the results can be used to determine the stability of stationary fronts.

4.1. The compatibility condition

The requirement that the potential eigenfunction Ψ , as given in (22), is in $H^2(\mathbb{R})$ (hence in $C^1(\mathbb{R})$) gives $2\nu + 4$ conditions involving the $2\nu + 3$ parameters, A_i , B_i and $\hat{k}$, by matching the function (value and derivative) at x_i , $i = 0, \dots, \nu + 1$. The conditions give $2\nu + 4$ linear equations for the $2\nu + 3$ parameters and lead to the compatibility condition stated below.

Lemma 4.1. *Let $u(x; g)$ be a stationary front solution of the wave equation (3) with one middle interval. If all zeroes of u_x are simple, then $\mathcal{L}$, the linearisation about the front u , has an eigenvalue zero with an eigenfunction in $H^2(\mathbb{R})$, if and only if*

$$\mathcal{B}_l + \mathcal{B}_r + \mathcal{B}_l \mathcal{B}_r C_\nu = 0.$$

Here $\mathcal{B}_l$ and $\mathcal{B}_r$ are the bifurcation functions defined in (13) and the constants C_ν are defined as follows (recall that ν is the number of turning points of the front u , or equivalent, the number of zeroes of u_x , in the middle interval)

- if $\nu = 0$ and $u_x(\pm L) \neq 0$ (u_x has no zero's in $[-L, L]$), then $C_0 = \int_{-L}^L \frac{d\xi}{u_x^2(\xi)}$;
- if $\nu \geq 1$ and $u_x(\pm L) \neq 0$, then

$$C_\nu = G_1(-L, M_0) + \sum_{i=1}^{\nu} G_i(M_{i-1}, M_i) + G_\nu(M_\nu, L), \quad \text{for } \nu \geq 1, \quad (23)$$

where we recall that $M_i = \frac{1}{2}(x_i + x_{i+1})$;

- if $u_x(-L) = 0$ or $u_x(L) = 0$ (hence $p_l = 0$ or $p_r = 0$), then $C_\nu = 0$ for any ν .

Note that if $u_x(-L) = 0$ or $u_x(L) = 0$, then the definition of C_ν does not matter for the compatibility condition of the lemma, as in this case $\mathcal{B}_l = 0$ or $\mathcal{B}_r = 0$. For $\nu > 0$, the definition of C_ν is a regularisation of the integral for C_0 .

Proof of Lemma 4.1. The proof is split in two parts. First we look at the case where $u_x(\pm L) \neq 0$.

As the front $u(x)$ solves the wave equation (10), u_{xx} satisfies

$$u_{xx} = -V'_l(u), \quad x < -L; \quad u_{xx} = -V'_m(u), \quad |x| < L; \quad u_{xx} = -V'_r(u), \quad x > L.$$

Using this and the definition of the parameters in the eigenfunction Ψ in (22), the continuity conditions for Ψ and Ψ_x at $x = -L$ give, after some rearranging,

$$B_0 = p_l[V'_m(u_l) - V'_l(u_l)] = \mathcal{B}_l \quad \text{and} \quad A_0 = 1 - B_0 \int_{M_0}^{-L} \frac{d\xi}{u_x^2(\xi)} = 1 + \mathcal{B}_l \int_{-L}^{M_0} \frac{d\xi}{u_x^2(\xi)}. \quad (24)$$

At the zeroes x_i , $i = 1, \dots, \nu$, we have that $u_x(x_i) = 0$. Since all the zeroes are simple, this implies $u_{xx}(x_i) \neq 0$. From Lemma A.1 (ii), it follows immediately that for any zero x_i , $i = 1, \dots, \nu$, we have

$$\lim_{\epsilon \downarrow 0} u_x(x_i - \epsilon) \int_{M_{i-1}}^{x_i - \epsilon} \frac{d\xi}{u_x^2(\xi)} = -\frac{1}{u_{xx}(x_i)} = \lim_{\epsilon \downarrow 0} u_x(x_i + \epsilon) \int_{M_i}^{x_i + \epsilon} \frac{d\xi}{u_x^2(\xi)}.$$

Using this, the continuity condition for Ψ at x_i gives

$$-\frac{B_{i-1}}{u_{xx}(x_i)} = -\frac{B_i}{u_{xx}(x_i)}, \quad \text{thus } B_i = B_{i-1} = \mathcal{B}_l, \text{ for } i = 1, \dots, \nu. \quad (25)$$

Again using Lemma A.1, it is seen that for $i = 1, \dots, \nu$

$$\lim_{\epsilon \downarrow 0} \left[u_{xx}(x_i - \epsilon) \int_{M_{i-1}}^{x_i - \epsilon} \frac{d\xi}{u_x^2(\xi)} + \frac{1}{u_x(x_i - \epsilon)} \right] = u_{xx}(x_i) \tilde{G}(M_{i-1}, x_i)$$

and

$$\lim_{\epsilon \downarrow 0} \left[u_{xx}(x_i + \epsilon) \int_{M_i}^{x_i + \epsilon} \frac{d\xi}{u_x^2(\xi)} + \frac{1}{u_x(x_i + \epsilon)} \right] = u_{xx}(x_i) \tilde{G}(M_i, x_i).$$

Thus the continuity condition for Ψ_x at x_i , $i = 1, \dots, \nu$, can be written as

$$A_{i-1} + B_{i-1} \tilde{G}(M_{i-1}, x_i) = A_i + B_i \tilde{G}(M_i, x_i).$$

From the definition of G_i and $B_i = \mathcal{B}_l$, $i = 1, \dots, \nu$, this implies the iterative relation

$$A_i = A_{i-1} + \mathcal{B}_l G_i(M_{i-1}, M_i) \quad \text{for } i = 1, \dots, \nu. \quad (26)$$

Finally, the continuity condition for Ψ at $x = L$ gives

$$\hat{k} = A_\nu + \mathcal{B}_l \int_{M_\nu}^L \frac{d\xi}{u_x^2(\xi)}$$

and the continuity condition for Ψ_x at $x = L$ gives that the compatibility condition for the existence of an eigenvalue zero is

$$\mathcal{B}_l = \hat{k} p_r [V'_m(u_r) - V'_r(u_r)] = -\hat{k} \mathcal{B}_r = -\mathcal{B}_r \left(A_\nu + \mathcal{B}_l \int_{M_\nu}^L \frac{d\xi}{u_x^2(\xi)} \right). \quad (27)$$

To re-write this as the expression in the lemma, we consider two cases:

- If u_x has no zeroes in the interval $[-L, L]$, i.e. $v = 0$, we have $x_0 = -L$, $x_1 = L$, $M_v = M_0 = \frac{-L+L}{2} = 0$ and $A_v = A_0$. Using the expression for A_0 the compatibility condition (27) becomes, upon rearrangement,

$$0 = \mathcal{B}_l + \mathcal{B}_r + \mathcal{B}_l \mathcal{B}_r \int_{-L}^L \frac{d\xi}{u_x^2(\xi)}.$$

- If u_x has one or more zero in the interval $[-L, L]$ then using the recursive relationship for A_i , (26), gives

$$A_v = \mathcal{B}_l \sum_{i=1}^v G_i(M_{i-1}, M_i) + A_0 = 1 + \mathcal{B}_l \left(\int_{-L}^{M_0} \frac{d\xi}{u_x^2(\xi)} + \sum_{i=1}^v G_i(M_{i-1}, M_i) \right). \quad (28)$$

Thus the compatibility condition (27) becomes

$$\mathcal{B}_l = -\mathcal{B}_r \left(1 + \mathcal{B}_l \left(\int_{-L}^{M_0} \frac{d\xi}{u_x^2(\xi)} + \sum_{i=1}^v G_i(M_{i-1}, M_i) + \int_{M_v}^L \frac{d\xi}{u_x^2(\xi)} \right) \right).$$

Using Lemma 3.1 to re-write the two integrals in this expression, the condition becomes

$$0 = \mathcal{B}_l + \mathcal{B}_r + \mathcal{B}_l \mathcal{B}_r \left(G_1(-L, M_0) + \sum_{i=1}^v G_i(M_{i-1}, M_i) + G_v(M_v, L) \right).$$

In both cases, the term multiplying $\mathcal{B}_l \mathcal{B}_r$ is defined as C_v , thus completing the proof in the case $u_x(\pm L) \neq 0$.

Next we consider the case when $u_x(-L) = 0$ or $u_x(L) = 0$, i.e., $p_l = 0$ or $p_r = 0$. Recall that if $u_x(-L) = 0$ then $V'_m(u(-L)) = -\lim_{\epsilon \downarrow 0} u_{xx}(-L + \epsilon) \neq 0$ and $V'_l(u(-L)) = -\lim_{\epsilon \downarrow 0} u_{xx}(-L - \epsilon) \neq 0$ as u_x only has simple zeroes. Similarly if $u_x(L) = 0$, $V'_m(u(L)) \neq 0 \neq V'_r(u(L))$. The proof is split up in to three parts:

- First the case when $p_l = 0$ and $p_r \neq 0$ is considered, thus $\mathcal{B}_l = 0$ and $V'_{m_1}(u_l) \neq 0$. As before, the continuity condition for Ψ at $-L$ implies that $B_0 = \mathcal{B}_l = 0$ (using Lemma A.1 (ii)) and the continuity condition for Ψ_x at $x = -L$ gives that $A_0 = \frac{V'_l(u_l)}{V'_m(u_l)}$. The continuity conditions for the other zeroes are as before and imply that $B_i = 0$ and $A_i = A_0$, for $i = 0, \dots, v$. Finally, the continuity conditions for Ψ and Ψ_x at $x = L$ give that $\hat{k} = A_0$ and $V'_r(u_r) = V'_m(u_r)$. Hence the compatibility condition becomes $\mathcal{B}_r = 0$, which can be written as $\mathcal{B}_l + \mathcal{B}_r + \mathcal{B}_l \mathcal{B}_r C_v = 0$ for any constant C_v (as $\mathcal{B}_l = 0$).
- Similarly if $p_r = 0$ and $p_l \neq 0$, hence $\mathcal{B}_r = 0$ and $V'_{m_1}(u_r) \neq 0$, $V'_r(u_r) \neq 0$, then $B_i = 0$ and $A_i = \frac{\hat{k} V'_r(u_r)}{V'_m(u_r)}$, for $i = 0, \dots, v$. The continuity conditions at $x = -L$ then imply that $\hat{k} = \frac{V'_m(u_r)}{V'_r(u_r)}$ and the compatibility condition is $V'_l(u_l) = V'_m(u_l)$, hence $\mathcal{B}_l = 0$.
- Finally if both $p_l = 0$ and $p_r = 0$, then $\mathcal{B}_l = 0 = \mathcal{B}_r$ and $V'_m(u_l) \neq 0$, $V'_r(u_r) \neq 0$. The two continuity conditions at L imply that $\hat{k} V'_r(u_r) = A_0 V'_m(u_r)$. This gives $\hat{k} = \frac{V'_m(u_r) V'_l(u_l)}{V'_r(u_r) V'_m(u_l)}$ and hence a non-trivial eigenfunction exists. $\square$

In the situation that one or both of the bifurcation functions vanishes, we can immediately draw a conclusion about the existence of the eigenvalue zero and its stability.

Corollary 4.2. *Under the conditions of Lemma 4.1: If exactly one of $\mathcal{B}_l$, $\mathcal{B}_r$ is zero then the linearisation $\mathcal{L}$ has no eigenvalue zero. If $\mathcal{B}_l = 0 = \mathcal{B}_r$ then the linearisation $\mathcal{L}$ has an eigenvalue zero whose eigenfunction is a multiple (with possibly a different multiplication factor in each interval) of $u_x(x)$. In this case, the eigenvalue zero is the largest eigenvalue if u_x has no zeroes. Thus we can conclude that if $\mathcal{B}_l = 0 = \mathcal{B}_r$ then the front u is linearly stable if and only if it is strictly monotonic (i.e. $\forall x \ u_x(x) \neq 0$).*

In the homogeneous case, $V_l = V_m = V_r$, so $\mathcal{B}_l = 0 = \mathcal{B}_r$. Thus this corollary recovers the well-known result that only monotonic fronts are stable in the homogeneous case (i.e. when there is no defect).

4.2. Variations of length

Next we will show that the constant C_v in Lemma 4.1 can be expressed differently by using the fact that the stationary fronts are not isolated, but part of a larger family parametrised by g . It will be shown that C_v is a multiple of $\frac{dL}{dg}$ and hence the linearisation about $u(x; \hat{g})$, i.e. $\mathcal{L}(\hat{g})$, has an eigenvalue zero if the length curve $L(g)$ has a critical point at $g = \hat{g}$.

For three intervals, the general expression for the length function as given in (15) becomes

$$2L(g) = \int_{u_l(g)}^{u(x_1(g); g)} \frac{du}{p(u, g)} + \sum_{i=1}^{v-1} \int_{u(x_i(g); g)}^{u(x_{i+1}(g); g)} \frac{du}{p(u, g)} + \int_{u(x_v(g); g)}^{u_r(g)} \frac{du}{p(u, g)}, \quad (29)$$

where as before $p^2(u, g) = 2[g - V_m(u)]$, for $u \in \{u(x; g): x \in \bar{I}_1\}$ and the sign of $p(u, g)$ is determined by the position of u on the orbit: if u is increasing then p is defined to be positive and if u is decreasing p is defined to be negative. In each of these integrals there are either one or two zeroes of p which only occur at the end points of integration. To analyse these expressions succinctly we further split an integral if an interval of integration contains two zeroes. Thus expressing the length function in terms of integrals with at most one zero of p at one of the end points of the interval of integration only. First we recall that all zeroes $x_i(g)$ are simple, so no pair can collide. Also the zeroes $x_1(g)$ and $x_v(g)$ can only leave the middle interval if $p_l(g) = 0$, $p_r(g) = 0$ respectively, i.e. at a bifurcation point. Thus for g away from a bifurcation point, the function $M_i(g) = \frac{x_i(g) + x_{i+1}(g)}{2}$ is smooth and lies between the two adjacent zeroes $x_i(g)$ and $x_{i+1}(g)$. Finally, $p \equiv u_x$ has a fixed sign for x between $x_i(g)$ and $M_i(g)$ and between $M_i(g)$ and $x_{i+1}(g)$. Thus the length functions can be written as a sum of integrals of the form

$$I_i^l(g) := \int_{u(M_i(g), g)}^{u(x_i(g), g)} \frac{du}{p(u, g)} \quad \text{and} \quad I_i^r(g) := \int_{u(M_i(g), g)}^{u(x_{i+1}(g), g)} \frac{du}{p(u, g)}. \quad (30)$$

The derivative of such integrals can be calculated using the following lemma.

Lemma 4.3. *If $x_i(g)$ is a simple zero of u_x and $y(g) \in (x_{i-1}(g), x_i(g))$ is a smooth function of g then*

$$I(g) := \int_{u(y(g), g)}^{u(x_i(g), g)} \frac{du}{p(u, g)}$$

is differentiable and

$$I'(g) = -\tilde{G}(y(g), x_i(g)) - \frac{1}{u_x(y(g), g)} \frac{d}{dg} u(y(g), g).$$

The proof of this lemma can be found in Appendix A.2. Applying this result to the integrals in (30), we get the derivatives for the integrals in the length function (29).

Lemma 4.4. For $i = 1, \dots, v - 1$ and $\tilde{g}$ not a bifurcation point

$$\frac{d}{dg} \int_{u(x_i; g)}^{u(x_{i+1}; g)} \frac{du}{p(u, g)} \Big|_{g=\tilde{g}} = \tilde{G}(M_i(\tilde{g}), x_i(\tilde{g})) - \tilde{G}(M_i(\tilde{g}), x_{i+1}(\tilde{g})).$$

Proof. If $\tilde{g}$ is not at a bifurcation point and is such that $x_i(\tilde{g})$ and $x_{i+1}(\tilde{g})$ exist for g near $\tilde{g}$, the zeroes $x_i(g)$ and $x_{i+1}(g)$ exist and are smooth functions of g . Thus Lemma 4.3 gives that

$$\frac{d}{dg} I_i^l(g) \Big|_{g=\tilde{g}} = -\tilde{G}(M_i(\tilde{g}), x_i(\tilde{g})) - \frac{1}{u_x(M_i(\tilde{g}), \tilde{g})} \frac{d}{dg} u(M_i(g), g) \Big|_{g=\tilde{g}}$$

and

$$\frac{d}{dg} I_i^r(g) \Big|_{g=\tilde{g}} = -\tilde{G}(M_i(\tilde{g}), x_{i+1}(\tilde{g})) - \frac{1}{u_x(M_i(\tilde{g}), \tilde{g})} \frac{d}{dg} u(M_i(g), g) \Big|_{g=\tilde{g}}.$$

As $\int_{u(x_i; g)}^{u(x_{i+1}; g)} \frac{du}{p(u, g)} = I_i^r(g) - I_i^l(g)$, this immediately implies the relation of the lemma. $\square$

These identities allow the existence condition for an eigenvalue zero in Lemma 4.1 to be related to the derivative of the length curve $L(g)$ with respect to g .

Theorem 4.5. Let the front $u(x; g)$ be a solution of Eq. (10), such that all zeroes of $u_x(x; g)$ are simple and the length of the middle interval of $u(x; g)$ is part of a smooth length curve $L(g)$. The linearisation operator $\mathcal{L}(g)$ associated with $u(x; g)$ has an eigenvalue zero if and only if

$$\mathcal{B}_l(g) \mathcal{B}_r(g) L'(g) = 0. \quad (31)$$

Since $L'(g)$ has a pole in a bifurcation point, i.e., if $\mathcal{B}_l = 0$ or $\mathcal{B}_r = 0$, this expression should be read as a limit in those cases. This gives

- if $\mathcal{B}_l(g) = 0$, then the condition becomes $\mathcal{B}_r(g) = 0$;
- if $\mathcal{B}_r(g) = 0$, then the condition becomes $\mathcal{B}_l(g) = 0$.

A stationary front $u(x; g)$ with a length of the middle interval that is not part of a smooth curve, but an isolated point instead, has a linearisation operator $\mathcal{L}(g)$ with an eigenvalue zero.

For all stationary fronts, if zero is an eigenvalue of the linearisation operator, then the eigenfunction is a multiple of $u_x(x; g)$ for $|x| > L$ (with possibly a different multiplication factor at each end).

Whilst on first inspection it might seem that (31) is automatically satisfied if g is a bifurcation point, this is not the case. It turns out (see Appendix A.3) that at a bifurcation point $L'(g)$ is unbounded. However it is still possible to take the limit of the bifurcation function multiplied by this derivative and this expression is bounded. For example, if $\hat{g}$ is a bifurcation point with $\mathcal{B}_l(\hat{g}) = 0$ then Lemma A.4 gives

$$\lim_{g \rightarrow \hat{g}} \mathcal{B}_l(g) L'(g) = -\frac{V'_l(u_l(\hat{g}))}{V'_m(u_l(\hat{g}))} \neq 0.$$

Note that this limit is a one-sided limit, as $\hat{g}$ is at the edge of an existence interval, see Lemma 2.1.

Theorem 4.5 and Corollary 4.2 immediately imply the following stability result.

Corollary 4.6. *Let $\{u(x; g)\}$ be a family of stationary fronts of Eq. (10) that form a smooth length curve $L(g)$. If all fronts $u(x; g)$ are such that their derivative $u_x(x; g)$ has only simple zeroes, then a change of stability can only occur at a critical point of the $L(g)$ curve or when $\mathcal{B}_l(g) = 0 = \mathcal{B}_r(g)$.*

Proof of Theorem 4.5. First we look at the generic case that $u(x; g)$ is such that the length of its middle interval is part of a smooth length curve $L(g)$ and that $\mathcal{B}_l(g) \neq 0 \neq \mathcal{B}_r(g)$. Hence g is not at the edge of the existence interval. By differentiating the relations in (12) for $u_l(g)$, $p_l(g)$, etc. with respect to g , and remembering that in this case h has been identified with g and $V_m = V_{m_1} = V_{m_2}$, we get

$$\begin{aligned} 0 &= p_l p_l'(g) + V_l'(u_l) u_l'(g), & 1 &= p_l p_l'(g) + V_m'(u_l) u_l'(g), \\ 1 &= p_r p_r'(g) + V_m'(u_r) u_r'(g), & 0 &= p_r p_r'(g) + V_r'(u_r) u_r'(g). \end{aligned}$$

Hence

$$1 = [V_m'(u_l) - V_l'(u_l)] u_l'(g) \quad \text{and} \quad 1 = [V_m'(u_r) - V_r'(u_r)] u_r'(g).$$

Furthermore, the function $p(u, g)$ satisfies $g = \frac{1}{2} p^2(u, g) + V_m(u)$; differentiating this relation with respect to g gives

$$1 = p(u, g) \frac{\partial p}{\partial g}(u, g).$$

Finally, the front values at the turning points $x_i(g)$ are denoted by $u_i(g) := u(x_i(g); g)$ and they satisfy $g - V_m(u_i(g)) = 0$ (using (11) and $u_x(x_i(g); g) = 0$). Differentiating this relation with respect to g gives

$$0 = 1 - V_m'(u_i(g)) u_i'(g), \quad \text{hence } u_i'(g) = \frac{1}{V_m'(u_i(g))} = -\frac{1}{u_{xx}(x_i(g))}.$$

As in the proof of the compatibility condition, we will consider different cases depending on the number of turning points of the front in the middle interval.

- In the case that the front $u(x; g)$ has no turning points in the middle interval, differentiating the expression for length $L(g)$, (29), with respect to g gives

$$2L'(g) = \frac{u_r'(g)}{p_r(g)} - \frac{u_l'(g)}{p_l(g)} - \int_{u_l(g)}^{u_r(g)} \frac{du}{p^2(u, g)} \frac{\partial p(u, g)}{\partial g}.$$

Substituting the expressions derived above for the various derivatives into this equation and changing the integration variable to x instead of u gives

$$2L'(g) = -\frac{1}{\mathcal{B}_r(g)} - \frac{1}{\mathcal{B}_l(g)} - \int_{-L(g)}^{L(g)} \frac{d\xi}{u_x^2(\xi, g)} = -\left(\frac{1}{\mathcal{B}_r(g)} + \frac{1}{\mathcal{B}_l(g)} + C_0 \right). \quad (32)$$

- In the case that the front $u(x; g)$ has $v \geq 1$ turning points in the middle interval, differentiating the length function (29) with respect to g and using Lemma 4.4 gives

$$\begin{aligned}
 2L'(g) &= -\frac{u'_l(g)}{p_l(g)} + \frac{u'_r(g)}{p_r(g)} - \tilde{G}(-L, x_1) + \tilde{G}(L, x_v) + \sum_{i=2}^v (\tilde{G}(M_{i-1}, x_{i-1}) - \tilde{G}(M_{i-1}, x_i)) \\
 &= -\frac{u'_l(g)}{p_l(g)} + \frac{u'_r(g)}{p_r(g)} - G_1(-L, M_0) - G_v(M_v, L) - \sum_{i=1}^v G_i(M_{i-1}, M_i) \\
 &= -\frac{u'_l(g)}{p_l(g)} + \frac{u'_r(g)}{p_r(g)} - C_v = -\left(\frac{1}{\mathcal{B}_l(g)} + \frac{1}{\mathcal{B}_r(g)} + C_v\right). \tag{33}
 \end{aligned}$$

In both of these cases multiplying through by $\mathcal{B}_l(g)\mathcal{B}_r(g)$ gives

$$2\mathcal{B}_l(g)\mathcal{B}_r(g)L'(g) = -(\mathcal{B}_l(g) + \mathcal{B}_r(g) + \mathcal{B}_l(g)\mathcal{B}_r(g)C_v),$$

which is zero if and only if $\mathcal{L}(g)$ has an eigenvalue zero by Lemma 4.1.

Next we look at the non-generic cases. If $g = \hat{g}$ is such that $\mathcal{B}_l(\hat{g}) = 0$ or $\mathcal{B}_r(\hat{g}) = 0$ then the result follows directly from Corollary 4.2. The fact that these results are limits of the general expression can be seen by using Lemma A.4 and noting that $\lim_{g \rightarrow \hat{g}} \mathcal{B}_l(g) \frac{dL(g)}{dg}$ and $\lim_{g \rightarrow \hat{g}} \mathcal{B}_r(g) \frac{dL(g)}{dg}$ are non-zero respectively.

If g is such that $u(x; g)$ is an isolated point, then both left and right bifurcation functions will vanish, i.e., $\mathcal{B}_l(g) = 0 = \mathcal{B}_r(g)$. Thus Corollary 4.2 implies that the linearisation has an eigenvalue zero.

The fact that if there is an eigenvalue zero, the associated eigenfunction is a multiple of $u_x(x; g)$ for $|x| > L$ follows immediately from the expression for the eigenfunction in (22). $\square$

4.3. Extension to a non-simple zero

So far we have focused on the case that the derivative of the front, u_x , has only simple zeroes. In this section we will consider the case that u_x has a non-simple zeroes. Recall that a non-simple zero in an interval implies that the front stays constant and corresponds to a fixed point of the dynamics in that interval. There are two cases that must be considered separately, namely, if there is a non-simple zero of u_x in the middle interval or not.

