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J. Differential Equations 256 (2014) 1133–1150

Journal of Differential Equations

www.elsevier.com/locate/jde

The Maximum principle of Alexandrov for very weak solutions

Teresa Radice, Gabriella Zecca *

Department of Mathematics "R. Caccioppoli", University of Naples "Federico II", Via Cintia – 80126, Napoli, Italy

Received 12 September 2012; revised 8 September 2013

Available online 1 November 2013

Abstract

We extend the classical maximal principle of Alexandrov, to very weak solutions of the elliptic equation $\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = f$.

MSC: 35J15; 35B50

Keywords: Second-order elliptic equations; Very weak solutions; Maximum principle

1. Introduction

We consider elliptic second-order partial differential operators of the form

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{1.1}$$

where the measurable coefficients $a_{ij} = a_{ji}$ are defined on a bounded C^1-smooth domain $\Omega \subset \mathbb{R}^n$. Let the equation

$$Lu = f \tag{1.2}$$

be satisfied almost everywhere in Ω for some function f.

* Corresponding author.
E-mail addresses: teresa.radice@unina.it (T. Radice), g.zecca@unina.it (G. Zecca).

0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2013.10.010

The operator L is elliptic in Ω if the coefficient matrix $\mathscr{A}(x) = [a_{ij}(x)]$ is positive in Ω. Precisely, if $\lambda(x)$, $\Lambda(x)$ denote the smallest and the largest eigenvalues of $\mathscr{A}(x)$ then

$$0 < \lambda(x)|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \Lambda(x)|\xi|^2$$

for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. If $K(x) = \Lambda(x)/\lambda(x)$ is bounded in Ω, we refer to the operator L as uniformly elliptic in Ω, see [6].

We abbreviate $\frac{\partial^2 u}{\partial x_i \partial x_j}$ to u_{ij} and write

$$Lu = \operatorname{Tr}\big[\mathscr{A}(x) D^2 u\big]$$

where $D^2 u$ is the Hessian matrix

$$D^2 u = \begin{bmatrix} u_{11} & \dots & u_{1n} \\ & \dots & \\ u_{n1} & \dots & u_{nn} \end{bmatrix}.$$

The Sobolev class $\mathscr{W}^{2,n}_{\mathrm{loc}}(\Omega)$ is considered the natural domain of definition of the operator L see [16] and [20]. For in this class the determinant of the Hessian matrix is locally integrable,

$$\mathscr{H} u = \det D^2 u \in \mathscr{L}^1_{\mathrm{loc}}(\Omega). \tag{1.3}$$

Observing that condition (1.3) is less restrictive than $u \in \mathscr{W}^{2,n}_{\mathrm{loc}}(\Omega)$ we are interested in studying the operator L in weaker domains of definition than that in $\mathscr{W}^{2,n}_{\mathrm{loc}}(\Omega)$.

We will let $\mathscr{D}$ denote the determinant of $\mathscr{A}$. Thus $\mathscr{D}^{\frac{1}{n}}$ is the geometric mean of the eigenvalues of $\mathscr{A}$,

$$0 < \lambda(x) \leqslant \mathscr{D}^{\frac{1}{n}}(x) \leqslant \Lambda(x).$$

The classical maximum principle of Alexandrov [3] reads as:

Theorem 1.1. *Let Ω be a bounded domain and $u \in C(\overline{\Omega}) \cap \mathscr{W}^{2,n}_{\mathrm{loc}}(\Omega)$. Then*

$$\sup_{\Omega} u \leqslant \sup_{\partial\Omega} u^+ + C \big\| Lu/\mathscr{D}^{1/n} \big\|_{\mathscr{L}^n(\Omega)} \tag{1.4}$$

where $u^+ = \max\{0, u\}$ and C is a constant that depends on n and $\operatorname{diam}\Omega$.

Numerous papers have been devoted to this estimate. In particular, Alexandrov [4] and Pucci [16] proved that, without any restriction on the ellipticity constant, it is not possible to replace the $\mathscr{L}^n(\Omega)$-norm of Lu by some $\mathscr{L}^p(\Omega)$-norm, $p < n$.