If there is a non-simple zero in one of the outer intervals (I_l or I_r), it will occur for an isolated value of g , say $g = \hat{g}$ (corresponding to the Hamiltonian of the orbit of the V_m dynamics that contains the fixed point of the V_l or V_r dynamics, i.e., $(u_{-\infty}, 0)$ or $(u_{\infty}, 0)$). From now on we shall assume that a non-simple zero occurs in the left interval. The relabelling symmetry $x \mapsto -x$ can then be used to extract results if a non-simple zero instead resides in the right interval.

A non-simple zero in the left interval at $g = \hat{g}$ implies that $p_l(\hat{g}) = 0$ and $u_l = u_{-\infty}$. From Lemma 2.1 we get that there are two smooth curves of left matching points $(u_l(g), p_l^{\pm}(g))$ nearby $(u_l(\hat{g}), p_l(\hat{g})) = (u_{-\infty}, 0)$. There is one issue however, in our analysis thus far we could assume that there is a constant number of zeroes of u_x in the middle interval (see the definition of the zeroes $x_i(g)$ at the start of Section 4). However, along the curves of left matching points $(u_l(g), p_l^{\pm}(g))$, the number of zeroes of u_x in the middle interval changes if g crosses $\hat{g}$ as $p_l^{\pm}(g)$ changes sign at $g = \hat{g}$. On the other hand, the curves $L(g)$ are smooth, so it is enough when calculating the derivative $\frac{dL}{dg}(\hat{g})$ to calculate the one sided limit $g \rightarrow \hat{g}$. This allows us to state the following extension to Theorem 4.5.

Theorem 4.7. *If $u(x; \hat{g})$ is a front solution of (10), such that all the zeroes of $u_x(x; \hat{g})$ in the middle interval are simple and either:*

- (i) there is a non-simple zero of $u_x(x; \hat{g})$ in the left interval I_l and $\mathcal{B}_r(\hat{g}) \neq 0$ or there is a non-simple zero of $u_x(x; \hat{g})$ in the right interval I_r and $\mathcal{B}_l(\hat{g}) \neq 0$; or
 (ii) there is a non-simple zero of $u_x(x; \hat{g})$ in both left and right intervals I_l and I_r ;

then the linearisation operator $\mathcal{L}(\hat{g})$ has an eigenvalue zero if and only if $L'(\hat{g}) = 0$.

If there is a non-simple zero of $u_x(x; \hat{g})$ in the left interval (but not in the right interval) and $\mathcal{B}_r(\hat{g}) = 0$ or there is a non-simple zero of $u_x(x; \hat{g})$ in the right interval (but not in the left interval) and $\mathcal{B}_l(\hat{g}) = 0$, then the linearisation operator $\mathcal{L}(\hat{g})$ does not have an eigenvalue zero.

Note that Theorem 4.7 is slightly different from Theorem 4.5. In Theorem 4.5, $\mathcal{B}_l(\hat{g}) = 0$ implied that $\hat{g}$ was the boundary of an existence interval and $L'(g)$ would diverge at $g = \hat{g}$. As seen in Lemma 2.1, a non-simple zero of u_x is not related to a boundary of existence if $V'_m(u_l) \neq V'_l(u_l) = 0$ as $u_l = u_{-\infty}$ at $g = \hat{g}$. We are interested in fronts which only have simple zeroes in the middle interval, hence $V'_m(u_l) \neq 0$. Thus in this case, even though $\mathcal{B}_l(\hat{g}) = 0$, there is no boundary of existence of the (u_l, p_l) points, and thus the derivative $L'(\hat{g})$ is bounded.

To prove Theorem 4.7 we use the same approach as for the Theorem 4.5: first we get a compatibility condition by constructing an eigenfunction for the eigenvalue zero and then link this result to $\frac{dL}{dg}$. We start with the compatibility condition.

Lemma 4.8. If $u(x; \hat{g})$ is a front solution of (10), such that the only zero of u_x is a non-simple zero occurring in the left interval I_l and $\mathcal{B}_r \neq 0$ then the linearisation operator $\mathcal{L}(\hat{g})$ has an eigenvalue zero if and only if

$$-\frac{1}{\mathcal{B}_r} + \tilde{G}(L, -L) + \frac{\sqrt{-V''_l(u_l)}}{[V'_m(u_l)]^2} = 0,$$

where $\tilde{G}(L, -L)$ is the regularisation of $-\int_{-L}^L \frac{d\xi}{u_x^2(\xi)}$ defined in (21).

If $u(x; \hat{g})$ is a front solution of (10), such that the only zeroes of u_x are non-simple zeroes in the left and right intervals I_l and I_r , then the linearisation operator $\mathcal{L}(\hat{g})$ has an eigenvalue zero if and only if

$$\tilde{G}(0, -L) - \tilde{G}(0, L) + \frac{\sqrt{-V''_l(u_l)}}{[V'_m(u_l)]^2} + \frac{\sqrt{-V''_r(u_r)}}{[V'_m(u_r)]^2} = 0.$$

If there is a non-simple zero of u_x in the left interval I_l , no other zeroes of u_x , except possibly a simple zero at $x = L$, and $\mathcal{B}_r(\hat{g}) = 0$ then the linearisation operator $\mathcal{L}(\hat{g})$ does not have an eigenvalue zero.

Recall that $\sqrt{-V''_l(u_l)} \in \mathbb{R}$ and $\sqrt{-V''_r(u_r)} \in \mathbb{R}$ as $u_{-\infty}$ and u_{∞} are local maxima of V_l and V_r respectively. This lemma considers a more restrictive case than Theorem 4.7, but it will be adequate to prove the entirety of the theorem, as we shall see later.

Proof of Lemma 4.8. To prove the first statement in the lemma note that the eigenfunction will be as in (22) in the intervals I_m and I_r . The only difference will occur in the interval I_l , the interval with the non-simple zero. As u is the constant function $u(x) = u_{-\infty}$ within this interval, the eigenfunction is given by

$$\Psi(x) = \begin{cases} e^{\alpha_l(x+L)}, & x < -L; \\ Au_x(x) + Bu_x(x) \int_0^x \frac{d\xi}{u_x^2(\xi)}, & -L < x < L; \\ \hat{k}u_x(x), & x > L; \end{cases}$$

where $\alpha_l = \sqrt{-V_l''(u_l)}$. We have only one term in the left interval because we require the eigenfunction to decay as $x \rightarrow \pm\infty$.

Matching the value of Ψ , and its derivative, at $x = -L$ (using Lemma A.1) gives

$$1 = \frac{B}{V_m'(u_l)}, \quad \alpha_l = -AV_m'(u_l) - BV_m'(u_l)\tilde{G}(0, -L), \quad (34)$$

meaning $A = -V_m'(u_l)[\frac{\alpha_l}{[V_m'(u_l)]^2} + \tilde{G}(0, -L)]$. The assumption $\mathcal{B}_l \neq 0$ implies $p_r \neq 0$ and matching at $x = L$ gives

$$\hat{k} = A + B \int_0^L \frac{d\xi}{u_x^2(\xi)} \quad (35)$$

and the compatibility condition

$$-\hat{k}V_r'(u_r) = -\left(A + B \int_0^L \frac{d\xi}{u_x^2(\xi)}\right)V_m'(u_r) + \frac{B}{p_r}.$$

That is

$$B = \hat{k}p_r[V_m'(u_r) - V_r'(u_r)] = -\hat{k}\mathcal{B}_r. \quad (36)$$

Using the expressions for A , B and $\hat{k}$ given in (34) and (35) gives

$$V_m'(u_l) = \mathcal{B}_r V_m'(u_l) \left[\tilde{G}(0, -L) - \int_0^L \frac{d\xi}{u_x^2(\xi)} + \frac{\alpha_l}{[V_m'(u_l)]^2} \right] = \mathcal{B}_r V_m'(u_l) \left[\tilde{G}(L, -L) + \frac{\alpha_l}{[V_m'(u_l)]^2} \right]$$

where the last equality follows by using Lemma 3.1. Dividing through by $V_m'(u_l) \neq 0$ and re-arranging gives the first part of the lemma.

Next we consider the last part of the lemma. That is, u has a non-simple zero in I_l and $\mathcal{B}_r = 0$, so either $p_r = 0$ or $V_m'(u_r) = V_r'(u_r)$. In both cases the eigenfunction stays the same but the matching conditions change. The compatibility condition (36). This contradicts $B = V_m'(u_l) \neq 0$ from (34), proving the final statement in the lemma.

Finally the middle statement of the lemma. If as well as a non-simple zero in the left interval I_l there is also a non-simple zero in the right interval I_r , then the eigenfunction is altered, changing for $x > L$ to $\hat{k}e^{\alpha_r(x-L)}$, where $\alpha_r = -\sqrt{-V_r''(u_r)}$. This means that the previous matching conditions at $x = L$ are replaced by

$$\hat{k} = \frac{B}{V_m'(u_r)}, \quad \alpha_r \hat{k} = -AV_m'(u_r) - BV_m'(u_r)\tilde{G}(0, L).$$

Substituting the expression for $\hat{k}$, along with the expressions for A and B in (34), into the second equality above and re-arranging gives the compatibility condition as stated in the middle part of the lemma. $\square$

Next we proceed with the proof of Theorem 4.7 using the notation $\hat{u} := u_l(\hat{g})$.

Proof of Theorem 4.7. It is important to note that in order to prove this theorem we do not need to consider every case. Instead it is sufficient to consider the case when the pinned fluxon $u(x, \hat{g})$ is such that $u_x(x, \hat{g})$ has no zeroes in the middle interval. If there are zeroes in the middle interval then the result is proved by combining what follows with Theorem 4.5 away from $x = -L$.

First we focus on the case (i) in Theorem 4.7, i.e., $\mathcal{B}_r(\hat{g}) \neq 0$. We have already discussed how $p_l(g)$ is defined for g near $\hat{g}$. For g on one side of $\hat{g}$, $p_l(g)$ will be positive whilst on the other side it will be negative. As p_l changes sign an extra zero of u_x will be introduced on one side of $\hat{g}$. We will consider the one-side limit of $L'(g)$, where g approaches $\hat{g}$ from the side such that there are no zeroes of u_x in the middle interval $-L < x < L$. For g on this side of $\hat{g}$, the length function can be written as

$$2L(g) = \int_{u_l(g)}^{u_r(g)} \frac{du}{p(u, g)} = \int_{u_0(g)}^{u_r(g)} \frac{du}{p(u, g)} + \int_{u_l(g)}^{u_0(g)} \frac{du}{p(u, g)}$$

where $u_0(g)$ is defined such that $p(u_0(g), g) = 0$ and $u_0(g) \rightarrow \hat{u}$ for $g \rightarrow \hat{g}$. Furthermore, $u(x, g)$ satisfies the middle dynamics in $(-L, L)$ and we define $\tilde{u}(x, g)$ to be the extension of this function, i.e., $\tilde{u}(x, g)$ satisfies the middle dynamics for all $x \in \mathbb{R}$. Finally we define $\xi_0(g) < -L(g)$ to be such that $\tilde{u}(\xi_0(g), g) = u_0(g)$ and $\xi_0 \rightarrow -L$ as $g \rightarrow \hat{g}$. Using this definition, Lemma 4.3 gives for $g \neq \hat{g}$

$$\begin{aligned} 2L'(g) &= \tilde{G}(L, \xi_0) + \frac{1}{p_r(g)} \frac{du_r}{dg}(g) - \tilde{G}(-L, \xi_0) - \frac{1}{p_l(g)} \frac{du_l}{dg}(g) \\ &= \tilde{G}(L, \xi_0) - \frac{1}{\mathcal{B}_r(g)} - \tilde{G}(-L, \xi_0) - \frac{1}{p_l(g)} \frac{du_l}{dg}(g). \end{aligned} \quad (37)$$

Taking the limit $g \rightarrow \hat{g}$, the first two terms in the above expression converge to $\tilde{G}(L, -L) - \frac{1}{\mathcal{B}_r(\hat{g})}$. The remaining two terms each blow up as $g \rightarrow \hat{g}$, however their sum doesn't, meaning that the limit can be taken. Indeed, from Lemma A.3, we get that for g near $\hat{g}$, i.e., p_l near 0

$$\begin{aligned} \tilde{G}(-L, \xi_0) + \frac{1}{p_l(g)} \frac{du_l}{dg}(g) &= \int_{-L}^{\xi_0} \left[\frac{1}{u_x^2(\xi)} - \frac{1}{u_{xx}^2(\xi_0)(\xi - \xi_0)^2} \right] d\xi \\ &\quad - \frac{1}{p_l V'_m(u_l)} + O(p_l) + \frac{1}{p_l [V'_m(u_l) - V'_l(u_l)]}. \end{aligned}$$

The term inside the integral is bounded in the limit $g \rightarrow \hat{g}$ (as we have performed a regularisation, see Lemma A.1) and the length of the region of integration tends to zero, so the integral term above vanishes in the limit $g \rightarrow \hat{g}$. Now, $V'_l(u_l) \rightarrow 0$ as $g \rightarrow \hat{g}$ so the dominant terms in the above expression will cancel, however we do still need how 'fast' $V'_l(u_l)$ goes to zero. In Lemma A.2 we have approximations for $u_l(g)$ and $p_l(g)$ near $u_l(\hat{g})$ and $p_l(\hat{g})$. In particular we have $u_l(g) - u_l(\hat{g}) = \frac{p_l(g)}{\sqrt{-V''_l(u_l(\hat{g}))}} + O(p_l(g)^2)$. Thus

$$\begin{aligned} V'_l(u_l(g)) &= V'_l(u_l(\hat{g})) + (u_l(g) - u_l(\hat{g})) V''_l(u_l(\hat{g})) + O((u_l(g) - u_l(\hat{g}))^2) \\ &= p_l \frac{V''_l(u_l(\hat{g}))}{\sqrt{-V''_l(u_l(\hat{g}))}} + O(p_l^2). \end{aligned}$$

Using this expression gives

$$\frac{1}{p_l V'_m(u_l)} - \frac{1}{p_l [V'_m(u_l) - V'_l(u_l)]} = \frac{\sqrt{-V''_l(u_l(\hat{g}))}}{[V'_m(u_l(\hat{g}))]^2} + O(p_l(g)).$$

Substituting this into the expression for $2L'(g)$, (37), and taking the limit $g \rightarrow \hat{g}$ ($p_l \rightarrow 0$) gives

$$2L'(\hat{g}) = \tilde{G}(L(\hat{g}), -L(\hat{g})) - \frac{1}{\mathcal{B}_r(\hat{g})} + \frac{\sqrt{-V''_l(u_l(\hat{g}))}}{[V'_m(u_l(\hat{g}))]^2},$$

which is exactly the term in the compatibility condition in Lemma 4.8.

Next we look at case (ii) of Theorem 4.7, i.e., there is a non-simple zero in both intervals I_l and I_r . Using the same ideas as before and the symmetry $l \leftrightarrow r$ gives

$$\begin{aligned} 2 \frac{dL}{dg}(\hat{g}) &= \frac{d}{dg} \left[\int_{u_l(g)}^{u_r(g)} \frac{du}{p(u, g)} \right]_{g=\hat{g}} = \frac{d}{dg} \left[\int_{u_l(g)}^{u(0, g)} \frac{du}{p(u, g)} \right]_{g=\hat{g}} - \frac{d}{dg} \left[\int_{u_r(g)}^{u(0, g)} \frac{du}{p(u, g)} \right]_{g=\hat{g}} \\ &= \tilde{G}(0, -L) + \frac{1}{p(0, \hat{g})} \frac{\partial u}{\partial g}(0, \hat{g}) + \frac{\sqrt{-V''_l(u_l(\hat{g}))}}{[V'_m(u_l(\hat{g}))]^2} \\ &\quad - \left[\tilde{G}(0, L) + \frac{1}{p(0, \hat{g})} \frac{\partial u}{\partial g}(0, \hat{g}) - \frac{\sqrt{-V''_r(u_r(\hat{g}))}}{[V'_m(u_r(\hat{g}))]^2} \right]. \end{aligned}$$

Again this is the term in the compatibility condition in the middle part of Lemma 4.8.

The final part of this theorem follows directly from Lemma 4.8. $\square$

If there is a non-simple zero in the middle interval for $g = \hat{g}$, then $u(x) = \hat{u}$, where $\hat{u}$ is a fixed point of the dynamics in the middle interval. Thus $p(u(x), \hat{g}) = 0$ for any x in the middle interval. So the length of the middle interval can no longer be expressed as the integral of $1/p(u, g)$. Moreover, if $\hat{g}$ is in the existence interval of both left and right matching points, then a stationary solution exists for any length L , i.e. there will be a vertical curve in the g - L plane at $g = \hat{g}$. We can determine the stability of such fronts, though obviously they cannot be linked to the derivative of the length function.

Lemma 4.9. *If $u(x)$ is a front solution of (10) such that $u(x) \equiv \hat{u}$ in the middle interval, then let $\alpha_m = V''_m(\hat{u})$. If $V'_l(\hat{u}) \neq 0 \neq V'_r(\hat{u})$ (i.e. there is not a non-simple zero in either the left or right interval) then*

- if $\alpha_m > 0$ the linearisation operator $\mathcal{L}$ has an eigenvalue zero if and only if $L = \frac{m\pi}{2\sqrt{\alpha_m}}$ for $m \in \mathbb{N}_0$;
- if $\alpha_m \leq 0$ the linearisation operator $\mathcal{L}$ has an eigenvalue zero if and only if $L = 0$.

In both these cases the eigenfunction associated with an eigenvalue zero has a zero, thus the front $u(x)$ is unstable.

If there is also a non-simple zero in either the left or right interval (hence $V'_l(\hat{u}) = 0$ or $V'_r(\hat{u}) = 0$) then the linearisation operator $\mathcal{L}$ does not have an eigenvalue zero for any $L \geq 0$ and the front $u(x)$ is stable.

Proof. If there is a non-simple zero only in the middle interval then the eigenfunction associated with the eigenvalue zero (22) becomes

$$\Psi(x) = \begin{cases} u_x(x), & x < -L; \\ A \cos(\sqrt{\alpha_m}(x+L)) + B \sin(\sqrt{\alpha_m}(x+L)), & |x| < L; \\ \hat{k}u_x(x), & x > L; \end{cases}$$

for $\alpha_m \neq 0$. If $\alpha_m > 0$ then in the middle interval we have a linear combination of cos- and sin-functions and if $\alpha_m < 0$ it is a linear combination of cosh- and sinh-functions. Matching Ψ and Ψ_x at $x = \pm L$ gives $\sin(2\sqrt{\alpha_m}L) = 0$ as the compatibility condition. If $\alpha_m > 0$ this condition holds if and only if $L = \frac{m\pi}{2\sqrt{\alpha_m}}$. If $\alpha_m < 0$ then the compatibility condition is equivalent to $\sinh(2\sqrt{-\alpha_m}L) = 0$ which only holds if $L = 0$.

If $\alpha_m = 0$ then the eigenfunction in the middle interval is replaced by $A + Bx$, matching Ψ and its derivative again gives $L = 0$ as a compatibility condition. In all of these cases $\Psi(\pm L) = 0$, that is, the eigenfunction has a zero. So by Sturm–Liouville theory there is a strictly positive eigenvalue for all $L \geq 0$. Hence $u(x)$ is unstable.

If there is a non-simple zero in either the left or right interval then the eigenfunction in that interval becomes $e^{\sqrt{-V_l''(\hat{u})(x+L)}} \hat{k}e^{-\sqrt{-V_r''(\hat{u})(x-L)}}$ respectively. We again match Ψ and its derivative to get a compatibility condition for the existence of an eigenvalue zero. If $\alpha_m \neq 0$ then the compatibility condition gives that L must be a positive multiple of an arctan or arctanh of a negative number. For instance, if there is a non-simple zero in interval l (and m) but not in r and $\alpha_m > 0$ then the compatibility condition is

$$L = \frac{1}{2\sqrt{\alpha_m}} \arctan\left(-\frac{\sqrt{\alpha_m}}{\sqrt{-V_l''(\hat{u})}}\right).$$

Which has no positive solution for L . If $\alpha_m = 0$ then a simple calculation shows that $L < 0$ for an eigenvalue zero to exist. Thus in all cases $L < 0$ for an eigenvalue zero to exist. $\square$

From the proof we can see that the condition $L = \frac{m\pi}{2\sqrt{\alpha_m}}$, $m \in \mathbb{N}_0$ is a resonance condition, as the eigenfunction in the middle interval is made up of a linear combination of cos- and sin-functions.

4.4. A long Josephson junction with a microresistor inhomogeneity

In [5], it is shown that various families of stationary fronts exist in long Josephson junctions with a microresistor or micro resonator inhomogeneity. These stationary fronts are usually called “pinned fluxons”. In this section we investigate the stability of the families of pinned fluxons in the case where the inhomogeneity is a micro resistor, i.e., the junction is locally thinned and there is less resistance for the Josephson supercurrent to go across the junction. The phase difference u of a Josephson junction can be described by a perturbed sine-Gordon equation:

$$u_{tt} = u_{xx} - D(x) \sin(u) + \gamma - \alpha u_t, \quad \text{with } D(x; L, d) = \begin{cases} d, & |x| < L, \\ 1, & |x| > L. \end{cases} \quad (38)$$

Here x and t are the spatial and temporal variable respectively; $\alpha \geq 0$ is the damping coefficient due to normal electron flow across the junction; and γ is the applied bias current. The function $D(x)$ represents the Josephson tunnelling critical current. For a microresistor, we have $0 \leq d < 1$. So in terms of our general set-up as presented in the previous sections, we have

$$V_l(u) = \cos u + \gamma u - H_0(\gamma) - 1 = V_r(u) \quad \text{and} \quad V_m(u) = d \cos u + \gamma u - H_0(\gamma) - d, \quad (39)$$

where $H_0(\gamma) = \gamma \arcsin \gamma + \sqrt{1 - \gamma^2} - 1 + 2\pi \gamma$ (this constant is chosen such that $V_r(u_\infty) = 0$). The maxima of the potentials $V_r = V_l$ in the outer intervals are $\arcsin \gamma + 2k\pi$, $k \in \mathbb{Z}$. We focus on the case that the asymptotic fixed point in the left interval is $u_{-\infty} = \arcsin \gamma$ and in the right interval it is

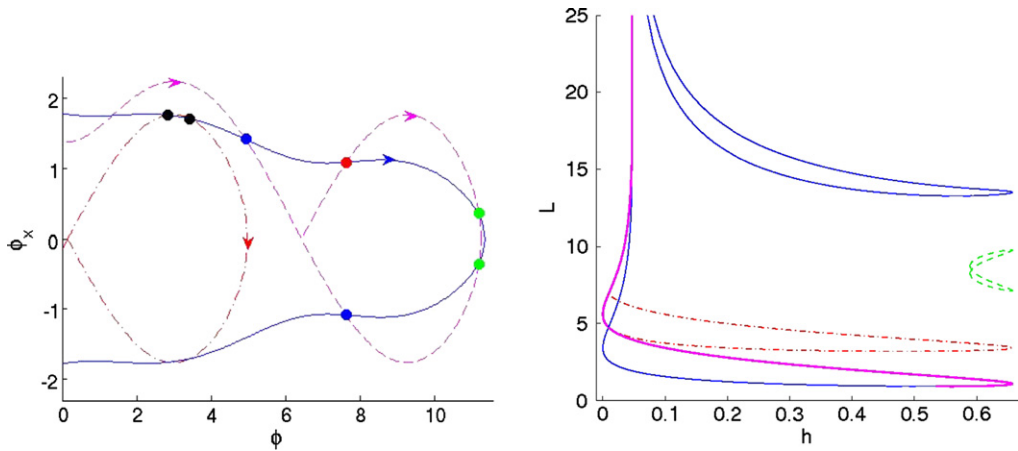


Fig. 3. Phase portrait (left) and g - L curves (right) for $d = 0.2$ and $\gamma = 0.15$. In the phase portrait on the left, the unstable manifold to $u_{-\infty}$ is denoted by the red dash-dotted line; the stable manifold to u_{∞} is denoted by the magenta dashed line; and the blue line is one of the g -level sets of the Hamiltonian in the middle interval. The two possible left matching points on a g -orbit are denoted by the black dots. The possible right matching points are denoted by blue, red, or green dots, corresponding to the colour used for the curves in the g - L diagram on the right. Part of the blue curve is coloured magenta, to indicate the stable pinned fluxons. Note that the changes of stability happen at the extremal points of L . (For interpretation of the references to colour, the reader is referred to the web version of this article.)

$u_{\infty} = 2\pi + \arcsin \gamma$. A full overview of the existence and construction of the pinned fluxons and their length curves can be found in [5], here we give a short overview of the main features. In [5], it is shown that for fixed $0 \leq d < 1$, various types of pinned fluxons exist if $0 \leq \gamma \leq \frac{1-d}{\pi}$. The parameter g , representing the values of the Hamiltonian in the middle interval, allows for pinned fluxons to exist if $0 \leq g \leq 2(1-d-\pi\gamma)$. A typical phase portrait and the related Hamiltonian-length (g - L) curves for $\gamma = 0.15$ and $d = 0.2$ are presented in Fig. 3. There are up to three length curves; in Fig. 3 they are denoted by a blue, red and green curve. The red curve exists for $0 \leq \gamma < \frac{4\pi(1-d)}{4\pi^2 + (1-d)^2}$. The green curve exists for $0 \leq \gamma \leq \gamma_1$, where γ_1 is the solution of $(1-d)(\cos u_{\max}(\gamma) + 1) = 2\pi\gamma$ and $u_{\max}(\gamma)$ is the maximal angle on the orbit homoclinic to $2\pi + \arcsin \gamma$ in the V_r dynamics, i.e., $u_{\max}(\gamma)$ satisfies $V_r(u_{\max}) = H_0(\gamma)$ and $u_{\max} \in (3\pi - \arcsin \gamma, 4\pi)$. Note that $\gamma_1 < \frac{4\pi(1-d)}{4\pi^2 + (1-d)^2} < \frac{1-d}{\pi}$.