However, at least in the plane, if one fixes the ellipticity constant $K = \operatorname{ess\,sup}_{x\in\Omega} K(x)$ there is a positive ε depending on K such that one can obtain estimates in $\mathscr{L}^p(\Omega)$ for all $p > 2 - \varepsilon$. The sharp range of values for p for which an estimate of the form (1.4) holds has been conjectured by Pucci [17] and proved by Astala, Iwaniec and Martin in [5].

In the present paper, we relax the assumption $u \in \mathscr{W}^{2,n}_{\mathrm{loc}}(\Omega)$ and obtain several results for such solutions which we refer to *very weak solutions*.

First we extend the maximum principle (1.4) to very weak solutions which are concave, i.e. solutions whose Hessian matrix is nonpositive on Ω.

Let $\mathscr{W}^{2,\Theta}(\Omega)$ denote the Orlicz–Sobolev space, $\Theta(t) = \frac{t^n}{\log^{\frac{1}{n}}(e+t)}$, of functions whose second order derivatives satisfy

$$\int_\Omega \frac{|D^2u|^n}{\log^{\frac{1}{n}}(e+|D^2u|)} < \infty.$$

Theorem 1.2. *For any concave function $u \in \mathscr{W}^{2,\Theta}(\Omega) \cap C(\overline{\Omega})$ we have*

$$\sup_\Omega u \leqslant \sup_{\partial\Omega} u^+ + C\|Lu/\mathscr{D}^{1/n}\|_{\mathscr{L}^n(\Omega)} \tag{1.5}$$

where C is a constant depending only on n and $\operatorname{diam}\Omega$.

Then, we deal with the case where the coefficient matrix $\mathscr{A}(x)$ verifies an exponential integrability condition

$$\int_\Omega \exp\left(\frac{|\mathscr{A}|^n}{\mathscr{D}}\right) dx < \infty. \tag{1.6}$$

This additional assumption on the coefficients allows us to consider solutions in somewhat weaker class than $\mathscr{W}^{2,n}_{\mathrm{loc}}(\Omega)$; namely, satisfying the following condition

$$\lim_{\varepsilon\to 0} \varepsilon^n \int_\Omega |D^2u|^{(1-\varepsilon)n} \log(e+|D^2u|) = 0. \tag{1.7}$$

Theorem 1.3. *Under the assumption (1.6) the estimate (1.4) remains true for any function $u \in C(\overline{\Omega}) \cap \mathscr{W}^{2,1}(\Omega)$ satisfying (1.7).*

Finally, we consider the uniformly elliptic case. In this case we obtain the estimate (1.4) for solutions u in the Grand–Sobolev spaces $\mathscr{W}^{2,n)}(\Omega) \supset \mathscr{W}^{2,n}(\Omega)$ (see Section 2 for definitions and basic properties of this space). Precisely,

Theorem 1.4. *Let L be uniformly elliptic and $u \in C(\overline{\Omega}) \cap \mathscr{W}^{2,n)}(\Omega)$. Then (1.4) holds.*

The paper is organized as follows: first we set up some notation and terminology concerning Orlicz spaces, and other function spaces. Next we discuss Hodge decomposition of matrix fields. Finally, in Section 6 the case L uniformly elliptic is treated.

2. Notations and preliminary results

In this section we recall some basic properties of the Orlicz–Zygmund spaces and their duals. The results we will state are well known, for the proofs we refer to [9,10,18].

An Orlicz function is a continuously increasing function

$$\Phi : [0,\infty) \to [0,\infty),$$
$$\Phi(0) = 0, \qquad \lim_{t\to\infty} \Phi(t) = \infty,$$

though in most of our applications Φ will be convex and in this case we call it a Young function.