In the phase portrait, there are two possible left matching points (u_l, p_l) , indicated by the black dots in Fig. 3 and up to five right matching points (u_r, p_r) , indicated by the blue, red or green dots. The colour of the right matching points is the same as the colour of the length curve for the corresponding pinned fluxon.

From the expressions (39) for the potentials V_i , it follows immediately that the left and right bifurcations functions are given by

$$\mathcal{B}_l = p_l(1-d)\sin u_l \quad \text{and} \quad \mathcal{B}_r = p_r(d-1)\sin u_r. \quad (40)$$

Theorems 4.5 and 4.7 can now be applied to determine which pinned fluxons have a linearisation operator with an eigenvalue zero. This lemma is quoted in [5] too, with a reference to this paper for its proof.

Lemma 4.10. *If $0 \leq d < 1$, then the linearisation operator $\mathcal{L}(g)$ for the pinned fluxon $u(x; g)$ has an eigenvalue zero if and only if*

- the length curve has a critical point at g , i.e., $\frac{dL}{dg}(g) = 0$;
- or $\gamma = \frac{1-d}{\pi}$ (this eigenvalue zero is the largest eigenvalue);

- or $\gamma = \gamma_1$ and the fluxon $u(x; g)$ is the one with a right matching point at $(2\pi + u_{\max}(\gamma), 0)$ (this eigenvalue zero is not the largest eigenvalue).

Before proving this lemma, we first consider the special cases in this lemma. As observed earlier, $\gamma = \frac{1-d}{\pi}$ is at the edge of the interval of existence for pinned fluxons; for $\gamma > \frac{1-d}{\pi}$ no pinned fluxons exist. At $\gamma = \frac{1-d}{\pi}$, there is exactly one value of the length L and g for which a pinned fluxon exists: the solid blue curve has degenerated to one point. In the other case ($\gamma = \gamma_1$), the dashed green curve has degenerated to a point. In this case, there are more pinned fluxons, but they are on the other (blue and red) curves. For these isolated pinned fluxons, the eigenfunction associated with the eigenvalue zero of the linearisation is either the derivative of the pinned fluxon or a combination of multiples of the derivative of the pinned fluxon.

Note that the stability in case $d = 0$ is analysed in full detail in [5]; it is substantially simpler than the general case as considered in this paper.

Proof of Lemma 4.10. Theorem 4.5 applies if the pinned fluxon doesn't have non-simple fixed points. So we first consider which of the pinned fluxons have non-simple fixed points. None of the pinned fluxons have a non-simple fixed point in the left interval as this would imply that $u_l = \arcsin \gamma$ and this does not happen. For a pinned fluxon with non-simple zeroes in the middle interval, we would need $u_l = u_r$ to be fixed points of $V_m(u)$ and $p_l = 0 = p_r$. There are no left matching points with $p_l = 0$, hence no pinned fluxons with a non-simple zero in the middle interval exist. There are two pinned fluxons with a non-simple zero in the right interval. These pinned fluxons have a right matching point at the fixed point of the right dynamics, i.e., at $(2\pi + \arcsin \gamma, 0)$, and $g = g_2 := (1-d)(1 - \sqrt{1 - \gamma^2})$. So we can apply Theorem 4.5 to all pinned fluxons, except for these two special pinned fluxons; we will consider them at the end of this proof.

Theorem 4.5 gives that the linearisation of the pinned fluxon has an eigenvalue zero if and only if $L'(g) = 0$ or if the left and right bifurcation functions are simultaneously zero. From the expressions in (40) and the observation that the (black) left matching points satisfy $\arcsin \gamma < u_l < 2\pi$ and $p_l > 0$, we can conclude that the left bifurcation function $\mathcal{B}_l$ vanishes only when $u_l = \pi$, i.e., at $g = 1/(1-d - \pi\gamma)$. This g -value is at the end of the existence interval, the two left matching points merge into one. At this g -value, the right bifurcation functions vanish simultaneously only if $\gamma = \gamma_1$; $\gamma = \frac{4\pi(1-d)}{4\pi^2 + (1-d)^2}$; or $\gamma = \frac{1-d}{\pi}$. If $\gamma = \gamma_1$, the right bifurcation function for the pinned fluxon with a right matching point at $(2\pi + u_{\max}(\gamma), 0)$, vanishes too (this is the pinned fluxon in the green curve; the right bifurcation point for the other pinned fluxon does not vanish). This is the last special case in the lemma and we can conclude that the linearisation of this pinned fluxon has an eigenvalue zero. If $\gamma = \frac{4\pi(1-d)}{4\pi^2 + (1-d)^2}$, the pinned fluxons with vanishing left and right bifurcation functions are the two special cases for which Theorem 4.5 does not apply. Finally, if $\gamma = \frac{1-d}{\pi}$, there is only one isolated pinned fluxon, which has $u_l = \pi$ and $u_r = 2\pi$. From (40), it follows immediately that both left and right bifurcations functions disappear, hence this pinned fluxon has an eigenvalue zero. This is the first special case in the lemma. Thus Theorem 4.5 gives the statement of the lemma, apart from the two pinned fluxons with a non-simple zero, which we will consider now.

If $g = g_2$, the two pinned fluxons connecting the left matching points to $(2\pi + \arcsin \gamma, 0)$ have a non-simple zero in the right interval and no other non-simple zeroes. These fluxons are part of the smooth solid blue curve and $\mathcal{B}_l(g_2) = 0$ only if $\gamma = \gamma_2 := \frac{4\pi(1-d)}{4\pi^2 + (1-d)^2}$, thus Theorem 4.7 gives that for $\gamma \neq \gamma_2$, the linearisation about the pinned fluxon has an eigenvalue zero if and only if $L'(g_2) = 0$. And if $\gamma = \gamma_2$, then $\mathcal{B}_l(g_2) = 0$, thus Theorem 4.7 gives that the pinned fluxon does not have an eigenvalue zero. Note that if $\gamma = \gamma_2$, g_2 is on the edge of the existence interval in the g parameter space, thus the length curve has a vertical derivative and hence its derivative does not vanish. $\square$

In [5] it is shown that Lemma 4.10 implies that there for each length L , there is exactly one stable fluxon. In the g - L curves in the right plot of Fig. 3, these stable pinned fluxons are indicated by the magenta curve.

5. Stability result in four intervals

Having analysed fronts in wave equations with potentials that have one middle interval ($n = 1$), we now focus on the case with two middle intervals ($n = 2$). We will show that the results for one middle interval can be extended to two middle intervals using similar ideas, though the analysis is considerably more complicated as there are two Hamiltonians, parameterised by g and h , and two length functions, L_1 and L_2 . As the methods employed are the same as in Section 4, we give the derivation of the results obtained in this section in Appendix A.4. Just as for one middle interval, this derivation involves the construction of the eigenfunction associated with an eigenvalue zero, leading to a compatibility condition in terms of regularisation functions like C_v , see (23), and then linking the compatibility condition to derivatives of the length functions L_1 and L_2 . After stating the results for two middle intervals, we discuss how they relate to the case with one middle interval.

5.1. The linearisation operator and eigenvalues zero

Due to the added complexity we state the counterpart to Theorem 4.5 in two parts. First the generic case that the middle bifurcation function $\mathcal{B}_m \neq 0$ followed by the case when $\mathcal{B}_m = 0$.

Theorem 5.1. *If $u(x; g, h)$ is a front solution of Eq. (10), such that all zeroes of $u_x(x; g, h)$ are simple and $\mathcal{B}_m(g, h) \neq 0$, then the linearisation operator $\mathcal{L}(g, h)$ has an eigenvalue zero if and only if*

$$\mathcal{B}_l(g) \mathcal{B}_m(g, h) \mathcal{B}_r(h) \det \begin{pmatrix} \frac{\partial L_1}{\partial g}(g, h) & \frac{\partial L_1}{\partial h}(g, h) \\ \frac{\partial L_2}{\partial g}(g, h) & \frac{\partial L_2}{\partial h}(g, h) \end{pmatrix} = 0. \quad (41)$$

Since $\frac{\partial}{\partial g} L_1(g, h)$ and $\frac{\partial}{\partial h} L_2(g, h)$ have poles in a bifurcation point, i.e., if $\mathcal{B}_l = 0$ or $\mathcal{B}_r = 0$, this expression should be read as a limit in those cases. This gives,

- if $\mathcal{B}_l(g) = 0$ and $\mathcal{B}_r(h) \neq 0$, then the condition becomes $\frac{\partial L_2}{\partial h}(g, h) = 0$;
- if $\mathcal{B}_r(h) = 0$ and $\mathcal{B}_l(g) \neq 0$, then the condition becomes $\frac{\partial L_1}{\partial g}(g, h) = 0$;
- if $\mathcal{B}_l(g) = 0 = \mathcal{B}_r(h)$, then there is no eigenvalue zero.

The proof of this theorem is given in Appendix A.4. Note that by (16) the Jacobian can be written as

$$\begin{pmatrix} \frac{\partial L_1}{\partial g}(g, h) & \frac{\partial L_1}{\partial h}(g, h) \\ \frac{\partial L_2}{\partial g}(g, h) & \frac{\partial L_2}{\partial h}(g, h) \end{pmatrix} = \begin{pmatrix} \frac{\partial L_1}{\partial g}(g, h) & \frac{1}{\mathcal{B}_m(g, h)} \\ \frac{1}{\mathcal{B}_m(g, h)} & \frac{\partial L_2}{\partial h}(g, h) \end{pmatrix}.$$

This is the form that will be generalised to the case of N middle intervals in Section 6.

Looking at the bifurcation points in Theorem 5.1, just as in Theorem 4.5, if $\mathcal{B}_l(\hat{g}) = 0$ then $\frac{\partial L_1}{\partial g}(\hat{g}, h)$ is unbounded but $\lim_{g \rightarrow \hat{g}} \mathcal{B}_l(g) \frac{\partial L_1}{\partial g}(\hat{g}, h)$ exists and does not vanish. In the case $\mathcal{B}_l = 0 = \mathcal{B}_r$, $\mathcal{B}_l \frac{\partial L_1}{\partial g}$ and $\mathcal{B}_r \frac{\partial L_2}{\partial h}$ are both non-zero in the limit whilst $\mathcal{B}_l \frac{\partial L_1}{\partial h}$ and $\mathcal{B}_r \frac{\partial L_2}{\partial g}$ both vanish. Thus, as $\mathcal{B}_m \neq 0$, (41) cannot be satisfied. Note also that if at $(g, h) = (\hat{g}, \hat{h})$ the function L_1 has a bounded critical point (i.e. $\frac{\partial L_1}{\partial g}(\hat{g}, \hat{h}) = 0 = \frac{\partial L_1}{\partial h}(\hat{g}, \hat{h})$) then (41) implies that there is an eigenvalue zero irrespective of the other function L_2 . Similarly, a critical point of L_2 implies the existence of an eigenvalue zero.

Corollary 5.2. *Comparing Theorem 5.1 (two middle intervals) with Theorem 4.5 (one middle interval), we observe that the determinant in Theorem 5.1 is the Jacobian of the vector function $(L_1, L_2)(g, h)$, which is the two-dimensional equivalent of $L'(g)$. Furthermore, fixing L_1 at a non-critical value $\hat{L}$ (not a saddle or extremum), defines a curve $\hat{h}(g)$ or $\hat{g}(h)$ in the existence region of the pinned fluxons. On this curve there are pinned fluxons with $L_1 = \hat{L}$. Applying Theorem 5.1 to this situation gives that an eigenvalue zero will occur at*

a constrained critical point of L_2 or when one of the left or right bifurcation vanishes; recovering the condition of Theorem 4.5.

Proof. If $\hat{L}$ is not a critical value for the L_1 lengths, then the implicit function theorem gives that $L_1(g, h) = \hat{L}$ defines a curve $\hat{h}(g)$ or $\hat{g}(h)$. We shall assume that the curve which is defined is $\hat{h}(g)$, i.e., $\frac{\partial L_1}{\partial h}(g, \hat{h}(g)) \neq 0$. If it is $\hat{g}(h)$ instead, then the proof is similar.

Differentiating $L_1(g, \hat{h}(g)) = \hat{L}$ with respect to g gives $(\frac{\partial L_1}{\partial g} + \frac{\partial L_1}{\partial h} \frac{d\hat{h}}{dg})(g, \hat{h}(g)) = 0$. This means that at $(g, h) = (g, \hat{h}(g))$

$$\begin{aligned} \det \begin{pmatrix} \frac{\partial L_1}{\partial g} & \frac{\partial L_1}{\partial h} \\ \frac{\partial L_2}{\partial g} & \frac{\partial L_2}{\partial h} \end{pmatrix} &= \frac{\partial L_1}{\partial g}(g, \hat{h}(g)) \frac{\partial L_2}{\partial h}(g, \hat{h}(g)) - \frac{\partial L_1}{\partial h}(g, \hat{h}(g)) \frac{\partial L_2}{\partial g}(g, \hat{h}(g)) \\ &= -\frac{\partial L_1}{\partial h}(g, \hat{h}(g)) \left[\frac{\partial L_2}{\partial g}(g, \hat{h}(g)) + \frac{\partial L_2}{\partial h}(g, \hat{h}(g)) \frac{dh}{dg}(g, \hat{h}(g)) \right] \\ &\quad + \frac{\partial L_2}{\partial h}(g, \hat{h}(g)) \left[\frac{\partial L_1}{\partial h}(g, \hat{h}(g)) \frac{dh}{dg}(g, \hat{h}(g)) + \frac{\partial L_1}{\partial g}(g, \hat{h}(g)) \right] \\ &= -\frac{\partial L_1}{\partial h}(g, \hat{h}(g)) \frac{dL_2}{dg}(g, \hat{h}(g)). \end{aligned}$$

If the point $(\tilde{g}, \hat{h}(\tilde{g}))$ is such that an eigenvalue zero exists then the above expression multiplied by $\mathcal{B}_l \mathcal{B}_r$ must be zero at this point. Since $\frac{\partial L_1}{\partial h}(\tilde{g}, \hat{h}(\tilde{g})) \neq 0$, the condition for the existence of an eigenvalue zero becomes $\mathcal{B}_l \mathcal{B}_r \frac{dL_2}{dg}(g, \hat{h}(g)) = 0$. Which is the condition in Theorem 4.5 for $L(g) = L_2(g, \hat{h}(g))$. $\square$

If $\mathcal{B}_m(\hat{g}, \hat{h}) = 0$ then all of the partial derivatives that we have used previously, i.e., $\frac{\partial L_1}{\partial g}$, etc., are unbounded. From Lemma 2.2 it follows that there exists a smooth bijective curve $\tilde{h}(g)$ with $\tilde{h}(\hat{g}) = \hat{h}$ such that $\mathcal{B}_m(g, \tilde{h}(g)) = 0$ for g in a neighbourhood of $\hat{g}$. As $\tilde{h}$ is bijective, it can be inverted near $(\hat{g}, \hat{h})$ and a curve $\tilde{g}(h)$ can be defined such that $\tilde{g}(\tilde{h}(g)) = g$ for g near $\hat{g}$ and $\tilde{h}(\tilde{g}(h)) = h$ for h near $\hat{h}$. In Appendix A.4, proof of Theorem 5.3, it is shown that $\frac{d}{dh} L_1(\tilde{g}(h), h)$ and $\frac{d}{dg} L_2(g, \tilde{h}(g))$ are well defined near $h = \hat{h}$, resp. near $g = \hat{g}$. This allows us to formulate the following theorem for $\mathcal{B}_m = 0$.

Theorem 5.3. If $u(x; \hat{g}, \hat{h})$ is a front solution of Eq. (10), such that all zeroes of $u_x(x; \hat{g}, \hat{h})$ are simple and $\mathcal{B}_m(\hat{g}, \hat{h}) = 0$, then one of the following cases holds for the linearisation operator $\mathcal{L}(\hat{g}, \hat{h})$:

- if $\mathcal{B}_l(\hat{g}) \neq 0 \neq \mathcal{B}_r(\hat{h})$ then $\mathcal{L}(\hat{g}, \hat{h})$ has an eigenvalue zero if and only if

$$0 = \mathcal{B}_l(\hat{g}) \mathcal{B}_r(\hat{h}) \left[\frac{dL_1}{dh}(\tilde{g}(h), h) \Big|_{h=\hat{h}} + \frac{dL_2}{dg}(g, \tilde{h}(g)) \Big|_{g=\hat{g}} \right];$$

- if exactly one of $\mathcal{B}_l(\hat{g})$, $\mathcal{B}_r(\hat{h})$ equals zero then $\mathcal{L}(\hat{g}, \hat{h})$ has no eigenvalue zero;
- if $\mathcal{B}_l(\hat{g}) = 0 = \mathcal{B}_r(\hat{h})$ then $\mathcal{L}(\hat{g}, \hat{h})$ has an eigenvalue zero.

Corollary 5.4. Theorem 5.3 can be thought of as a limiting case of Theorem 5.1.

Proof. In order to link Theorem 5.3 ($\mathcal{B}_m = 0$) to Theorem 5.1 ($\mathcal{B}_m \neq 0$) we will show that for $h(g, \epsilon) := \tilde{h}(g) + \epsilon$ (with the sign of ϵ such that $h(g, \epsilon)$ is inside the existence region of the stationary fronts)

$$\lim_{\epsilon \rightarrow 0} \mathcal{B}_m(g, h(g, \epsilon)) \frac{\partial L_1}{\partial h}(g, h(g, \epsilon)) = 1$$

and

$$\lim_{\epsilon \rightarrow 0} \mathcal{B}_m(g, h(g, \epsilon)) \frac{\partial L_2}{\partial h}(g, h(g, \epsilon)) = -\frac{1}{\tilde{h}'(g)}. \quad (42)$$

The first of these relations is straightforward as (16) gives that $\mathcal{B}_m \frac{\partial L_1}{\partial h} = 1$ for $\epsilon \neq 0$. The second relation is a bit more complicated. In Appendix A, Lemma A.6, it is shown that for any g near $\hat{g}$, $\mathcal{B}_m \frac{\partial L_2}{\partial h} \rightarrow -\frac{V'_{m_1}(u_m)}{V'_{m_2}(u_m)}$ for $h \rightarrow \tilde{h}(g)$ and in Lemma A.9, it is shown that $\tilde{h}'(g) = \frac{V'_{m_2}(u_m)}{V'_{m_1}(u_m)}$. Together this implies the second relation in (42).

Thus for g near $\hat{g}$:

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \mathcal{B}_m \det \begin{pmatrix} \frac{\partial L_1}{\partial g} & \frac{\partial L_1}{\partial h} \\ \frac{\partial L_2}{\partial g} & \frac{\partial L_2}{\partial h} \end{pmatrix} \Big|_{(g,h)=(g,h(g,\epsilon))} \\ &= \lim_{\epsilon \rightarrow 0} \left[\mathcal{B}_m \frac{\partial L_2}{\partial h} \left[\frac{\partial L_1}{\partial g} + \frac{\partial L_1}{\partial h} \frac{dh}{dg} \right] - \mathcal{B}_m \frac{\partial L_1}{\partial h} \left[\frac{\partial L_2}{\partial g} + \frac{\partial L_2}{\partial h} \frac{dh}{dg} \right] \right] \Big|_{(g,h)=(g,h(g,\epsilon))} \\ &= \lim_{\epsilon \rightarrow 0} \left[\mathcal{B}_m \frac{\partial L_2}{\partial h} \right] \lim_{\epsilon \rightarrow 0} \frac{d}{dg} L_1(g, h(g, \epsilon)) - \lim_{\epsilon \rightarrow 0} \left[\mathcal{B}_m \frac{\partial L_1}{\partial h} \right] \lim_{\epsilon \rightarrow 0} \frac{d}{dg} L_2(g, h(g, \epsilon)) \\ &= -\frac{1}{\tilde{h}'(g)} \left[\frac{d}{dg} L_1(g, \tilde{h}(g)) + \tilde{h}'(g) \frac{d}{dg} L_2(g, \tilde{h}(g)) \right] \\ &= -\frac{1}{\tilde{h}'(g)} \left[\frac{d}{dg} L_1(g, \tilde{h}(g)) + \frac{d}{dh} L_2(\tilde{g}(h), h) \Big|_{h=\tilde{h}(g)} \right]. \end{aligned}$$

In the last step, we use that $g = \tilde{g}(\tilde{h}(g))$ thus

$$\frac{d}{dg} L_2(g, \tilde{h}(g)) = \frac{d}{dg} [L_2(\tilde{g}(h), h) \Big|_{h=\tilde{h}(g)}] = \frac{d}{dh} (L_2(\tilde{g}(h), h)) \Big|_{h=\tilde{h}(g)} \cdot \tilde{h}'(g). \quad \square \quad (43)$$

To illustrate the stability results presented in this section, we consider an example of the $0-\pi$ long Josephson junction with a defect and show that a defect can stabilise an unstable non-monotonic pinned fluxon.

5.2. Defect in a $0-\pi$ long Josephson junction

By layering two long Josephson junctions with different material properties, a front joining states π apart (rather than 2π apart) can be made [28,25]. This is called a $0-\pi$ junction and is modelled by an inhomogeneous sine-Gordon-type equation with forcing and dissipation, see for example [4,9,29]:

$$u_{tt} = u_{xx} - \theta(x) \sin(u) + \gamma - \alpha u_t \quad \text{where } \theta(x; L_2) = \begin{cases} 1, & x < L_2; \\ -1, & L_2 < x. \end{cases} \quad (44)$$

Pinned semi-fluxons are stationary solutions that join the asymptotic states $\arcsin(\gamma)$ in the left interval and $\pi + \arcsin(\gamma)$ in the right interval. In [4,29], the existence and stability of such solutions are studied and three types of stationary semi-fluxons are found for $\gamma \in [0, \frac{2}{\sqrt{4+\pi^2}})$, a typical representation of these solutions for $L_2 = 0$ is given in Fig. 4. The three types of stationary semi-fluxons consist

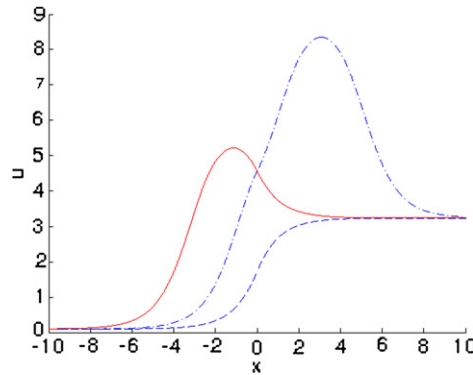


Fig. 4. The three stationary fluxons of (44) with $\gamma = 0.1$ and $L_2 = 0$. The monotonic fluxon is stable whilst the other two are unstable. (For interpretation of the references to colour, the reader is referred to the web version of this article.)

of a monotonic one and two non-monotonic ones, one with a small “hump” and the other with a big one. The monotonic semi-fluxon is stable whilst the other two are unstable. The linearisation about the semi-fluxon with the smallest “hump” (depicted by the solid red curve) has one positive eigenvalue which is very small for γ small. That is, this semi-fluxon is only marginally unstable. In this section we will show that it is possible to stabilise this semi-fluxon by introducing a defect in the left interval.

We model a $0-\pi$ junction with defect by

$$u_{tt} = u_{xx} - \theta(x) \sin(u) + \gamma - \alpha u_t \quad \text{where } \theta(x; L_1, L_2) = \begin{cases} 1, & x < -L_1; \\ 0, & -L_1 < x < 0; \\ 1, & 0 < x < L_2; \\ -1, & L_2 < x. \end{cases} \quad (45)$$

Hence the defect has length L_1 and is placed on the left of the $0-\pi$ junction at distance L_2 . So taking $V(u, x; L_1, L_2) = \theta(x; L_1, L_2) \cos(u) + \gamma u$ allows the equation to be re-written as (3), where the potentials are those shown in Fig. 1. The dynamics in the intervals $I_1 = (-L_1, 0)$ and $I_2 = (0, L_2)$ can be parameterised in terms of the values of the Hamiltonians g and h respectively and represented via a phase portrait, where $p = u_x$, see Fig. 5. If $L_1 = 0$, then we recover the $0-\pi$ junction without a defect. The “small hump” pinned semi-fluxon in the defect-less system is depicted by the bold dashed red curve. It is on the h -level set which contains the fixed point at $-\infty$, i.e., $(u_{-\infty}, 0)$. As $V_l = V_{m_2}$, the condition $L_1 = 0$ confirms that the V_{m_1} dynamics does not play a role for the defect-less semi-fluxon. So any g -value that crosses the unstable manifold of $(u_{-\infty}, 0)$ can be used to represent the defect-less “small hump” pinned semi-fluxon. By varying g , the L_2 value in the description of the defect-less “small hump” pinned semi-fluxon gets modified. The value of L_2 does not play a significant role in the defect-less system as it can be changed by a shift in the x -coordinate. However, in the system with a defect it is significant as we have used the spatial translation invariance to fix the transition from the V_{m_1} -dynamics to the V_{m_2} -dynamics at $x = 0$.