Let Ω be a C^1-smooth domain. The Orlicz space, denoted by $\mathscr{L}^\Phi(\Omega)$, consists of all measurable functions f on Ω such that

$$\int_\Omega \Phi\big(k^{-1}|f|\big) < \infty, \quad \text{for some } k = k(f) > 0.$$

$\mathscr{L}^\Phi(\Omega)$ is a complete linear metric space with respect to the following distance function:

$$\mathrm{dist}_{\Phi(f,g)} = \inf\left\{k > 0\colon \int_\Omega \Phi\big(k^{-1}|f-g|\big) \leqslant 1\right\}.$$

There is also a homogeneous nonlinear functional on $\mathscr{L}^\Phi(\Omega)$ called the Luxemburg functional:

$$\|f\|_{\mathscr{L}^\Phi} = \|f\|_\Phi = \inf\left\{k > 0;\ \int_\Omega \Phi\big(k^{-1}|f|\big) \leqslant \Phi(1)\right\} \tag{2.1}$$

in the case when Φ is a Young function, the expression $\|\ \|_\Phi$ is a norm and $\mathscr{L}^\Phi(\Omega)$ becomes a Banach space.

The Zygmund spaces, denoted by $\mathscr{L}^p \log^\alpha \mathscr{L}(\Omega)$, correspond to the Orlicz functions $\Phi(t) = t^p \log^\alpha(a+t)$ with $1 \leqslant p < \infty$; $\alpha \in \mathbb{R}$.

The defining function $\Phi(t) = t^p \log^\alpha(e+t)$, $1 \leqslant p < \infty$ is a Young function when $\alpha \geqslant 1-p$ and we have the following estimates

$$\|f\|_{\mathscr{L}^p \log^{-1} \mathscr{L}} \leqslant \|f\|_p \leqslant \|f\|_{\mathscr{L}^p \log \mathscr{L}}$$

and

$$\|f\|_{\mathscr{L}^p \log \mathscr{L}} \leqslant \left[\int |f|^p \log\left(e + \frac{|f|}{\|f\|_p}\right)\right]^{\frac{1}{p}} \leqslant 2\|f\|_{\mathscr{L}^p \log \mathscr{L}}. \tag{2.2}$$

For $p \geqslant 1$ and $\alpha \geqslant 0$ the nonlinear functional

$$[[f]]_{p,\alpha} = \left[\int_\Omega |f|^p \log^\alpha\left(e + \frac{|f|}{\|f\|_p}\right)\right]^{\frac{1}{p}}$$

is comparable with the Luxemburg norm, given at (2.1).

The following estimates are straightforward

$$\|f\|_{\mathscr{L}^p \log^{-1} \mathscr{L}} \leqslant \|f\|_{\mathscr{L}^p} \leqslant \|f\|_{\mathscr{L}^p \log^{\alpha} \mathscr{L}} \leqslant [[f]]_{p,\alpha} \leqslant 2\|f\|_{\mathscr{L}^p \log^{\alpha} \mathscr{L}}$$

for $\alpha > 0$. We have the Hölder-type inequalities

$$\|fg\|_{\mathscr{L}^c \log^{\gamma} \mathscr{L}} \leqslant C_{\alpha\beta}(a,b)\|f\|_{\mathscr{L}^a \log^{\alpha} \mathscr{L}} \cdot \|g\|_{\mathscr{L}^b \log^{\beta} \mathscr{L}}$$

whenever $a, b > 1$ and $\alpha, \beta \in \mathbb{R}$ are coupled by the relationships

$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}, \qquad \frac{\gamma}{c} = \frac{\alpha}{a} + \frac{\beta}{b}.$$

Proposition 2.1. *For any $\alpha > 0$ and $p \geqslant 1$,*

$$c(\alpha, p)\|f\|^{\alpha}_{\mathscr{L}^{\alpha p} \log \mathscr{L}} \leqslant \big\| |f|^{\alpha} \big\|_{\mathscr{L}^p \log \mathscr{L}} \leqslant C(\alpha, p)\|f\|^{\alpha}_{\mathscr{L}^{\alpha p} \log \mathscr{L}}.$$

Proof. By the obvious inequality

$$c(\gamma)\log(e+t) \leqslant \log\big(e+t^{\gamma}\big) \leqslant C(\gamma)\log(e+t) \quad \forall \gamma, t > 0, \tag{2.3}$$

and by (2.2) we have

$$\begin{aligned}
\big\| |f|^{\alpha} \big\|^p_{\mathscr{L}^p \log \mathscr{L}} &\leqslant \int_{\Omega} |f|^{\alpha p} \log\left(e + \frac{|f|^{\alpha}}{\||f|^{\alpha}\|_p}\right) \\
&\leqslant c(\alpha) \int_{\Omega} |f|^{\alpha p} \log\left(e + \frac{|f|}{\|f\|_{\alpha p}}\right) \\
&\leqslant c(\alpha, p)\|f\|^{\alpha p}_{\mathscr{L}^{\alpha p} \log \mathscr{L}}.
\end{aligned}$$