The example of the long Josephson junction in Section 4.4 shows the existence of a plethora of stationary fluxons if a defect is added. So it can be expected that many stationary semi-fluxons can be found for a $0-\pi$ junction with defect too. We will not study all possible stationary semi-fluxons, but restrict to a family that shows how the defect-less “small hump” pinned semi-fluxon can be stabilised by a defect. To be explicit, we consider pinned semi-fluxons with the matching points at $x = -L_1$, $x = 0$ and $x = L_2$ given by

$$u_l = \arccos(V_- - g), \quad u_m = 2\pi - \arccos(h - g) \quad \text{and} \quad u_r = 2\pi - \arccos\left(\frac{h - V_+}{2}\right),$$

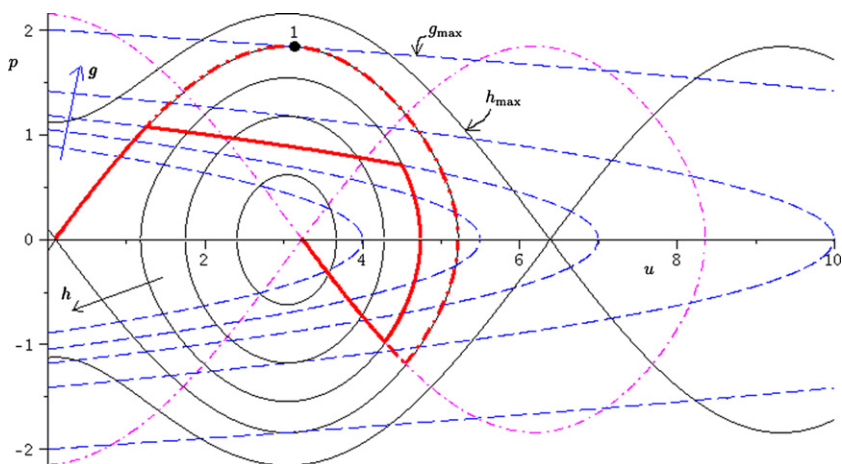


Fig. 5. A phase portrait for the sine-Gordon equation (45), with different g and h values and $\gamma = 0.1$. The dynamics in the first middle interval $(L_1, 0)$ is depicted by the dashed blue lines. The dynamics in the second middle interval $(0, L_2)$ is depicted by the solid black lines. The magenta dash-dotted curves depict the stable manifolds of $(u_\infty, 0)$. The g and h arrows depict how g and h increase across level sets. The bold dashed red line depicts the “small bump” defect-less semi-fluxon. The bold solid red line depicts an example of a semi-fluxion with defect. The black dot indicated with the number 1 indicates the left/middle matching point for the “small bump” defect-less semi-fluxon. (For interpretation of the references to colour, the reader is referred to the web version of this article.)

$$p_l = \sqrt{g - V_{m_1}(u_l)}, \quad p_m = \sqrt{g - V_{m_1}(u_m)} \quad \text{and} \quad p_r = -\sqrt{h - V_{m_2}(u_r)},$$

where

$$V_- = \sqrt{1 - \gamma^2} + \gamma \arcsin(\gamma) \quad \text{and} \quad V_+ = \sqrt{1 - \gamma^2} + \gamma \arcsin(\gamma) + \gamma\pi.$$

This means that the left matching point has $p_l > 0$ and $\arcsin \gamma < u_l \leq \pi$, the middle matching point has $p_m > 0$ and $\pi \leq u_m < 2\pi + \arcsin \gamma$, and the right matching point has $p_r < 0$ and $\pi + \arcsin \gamma < u_r \leq 2\pi - \arccos(\gamma\pi/2)$. The bold solid red curve in Fig. 5 shows an example of such pinned semi-fluxion with defect. Fig. 5 also shows that all of the semi-fluxions that we consider are non-monotonic with the zero of u_x in interval I_2 . The defect-less “small hump” pinned semi-fluxion has left and middle matching points with $u_l = u_m = \pi$; in Fig. 5 these points are indicated by the dot with the number 1. The associated g value is the maximal possible g value, hence the dashed blue curve which touches the solid black unstable manifold. With Fig. 5 we can also find the boundaries of the existence region of those matching points. The maximal possible g value is the value associated with the dashed blue curve which touches the solid black unstable manifold, indicated by $g_{\max}$ in Fig. 5. For $g = g_{\max}$ fixed, the h value of the defect-less “small hump” pinned semi-fluxion is the smallest possible h value and h can increase to the h value of the stable manifold to $2\pi + \arcsin \gamma$. This h value is indicated by $h_{\max}$ in Fig. 5 as it is the maximal h value for all g values. (It is possible to find pinned fluxions with $h > h_{\max}$, but we do not consider those here.) On the boundary $g = g_{\max}$, we have $\mathcal{B}_l = 0$ with $V'_1(u_l) = V'_{m_1}(u_l)$ with $u_l = \pi$. And $h = h_{\max}$ is the maximal h value for any g less than $g_{\max}$. For $h \uparrow h_{\max}$, the length $L_2(g, h)$ will blow up. If $g < g_{\max}$ is fixed, the smallest possible h value is the one associated with the closed curve that touches this g curve. At this point, we have $\mathcal{B}_m = 0$ with $V'_{m_1}(u_m) = V'_{m_2}(u_m)$. Finally, if $h < g_{\max}$ is fixed, the smallest possible g value is the one associated with the g orbit that touches the h orbit at $p = 0$. So here we have $\mathcal{B}_m = 0$ with $p_m = 0$. The g - h region described above is shown in the left plot of Fig. 6. Note that at the bottom left point $(g_{\min}(\gamma), h_{\min}(\gamma))$, we have $u_m = u_r$ and $p_m = p_r = 0$, meaning $L_2(g_{\min}, h_{\min}) = 0$.

Now we have established the existence of the pinned semi-fluxions, we can look at their stability. Theorem 5.1 gives that $\mathcal{B}_l \mathcal{B}_m \mathcal{B}_r \det(J) = 0$ determines the curve along which the linearisation about

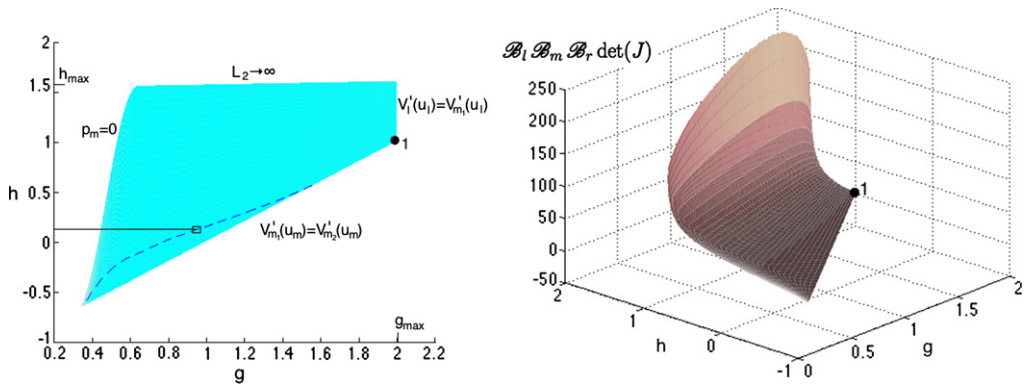


Fig. 6. The left plot shows the g - h region for which our left, middle and right matching points are well defined. At the boundaries the relevant bifurcation criteria are shown. The right plot gives $\mathcal{B}_1 \mathcal{B}_m \mathcal{B}_r \det(J)$ surface in g - h plane for $\gamma = 0.1$. The curve at which $\det(J) = 0$ is indicated by the dashed blue line in the left plot (due to the scale it can't be seen in right figure). The solid circle 1 indicates the small defect-less semi-fluxon (as in Fig. 5), while the open square is the point with $g = 0.924$ and $h = 0.1$. (For interpretation of the references to colour, the reader is referred to the web version of this article.)

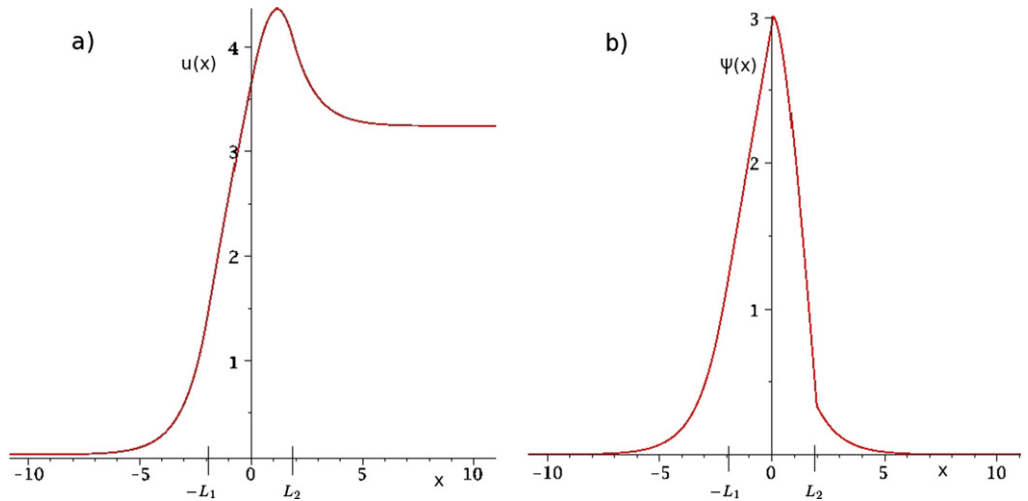


Fig. 7. In the case when $g = 0.924$, $h = 0.1$ and $\gamma = 0.1$, $L_1 \approx 1.966$ and $L_2 \approx 1.904$: (a) The pinned semi-fluxon $u(x; g, h)$ and (b) the eigenfunction associated with the eigenvalue zero.

the pinned semi-fluxon has an eigenvalue zero. Here J denotes the Jacobian of $(L_1, L_2)(g, h)$. The surface $\mathcal{B}_1 \mathcal{B}_m \mathcal{B}_r \det(J)$ for $\gamma = 0.1$ is shown in the right plot of Fig. 6. Of most interest is the curve $\det(J) = 0$ which is indicated in the left plot of Fig. 6 by the dashed blue curve. From [4], it follows that the defect-less pinned semi-fluxon is unstable and its linearisation has one negative eigenvalue. So all nearby pinned semi-fluxon have this property and the number of eigenvalues of the linearisation can only change at the curve with $\det(J) = 0$. To determine whether on this curve the positive eigenvalue becomes zero or an extra positive eigenvalue will be gained, we choose a value of g and h on the $\det(J) = 0$ curve and construct the eigenfunction associated with this eigenvalue zero. This is possible as the proof of Lemma A.7 is constructive and includes expressions for all of the coefficients in (19). We choose $g = 0.924$ and $h = 0.1$, indicated by the dot in Fig. 6. Both the non-monotonic semi-fluxon and its eigenfunction are shown in Fig. 7.

We can see that the eigenfunction is strictly positive, thus, by Sturm-Liouville theory, the eigenvalue zero is the largest one. This means that as g or h moves from above the $\det(J) = 0$ curve to

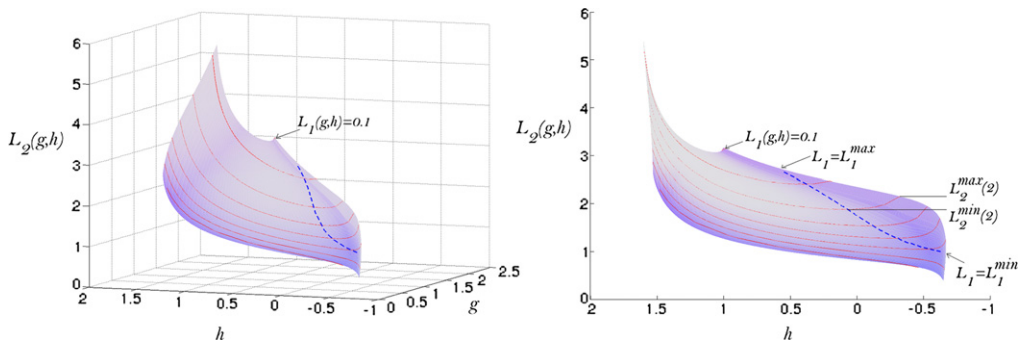


Fig. 8. The surface $L_2(g, h)$. The solid red curves correspond to level sets of L_1 , the upper curve has $L_1 = 0.1$, then $L_1 = 1, 2, \dots, 7$. The dashed blue line is the curve for which the Jacobian vanishes, i.e. for these values there is an eigenvalue zero. The diagram on the right is the projection of $L_2(g, h)$ on the $g = 0$ plane and it shows that the curve of these points intersect the L_1 level sets at the minimal possible value of L_2 . (For interpretation of the references to colour, the reader is referred to the web version of this article.)

below, the linearisation about the semi-fluxon loses the positive eigenvalue. Hence the semi-fluxons below the $\det(J) = 0$ curve are stable. In other words, the defect has indeed stabilised the semi-fluxon.

To illustrate for which values of L_1 and L_2 this stable non-monotonic semi-fluxon exists, Fig. 8 shows the L_2 surface, with fixed L_1 level sets in solid red curves and the $g-h$ curve at which $\det(J) = 0$ as a dashed blue curve. From Corollary 5.2 we know that the change of stability occurs at an extremal value of L_2 for L_1 fixed. In Fig. 8, we see that the extremum is a minimum. Thus for $\gamma = 0.1$, there is an interval $(L_1^{\min}, L_1^{\max})$ such that for each $L_1 \in (L_1^{\min}, L_1^{\max})$, there is a minimal and a maximal L_2 value, say $L_2^{\min}(L_1)$ and $L_2^{\max}(L_1)$ such that for $L_2^{\min}(L_1) < L_2 < L_2^{\max}(L_1)$, there exists both a stable and an unstable pinned semi-fluxon. For $L_1 < L_1^{\min}$ or $L_1 > L_1^{\max}$, all pinned semi-fluxons are unstable. Using continuity in γ , we can conclude the following.

Lemma 5.5. *For γ near 0.1, there exist some $0 < L_1^{\min}(\gamma) < L_1^{\max}(\gamma)$ such that for each $L_1 \in (L_1^{\min}(\gamma), L_1^{\max}(\gamma))$, there exists an interval of L_2 values such that the $0-\pi$ junction with a defect of length L_1 , placed at distance L_2 from the $0-\pi$ junction, has a stable non-monotonic semi-fluxon.*

To illustrate that the change of stability occurs at a minimal value of L_2 , we have numerically determined the largest eigenvalue for the linearisation about the pinned semi-fluxons for a defect with length $L_1 = 1$ and $\gamma = 0.1$, $\alpha = 0$ as function of the distance L_2 , see Fig. 9. The numerical procedure is the same as that in [4]. The presence of two branches of curves corresponding to two different solutions can be easily noted in the figure. One of the solutions, i.e. the lower branch, is stable indicated by $\Lambda < 0$. The branches disappear and collide at a saddle-node bifurcation at a critical value $L_2^{\min}$, which for the parameter values used in the figure is $L_2 \approx 2.5$. This is in agreement with Lemma 5.5.

Finally, we also depict in Fig. 10 several profiles of the numerically obtained pinned fluxons along the two branches at the parameter values of L_2 indicated as big dots in Fig. 9. All the pinned fluxons are clearly non-monotonic. We also plot in the insets the eigenvalues of the fluxons in the complex plane, where stability is indicated by the absence of eigenvalues with non-zero real parts, i.e. $\Lambda < 0$. It is clear that in each panel in Fig. 10, only one of the fluxon pairs has no eigenvalues with non-zero real parts, confirming that there is one non-monotonous stable pinned fluxon.

6. Stability result with N middle intervals

The results presented so far are for potentials with one or two middle intervals. Whilst there are lots of technicalities in the proofs, the main ideas can be extended to the general case of N middle

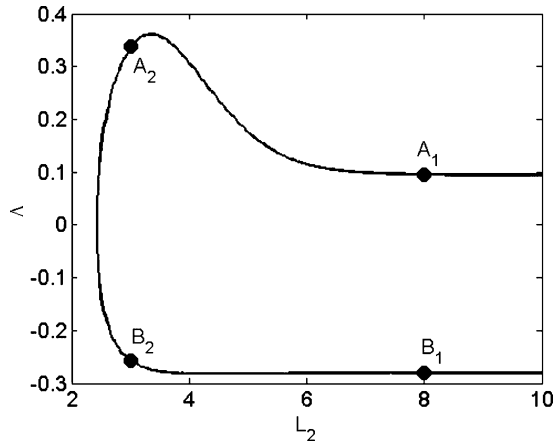


Fig. 9. A numerical evaluation of the critical eigenvalue λ versus the position of the inhomogeneity L_2 with the length of the defect $L_1 = 1$ and $\gamma = 0.1$, $\alpha = 0$. Note the critical eigenvalue indeed crosses through zero at the minimal value of L_2 . In Fig. 10, the semi-fluxons associated with the points A_i and B_i are visualised together with their temporal eigenvalues.

intervals. The previous results that we presented for one and two middle intervals (Theorems 4.5 and 5.1) used the Jacobian of the lengths of the intervals as functions of Hamiltonians and the bifurcation points. To generalise these results to N middle intervals, with potentials $V_{m_1}, \dots, V_{m_N}$, we must also generalise these functions. Generalising the bifurcation points is straightforward: If the N middle intervals I_i are defined as $I_i = (\chi_{i-1}, \chi_i)$, then

$$\mathcal{B}_i = p(\chi_i) [V'_{m_{i+1}}(u(\chi_i)) - V'_{m_i}(u(\chi_i))] \quad \text{for } i = 1, \dots, N-1$$

whilst for the left and right bifurcation functions we have

$$\mathcal{B}_0 := \mathcal{B}_l = p(\chi_0) [V'_{m_1}(u(\chi_0)) - V'_l(u(\chi_0))]$$

and

$$\mathcal{B}_N := \mathcal{B}_r = p(\chi_N) [V'_r(u(\chi_N)) - V'_{m_N}(u(\chi_N))].$$

The length of each of the middle intervals is defined as $L_i := \chi_i - \chi_{i-1}$. In each interval I_i there will be a value of the Hamiltonian h_i , (11). The length L_i will depend on h_i as well as the values of the Hamiltonians in the adjacent intervals via χ_{i-1} and χ_i . Specifically, we have the following dependence of L_i on h_j :

$$L_1 = L_1(h_1, h_2), \quad L_i = L_i(h_{i-1}, h_i, h_{i+1}), \text{ for } i = 2, \dots, N-1, \quad \text{and} \quad L_N = L_N(h_{N-1}, h_N).$$

This means that the determinant of the Jacobian $\frac{\partial(L_1, \dots, L_N)}{\partial(h_1, \dots, h_N)}$ can be written as Γ_N , where Γ_i is the determinant of an $i \times i$ symmetric triband matrix:

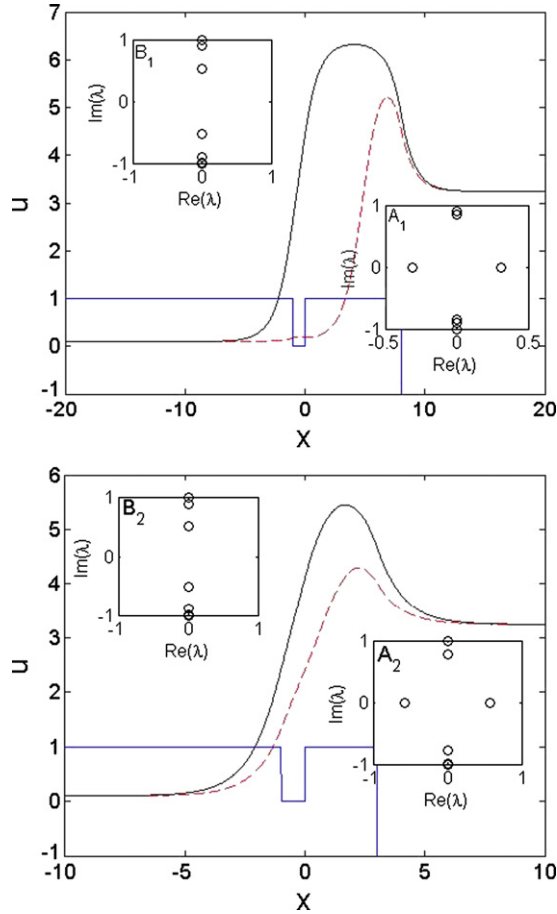


Fig. 10. Two pairs of pinned fluxons admitted by the Josephson system corresponding to the points A_j and B_j , $j = 1, 2$, in Fig. 9. Recall that we fix $\alpha = 0$, $\gamma = 0.1$ and $L_1 = 1$. Unstable profiles are shown in dashed lines. The function $D(x)$ multiplying the sine term in the governing equation is also presented (blue lines). The insets show the eigenvalues of each fluxon. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$\Gamma_i := \det \begin{pmatrix} \frac{\partial L_1}{\partial h_1} & \frac{1}{\mathcal{B}_1} & 0 & 0 & \dots & 0 \\ \frac{1}{\mathcal{B}_1} & \frac{\partial L_2}{\partial h_2} & \frac{1}{\mathcal{B}_2} & 0 & \ddots & 0 \\ 0 & \frac{1}{\mathcal{B}_2} & \frac{\partial L_3}{\partial h_3} & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \frac{1}{\mathcal{B}_{i-1}} \\ 0 & 0 & 0 & 0 & \frac{1}{\mathcal{B}_{i-1}} & \frac{\partial L_i}{\partial h_i} \end{pmatrix}, \quad i \geqslant 3.$$

We used here that $\frac{\partial L_i}{\partial h_{i-1}} = \frac{1}{\mathcal{B}_{i-1}}$, $i = 2, \dots, N$ and $\frac{\partial L_i}{\partial h_{i+1}} = \frac{1}{\mathcal{B}_i}$, $i = 1, \dots, N - 1$, see (42). Note that we can extend the definition of Γ_i naturally to

$$\Gamma_2 = \det \begin{pmatrix} \frac{\partial L_1}{\partial h_1} & \frac{1}{\mathcal{B}_1} \\ \frac{1}{\mathcal{B}_1} & \frac{\partial L_2}{\partial h_2} \end{pmatrix} \quad \text{and} \quad \Gamma_1 = \frac{\partial L_1}{\partial h_1},$$

which are the definition of the Jacobian resp. derivative used in the results for the two and one middle intervals case. In the upcoming theorem, we will restrict ourselves to a generalisation of Theorems 4.5 and 5.1, i.e., for the case that none of the bifurcation functions vanishes, and show that there is a direct relation between having a zero eigenvalue for the linearised stability problem and having a zero in the determinant Γ_N . We refrain from going into the details associated to having zeroes in the bifurcation functions.

We have already shown for one or two middle intervals ($N = 1$ or $N = 2$) that these cases can be understood by reading the results as a limit. This was quite a technical enterprise, which will only become more technical and so will be hard to repeat for general N . We conjecture that with the methods presented here, the following result also holds at bifurcation points when read in a limit (as in Theorems 4.5 and 5.1).

Theorem 6.1. *Let $u(x)$ be a stationary front solution of the inhomogeneous non-linear wave equation (3), i.e.,*

$$u_{xx} = -\frac{\partial V(u, x; I_r, I_1, \dots, I_N, I_r)}{\partial u},$$

such that the bifurcation points $\mathcal{B}_i \neq 0$ for $i = 0, \dots, N$. The linearisation about the front has an eigenvalue zero if and only if

$$\Gamma_N \prod_{i=0}^N \mathcal{B}_i = 0.$$

Here Γ_N is the determinant of the Jacobian $\frac{\partial(L_1, \dots, L_N)}{\partial(h_1, \dots, h_N)}$.

Proof. The statement has already been proved for $N = 1$ and $N = 2$ in Theorem 4.5 and Theorem 5.1 respectively. Thus, in the remainder of the proof we will assume that $N \geq 3$. Also we assume that $\mathcal{B}_i \neq 0$ for $i = 0, \dots, N$.