On the other hand, using again (2.3) we have

$$\begin{aligned}
\|f\|^{\alpha p}_{\mathscr{L}^{\alpha p} \log \mathscr{L}} &\leqslant \int_{\Omega} |f|^{\alpha p} \log\left(e + \frac{|f|}{\|f\|_{\alpha p}}\right) \\
&\leqslant c(\alpha, p) \int_{\Omega} |f|^{\alpha p} \log\left(e + \left(\frac{|f|}{\|f\|_{\alpha p}}\right)^{\alpha}\right) \\
&\leqslant c(\alpha, p)\big\| |f|^{\alpha} \big\|^p_{\mathscr{L}^p \log \mathscr{L}}. \qquad \Box
\end{aligned}$$

A pair of Orlicz function (Φ, Ψ) are called Hölder conjugate couple, or Young complementary functions, if we have Hölder's inequality

$$\left| \int_{\Omega} fg \right| \leqslant C_{\Phi\Psi}\|f\|_{\Phi}\|g\|_{\Psi}$$

for $f \in \mathscr{L}^{\Phi}(\Omega)$ and $g \in \mathscr{L}^{\Psi}(\Omega)$. Our basic example is the Hölder conjugate couple $\Phi(t) = t\log(e+t)$ and $\Psi(t) = e^t - 1$ which define the Zygmund and exponential classes. In this case we have the estimate

$$\left|\int_\Omega fg\right| \leqslant 4\|f\|_{\mathscr{L}\log\mathscr{L}}\|g\|_{\mathrm{Exp}}.$$

Thus $\mathrm{Exp}(\Omega)$ is the dual space to the Zygmund space $\mathscr{L}\log\mathscr{L}(\Omega)$.

The Orlicz–Sobolev space $\mathscr{W}^{k,\Phi}(\Omega)$, $k \in \mathbb{N}$ is the space of all functions u such that u and its distributional derivatives up to order k lie in $\mathscr{L}^{\Phi}(\Omega)$. Note that $\mathscr{W}^{1,\Phi}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{\mathscr{W}^{1,\Phi}(\Omega)} = \|u\|_{\mathscr{L}^{\Phi}(\Omega)} + \|\nabla u\|_{\mathscr{L}^{\Phi}(\Omega)}.$$

Also $\mathscr{W}^{2,\Phi}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{\mathscr{W}^{2,\Phi}(\Omega)} = \|u\|_{\mathscr{L}^{\Phi}(\Omega)} + \left\|D^2u\right\|_{\mathscr{L}^{\Phi}(\Omega)}$$

where D^2u denotes the Hessian matrix of u.

Here we confine ourselves to the essential matters and refer to [1,10,23] for further informations on Orlicz–Sobolev spaces.

Let Ω be any measurable set of finite positive measure $|\Omega| < \infty$, and f an integrable function, the notation $f_\Omega = ⨍_\Omega f(x)dx$ stands for the integral mean of f over Ω.

The class $\mathscr{L}^{q)}(\Omega)$. For $q \geqslant 1$, following [12] and [13], we recall the class $\mathscr{L}^{q)}(\Omega)$ of all measurable functions $h : \Omega \to \mathbb{R}$ such that

$$\sup_{0<\varepsilon\leqslant q} \varepsilon \int_\Omega \left|h(x)\right|^{q-\varepsilon} dx < \infty.$$

If $q > 1$, then $\mathscr{L}^{q)}(\Omega)$ becomes a Banach space with the norm

$$\|h\|_{q)} = \sup_{0<\varepsilon\leqslant q-1} \left\{\varepsilon ⨍_\Omega \left|h(x)\right|^{q-\varepsilon} dx\right\}^{\frac{1}{q-\varepsilon}}.$$

Clearly $\mathscr{L}^{q)}(\Omega) \subset \bigcap_{1\leqslant p<q} \mathscr{L}^p(\Omega)$, for $q > 1$. For a measurable function h on Ω we shall also consider the quantity:

$$(h)_q = \limsup_{\varepsilon\to 0^+} \left\{\varepsilon ⨍_\Omega \left|h(x)\right|^{q-\varepsilon} dx\right\}^{\frac{1}{q-\varepsilon}}.$$

Hence $h \in \mathscr{L}^{q)}(\Omega)$ if and only if $(h)_q < \infty$.