If $u(x)$ is monotonic for $x \in [\chi_0, \chi_N]$ (i.e. $u_x(x)$ has no zeroes in the middle intervals) then the eigenfunction associated with the eigenvalue zero is

$$\psi = \begin{cases} u_x, & x < \chi_0; \\ a_i u_x + b_i u_x \int_{\chi_{i-1}}^x \frac{d\xi}{u_x^2(\xi)}, & \chi_{i-1} < x < \chi_i \text{ for } i = 1, \dots, N; \\ \hat{k} u_x, & x > \chi_N. \end{cases}$$

Matching the value of ψ and its derivative at $x = \chi_0$ gives $a_1 = 1$ and $b_1 = \mathcal{B}_0$. Matching at $x = \chi_i$ for $i = 1, \dots, N - 1$ gives

$$a_{i+1} = a_i + b_i \int_{\chi_{i-1}}^{\chi_i} \frac{d\xi}{u_x^2(\xi)} \quad \text{and} \quad b_{i+1} = b_i + \mathcal{B}_i a_{i+1}, \quad i = 1, \dots, N - 1. \quad (46)$$

Finally matching at $x = \chi_N$ gives the compatibility criterion for the existence of an eigenvalue zero

$$b_N = -\mathcal{B}_N \left[a_N + b_N \int_{\chi_{N-1}}^{\chi_N} \frac{d\xi}{u_x^2(\xi)} \right].$$

As before it can be shown that

$$\frac{\partial L_i}{\partial h_i} = -\frac{1}{\mathcal{B}_{i-1}} - \frac{1}{\mathcal{B}_i} - \int_{\chi_{i-1}}^{\chi_i} \frac{d\xi}{u_x^2(\xi)}, \quad i = 1, \dots, N, \quad (47)$$

see (32). Furthermore, (46) gives $a_i = \frac{b_i - b_{i-1}}{\mathcal{B}_{N-1}}$, for $i = 2, \dots, N$. Using these two relations with $i = N$ shows that the compatibility condition can be re-written as

$$0 = -\mathcal{B}_N \mathcal{B}_{N-1} \left[\frac{\partial L_N}{\partial h_N} \frac{b_N}{\mathcal{B}_{N-1}} + \left(\frac{1}{\mathcal{B}_{N-1}} \right)^2 b_{N-1} \right]. \quad (48)$$

Similarly, using the second relation in (46) to remove a_{i+1} and a_i in the first relation of (46) and using (47) to remove the integral in the first relation (46) gives a second order recursion relation for b_i :

$$\begin{aligned} b_i &= -\mathcal{B}_{i-1} \mathcal{B}_{i-2} \left[\frac{\partial L_{i-1}}{\partial h_{i-1}} \frac{b_{i-1}}{\mathcal{B}_{i-2}} + \left(\frac{1}{\mathcal{B}_{i-2}} \right)^2 b_{i-2} \right], \quad i = 3, \dots, N; \\ b_2 &= -\mathcal{B}_1 \mathcal{B}_0 \frac{\partial L_1}{\partial h_1}; \quad b_1 = \mathcal{B}_0. \end{aligned} \quad (49)$$

To link the compatibility condition (48) with b_i as given recursively by (49), we consider the determinants Γ_i and show that they satisfy a similar recursion relation. Due to the symmetric triband nature of Γ_i it is possible to write Γ_i in terms of sub-determinants, that is, in terms of Γ_{i-1} and Γ_{i-2} . We have the following recursion relation:

$$\Gamma_i = \frac{\partial L_i}{\partial h_i} \Gamma_{i-1} - \left(\frac{1}{\mathcal{B}_{i-1}} \right)^2 \Gamma_{i-2}, \quad i \geq 2; \quad \Gamma_1 = \frac{\partial L_1}{\partial h_1} \quad \text{and} \quad \Gamma_0 = 1.$$

This looks very similar to the previous recursion relation. In fact if we define

$$c_i = (-1)^{i-1} \Gamma_{i-1} \prod_{j=0}^{i-1} \mathcal{B}_j, \quad i \geq 1,$$

then

$$c_1 = \Gamma_0 \mathcal{B}_0 = \mathcal{B}_0; \quad c_2 = -\Gamma_1 \mathcal{B}_0 \mathcal{B}_1 = -\mathcal{B}_0 \mathcal{B}_1 \frac{\partial L_1}{\partial h_1}.$$

Writing the relation between c_i and Γ_{i-1} as $\Gamma_{i-1} = (-1)^{i-1} c_i / \prod_{j=0}^{i-1} \mathcal{B}_j$ shows that the recursion for c_i is

$$c_i = -\mathcal{B}_{i-1} \mathcal{B}_{i-2} \left[\frac{\partial L_{i-1}}{\partial h_{i-1}} \frac{c_{i-1}}{\mathcal{B}_{i-2}} + \left(\frac{1}{\mathcal{B}_{i-2}} \right)^2 c_{i-2} \right], \quad i \geq 3.$$

In other words, c_i and b_i satisfy exactly the same recursion relation, hence $b_i = c_i = (-1)^{i-1} \Gamma_{i-1} \times \prod_{j=0}^{i-1} \mathcal{B}_j$, for $i \geq 1$. The compatibility condition (48) gives that there is an eigenvalue zero if and only if

$$\begin{aligned}
0 &= -\mathcal{B}_N \mathcal{B}_{N-1} \left[\frac{\partial L_N}{\partial h_N} \frac{c_N}{\mathcal{B}_{N-1}} + \left(\frac{1}{\mathcal{B}_{N-1}} \right)^2 c_{N-1} \right] \\
&= -\mathcal{B}_N \mathcal{B}_{N-1} \prod_{j=0}^{N-2} \mathcal{B}_j \left[\frac{\partial L_N}{\partial h_N} \Gamma_{N-1} - \left(\frac{1}{\mathcal{B}_{N-1}} \right)^2 \Gamma_{N-2} \right] \\
&= \Gamma_N \prod_{j=0}^N \mathcal{B}_j,
\end{aligned}$$

with the recursion relation for Γ_N . So we have proved our theorem for a monotonic front $u(x)$.

Next we consider the case that u is not monotonic, i.e., in each middle interval I_j , $j = 1, \dots, N$, the function $u_x(x)$ has ν^j zeroes at position x_k^j , $k = 1, \dots, \nu^j$. The eigenfunction $\psi(x)$ has to be altered in each interval I_j . Instead of

$$a_j u_x + b_j u_x \int_{\chi_{j-1}}^x \frac{d\xi}{u_x^2(\xi)} \quad \text{for } \chi_{j-1} < x < \chi_j$$

the eigenfunction in the interval I_j , $j = 1, \dots, N$, becomes

$$A_k^j u_x(x) + B_k^j u_x(x) \int_{M_k^j}^x \frac{d\xi}{u_x^2(\xi)} \quad \text{for } x_k^j < x < x_{k+1}^j \text{ and } k = 0, \dots, \nu^j;$$

where $x_0^j = \chi_{j-1}$, $x_{\nu^j+1}^j = \chi_j$ and $M_k^j = \frac{x_k^j + x_{k+1}^j}{2}$. By matching ψ and its derivative at the points x_k^j for $k = 1, \dots, \nu^j$ we have that for $j = 1, \dots, N$ (see (25) and (28))

$$B_k^j = B_{k-1}^j =: b_j, \quad \text{for } k = 1, \dots, \nu^j, \quad \text{and} \quad A_{\nu^j}^j = A_0^j + b_j \left[\sum_{k=1}^{\nu^j} G_k(M_{k-1}^j, M_k^j) \right].$$

Note that we have introduced the notation b_j for the value of the parameters B_k^j in the interval I_j . If $j = 1$, then matching at $\chi_0 = -L$ gives $A_0^1 = 1$ and $b_1 = \mathcal{B}_0$. If $1 < j \leq N$ then we note that matching at $x = \chi_{j-1}$ is similar to matching at the middle matching point in the two intervals case, see (63) and (64). So we get

$$\begin{aligned}
A_{\nu^{j-1}}^{j-1} + b_{j-1} \int_{M_{\nu^{j-1}}^{j-1}}^{\chi_{j-1}} \frac{d\xi}{u_x^2(\xi)^2} &= A_0^j + b_j \int_{M_0^j}^{\chi_{j-1}} \frac{d\xi}{u_x^2(\xi)^2}, \quad j = 2, \dots, N; \\
b_j &= b_{j-1} + \mathcal{B}_{j-1} \left[A_0^j + b_j \int_{M_0^j}^{\chi_{j-1}} \frac{d\xi}{u_x^2(\xi)^2} \right], \quad j = 2, \dots, N.
\end{aligned} \tag{50}$$

Finally we match $\psi(x)$ at the right matching point $x = \chi_N$. This gives the compatibility condition

$$\begin{aligned}
b_N &= -\mathcal{B}_N \left[A_{\nu_N}^N + b_N \int_{M_{\nu_N}^N}^{\chi_N} \frac{d\xi}{u_x^2(\xi)^2} \right] \\
&= -\mathcal{B}_N \left[A_0^N + b_N \left[\sum_{k=1}^{\nu_N} G_k(M_{k-1}^N, M_k^N) + \int_{M_{\nu_N}^N}^{\chi_N} \frac{d\xi}{u_x^2(\xi)^2} \right] \right] \\
&= -\mathcal{B}_N \left[\frac{b_N - b_{N-1}}{\mathcal{B}_{N-1}} + b_N \left[\int_{\chi_{N-1}}^{M_0^N} \frac{d\xi}{u_x^2(\xi)^2} + \sum_{k=1}^{\nu_N} G_k(M_{k-1}^N, M_k^N) + \int_{M_{\nu_N}^N}^{\chi_N} \frac{d\xi}{u_x^2(\xi)^2} \right] \right],
\end{aligned}$$

where we used (50) with $j = N$ to get an expression for A_0^N in terms of b_N and b_{N-1} .

From (33) we can see that for $j = 1, \dots, N$,

$$\frac{\partial L_j}{\partial h_j} = -\frac{1}{\mathcal{B}_j} - \frac{1}{\mathcal{B}_{j-1}} - \left[\int_{\chi_{j-1}}^{M_0^j} \frac{d\xi}{u_x^2(\xi)^2} + \sum_{k=1}^{\nu_j} G_k(M_{k-1}^j, M_k^j) + \int_{M_{\nu_j}^j}^{\chi_j} \frac{d\xi}{u_x^2(\xi)^2} \right].$$

Using this in the compatibility condition shows that the compatibility equation can be written as (48). Similarly, using it in the relations for A_0^j , $A_{\nu_j}^j$ and b_j shows that b_j satisfies the recursion relation (49). Hence this completes the proof for a general front $u(x)$ with $\mathcal{B}_i \neq 0$. $\square$

7. Conclusion and discussion

In this paper we have studied a non-linear wave equation with spatial inhomogeneity in the form of a step function with $N + 2$ intervals. We have considered the existence and stability of front-like C^1 -smooth stationary solutions. Our main result concerns an explicit relation between occurrence of zero eigenvalues in the linearised stability problem associated to the stability of the front and the zeroes of the product of the determinant Γ_N of a Jacobian associated to the length functions of the inner N intervals with $N + 1$ bifurcations curves $\mathcal{B}_i$ (Theorem 6.1). The method by which we have obtained this result is completely constructive, in the sense that we have explicitly constructed the eigenfunction corresponding to the zero eigenvalue. As a consequence, and with the help of classical Sturm–Liouville theory, this result can be used to also determine whether the zero eigenvalue indeed marks the transition from a stable state to an unstable one, or vice versa – in that case the zero eigenvalue is the critical eigenvalue and its corresponding eigenfunction thus cannot have any zeroes.

There are several ways in which the insights of the present work can be further extended. First of all, our analysis does not give direct information about whether there indeed is an eigenvalue that crosses through zero as the product of Γ_N and the $\mathcal{B}_i$'s is zero: the zero may be degenerate and the eigenvalue may not change sign. This does not occur in the explicit examples – based on Josephson junctions and with $N = 1$ or 2 – we have considered here, but in general this is likely to be possible. The precise relation between such a possible degeneracy and the structure of the length functions, and especially the implications of such a degeneracy for the stability of front-type solutions near it and the potential for hysteresis loops, is the subject of work in progress.

Another natural question is whether a result like Theorem 6.1 may also be formulated for systems with a spatial inhomogeneity of stepfunction type that are not of the form (3). In fact, spatial inhomogeneities of stepfunction type have recently also been considered in the context of systems of coupled non-linear wave equations, non-linear Schrödinger equations and reaction–diffusion equations [6,12,

14–16,21,22,34] by methods that are similar to the methods employed here. However, a general, abstract, result like Theorem 6.1 has not been obtained in a setting beyond that of the non-linear wave equation (3). It would be very interesting to investigate whether an equivalent of Theorem 6.1 could be derived for more general and/or complex systems than the non-linear wave equation (3).

Acknowledgment

Christopher Knight was partially supported in this work by an EPSRC Doctoral Training Grant: EP/P50404X/1.

Appendix A

A.1. Observations on solutions of planar Hamiltonian systems

In this section we consider a Hamiltonian ODE in the plane:

$$u_{xx} + V'(u) = 0, \quad (51)$$

where V is a smooth C^3 function. We will derive some properties of solutions and of integrals of solutions.

Lemma A.1. *Let $u(x)$ be a solution of the Hamiltonian system and let x_0 and $\delta > 0$ be such that*

$$u_x(x_0) = 0, \quad u_{xx}(x_0) \neq 0 \quad \text{and} \quad u_x(x) \neq 0 \quad \text{for all } x \in [x_0 - \delta, x_0 + \delta] \setminus \{x_0\}.$$

Then

- (i) *for all $|\xi| < \delta$, $u(x_0 - \xi) = u(x_0 + \xi)$ and $u_x(x_0 - \xi) = -u_x(x_0 + \xi)$;*
- (ii) *there is some $K > 0$ such that $|\frac{1}{u_x^2(\xi)} - \frac{1}{u_{xx}^2(x_0)(\xi - x_0)^2}| \leq K$ for all $|\xi - x_0| < \delta$;*
- (iii) $\lim_{\epsilon \downarrow 0} u_x(x_0 - \epsilon) \int_{x_0 - \delta}^{x_0 - \epsilon} \frac{d\xi}{u_x^2(\xi)} = -\frac{1}{u_{xx}(x_0)} = \lim_{\epsilon \downarrow 0} u_x(x_0 + \epsilon) \int_{x_0 + \delta}^{x_0 + \epsilon} \frac{d\xi}{u_x^2(\xi)}$;
- (iv) $\lim_{\epsilon \downarrow 0} \int_{x_0 - \delta}^{x_0 - \epsilon} [\frac{1}{u_x^2(\xi)} - \frac{1}{u_{xx}^2(x_0)(\xi - x_0)^2}] d\xi = \tilde{G}(x_0 - \delta, x_0) + \frac{1}{\delta u_{xx}^2(x_0)}$;
- (v) $\lim_{\epsilon \downarrow 0} [\int_{x_0 - \delta}^{x_0 - \epsilon} \frac{d\xi}{u_x^2(\xi)} + \frac{1}{u_x(x_0 - \epsilon)u_{xx}(x_0)}] = \tilde{G}(x_0 - \delta, x_0)$;
- (vi) $\lim_{\epsilon \downarrow 0} [\int_{x_0 + \delta}^{x_0 + \epsilon} \frac{d\xi}{u_x^2(\xi)} + \frac{1}{u_x(x_0 + \epsilon)u_{xx}(x_0)}] = \tilde{G}(x_0 + \delta, x_0)$.

Proof. (i) The first observation follows directly from the existence and uniqueness theorem for ODEs as both $v_+(\xi) = u(x_0 + \xi)$ and $v_-(\xi) = u(x_0 - \xi)$ are solutions of the initial value problem

$$v_{xx} + V'(v) = 0, \quad v(0) = u(x_0), \quad v_x(0) = 0.$$

Hence $v_+(\xi) = v_-(\xi)$ and $v'_+(\xi) = -v'_-(\xi)$, which implies $u_x(x_0 + \xi) = -u_x(x_0 - \xi)$.

To prove the rest of the statements, the Taylor expansion for u_x about x_0 is used. For any $|\xi| < \delta$, there is some $0 < c(\xi) < 1$ such that

$$\begin{aligned} u_x(x_0 + \xi) &= \xi u_{xx}(x_0) + \frac{1}{2} \xi^2 u_{xxx}(x_0) + \frac{1}{6} \xi^3 u_{xxxx}(x_0 + c\xi) \\ &= \xi u_{xx}(x_0) + \frac{1}{6} \xi^3 u_{xxxx}(x_0 + c\xi), \end{aligned}$$

as $u_{xxx}(x_0) = V''(u(x_0))u_x(x_0) = 0$. Furthermore, $u_{xxxx} = -V'''(u)u_x^2 + V''(u)V'(u)$ and $V \in C^3$. This means that there is some $K > 0$ such that for all $|\xi| < \delta$

$$|u_x(x_0 + \xi) - \xi u_{xx}(x_0)| \leq K\xi^3, \quad \left| \frac{1}{u_x(x_0 + \xi)} - \frac{1}{\xi u_{xx}(x_0)} \right| \leq K\xi$$

and

$$\left| \frac{1}{u_x^2(x_0 + \xi)} - \frac{1}{\xi^2 u_{xx}^2(x_0)} \right| \leq K.$$

(ii) This follows immediately from above.

(iii) Using the estimates above, it follows for small $\epsilon > 0$ that

$$\int_{-\delta}^{-\epsilon} \left| \frac{1}{u_x^2(x_0 + \xi)} - \frac{1}{\xi^2 u_{xx}^2(x_0)} \right| d\xi \leq \int_{-\delta}^{-\epsilon} K d\xi = K(-\epsilon + \delta)$$

and

$$\begin{aligned} u_x(x_0 - \epsilon) \int_{x_0 - \delta}^{x_0 - \epsilon} \frac{d\xi}{u_x^2(\xi)} &= (-\epsilon u_{xx}(x_0) + O(\epsilon^3)) \left(\int_{-\delta}^{-\epsilon} \frac{d\xi}{\xi^2 u_{xx}^2(x_0)} + O(1) \right) \\ &= -\epsilon u_{xx}(x_0) \left(\frac{1}{\epsilon u_{xx}^2(x_0)} - \frac{1}{\delta u_{xx}^2(x_0)} \right) + O(\epsilon) \\ &= -\frac{1}{u_{xx}(x_0)} + O(\epsilon). \end{aligned}$$

Since $u_x(x_0 + \xi)$ is even in ξ , the expression for the other integral follows immediately.

(iv) The uniform boundedness of $\frac{1}{u_x^2(x_0 + \xi)} - \frac{1}{u_{xx}^2(x_0)\xi^2}$ implies the existence of the limit

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \int_{-\delta}^{-\epsilon} \left[\frac{1}{u_x^2(x_0 + \xi)} - \frac{1}{u_{xx}^2(x_0)\xi^2} \right] d\xi &= \lim_{\epsilon \downarrow 0} \int_{x_0 - \delta}^{x_0 - \epsilon} \left[\frac{1}{u_x^2(\xi)} - \frac{1}{u_{xx}^2(x_0)(\xi - x_0)^2} \right] d\xi \\ &= \tilde{G}(x_0 - \delta, x_0) + \frac{1}{\delta u_{xx}^2(x_0)}, \end{aligned}$$

where the last line follows from the definition of $\tilde{G}$, (21).

(v) We have

$$\begin{aligned} \int_{x_0 - \delta}^{x_0 - \epsilon} \frac{d\xi}{u_x^2(\xi)} + \frac{1}{u_{xx}(x_0)u_x(x_0 - \epsilon)} &= \int_{x_0 - \delta}^{x_0 - \epsilon} \frac{d\xi}{u_x^2(\xi)} - \left(\frac{1}{\epsilon u_{xx}^2(x_0)} - \frac{1}{\delta u_{xx}^2(x_0)} \right) - \frac{1}{\delta u_{xx}^2(x_0)} \\ &\quad + \frac{1}{u_{xx}(x_0)} \left(\frac{1}{\epsilon u_{xx}(x_0)} + \frac{1}{u_x(x_0 - \epsilon)} \right) \\ &= \int_{x_0 - \delta}^{x_0 - \epsilon} \left[\frac{1}{u_x^2(\xi)} - \frac{1}{u_{xx}^2(x_0)(\xi - x_0)^2} \right] d\xi - \frac{1}{\delta u_{xx}^2(x_0)} \\ &\quad + \frac{1}{u_{xx}(x_0)} \left(\frac{1}{\epsilon u_{xx}(x_0)} + \frac{1}{u_x(x_0 - \epsilon)} \right). \end{aligned}$$

Thus

$$\left| \int_{x_0-\delta}^{x_0-\epsilon} \frac{d\xi}{u_x^2(\xi)} + \frac{1}{u_{xx}(x_0)u_x(x_0-\epsilon)} - \int_{x_0-\delta}^{x_0-\epsilon} \left[\frac{1}{u_x^2(\xi)} - \frac{1}{u_{xx}^2(x_0)(\xi-x_0)^2} \right] d\xi + \frac{1}{\delta u_{xx}^2(x_0)} \right| \\ \leq \left| \frac{1}{u_{xx}(x_0)} \left(\frac{1}{\epsilon u_{xx}(x_0)} + \frac{1}{u_x(x_0-\epsilon)} \right) \right| \leq \frac{K\epsilon}{|u_{xx}(x_0)|}.$$

Therefore, by part (iii),

$$\lim_{\epsilon \downarrow 0} \left[\int_{x_0-\delta}^{x_0-\epsilon} \frac{d\xi}{u_x^2(\xi)} + \frac{1}{u_{xx}(x_0)u_x(x_0-\epsilon)} \right] = \tilde{G}(x_0-\delta, x_0).$$

(vi) Using that $u_x(x_0+\xi) = -u_x(x_0-\xi)$ for $|\xi| < \delta$ we have the last equality. $\square$

A.2. Proof of Lemma 4.3

In this section we consider the function $I(g)$ defined in (30), i.e.,

$$I(g) := \int_{u(y(g), g)}^{u(x_i(g), g)} \frac{du}{p(u, g)},$$

where $x_i(g)$ and $y(g)$ are in the middle interval, $x_i(g)$ is a zero of u_x , $y(g)$ is a smooth function of g such that $u_x(y(g), g) \neq 0$, and u_x has fixed sign for x between $y(g)$ and $x_i(g)$. The statement in Lemma 4.3 is

$$I'(g) = -\tilde{G}(y(g), x_i(g)) - \frac{1}{u_x(y(g), g)} \frac{d}{dg} u(y(g), g)$$

which is proved as follows.

Proof of Lemma 4.3. We consider two ways of representing the governing equation for u , (10), using different independent variables.

- (i) $\frac{1}{2}u_x^2(x, g) = g - V_m(u(x, g))$, in this representation the independent variables are x and g ;
- (ii) $\frac{1}{2}p^2(u, g) = g - V_m(u)$ whilst in this representation the independent variables are u and g .

During this proof we will use both of these representations so care must be taken that the uses are consistent. First we derive some relations from these representations of the governing equation.

(i) Using the first representation, since x_i is a zero of u_x , $u_x(x_i(g), g) = 0$ for all g . Thus

$$\frac{\partial u_x}{\partial g}(x_i(g), g) + \frac{\partial^2 u}{\partial x^2}(x_i(g), g)x'_i(g) = \frac{du_x}{dg}(x_i(g), g) = 0 \\ \Rightarrow x'_i(g) = \frac{1}{V'_m(u_i(g))} \frac{\partial u_x}{\partial g}(x_i(g), g). \quad (52)$$

Further differentiating the first representation with respect to g gives, for all x between $x_i(g)$ and $y(g)$,

$$u_x(x, g) \frac{\partial u_x}{\partial g}(x, g) = 1 - V'_m(u(x, g)) \frac{\partial u}{\partial g}(x, g). \quad (53)$$

(ii) The second relation can also be differentiated in this manner, remembering that u is now considered independent of g to give, for all u ,

$$p(u, g) \frac{\partial p}{\partial g}(u, g) = 1. \quad (54)$$

We proceed by using the fundamental law of calculus to conclude that for any x between $y(g)$ and $x_i(g)$

$$x - y(g) = \int_{y(g)}^x dx = \int_{u(y(g), g)}^{u(x, g)} \frac{dv}{p(v, g)}. \quad (55)$$

From the definition of $I(g)$ it can be seen that $I(g) = x_i(g) - y(g)$. Thus $I'(g) = x'_i(g) - y'(g)$.

Differentiating (55) with respect to g and using (54) we obtain, for any x between $x_i(g)$ and $y(g)$, that

$$\begin{aligned} -y'(g) &= \frac{1}{p(u(x, g), g)} \frac{\partial u}{\partial g}(x, g) - \frac{1}{p(u(y(g), g), g)} \left[u_x(y(g), g) y'(g) + \frac{\partial u}{\partial g}(y(g), g) \right] \\ &\quad - \int_{u(y(g), g)}^{u(x, g)} \frac{dv}{p^3(v, g)}. \end{aligned}$$

This equation can be re-written using the transformation of independent variables $(u, g) \rightarrow (x, g)$.

$$-y'(g) = \frac{1}{u_x(x, g)} \frac{\partial u}{\partial g}(x, g) - \frac{1}{u_x(y(g), g)} \left[u_x(y(g), g) y'(g) + \frac{\partial u}{\partial g}(y(g), g) \right] - \int_{y(g)}^x \frac{d\xi}{u_x^2(\xi, g)}.$$

Notice that the two terms involving $y'(g)$ cancel, leaving

$$\frac{\partial u}{\partial g}(x, g) = u_x(x, g) \left(\frac{1}{u_x(y(g), g)} \frac{\partial u}{\partial g}(y(g), g) + \int_{y(g)}^x \frac{d\xi}{u_x^2(\xi, g)} \right).$$

Combining this with expression (53) for $\frac{\partial u_x}{\partial g}(x, g)$ and taking the limit for $x \rightarrow x_i(g)$, using Lemma A.1 (iv)/(v), gives that

$$\frac{\partial u_x}{\partial g}(x_i(g), g) = -V'_m(u(x_i(g), g)) \left(\frac{1}{u_x(y(g), g)} \frac{\partial u}{\partial g}(y(g), g) + \tilde{G}(y(g), x_i(g)) \right).$$

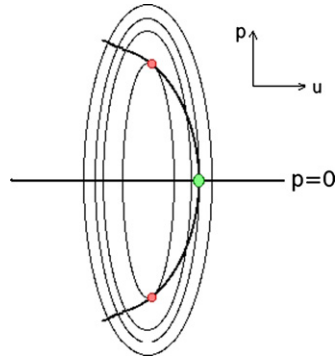


Fig. 11. The bold curve represents part of the unstable manifold of $u_{-\infty}$ in the left interval whilst the other curves represent the orbits of the dynamics in the first middle interval I_1 parametrised by g . The intersection of the unstable manifold with the orbits in the middle interval creates a curve of left intersection points $(u_l(g), p_l(g))$. Two bifurcation points are highlighted: $\hat{p} = 0$ is shown in green; whilst the points where the dynamics in the two intervals have the same curvature at their intersection, $V'_{m_1}(\hat{u}) = V'_l(\hat{u})$, are shown in red. (For interpretation of the references to colour, the reader is referred to the web version of this article.)

Finally (52) gives

$$x'_i(g) = -\tilde{G}(y(g), x_i(g)) - \frac{1}{u_x(y(g), g)} \frac{\partial u}{\partial g}(y(g), g).$$

Hence we have the following expression for $I'(g) = x'_i(g) - y'(g)$,

$$I'(g) = -\tilde{G}(y(g), x_i(g)) - \frac{1}{u_x(y(g), g)} \frac{d}{dg} u(y(g), g). \quad \square$$

A.3. Continuation and bifurcations of matching points

First we consider the left matching point and assume that there are some $\hat{g}$, $\hat{u}$ and $\hat{p}$ such that the first set of equations of (12) are satisfied. As V_l and V_{m_1} are smooth functions, we can expand the expressions in the first set of relations of (12):

$$\begin{aligned} g - V_- + V_l(u) - V_{m_1}(u) &= g - \hat{g} + [V'_l(\hat{u}) - V'_{m_1}(\hat{u})](u - \hat{u}) + \frac{1}{2}[V''_l(\hat{u}) - V''_{m_1}(\hat{u})](u - \hat{u})^2 \\ &\quad + \mathcal{O}(|u - \hat{u}|^3); \\ \frac{1}{2}p^2 - V_- + V_l(u) &= \hat{p}(p - \hat{p}) + \frac{1}{2}(p - \hat{p})^2 + V'_l(\hat{u})(u - \hat{u}) + \frac{1}{2}V''_l(\hat{u})(u - \hat{u})^2 \\ &\quad + \mathcal{O}(|u - \hat{u}|^3). \end{aligned} \quad (56)$$

So if $V'_l(\hat{u}) - V'_{m_1}(\hat{u}) \neq 0$, the implicit function theorem gives the local existence of a unique smooth curve $u_l(g)$ satisfying $g - V_- + V_l(u_l(g)) - V_{m_1}(u_l(g)) = 0$ and $u_l(\hat{g}) = \hat{u}$. And if $\hat{p} \neq 0$, the second equation gives the local existence of a unique smooth curve $p_l(g)$ satisfying $\frac{1}{2}p_l^2(g) - V_- + V_l(u_l(g)) = 0$ and $p_l(\hat{g}) = \hat{p}$. In other words, we have shown the first part of Lemma 2.1.

We continue with the case that there is a bifurcation point, that is, we also assume that $\hat{g}$ is such that $\mathcal{B}_l(\hat{g}) = 0$. There are two ways to get $\mathcal{B}_l(\hat{g}) = 0$: the first is that $p_l(\hat{g}) = 0$ (green point in Fig. 11, we have assumed that $\hat{u} \neq u_{-\infty}$) and the second is that $V'_{m_1}(\hat{u}) = V'_l(\hat{u})$ (red points in Fig. 11). Fig. 11 suggests that these bifurcation points occur at minimal or maximal values of g . This means that any limit for $g \rightarrow \hat{g}$ is a one-sided limit. Furthermore we can see from Fig. 11 that if

a left matching points exists for g close to $\hat{g}$ then two left matching points exist. From the second equation in (56), we can see that if $(\hat{u}, \hat{p})$ is such that $\hat{p} = 0$ and $V'_l(\hat{u}) = 0$, then we get a degenerate bifurcation equation for p_l . On the other hand, $\hat{p} = 0$ and $V'_l(\hat{u}) = 0$ means that $(\hat{u}, \hat{p})$ is a fixed point of the V_l -dynamics. As $(\hat{u}, \hat{p})$ has to represent a left matching point, hence connect to $(u_{-\infty}, 0)$, this implies that $(\hat{u}, \hat{p})$ must be $(u_{-\infty}, 0)$. We formalise these observations in the following lemma.

Lemma A.2. Assume that there is a point $(\hat{u}, \hat{p})$ such that the first set of equations of (12) are satisfied for some $g = \hat{g}$, $\mathcal{B}_l(\hat{g}) = 0$, then we have the following unfoldings of the left matching points (u_l, p_l) for g near $\hat{g}$.

- (i) If $\hat{p} \neq 0$ and $V'_{m_1}(\hat{u}) = V'_l(\hat{u})$, then there is a unique curve of left matching points in the (u, u_x) plane nearby $(\hat{u}, \hat{p})$; this curve can be parametrised and related to the g -orbits with a (small) parameter η as follows

$$u_l = \hat{u} + \eta, \quad p_l(g) = \hat{p} + O(\eta), \quad g = \hat{g} + \frac{\eta^2}{2} [V''_{m_1}(\hat{u}) - V''_l(\hat{u})] + O(\eta^3).$$

- (ii) If $\hat{p} = 0$ and $\hat{u} \neq u_{-\infty}$, then there is a unique curve of left matching points in the (u, u_x) plane nearby $(\hat{u}, \hat{p})$; this curve can be parametrised and related to the g -orbits with a (small) parameter η as follows

$$\begin{aligned} p_l(g) &= \eta, \quad u_l = \hat{u} - \frac{\eta^2}{2V'_l(\hat{u})} + O(\eta^4), \\ g &= \hat{g} - \frac{\eta^2 [V'_{m_1}(\hat{u}) - V'_l(\hat{u})]}{2V'_l(\hat{u})} + O(\eta^4), \quad \text{if } V'_{m_1}(\hat{u}) - V'_l(\hat{u}) \neq 0; \\ g &= \hat{g} + \frac{\eta^4}{8[V'_l(\hat{u})]^2} [V''_{m_1}(\hat{u}) - V''_l(\hat{u})] + O(\eta^6), \quad \text{if } V'_{m_1}(\hat{u}) - V'_l(\hat{u}) = 0. \end{aligned}$$

- (iii) If $\hat{p} = 0$ and $\hat{u} = u_{-\infty}$, then there are two smooth curves of left matching points in the (u, u_x) plane nearby $(\hat{u}, \hat{p})$; these curves can be parametrised and related to the g -orbits with a (small) parameter η as follows

$$\begin{aligned} u_l &= \hat{u} + \eta, \quad p_l^\pm(g) = \pm \sqrt{-V''_l(u_{-\infty})} \eta + O(\eta^2), \\ g &= \hat{g} - [V'_l(\hat{u}) - V'_{m_1}(\hat{u})] \eta - \frac{1}{2} [V''_l(\hat{u}) - V''_{m_1}(\hat{u})] \eta^2 + O(\eta^3). \end{aligned}$$

Note that if $V'_l(\hat{u}) - V'_{m_1}(\hat{u}) = 0$, hence $(\hat{u}, \hat{p})$ is a fixed point of the V_{m_1} -dynamics, then the unfolding in g is of order η^2 .

Proof. (i) If $\hat{p} \neq 0$, the implicit function theorem applied to $\frac{1}{2}p^2 + V_l(u) - V_{-\infty} = 0$ gives that there is a unique smooth curve $\tilde{p}_l(u)$ for u near $\hat{u}$ and $\tilde{p}_l(\hat{u}) = \hat{p}$. Substituting $u_l = \hat{u} + \eta$ into (56) and using $V'_{m_1}(\hat{u}) = V'_l(\hat{u})$ gives immediately the relation with g -orbits and the unfolding

$$\begin{aligned} g - \hat{g} &= \frac{\eta^2}{2} [V''_{m_1}(\hat{u}) - V''_l(\hat{u})] + O(\eta^3), \\ p_l(g)^2 - p_l(\hat{g})^2 &= -2V'_l(\hat{u})\eta + O(\eta^2). \end{aligned}$$

(ii) If $\hat{p} = 0$ and $\hat{u} \neq u_{-\infty}$ (thus $V'_l(\hat{u}) \neq 0$), the implicit function theorem applied to $p^2 = 2V_{-\infty} - 2V_l(u)$ gives that there is a unique smooth curve $\tilde{u}_l(p)$ for p near $\hat{p} = 0$ and $\tilde{u}_l(\hat{g}) = \hat{u}$. Substituting

$p_l = \hat{p} + \eta = \eta$ into (56) and using $V'_l(\hat{u}) \neq 0$, gives immediately the unfolding and the relation with g -orbits

$$u - \hat{u} = -\frac{1}{2V'_l(\hat{u})}\eta^2 + \mathcal{O}(\eta^4),$$

$$g - \hat{g} = -\frac{V'_l(\hat{u}) - V'_{m_1}(\hat{u})}{2V'_l(\hat{u})}\eta^2 - \frac{1}{2}[V''_l(\hat{u}) - V''_{m_1}(\hat{u})]\eta^4 + [V'_l(\hat{u}) - V'_{m_1}(\hat{u})]\mathcal{O}(\eta^4) + \mathcal{O}(\eta^6).$$

(iii) If $\hat{p} = 0$ and $\hat{u} = u_{-\infty}$ (thus $V'_l(\hat{u}) = 0$ and $V''_l(\hat{u}) = V''_l(u_{-\infty}) < 0$), then $V_- - V_l(u) \geq 0$ for u near $\hat{u}$. Hence there will be two smooth curves of left matching points $\tilde{p}_l(u)$ which satisfy $p^2 = 2V_- - 2V_l(u)$ and $\tilde{p}_l(\hat{u}) = \hat{p} = 0$. Substituting $u_l = \hat{u} + \eta$ into (56) and the relation above gives the relation with g -orbits and the unfolding

$$g - \hat{g} = -[V'_l(\hat{u}) - V'_{m_1}(\hat{u})]\eta - \frac{1}{2}[V''_l(\hat{u}) - V''_{m_1}(\hat{u})]\eta^2 + \mathcal{O}(\eta^3);$$

$$p^2 = -V''_l(u_{-\infty})\eta^2 + \mathcal{O}(\eta^3). \quad \square$$

The right matching points can be analysed in the same way as the left matching points and this leads to analogue results.

The expressions in Lemma A.2 tell us the relative behaviour of p_l , u_l and g in each of the above cases. They can be used to calculate the behaviour of $\mathcal{B}_l(g)$ and $\frac{\partial L_1}{\partial g}$ for g near a bifurcation point. As seen before, if the bifurcation point is at the edge of the g existence interval, say at $g = \hat{g}$, then $\frac{\partial L_1}{\partial g}(\hat{g}, h)$ is unbounded, however it turns out that

$$\mathcal{B}_l(g) \frac{\partial L_1}{\partial g}(g, h)$$

is bounded in the limit $g \rightarrow \hat{g}$. In order to analyse this expression in case $p_l(\hat{g}) = 0$, we observe that Lemma A.2 gives that near $\hat{g}$ one should parametrise g by using p_l . We will use $g(p_l)$ to denote this parametrisation. And we define a function $\tilde{u}(x, p_l)$ such that $\tilde{u}(x, p_l)$ satisfies $g(p_l) = \frac{1}{2}\tilde{u}_x^2 + V_{m_1}(\tilde{u})$ and equals the front $u(x, g(p_l))$ on the first middle interval $(-L_1(g(p_l)), 0)$. Since $p_l(\hat{g}) = 0$, we get immediately that $\tilde{u}_x(x, 0)$ has a zero at $x = -L_1(\hat{g})$. And continuity of the curves $\tilde{u}$ in the parameter p_l gives that for p_l near zero, there exist x -values $\xi_0(p_l)$ such that $\tilde{u}_x(\xi_0(p_l), p_l) = 0$, $\tilde{u}_x(x, p_l) \neq 0$ for x and $\xi_0(p_l) \rightarrow -L(\hat{g})$ for $p_l \rightarrow 0$. We first analyse the regularisation function $\tilde{G}(\xi_0(p_l), -L_1(p_l))$ for p_l near 0 (hence g near $\hat{g}$).

Lemma A.3. *Let $\hat{g}$ be such that $p_l(\hat{g}) = 0$ and let $g(p_l)$ be the parametrisation as given by Lemma A.2. Define $\tilde{u}(x, p_l)$ to be such that $g(p_l) = \frac{1}{2}\tilde{u}_x^2 + V_{m_1}(\tilde{u})$ for all $x \in \mathbb{R}$ and define $\xi_0(p_l)$ to be such that $\tilde{u}_x(\xi_0(p_l), p_l) = 0$, $\tilde{u}_x(x, p_l) \neq 0$ for x between $-L(g(p_l))$ and $\xi_0(p_l)$, and $\xi_0 \rightarrow -L$ as $p_l \rightarrow 0$. Then for p_l near 0*

$$\tilde{G}(-L, \xi_0) + \frac{u'_l(g(p_l))}{p_l} = \int_{-L}^{\xi_0} \left[\frac{1}{\tilde{u}_x^2(\xi)} - \frac{1}{\tilde{u}_{xx}^2(\xi_0)(\xi - \xi_0)^2} \right] d\xi$$

$$- \frac{1}{p_l V'_m(u_l)} + \mathcal{O}(p_l) + \frac{1}{p_l [V'_m(u_l) - V'_l(u_l)]}.$$

Proof. From the definition of $\tilde{G}$, (21),

$$\begin{aligned} \tilde{G}(-L, \xi_0) + \frac{u'_l(g(p_l))}{p_l} &= \int_{-L}^{\xi_0} \left[\frac{1}{\tilde{u}_x^2(\xi)} - \frac{1}{\tilde{u}_{xx}^2(\xi_0)(\xi - \xi_0)^2} \right] d\xi \\ &\quad - \frac{1}{(\xi_0 + L)\tilde{u}_{xx}^2(\xi_0)} + \frac{1}{p_l[V'_m(u_l) - V'_l(u_l)]}. \end{aligned}$$

By definition of $\xi_0(p_l)$, we have

$$\begin{aligned} 0 &= \tilde{u}_x(-L + (\xi_0 + L), g(p_l)) \\ &= \tilde{u}_x(-L, p_l) + (\xi_0 + L)\tilde{u}_{xx}(-L, p_l) + \frac{(\xi_0 + L)^2}{2}\tilde{u}_{xxx}(-L, p_l) + O((\xi_0 + L)^3) \\ &= p_l - (\xi_0 + L)V'_m(u_l) - \frac{(\xi_0 + L)^2}{2}V''_m(u_l)p_l + O((\xi_0 + L)^3) \end{aligned}$$

as $\tilde{u}_{xx} + V'_{m_1}(\tilde{u}) = 0$. This implies that for g near $\hat{g}$

$$\xi_0(p_l) + L(g(p_l)) = \frac{p_l}{V'_{m_1}(u_l(p_l))} + O(p_l^3(g)),$$

with $u_l(p_l)$ given by Lemma A.2. Furthermore, if we define $u_0(p_l) := \tilde{u}(\xi_0(p_l), p_l)$, then

$$\begin{aligned} u_0(p_l) &= \tilde{u}(-L + (\xi_0 + L), p_l) \\ &= u_l + (\xi_0 + L)p_l - \frac{(\xi_0 + L)^2}{2}V'_m(u_l) - \frac{(\xi_0 + L)^3}{6}p_lV''_m(u_l) + O((\xi_0 + L)^4) \\ &= u_l + \frac{p_l^2}{2V'_m(u_l)} + O(p_l^4). \end{aligned}$$

Using the two expansions, we get for p_l near 0

$$(\xi_0 + L)\tilde{u}_{xx}^2(\xi_0, p_l) = (\xi_0 + L)V'_m(u_0)^2 = \frac{p_l}{V'_m(u_l)}V'_m(u_l)^2 + O(p_l^3) = p_lV'_m(u_l) + O(p_l^3).$$

Hence

$$\frac{1}{(\xi_0 + L)\tilde{u}_{xx}^2(\xi_0)} = \frac{1}{p_lV'_m(u_l)} + O(p_l). \quad \square$$

Lemma A.4. If $\hat{g}$ is such that the front $u(x; \hat{g}, h)$ is a solution of Eq. (10), for which all zeroes of $u_x(x; \hat{g}, h)$ are simple and $\mathcal{B}_l(\hat{g}) = 0$, then $\hat{g}$ is at the edge of the existence interval for the left matching points (u_l, p_l) . Furthermore, if $\mathcal{B}_m(\hat{g}, h) \neq 0$ then

$$\lim_{g \rightarrow \hat{g}} \left[\mathcal{B}_l(g) \frac{\partial L_1}{\partial g}(g, h) \right] = -\frac{V'_l(u_l(\hat{g}))}{V'_{m_1}(u_l(\hat{g}))}.$$

Similarly, if $\hat{h}$ is such that the front $u(x; g, \hat{h})$ is a solution of Eq. (10), for which all zeroes of $u_x(x; g, \hat{h})$ are simple and $\mathcal{B}_r(\hat{h}) = 0$, then $\hat{h}$ is at the edge of the existence interval for the right matching points (u_r, p_r) . Furthermore, if $\mathcal{B}_m(g, \hat{h}) \neq 0$ then

$$\lim_{h \rightarrow \hat{h}} \left[\mathcal{B}_r(h) \frac{\partial L_2}{\partial h}(g, h) \right] = - \frac{V'_r(u_r(\hat{h}))}{V'_{m_2}(u_r(\hat{h}))}.$$

Proof. We focus on the left matching points, as the proof for the right matching points is analogous. As we have seen in Section 2, the front has a non-simple zero in the derivative u_x at some point in an interval I_l if and only if the front is constant on the full interval I_l and in the phase portrait, it stays at a fixed point of the related dynamics. So the conditions of the lemma imply that we are in case (i) or (ii) of Lemma A.2 and this immediately implies that $\hat{g}$ is on the edge of the existence interval for the left matching points. Furthermore, it shows that two smooth curves of left matching points are emerging from $\hat{g}$ and we will show that both curves have the same limit for $\mathcal{B}_l(g) \frac{\partial L_1}{\partial g}(g, h)$ if $g \rightarrow \hat{g}$.

In this proof, we will use the shorthand $\hat{u} = u_l(\hat{g})$ and $\hat{p} = p_l(\hat{g})$. Expanding $u_l(g)$ about $\hat{u}$, it follows immediately that $\mathcal{B}_l(g)$ can be written as

$$\begin{aligned} \mathcal{B}_l(g) &= p_l(g) [V'_{m_1}(u_l) - V'_l(u_l)] \\ &= p_l(g) [V'_{m_1}(\hat{u}) - V'_l(\hat{u}) + (u_l - \hat{u}) [V''_{m_1}(\hat{u}) - V''_l(\hat{u})] + O((u_l - \hat{u})^2)]. \end{aligned}$$

Using Lemma A.2, we get for g near $\hat{g}$ that $\mathcal{B}_l(g) = \eta \hat{p} [V''_{m_1}(\hat{u}) - V''_l(\hat{u})] + O(\eta^3)$ if $\hat{p} \neq 0$; and $\mathcal{B}_l(g) = \eta [V'_{m_1}(\hat{u}) - V'_l(\hat{u})] + O(\eta^3)$ if $\hat{p} = 0$. In both cases, the relation between $g - \hat{g}$ and η is given in Lemma A.2.

If $p_l(\hat{g}) \neq 0$, then for g nearby $\hat{g}$, also $\hat{p} \neq 0$. If we split the (first) middle interval I_1 in to two parts: one with boundary $-L_1$ containing no zeroes of u_x in its interior and at the boundaries and the other containing all of the interior zeroes, then Lemma 4.3 can be used to show that the second interval can still make no contribution to the unboundedness of the derivative of the length function. Thus the unbounded part of $\frac{\partial L_1}{\partial g}$ must arise solely due to the interval with no zeroes of u_x . In other words to prove the lemma for $p_l(\hat{g}) \neq 0$, it is enough to prove it in the case when there are no zeroes of u_x in the interior of I_1 for $g = \hat{g}$.

In this case, we get

$$\frac{\partial L_1}{\partial g}(g, h) = -\frac{1}{\mathcal{B}_l(g)} - \frac{1}{\mathcal{B}_m(g, h)} - \int_{-L_1(g)}^0 \frac{d\xi}{u_x^2(\xi, g)}.$$

Thus if $\hat{p} \neq 0$ we have

$$\mathcal{B}_l(g) \frac{\partial L_1}{\partial g}(g, h) = -1 + O(\mathcal{B}_l(g)) = -\frac{V'_l(\hat{u})}{V'_{m_1}(\hat{u})} + O(\eta),$$

as $V'_l(\hat{u}) = V'_{m_1}(\hat{u})$ ($\mathcal{B}_l(\hat{g}) = 0$ and $\hat{p} \neq 0$).

If $\hat{p} = 0$ then calculating $\frac{\partial L_1}{\partial g}(g, h)$ is more complicated as $\frac{\partial L_1}{\partial g}(\hat{g}, h)$ is unbounded, although, as in the case $\hat{p} \neq 0$, we can reason that a zero of u_x in the interior of I_1 for g nearby $\hat{g}$ and still in the interior in the limit $g \rightarrow \hat{g}$ cannot contribute to the unbounded part of $\frac{\partial L_1}{\partial g}(g, h)$. Therefore we assume that there is no zero in the interior of I_1 for $g = \hat{g}$. To proceed $L_1(g, h)$ is split into two parts and Lemma A.3 is used to give a leading order expansion in terms of p_l . We write

$$L_1(g, h) = \int_{u_l(g)}^{u_m(g, h)} \frac{du}{p(u, g)} = \int_{u_0(g)}^{u_m(g, h)} \frac{du}{p(u, g)} + \int_{u_l(g)}^{u_0(g)} \frac{du}{p(u, g)},$$

where $u_0(g)$ is defined such that $p(u_0(g), g) = 0$ and $u_0(g) \rightarrow \hat{u}$ for $g \rightarrow \hat{g}$. Furthermore, $u(x, g)$ satisfies the middle dynamics in $(-L_1, 0)$ and we define $\tilde{u}(x, g)$ to be the extension of this function, i.e., $\tilde{u}(x, g)$ satisfies the middle dynamics for all $x \in \mathbb{R}$. Finally we define $\xi_0(g)$ to be such that $\tilde{u}(\xi_0(g), g) = u_0(g)$ and $\xi_0 \rightarrow -L_1$ as $g \rightarrow \hat{g}$. Using this definition, Lemma 4.3 gives for $g \neq \hat{g}$

$$\begin{aligned} \frac{\partial L_1}{\partial g}(g, h) &= \tilde{G}(L, \xi_0) + \frac{1}{p_r(g)} \frac{du_r}{dg}(g) - \tilde{G}(-L, \xi_0) - \frac{1}{p_l(g)} \frac{du_l}{dg}(g) \\ &= \tilde{G}(L, \xi_0) - \frac{1}{\mathcal{B}_r(g)} - \tilde{G}(-L, \xi_0) - \frac{1}{p_l(g)} \frac{du_l}{dg}(g). \end{aligned} \quad (57)$$

Taking the limit $g \rightarrow \hat{g}$, the first two terms in the above expression are bounded whilst by Lemma A.3 the remaining two terms diverge like $\frac{1}{\mathcal{B}_l(g)}$ as $g \rightarrow \hat{g}$. Thus from Lemma A.3

$$\begin{aligned} \mathcal{B}_l(g) \frac{\partial L_1}{\partial g}(g, h) &= O(\mathcal{B}_l(g(\eta))) + \mathcal{B}_l(g) \left[\frac{1}{p_l V'_m(u_l)} - \frac{1}{p_l [V'_m(u_l) - V'_l(u_l)]} \right] \\ &= \frac{V'_m(u_l) - V'_l(u_l)}{V'_m(u_l)} - 1 + O(\mathcal{B}_l(g(\eta))) = -\frac{V'_l(u_l)}{V'_m(u_l)} + O(\mathcal{B}_l(g(\eta))). \end{aligned}$$

Taking the limit $g \rightarrow \hat{g}$ gives the desired result, as $\mathcal{B}_l(g(\eta)) \rightarrow 0$ for $g \rightarrow \hat{g}$ ($\eta \rightarrow 0$). $\square$

The middle matching points need further study, for one, they depend on the orbit parameters for two Hamiltonian systems instead of just one. To study the middle matching points, we assume that there are some $\hat{g}$, $\hat{h}$, $\hat{u}$ and $\hat{p}$ such that the middle set of equations of (12) are satisfied. As V_{m_1} and V_{m_2} are smooth functions, we can expand the expressions in those relations as in (56):

$$\begin{aligned} g - h + V_{m_2}(u) - V_{m_1}(u) &= g - \hat{g} + h - \hat{h} + [V'_{m_2}(\hat{u}) - V'_{m_1}(\hat{u})](u - \hat{u}) \\ &\quad + \frac{1}{2}[V''_{m_2}(\hat{u}) - V''_{m_1}(\hat{u})](u - \hat{u})^2 + \mathcal{O}(|u - \hat{u}|^3); \\ \frac{1}{2}p^2 - g + V_{m_1}(u) &= \hat{p}(p - \hat{p}) + \frac{1}{2}(p - \hat{p})^2 - g + \hat{g} + V'_{m_1}(\hat{u})(u - \hat{u}) \\ &\quad + \frac{1}{2}V''_{m_1}(\hat{u})(u - \hat{u})^2 + \mathcal{O}(|u - \hat{u}|^3). \end{aligned} \quad (58)$$

So if $V'_{m_2}(\hat{u}) - V'_{m_1}(\hat{u}) \neq 0$, the implicit function theorem gives the local existence of a unique smooth function $u_m(g, h)$, for (g, h) in a neighbourhood of $(\hat{g}, \hat{h})$, satisfying $g - h + V_{m_2}(u_m(g, h)) - V_{m_1}(u_l(g, h)) = 0$ and $u_m(\hat{g}, \hat{h}) = \hat{u}$. And if $\hat{p} \neq 0$, the second equation gives the local existence of a unique smooth function $p_m(g, h)$, for (g, h) in a neighbourhood of $(\hat{g}, \hat{h})$, satisfying $\frac{1}{2}p_m^2(g, h) - g + V_{m_1}(u_m(g, h)) = 0$ and $p_m(\hat{g}, \hat{h}) = \hat{p}$. In other words, we have shown the first part of Lemma 2.2.

We continue with the case that there is a bifurcation point, that is, we also assume that $\hat{g}$ and $\hat{h}$ are such that $\mathcal{B}_m(\hat{g}, \hat{h}) = 0$. Usually such points are not isolated. In our analysis, we only look at cases where the middle matching points are not fixed points of the V_{m_1} - or V_{m_2} -dynamics (i.e., if $\hat{p} = 0$, then $V'_{m_1}(\hat{u}) \neq 0$ and $V'_{m_2}(\hat{u}) \neq 0$), so we restrict here to these cases too. The equation $\mathcal{B}_m(g, h) = 0$ implies that $p_m = 0$ or $V'_{m_1}(u_m) = V'_{m_2}(u_m)$ (or both). We consider each case separately.

- Assume that $\hat{p} = 0$. If we continue with $p_m = 0$, the middle set of equations of (12) are satisfied if

$$0 = F(u, h; g) = \begin{pmatrix} V_{m_1}(u) - g \\ h - g + V_{m_1}(u) - V_{m_2}(u) \end{pmatrix}.$$

Differentiating F with respect to u and h and evaluating at $u = \hat{u}$ and $h = \hat{h}$ gives

$$DF(\hat{u}, \hat{h}; g) = \begin{pmatrix} V'_{m_1}(\hat{u}) & 0 \\ V'_{m_1}(\hat{u}) - V'_{m_2}(\hat{u}) & 1 \end{pmatrix}.$$

This matrix is invertible for any g as we assumed that the middle matching point is not a fixed point of the V_{m_1} -dynamics, hence $V'_{m_1}(\hat{u}) \neq 0$. So the implicit function theorem gives the existence of curves $\tilde{u}_m(g)$, $\tilde{h}(g)$ for g near $\tilde{g}$ such that $F(\tilde{u}_m(g), \tilde{h}(g); g) = 0$. Hence by defining $\tilde{p}_m(g) = 0$ (as we continued with $p_m = 0$), this implies that the middle set of equations of (12) are satisfied. Furthermore, as $\tilde{p}_m(g) = 0$, we also satisfy $\mathcal{B}_m(g, \tilde{h}(g)) = 0$.

- Assume that $V'_{m_1}(\hat{u}) = V'_{m_2}(\hat{u}) = 0$ and $\hat{p} \neq 0$ (we already considered the case $\hat{p} = 0$ above). The middle set of equations of (12) and $\mathcal{B}_m(g) = 0$ are satisfied if

$$0 = F(u, p, h; g) = \begin{pmatrix} p^2/2 - V_{m_1}(u) + g \\ h - g + V_{m_1}(u) - V_{m_2}(u) \\ p(V'_{m_1}(u) - V'_{m_2}(u)) \end{pmatrix}.$$

Differentiating F with respect to u , p , and h and evaluating at $u = \hat{u}$, $p = \hat{p}$ and $h = \hat{h}$ gives

$$DF(\hat{u}, \hat{p}, \hat{h}; g) = \begin{pmatrix} -V'_{m_1}(\hat{u}) & \hat{p} & 0 \\ 0 & 0 & 1 \\ \hat{p}(V''_{m_1}(\hat{u}) - V''_{m_2}(\hat{u})) & 0 & 0 \end{pmatrix}.$$

This matrix is invertible for any g as $\hat{p} \neq 0$ and $V''_{m_1}(\hat{u}) - V''_{m_2}(\hat{u}) \neq 0$. So again, the implicit function theorem gives the existence of curves $\tilde{u}_m(g)$, $\tilde{p}_m(g)$, $\tilde{h}(g)$, for g near $\tilde{g}$, such that the middle set of equations of (12) and $\mathcal{B}_m(g, \tilde{h}(g)) = 0$ are satisfied.

Finally we consider the continuation nearby a bifurcation point.

Lemma A.5. Assume that there is a point $(\hat{u}, \hat{p})$ such that the middle set of equations of (12) are satisfied for some $(g, h) = (\hat{g}, \hat{h})$ and $\mathcal{B}_m(\hat{g}, \hat{h}) = 0$. If $(\hat{u}, \hat{p})$ is not a fixed point of the V_{m_1} - or V_{m_2} -dynamics, then for $g = \hat{g}$ fixed, there is a unique curve of middle matching points in the (u, u_x) plane nearby $(\hat{u}, \hat{p})$ which can be related to the h -orbits as follows:

- (i) If $\hat{p} \neq 0$, hence $V'_{m_1}(\hat{u}) = V'_{m_2}(\hat{u})$, then we have for a small parameter η

$$u_m = \hat{u} + \eta, \quad p_m(g) = \hat{p} + O(\eta), \quad h = \hat{h} + \frac{\eta^2}{2} [V''_{m_2}(\hat{u}) - V''_{m_1}(\hat{u})] + O(\eta^3).$$

- (ii) If $\hat{p} = 0$, then we have for a small parameter η

$$p = \eta, \quad u_m = \hat{u} - \frac{\eta^2}{2V'_{m_1}(\hat{u})} + O(\eta^4), \quad h = \hat{h} - \frac{\eta^2(V'_{m_2}(\hat{u}) - V'_{m_1}(\hat{u}))}{2V_{m_1}(\hat{u})} + O(\eta^4).$$

This implies that the curve $\tilde{h}(g)$ is an edge of the existence region for middle matching points.

Proof. With $g = \hat{g}$ fixed, we are in a similar situation as for the left matching point and hence the results above about the curve of matching points and the approximations follow immediately.

The curve $\tilde{h}(g)$ is a curve of middle bifurcation points, so for each value of g , there are expansions as above, which shows that h only extends to one side of the curve. $\square$

Along the curve $\tilde{h}(g)$, we have similar results to Lemma A.4.

Lemma A.6. Assume that along the curve $\tilde{h}(g)$, we have front solutions $u(x; g, \tilde{h}(g))$ which satisfy Eq. (10) and for which all zeroes of $u_x(x; g, \tilde{h}(g))$ are simple. If $\mathcal{B}_1(g) \neq 0$, then for every g

$$\lim_{h \rightarrow \tilde{h}(g)} \left[\mathcal{B}_m(g, h) \frac{\partial L_2}{\partial h}(g, h) \right] = - \frac{V'_{m_1}(u_m(g, \tilde{h}(g)))}{V'_{m_2}(u_m(g, \tilde{h}(g)))}.$$

To prove this, we observe that for g fixed, we are in a similar situation as for the left matching points and hence we can use Lemma A.4 with the appropriate change of indices.

A.4. Details of results related to Section 5.1

In this section, we will give the proofs of Theorems 5.1 and 5.3. These proofs follow the same layout as the proofs for the corresponding theorems for one middle interval: first we get a compatibility condition for the existence of an eigenvalue zero of the linearisation operator around the front in the same way that Lemma 4.1 was proved. This result is then used to prove Theorems 5.1 and 5.3.

In order to state the counterpart of Lemma 4.1 we introduce functions $E_{l,v}$, $E_{r,\mu}$, $F_{l,v}$ and $F_{r,\mu}$, which play a similar role to C_v , (23). The subscripts l and r denote the left and right middle interval respectively, whilst v and μ are the number of simple zeroes of u_x in each respective interval. The functions $E_{l,v}$ and $E_{r,\mu}$ are only defined if $p_m \neq 0$, whilst $F_{l,v}$ and $F_{r,\mu}$ are only defined for $p_m = 0$.

If $p_l = 0$ and $p_m \neq 0$, then $E_{l,v} := 0$. If $p_l \neq 0$ and $p_m \neq 0$, then $E_{l,0} := \int_{-L_1}^0 \frac{d\xi}{u_x^2(\xi)}$ and

$$E_{l,v} := G_1(-L_1, M_0) + \sum_{i=1}^v G_i(M_{i-1}, M_i) + G_v(M_v, 0), \quad v \geq 1,$$

where we recall that G_i is a regularisation of $\int \frac{d\xi}{u_x^2(\xi)}$, see (21). Similarly if $p_r = 0$ and $p_m \neq 0$, then $E_{r,\mu} := 0$. If $p_r \neq 0$ and $p_m \neq 0$, then $E_{r,0} := \int_0^{L_2} \frac{d\xi}{u_x^2(\xi)}$ and

$$E_{r,\mu} := G_{v+2}(0, M_{v+1}) + \sum_{i=v+2}^{\mu+v+1} G_i(M_{i-1}, M_i) + G_{\mu+v+1}(M_{\mu+v+1}, L_2), \quad \mu \geq 1.$$

If $p_l = 0$ and $p_m = 0$, then $F_{l,v} := 0$. If $p_l \neq 0$ and $p_m = 0$, then $F_{l,0} := \tilde{G}(-L_1, 0)$ and

$$F_{l,v} := G_1(-L_1, M_0) + \sum_{i=1}^v G_i(M_{i-1}, M_i) + \tilde{G}(M_v, 0), \quad v \geq 1.$$

Similarly if $p_r = 0$ and $p_m = 0$, then $F_{r,\mu} := 0$. If $p_r \neq 0$ and $p_m = 0$, then $F_{r,0} := -\tilde{G}(L_2, 0)$ and

$$F_{r,\mu} := -\tilde{G}(M_{v+1}, 0) + \sum_{i=v+2}^{\mu+v+1} G_i(M_{i-1}, M_i) + G_{\mu+v+1}(M_{\mu+v+1}, L_2), \quad \mu \geq 1.$$

Lemma A.7. Let $u(x)$ be a stationary front solution of the wave equation (10) with two middle intervals. If all zeroes of u_x are simple then

(i) if $p_m \neq 0$, the linearisation about the front u , $\mathcal{L}$, has an eigenvalue zero if and only if

$$\mathcal{B}_l + \mathcal{B}_m + \mathcal{B}_r + \mathcal{B}_l \mathcal{B}_m E_{l,v} + \mathcal{B}_m \mathcal{B}_r E_{r,\mu} + \mathcal{B}_l \mathcal{B}_r [E_{l,v} + E_{r,\mu}] + \mathcal{B}_l \mathcal{B}_m \mathcal{B}_r E_{l,v} E_{r,\mu} = 0;$$

(ii) if $p_m = 0$, hence $\mathcal{B}_m = 0$, the linearisation about the front u , $\mathcal{L}$, has an eigenvalue zero if and only if

$$\mathcal{B}_l [V'_{m_2}(u_m)]^2 + \mathcal{B}_r [V'_{m_1}(u_m)]^2 + \mathcal{B}_l \mathcal{B}_r ([V'_{m_1}(u_m)]^2 F_{l,v} + [V'_{m_2}(u_m)]^2 F_{r,\mu}) = 0.$$

Proof. The proof of this lemma is broken up into several parts determined by whether or not $p_i = 0$, $i = l, m, r$. However, the main argument in each case is the same: in order to create an eigenfunction in $H^2(\mathbb{R})$ we must make sure it is continuously differentiable by matching at all zeroes of u_x . This then leads the compatibility condition presented in the lemma.

Using the expression for the eigenfunction Ψ in (19), continuity of Ψ and Ψ_x at the zeroes x_i , for $i = 1, \dots, \nu$, gives exactly the same relations as in Lemma 4.1, namely

$$B_i = B_0 \quad \text{and} \quad A_i = A_{i-1} + B_0 G_i(M_{i-1}, M_i) \quad \text{for } i = 1, \dots, \nu.$$

Meaning that for $\nu \geq 1$

$$A_\nu = A_0 + B_0 \sum_{i=1}^{\nu} G_i(M_{i-1}, M_i). \quad (59)$$

The continuity conditions at x_i for $i = \nu + 2, \dots, \mu + \nu + 1$ are found in exactly the same manner

$$B_i = B_{\nu+1} \quad \text{and} \quad A_i = A_{i-1} + B_{\nu+1} G_i(M_{i-1}, M_i) \quad \text{for } i = \nu + 2, \dots, \mu + \nu + 1.$$

As in the first middle interval, this recursive relationship can be used to calculate $A_{\mu+\nu+1}$ in terms of $A_{\nu+1}$ for $\mu \geq 1$

$$A_{\mu+\nu+1} = A_{\nu+1} + B_{\nu+1} \sum_{i=\nu+2}^{\mu+\nu+1} G_i(M_{i-1}, M_i). \quad (60)$$

The continuity conditions at $x = -L_1, 0, L_2$ depend on whether or not $u_x(x) = 0$ for $x = -L_1, 0, L_2$.

- Matching at $x = -L_1$ shows that if $u_x(-L_1) = p_l \neq 0$, then

$$B_0 = \mathcal{B}_l \quad \text{and} \quad A_0 = 1 + \mathcal{B}_l \int_{-L_1}^{M_0} \frac{d\xi}{u_x^2(\xi)}. \quad (61)$$

And if $p_l = 0$, then

$$B_0 = 0 \quad \text{and} \quad A_0 = \frac{V'_l(u_l)}{V'_{m_1}(u_l)} \neq 0 \quad (62)$$

as $p_l = 0$ and u_x has only simple zeroes.

- Matching at the middle point $x = 0$ shows that if $p_m \neq 0$

$$B_{v+1} = \mathcal{B}_m \left[A_v + B_0 \int_{M_v}^0 \frac{d\xi}{u_x^2(\xi)} \right] + B_0 \quad \text{and} \quad A_{v+1} = A_v + B_0 \int_{M_v}^0 \frac{d\xi}{u_x^2(\xi)} + B_{v+1} \int_0^{M_{v+1}} \frac{d\xi}{u_x^2(\xi)}.$$

Using the expression (59) for A_v gives

$$B_{v+1} = \mathcal{B}_m \left[A_0 + B_0 \left(\sum_{i=1}^v G_i(M_{i-1}, M_i) + \int_{M_v}^0 \frac{d\xi}{u_x^2(\xi)} \right) \right] + B_0 \quad (63)$$

and

$$A_{v+1} = A_0 + B_0 \left(\sum_{i=1}^v G_i(M_{i-1}, M_i) + \int_{M_v}^0 \frac{d\xi}{u_x^2(\xi)} \right) + B_{v+1} \int_0^{M_{v+1}} \frac{d\xi}{u_x^2(\xi)}. \quad (64)$$

If $p_m = 0$ then, again by using Lemma A.1, the relations can be derived in the same manner as those for the zeroes x_i . The only difference is that there is a different potential on either side of this zero of u_x :

$$B_{v+1} = \frac{V'_{m_2}(u_m)}{V'_{m_1}(u_m)} B_0 \quad (65)$$

and

$$A_{v+1} = \frac{V'_{m_1}(u_m)}{V'_{m_2}(u_m)} [A_v + B_0 \tilde{G}(M_v, 0)] - B_{v+1} \tilde{G}(M_{v+1}, 0).$$

As before, using (59) to replace A_v gives

$$A_{v+1} = \frac{V'_{m_1}(u_m)}{V'_{m_2}(u_m)} \left[A_0 + B_0 \left(\sum_{i=1}^v G_i(M_{i-1}, M_i) + \tilde{G}(M_v, 0) \right) \right] - B_{v+1} \tilde{G}(M_{v+1}, 0). \quad (66)$$

- Finally, matching at $x = L_2$ for $p_r \neq 0$ gives

$$\hat{k} = A_{\mu+v+1} + B_{v+1} \int_{M_{\mu+v+1}}^{L_2} \frac{d\xi}{u_x^2(\xi)} = A_{\mu+v+1} + B_{v+1} G_{\mu+v+1}(M_{\mu+v+1}, L_2),$$

and the compatibility condition for the existence of an eigenvalue zero is $B_{v+1} = -\hat{k} \mathcal{B}_r$. With the expression (60) for $A_{\mu+v+1}$ the compatibility condition becomes

$$B_{v+1} = -\mathcal{B}_r \left[A_{v+1} + B_{v+1} \left(\sum_{i=v+2}^{\mu+v+1} G_i(M_{i-1}, M_i) + G_{\mu+v+1}(M_{\mu+v+1}, L_2) \right) \right]. \quad (67)$$

If $p_r = 0$ we get $A_{\mu+\nu+1} = \hat{k} \frac{V'_r(u_r)}{V'_{m_2}(u_r)}$ and the compatibility condition

$$B_{\mu+\nu+1} = 0 \quad (\Rightarrow B_{\nu+1} = 0).$$

Next we rewrite the compatibility condition for the various cases.

(i) If $p_m \neq 0$, $p_l \neq 0$, $p_r \neq 0$, then using (61) for A_0 and B_0 into (64) shows for $\nu \geq 1$

$$\begin{aligned} A_{\nu+1} &= 1 + \mathcal{B}_l \left(G_1(-L_1, M_0) + \sum_{i=1}^{\nu} G_i(M_{i-1}, M_i) + \int_{M_{\nu}}^0 \frac{d\xi}{u_x^2(\xi)} \right) + B_{\nu+1} \int_0^{M_{\nu+1}} \frac{d\xi}{u_x^2(\xi)} \\ &= 1 + \mathcal{B}_l E_{l,\nu} + B_{\nu+1} G_{\nu+2}(0, M_{\nu+1}). \end{aligned}$$

The same results also hold if $\nu = 0$, by Lemma 3.1. Substituting this into the compatibility condition (67) gives

$$\begin{aligned} B_{\nu+1} &= -\mathcal{B}_r \left[1 + \mathcal{B}_l E_{l,\nu} + B_{\nu+1} \left(G_{\nu+2}(0, M_{\nu+1}) + \sum_{i=\nu+2}^{\mu+\nu+1} G_i(M_{i-1}, M_i) \right. \right. \\ &\quad \left. \left. + G_{\mu+\nu+1}(M_{\mu+\nu+1}, L_2) \right) \right] \\ &= -\mathcal{B}_r [1 + \mathcal{B}_l E_{l,\nu} + B_{\nu+1} E_{r,\mu}]. \end{aligned}$$

Using (61) for A_0 and B_0 into (63) gives for $\nu \geq 1$

$$\begin{aligned} B_{\nu+1} &= \mathcal{B}_m \left[1 + \mathcal{B}_l \left(G_1(-L_1, M_0) + \sum_{i=1}^{\nu} G_i(M_{i-1}, M_i) + \int_{M_{\nu}}^0 \frac{d\xi}{u_x^2(\xi)} \right) \right] + \mathcal{B}_l \\ &= \mathcal{B}_m \left[1 + \mathcal{B}_l \left(G_1(-L_1, M_0) + \sum_{i=1}^{\nu} G_i(M_{i-1}, M_i) + G_{\nu}(M_{\nu}, 0) \right) \right] + \mathcal{B}_l \\ &= \mathcal{B}_m [1 + \mathcal{B}_l E_{l,\nu}] + \mathcal{B}_l. \end{aligned} \tag{68}$$

Again with Lemma 3.1 we can conclude that the same result holds if $\nu = 0$. Substituting this into the compatibility condition above gives

$$0 = \mathcal{B}_m [1 + \mathcal{B}_l E_{l,\nu}] + \mathcal{B}_l + \mathcal{B}_r [1 + \mathcal{B}_l E_{l,\nu} + (\mathcal{B}_m [1 + \mathcal{B}_l E_{l,\nu}] + \mathcal{B}_l) E_{r,\mu}],$$

which can be re-arranged to give the matching condition in the lemma.

(ii) If $p_m \neq 0$, $p_l = 0 = p_r$, then using (62) for A_0 and B_0 into (63) gives

$$B_{\nu+1} = \mathcal{B}_m \frac{V'_l(u_l)}{V'_{m_1}(u_l)}.$$

Now the compatibility condition is $B_{\nu+1} = 0$, which is true if and only if $\mathcal{B}_m = 0$, which is the desired result.

(iii) If $p_m \neq 0$, $p_l = 0$, $p_r \neq 0$ then (62) gives $B_0 = 0$, hence (63) and (64) become

$$B_{v+1} = \mathcal{B}_m A_0 \quad \text{and} \quad A_{v+1} = A_0 + B_{v+1} \int_0^{M_{v+1}} \frac{d\xi}{u_x^2(\xi)} = A_0 + B_{v+1} G_{v+2}(0, M_{v+1}).$$

Using this in the compatibility condition (67) gives

$$\mathcal{B}_m A_0 = -\mathcal{B}_r \left[A_0 + \mathcal{B}_m A_0 \left(G_{v+2}(0, M_{v+1}) + \sum_{i=v+2}^{\mu+v+1} G_i(M_{i-1}, M_i) + G_{\mu+v+1}(M_{\mu+v+1}, L_2) \right) \right].$$

Using the definition of $E_{r,\mu}$ and dividing through by $A_0 \neq 0$ (see (62)) gives $\mathcal{B}_m = -\mathcal{B}_r[1 + \mathcal{B}_m E_{r,\mu}]$ which is the desired result.

(iv) If $p_m \neq 0$, $p_l \neq 0$, $p_r = 0$, then the compatibility condition is $B_{v+1} = 0$. Using (68), this gives

$$0 = B_{v+1} = \mathcal{B}_m[1 + \mathcal{B}_l E_{l,v}] + \mathcal{B}_l,$$

which again gives the desired result.

(v) If $p_m = 0$, $p_l \neq 0$, $p_r \neq 0$ then substituting (65) and (66) into the compatibility condition (67), using $B_0 = \mathcal{B}_l$ and the expression for A_0 as given by (61), and multiplying through by $V'_{m_1}(u_m)V'_{m_2}(u_m) \neq 0$ gives

$$\begin{aligned} & \mathcal{B}_l [V'_{m_2}(u_m)]^2 \\ &= -\mathcal{B}_r \left[[V'_{m_1}(u_m)]^2 \left(1 + \mathcal{B}_l G_1(-L_1, M_0) + \mathcal{B}_l \left(\sum_{i=1}^v G_i(M_{i-1}, M_i) + \tilde{G}(M_v, 0) \right) \right) \right. \\ & \quad \left. + \mathcal{B}_l [V'_{m_2}(u_m)]^2 \left(-\tilde{G}(M_{v+1}, 0) + \sum_{i=v+2}^{\mu+v+1} G_i(M_{i-1}, M_i) + G_{\mu+v+1}(M_{\mu+v+1}, L_2) \right) \right] \\ &= -\mathcal{B}_r ([V'_{m_1}(u_m)]^2 (1 + \mathcal{B}_l F_{l,v}) + \mathcal{B}_l [V'_{m_2}(u_m)]^2 F_{r,\mu}), \end{aligned}$$

which is the desired result.

(vi) If $p_m = 0$, $p_l = 0 = p_r$ then (62), (65), and (66) give

$$B_0 = 0, \quad A_0 = \frac{V'_l(u_l)}{V'_{m_1}(u_l)}, \quad B_{v+1} = \frac{V'_{m_2}(u_m)}{V'_{m_1}(u_m)} B_0 = 0, \quad A_{v+1} = \frac{V'_{m_1}(u_m)}{V'_{m_2}(u_m)} \frac{V'_l(u_l)}{V'_{m_1}(u_l)}.$$

This is consistent with the compatibility condition $B_{v+1} = 0$ and from $A_{\mu+v+1} = \hat{k} \frac{V'_r(u_r)}{V'_{m_2}(u_r)}$, we get that there is an eigenvalue zero with

$$\hat{k} = \frac{V'_l(u_l)}{V'_{m_1}(u_l)} \frac{V'_{m_1}(u_m)}{V'_{m_2}(u_m)} \frac{V'_{m_2}(u_r)}{V'_r(u_r)}.$$

(vii) If $p_m = 0$, $p_l = 0$, $p_r \neq 0$, then substituting (62) into (65), and (66) gives $B_{v+1} = 0$, $A_{v+1} = \frac{V'_{m_1}(u_m)}{V'_{m_2}(u_m)} \frac{V'_l(u_l)}{V'_{m_1}(u_l)}$. Substituting this into the compatibility condition (67) shows that there is an eigenvalue zero iff

$$0 = B_{v+1} = -\frac{V'_{m_1}(u_m)}{V'_{m_2}(u_m)} \frac{V'_l(u_l)}{V'_{m_1}(u_l)} \mathcal{B}_r.$$

Since p_l and p_m are simple zeroes this means that there is an eigenvalue zero if and only if $\mathcal{B}_r = 0$, which is desired result.

(viii) If $p_m = 0$, $p_l \neq 0$, $p_r = 0$, then substituting $B_0 = \mathcal{B}_l$ from (61) into (65) gives $B_{v+1} = \frac{V'_{m_2}(u_m)}{V'_{m_1}(u_m)} \mathcal{B}_l$. Thus the compatibility condition $B_{v+1} = 0$ implies that $\mathcal{B}_l = 0$. So there is an eigenvalue zero if and only if $\mathcal{B}_l = 0$, which is the desired result. $\square$

This lemma has the following obvious corollary:

Corollary A.8. *If exactly two of $\mathcal{B}_l$, $\mathcal{B}_m$, $\mathcal{B}_r$ are zero then the linearisation operator $\mathcal{L}$ does not have an eigenvalue zero. If $\mathcal{B}_l = \mathcal{B}_m = \mathcal{B}_r = 0$ then the linearisation operator $\mathcal{L}$ has an eigenvalue zero and the eigenfunction is a multiple (possibly different in each interval) of $u_x(x)$.*

We can now prove the theorems stated in Section 5.1. We will relate $\frac{\partial L_1}{\partial g}$ with $E_{l,v}$ or $F_{l,v}$ and $\frac{\partial L_2}{\partial h}$ with $E_{r,\mu}$ or $F_{r,\mu}$ and then rewrite the compatibility condition in Lemma A.7 in terms of these partial derivatives.

Proof of Theorem 5.1. From the definitions of L_1 and L_2 , (15), we can see that the only h/g dependence respectively, appears via $u_m(g, h)$ in one of the limits of integration. Thus if (g, h) is such that $\mathcal{B}_m(g, h) \neq 0$ we have

$$\frac{\partial L_1}{\partial h}(g, h) = \frac{1}{\mathcal{B}_m(g, h)} = \frac{\partial L_2}{\partial g}(g, h).$$

If $\mathcal{B}_l = 0$ or $\mathcal{B}_r = 0$ then $\frac{\partial L_1}{\partial g}$ respectively $\frac{\partial L_2}{\partial h}$ is unbounded. From Lemma A.4 we can see that if $\hat{g}$ is such that $\mathcal{B}_l(\hat{g}) = 0$, then

$$\lim_{g \rightarrow \hat{g}} \left[\mathcal{B}_l(g) \frac{\partial L_1}{\partial g}(g, h) \right] = -\frac{V'_l(u_l(\hat{g}))}{V'_{m_1}(u_l(\hat{g}))} \neq 0. \quad (69)$$

Similarly, Lemma A.6 gives that if $\hat{h}$ is such that $\mathcal{B}_r(\hat{h}) = 0$, then

$$\lim_{h \rightarrow \hat{h}} \left[\mathcal{B}_r(h) \frac{\partial L_2}{\partial h}(g, h) \right] = -\frac{V'_r(u_r(\hat{h}))}{V'_{m_2}(u_r(\hat{h}))} \neq 0.$$

The rest of this proof is based on the matching conditions derived in Lemma A.7, and is split up in to four parts determined by whether $\mathcal{B}_l$ or $\mathcal{B}_r$ is 0.

- If g and h are such that $\mathcal{B}_l(g) \neq 0 \neq \mathcal{B}_r(h)$ then using the same method as in the proof of Theorem 4.5 it is straightforward to derive the following relations for the partial derivatives:

$$\begin{aligned}\mathcal{B}_l \mathcal{B}_m \frac{\partial L_1}{\partial g} &= -\mathcal{B}_l - \mathcal{B}_m - \mathcal{B}_l \mathcal{B}_m E_{l,v}, \\ \mathcal{B}_l \mathcal{B}_m \frac{\partial L_1}{\partial h} &= \mathcal{B}_l, \\ \mathcal{B}_m \mathcal{B}_r \frac{\partial L_2}{\partial g} &= \mathcal{B}_r, \\ \mathcal{B}_m \mathcal{B}_r \frac{\partial L_2}{\partial h} &= -\mathcal{B}_m - \mathcal{B}_r - \mathcal{B}_m \mathcal{B}_r E_{r,\mu},\end{aligned}$$

where the arguments have been suppressed for ease of notation. Hence

$$\begin{aligned}\mathcal{B}_l \mathcal{B}_m \mathcal{B}_r \det \begin{pmatrix} \frac{\partial L_1}{\partial g} & \frac{\partial L_1}{\partial h} \\ \frac{\partial L_2}{\partial g} & \frac{\partial L_2}{\partial h} \end{pmatrix} &= \mathcal{B}_l + \mathcal{B}_m + \mathcal{B}_r + \mathcal{B}_l \mathcal{B}_m E_{l,v} + \mathcal{B}_m \mathcal{B}_r E_{r,\mu} \\ &\quad + \mathcal{B}_l \mathcal{B}_r [E_{l,v} + E_{r,\mu}] + \mathcal{B}_l \mathcal{B}_m \mathcal{B}_r E_{l,v} E_{r,\mu}\end{aligned}$$

which vanishes if and only if an eigenvalue zero exists due to the compatibility condition of Lemma A.7.

- If $\hat{g}$ and h are such that $\mathcal{B}_l(\hat{g}) = 0$ and $\mathcal{B}_r(h) \neq 0$ then following the same method as in the previous case gives

$$\mathcal{B}_m \mathcal{B}_r \frac{\partial L_2}{\partial h} = -\mathcal{B}_m - \mathcal{B}_r - \mathcal{B}_m \mathcal{B}_r E_{r,\mu},$$

which vanishes if and only if there is an eigenvalue zero by Lemma A.7.

The theorem also states that this is the same as reading the determinant condition as a limit. To see this, note that we also have

$$\lim_{g \rightarrow \hat{g}} \left[\mathcal{B}_l(g) \frac{\partial L_1}{\partial g}(g, h) \right] \neq 0, \quad \mathcal{B}_l(\hat{g}) \frac{\partial L_1}{\partial h}(\hat{g}, h) = 0, \quad \mathcal{B}_m(\hat{g}, h) \mathcal{B}_r(h) \frac{\partial L_2}{\partial g}(\hat{g}, h) = \mathcal{B}_r(h).$$

Thus

$$\begin{aligned}\lim_{g \rightarrow \hat{g}} \left[\mathcal{B}_l(g) \mathcal{B}_m(g, h) \mathcal{B}_r(h) \det \begin{pmatrix} \frac{\partial L_1}{\partial g} & \frac{\partial L_1}{\partial h} \\ \frac{\partial L_2}{\partial g} & \frac{\partial L_2}{\partial h} \end{pmatrix} \right] \\ = \det \begin{pmatrix} \lim_{g \rightarrow \hat{g}} [(\mathcal{B}_l \frac{\partial L_1}{\partial g})|_{(g,h)}] & \lim_{g \rightarrow \hat{g}} [(\mathcal{B}_l \frac{\partial L_1}{\partial h})|_{(g,h)}] \\ (\mathcal{B}_m \mathcal{B}_r \frac{\partial L_2}{\partial g})|_{(\hat{g},h)} & (\mathcal{B}_m \mathcal{B}_r \frac{\partial L_2}{\partial h})|_{(\hat{g},h)} \end{pmatrix} \\ = \det \begin{pmatrix} \lim_{g \rightarrow \hat{g}} [(\mathcal{B}_l \frac{\partial L_1}{\partial g})|_{(g,h)}] & 0 \\ \mathcal{B}_r(h) & (\mathcal{B}_m \mathcal{B}_r \frac{\partial L_2}{\partial h})|_{(\hat{g},h)} \end{pmatrix}.\end{aligned}$$

Hence the limit of the general condition in the theorem is equivalent to $\mathcal{B}_m \mathcal{B}_r \frac{\partial L_2}{\partial h} = 0$.

- If g and $\hat{h}$ are such that $\mathcal{B}_l(g) \neq 0$, $\mathcal{B}_r(\hat{h}) = 0$ then the result follows in a similar way to the previous case.
- If $\hat{g}$ and $\hat{h}$ are such that $\mathcal{B}_l(\hat{g}) = 0 = \mathcal{B}_r(\hat{h})$ then the statement of the theorem follows directly from Corollary A.8. $\square$

Before proving Theorem 5.3, we first derive an expression for the derivative of the curve $\tilde{h}(g)$.

Lemma A.9. For any g for which the curve $\tilde{h}(g)$ exists and $V'_{m_1}(u_m(g, \tilde{h}(g))) \neq 0$, we have

$$\frac{d\tilde{h}}{dg}(g) = \frac{V'_{m_2}(u_m(g, \tilde{h}(g)))}{V'_{m_1}(u_m(g, \tilde{h}(g)))}.$$

Proof. As seen in the proof of Lemma 2.2, along the curve $\tilde{h}(g)$, we have $p_m(g, \tilde{h}(g)) = 0$ or $V'_{m_1}(u_m(g, \tilde{h}(g))) = V'_{m_2}(u_m(g, \tilde{h}(g)))$.

If $p_m(g, \tilde{h}(g)) = 0$, then (12) implies that $g = V_{m_1}(u_m(g, \tilde{h}(g)))$ and $\tilde{h}(g) = V_{m_2}(u_m(g, \tilde{h}(g)))$. Differentiating the first relation shows that $1 = V'_{m_1}(u_m(g, \tilde{h}(g))) \frac{d}{dg}[u_m(g, \tilde{h}(g))]$. Differentiating the second relation gives

$$\frac{d\tilde{h}}{dg}(g) = V'_{m_2}(u_m(g, \tilde{h}(g))) \frac{d}{dg}[u_m(g, \tilde{h}(g))] = \frac{V'_{m_2}(u_m(g, \tilde{h}(g)))}{V'_{m_1}(u_m(g, \tilde{h}(g)))}.$$

If $V'_{m_1}(u_m(g, \tilde{h}(g))) = V'_{m_2}(u_m(g, \tilde{h}(g)))$ then by (12) we have $g - \tilde{h}(g) = V_{m_1}(u_m(g, \tilde{h}(g))) - V_{m_2}(u_m(g, \tilde{h}(g)))$. Differentiating this with respect to g and using that $V'_{m_1}(u_m(g, \tilde{h}(g))) = V'_{m_2}(u_m(g, \tilde{h}(g)))$, we get

$$1 - \frac{d\tilde{h}}{dg}(g) = 0, \quad \text{hence } \frac{d\tilde{h}}{dg}(g) = 1 = \frac{V'_{m_2}(u_m(g, \tilde{h}(g)))}{V'_{m_1}(u_m(g, \tilde{h}(g)))}. \quad \square$$

Proof of Theorem 5.3. First of all note that from (12) we have

$$\begin{aligned} -1 &= [V'_l(u_l) - V'_{m_1}(u_l)] \frac{\partial u_l}{\partial g}; & 0 &= \frac{\partial u_l}{\partial h}; \\ 1 &= [V'_{m_1}(u_m) - V'_{m_2}(u_m)] \frac{\partial u_m}{\partial g}; & -1 &= [V'_{m_1}(u_m) - V'_{m_2}(u_m)] \frac{\partial u_m}{\partial h}; \\ 0 &= \frac{\partial u_r}{\partial g}; & -1 &= [V'_r(u_r) - V'_{m_2}(u_r)] \frac{\partial u_r}{\partial h}. \end{aligned}$$

As $\mathcal{B}_m = 0$ all of the partial derivatives $\frac{\partial L_i}{\partial g}$ and $\frac{\partial L_i}{\partial h}$ are unbounded. However, we will show that if $\mathcal{B}_l(\hat{g}) \neq 0 \neq \mathcal{B}_r(\hat{h})$, then along the curve $\tilde{h}(g)$, the functions $\frac{d}{dg}L_1(g, \tilde{h}(g))$ and $\frac{d}{dg}L_2(g, \tilde{h}(g))$ are well defined and along its inverse $\tilde{g}(h)$, the functions $\frac{d}{dh}L_1(\tilde{g}(h), h)$ and $\frac{d}{dh}L_2(\tilde{g}(h), h)$ are well defined. In the derivation of $\tilde{h}(g)$, we have seen that along the curve $\tilde{h}(g)$, either $p_m(g, \tilde{h}(g)) = 0$ or $V_{m_1}(u_m(g, \tilde{h}(g))) = V_{m_2}(u_m(g, \tilde{h}(g)))$.

- First we consider the case that the curve $\tilde{h}(g)$ is such that $p_m(\tilde{g}(h), h) = 0$ and $\mathcal{B}_l \neq 0 \neq \mathcal{B}_r$. Using the definitions of L_1 , G_i , $\tilde{G}$, and Lemma 4.3, we get

$$\begin{aligned} \frac{d}{dg}L_1(g, \tilde{h}(g)) &= \frac{d}{dg} \left[\int_{u_l(g)}^{u(x_1; g)} \frac{du}{p(u, g)} + \sum_{i=2}^v \left[\int_{u(M_{i-1}; g)}^{u(x_i; g)} \frac{du}{p(u, g)} - \int_{u(M_{i-1}; g)}^{u(x_{i-1}; g)} \frac{du}{p(u, g)} \right] \right. \\ &\quad \left. - \int_{u(M_v; g)}^{u(x_v; g)} \frac{du}{p(u, g)} + \int_{u(M_v; g)}^{u_m(g, \tilde{h}(g))} \frac{du}{p(u, g)} \right] \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\mathcal{B}_l(g)} - G_1(-L_1, M_0) - \sum_{i=1}^v G_i(M_{i-1}, M_i) - \tilde{G}(M_v, 0) \\
&= -\frac{1}{\mathcal{B}_l(g)} - F_{l,v}(g).
\end{aligned}$$

Note that Lemma 4.3 is also used for the calculation of $\int_{u(M_v;g)}^{u_m(g,\tilde{h}(g))} \frac{du}{p(u,g)}$ as $\tilde{h}(g)$ is a curve such that $p(u_m(g, \tilde{h}(g)), g) = 0$ for all g near $\hat{g}$. Using (43), the expression for $\frac{d}{dg}L_1(g, \tilde{h}(g))$ gives immediately

$$\left. \frac{d}{dh}L_1(\tilde{g}(h), h) \right|_{h=\tilde{h}(g)} = \frac{1}{\tilde{h}(g)} \frac{d}{dg}L_1(g, \tilde{h}(g)) = -\frac{1}{\tilde{h}'(g)} \left(\frac{1}{\mathcal{B}_l(g)} + F_{l,v}(g) \right).$$

Substituting $g = \tilde{g}(h)$ and using $\tilde{h}'(\tilde{g}(h)) = \frac{1}{\tilde{g}'(h)}$, we get

$$\frac{d}{dh}L_1(\tilde{g}(h), h) = -\tilde{g}'(h) \left(\frac{1}{\mathcal{B}_l(\tilde{g}(h))} + F_{l,v}(\tilde{g}(h)) \right).$$

In a similar way it can be shown that

$$\frac{d}{dh}L_2(\tilde{g}(h), h) = -\frac{1}{\mathcal{B}_r(h)} - F_{r,\mu}(h); \quad \frac{d}{dg}L_2(g, \tilde{h}(g)) = -\tilde{h}'(g) \left(\frac{1}{\mathcal{B}_r(\tilde{h}(g))} + F_{r,\mu}(\tilde{h}(g)) \right).$$

To link these expressions to the compatibility condition in Lemma A.7, we note that with Lemma A.9 it follows that along the curve $\tilde{h}(g)$, the compatibility condition in Lemma A.7 is

$$\begin{aligned}
0 &= \mathcal{B}_l \frac{V'_{m_2}(u_m)}{V'_{m_1}(u_m)} + \mathcal{B}_r \frac{V'_{m_1}(u_m)}{V'_{m_2}(u_m)} + \mathcal{B}_l \mathcal{B}_r \left(\frac{V'_{m_1}(u_m)}{V'_{m_2}(u_m)} F_{l,v} + \frac{V'_{m_2}(u_m)}{V'_{m_1}(u_m)} F_{r,\mu} \right) \\
&= \mathcal{B}_l \tilde{h}'(g) + \mathcal{B}_r / \tilde{h}'(g) + \mathcal{B}_l \mathcal{B}_r (F_{l,v} / \tilde{h}'(g) + F_{r,\mu} \tilde{h}(g)) \\
&= \mathcal{B}_l \mathcal{B}_r \left(\tilde{h}'(g) \left[\frac{1}{\mathcal{B}_r} + F_{r,\mu} \right] + \left[\frac{1}{\mathcal{B}_l} + F_{r,\mu} \right] / \tilde{h}'(g) \right) \\
&= -\mathcal{B}_l \mathcal{B}_r \left(\frac{d}{dg}L_2(g, \tilde{h}(g)) + \frac{d}{dh}L_1(\tilde{g}(h), h) \right) \Big|_{h=\tilde{h}(g)}.
\end{aligned}$$

- Next we consider the case that $\tilde{h}(g)$ is such that $V'_{m_1}(u_m(\tilde{g}(h), h)) = V'_{m_2}(u_m(\tilde{g}(h), h))$ and $p_m(\tilde{g}(h), h) \neq 0$, and $\mathcal{B}_l \neq 0 \neq \mathcal{B}_r$. Hence $\tilde{h}'(g) = 1$ and $\tilde{g}'(h) = 1$, see Lemma A.9. Using the same ideas as in the previous case, we get

$$\frac{d}{dh}L_1(\tilde{g}(h), h) = -\frac{1}{\mathcal{B}_l(\tilde{g}(h))} - E_{l,v}(\tilde{g}(h))$$

and

$$\frac{d}{dg}L_2(g, \tilde{h}(g)) = -\frac{1}{\mathcal{B}_r(\tilde{h}(g))} - E_{r,\mu}(\tilde{h}(g)).$$

Thus

$$\begin{aligned}\mathcal{B}_l \mathcal{B}_r \left[\frac{dL_1}{dh} + \frac{dL_2}{dg} \right] &= -\mathcal{B}_l \mathcal{B}_r \left(\frac{1}{\mathcal{B}_l} + E_{l,v} + \frac{1}{\mathcal{B}_r} + E_{r,\mu} \right) \\ &= -(\mathcal{B}_l + \mathcal{B}_r + \mathcal{B}_l \mathcal{B}_r (E_{l,v} + E_{r,\mu})).\end{aligned}$$

Which again is zero if and only if an eigenvalue zero exists by Lemma A.7 (ii).

The last two statements in the theorem, concerning the cases when $\mathcal{B}_l = 0$ or $\mathcal{B}_r = 0$, are direct re-statements of Corollary A.8. $\square$

References

- [1] H. Akoh, S. Sakai, A. Yagi, H. Hayakawa, Real time fluxon dynamics in Josephson transmission line, *IEEE Trans. Magn.* 21 (1985) 737–740.
- [2] A.R. Bishop, T. Schneider (Eds.), *Solitons and Condensed Matter Physics: Proceedings of a Symposium Held, June 27–29, 1978*, Springer-Verlag, 1978.
- [3] A.S. Davydov, *Solitons in Molecular Systems*, Reidel, Dordrecht, Netherlands, 1985.
- [4] G. Derks, A. Doelman, S.A. van Gils, H. Susanto, Stability analysis of π -kinks in a $0-\pi$ Josephson junction, *SIAM J. Appl. Dyn. Syst.* 6 (2007) 99–141.
- [5] G. Derks, A. Doelman, C.J.K. Knight, H. Susanto, Analysis of pinned fluxons in a Josephson junction with a finite-length inhomogeneity, *European J. Appl. Math.* 23 (2012) 201–244.
- [6] G. Derks, G. Gaeta, A minimal model of DNA dynamics in interaction with RNA-Polymerase, *Phys. D* 240 (2011) 1805–1817.
- [7] R.K. Dodd, J.C. Eilbeck, J.D. Gibbon, H.C. Morris, *Solitons and Nonlinear Wave Equations*, Academic Press, 1982.
- [8] J.D. Gibbon, I.N. James, I.M. Moroz, The sine-Gordon equation as a model for a rapidly rotating baroclinic fluid, *Phys. Scr.* 20 (1979) 402–408.
- [9] E. Goldobin, D. Koelle, R. Kleiner, Semi-fluxons in long Josephson $0-\pi$ -junctions, *Phys. Rev. B* 66 (2002) 100508.
- [10] R.H. Goodman, R. Haberman, Interaction of sine-Gordon kinks with defects: the two bounce resonance, *Phys. D* 195 (2004) 303–323.
- [11] R.H. Goodman, R. Haberman, Chaotic scattering and the n -bounce resonance in solitary-wave interactions, *Phys. Rev. Lett.* 98 (2007) 104103.
- [12] P.J.A. van Heijster, A. Doelman, T.J. Kaper, Y. Nishiura, K.I. Ueda, Pinned fronts in heterogeneous media of jump type, *Nonlinearity* 24 (2011) 127–157.
- [13] H. Hilgenkamp, π -phase shift Josephson structures, *Supercond. Sci. Technol.* 21 (2008) 034001.
- [14] R.K. Jackson, R. Marangell, H. Susanto, An instability criterion for standing waves on nonzero backgrounds, preprint, 2012.
- [15] C.K.R.T. Jones, Instability of standing waves for non-linear Schrödinger-type equations, *Ergodic Theory Dynam. Systems* 8 (1988) 119–138.
- [16] C.K.R.T. Jones, J.V. Moloney, Instability of standing waves in nonlinear optical waveguides, *Phys. Lett. A* 117 (1986) 175–180.
- [17] Y.S. Kivshar, A.M. Kosevich, O.A. Chubykalo, Finite-size effects in fluxon scattering by an inhomogeneity, *Phys. Lett. A* 129 (1988) 449–452.
- [18] Y.S. Kivshar, B.A. Malomed, Dynamics of solitons in nearly integrable systems, *Rev. Modern Phys.* 61 (1989) 763–915, *Rev. Modern Phys.* 63 (1991) 211 (Addendum).
- [19] Y.S. Kivshar, Zhang Fei, L. Vázquez, Resonant soliton-impurity interactions, *Phys. Rev. Lett.* 67 (1991) 1177.
- [20] C.J.K. Knight, Microresistor pinning of $2k\pi$ -fluxons in long Josephson junctions, <http://personal.maths.surrey.ac.uk/pg/Christopher.Knight/masters-thesis.pdf>, 2008.
- [21] R. Marangell, C.K.R.T. Jones, H. Susanto, Localized standing waves in inhomogeneous Schrödinger equations, *Nonlinearity* 23 (2010) 2059.
- [22] R. Marangell, H. Susanto, C.K.R.T. Jones, Unstable gap solitons in inhomogeneous Schrödinger equations, *J. Differential Equations* 253 (4) (2012) 1191–1205.
- [23] D.W. McLaughlin, A.C. Scott, Perturbation analysis of fluxon dynamics, *Phys. Rev. A* 18 (1978) 1652–1679.
- [24] B. Piette, W.J. Zakrzewski, Dynamical properties of a soliton in a potential well, *J. Phys. A* 40 (2007) 329–346.
- [25] V.V. Ryazanov, V.A. Oboznov, A.Yu. Rusanov, A.V. Veretennikov, A.A. Golubov, J. Aarts, Coupling of two superconductors through a ferromagnet: evidence for a π junction, *Phys. Rev. Lett.* 86 (2001) 2427–2430.
- [26] S. Sakai, H. Akoh, H. Hayakawa, Fluxon transfer devices, *Jpn. J. Appl. Phys.* 24 (1985) L771.
- [27] I.L. Serpuchenko, A.V. Ustinov, Experimental observation of the fine structure on the current-voltage characteristics of long Josephson junctions with a lattice of inhomogeneities, *Sov. Phys. JETP Lett.* 46 (1987) 549, *Pisma Zh. Eksp. Teor. Fiz.* 46 (1987) 435.
- [28] H. Susanto, Josephson junctions with phase shifts, stability analysis of fractional fluxons, PhD thesis, Enschede, 2006.
- [29] H. Susanto, S.A. van Gils, T.P.P. Visser Ariando, H.J.H. Smilde, H. Hilgenkamp, Static semifluxons in a long Josephson junction with π -discontinuity points, *Phys. Rev. B* 68 (2003) 104501–104508.
- [30] E.C. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations*, second ed., Oxford University Press, 1962.

- [31] A.N. Vystavkin, Yu.F. Drachevskii, V.P. Koshelets, I.L. Serpuchenko, First observation of static bound states of fluxons in long Josephson junctions with inhomogeneities, *Sov. J. Low Temp. Phys.* 14 (1988) 357–358, *Fiz. Nizk. Temp.* 14 (1988) 646–649.
- [32] G.B. Whitham, *Linear and Nonlinear Waves*, Wiley–Interscience, 1974.
- [33] L.V. Yakushevich, *Nonlinear Physics of DNA*, Wiley Ser. Nonlinear Sci., 1998.
- [34] X. Yuan, T. Teramoto, Y. Nishiura Y, Heterogeneity-induced defect bifurcation and pulse dynamics for a three-component reaction-diffusion system, *Phys. Rev. E* 75 (2007) 036220.