The class $\Sigma^q(\Omega)$. Let $q \geqslant 1$. By $\Sigma^q(\Omega)$ we denote the subclass of $\mathscr{L}^{q)}(\Omega)$ consisting of all functions h such that $(h)_q = 0$; i.e.,

$$h \in \Sigma^q(\Omega) \quad \Longleftrightarrow \quad \lim_{\varepsilon \to 0^+} \varepsilon \int_\Omega |h|^{q-\varepsilon}\, dx = 0.$$

The class $\Sigma^q(\Omega)$ was introduced by L. Greco in [7] where many properties of this class are studied. In particular the following relations between $\Sigma^q(\Omega)$ and the classes of functions introduced above are established.

Theorem 2.2. *(See [7].) For any $q \geqslant 1$ the following inclusions hold*

$$\mathscr{L}^q(\Omega) \subsetneqq \frac{\mathscr{L}^q}{\log \mathscr{L}}(\Omega) \subsetneqq \Sigma^q(\Omega) \subsetneqq \mathscr{L}^{q)}(\Omega) \subsetneqq \bigcap_{\beta > 1} \frac{\mathscr{L}^q}{\log^\beta \mathscr{L}}(\Omega). \tag{2.4}$$

Moreover, $\Sigma^q(\Omega)$ is a closed subspace of $\mathscr{L}^{q)}(\Omega)$, provided $q > 1$.

Our next purpose is to introduce another class of function, denoted by $\Sigma^q_\alpha(\Omega)$ and establish some relations.

The class $\Sigma^q_\alpha(\Omega)$. Let $q \geqslant 1$, $0 < \alpha \leqslant 1$. $\Sigma^q_\alpha(\Omega)$ is the subclass of $\mathscr{L}^{q)}(\Omega)$ consisting of all functions h such that

$$h \in \Sigma^q_\alpha(\Omega) \quad \Longleftrightarrow \quad \lim_{\varepsilon \to 0^+} \varepsilon^\alpha \int_\Omega |h|^{(1-\varepsilon)q} dx = 0.$$

Clearly if $\alpha = 1$ we obtain exactly the space $\Sigma^q(\Omega)$.

Theorem 2.3. *The following inclusion $\frac{\mathscr{L}^q}{\log^\alpha \mathscr{L}}(\Omega) \subset \Sigma^q_\alpha(\Omega)$ holds, for any $q \geqslant 1$.*

Proof. Let $h \in \frac{\mathscr{L}^q}{\log^\alpha \mathscr{L}}(\Omega)$. We may assume that $h \geqslant 1$ by considering $\max\{|h|, 1\}$ if necessary. Then by the elementary inequality $(e+t)^\varepsilon < e + t^\varepsilon$ $(0 < \varepsilon < 1, t \geqslant 0)$ we obtain

$$\begin{aligned}
\varepsilon^\alpha h^{(1-\varepsilon)q} &= \frac{h^q}{\log^\alpha(e+h)} \cdot \frac{\log^\alpha[(e+h)^\varepsilon]}{h^{q\varepsilon}} \\
&\leqslant \frac{h^q}{\log^\alpha(e+h)} \cdot \frac{\log^\alpha(e+h^\varepsilon)}{h^{q\varepsilon}} \\
&\leqslant \left[\sup_{t \geqslant 1} \frac{\log(e+t)}{t}\right]^q \cdot \frac{h^q}{\log^\alpha(e+h)} \\
&= \left[\log(e+1)\right]^q \frac{h^q}{\log^\alpha(e+h)}.
\end{aligned}$$

Therefore, since $\frac{h}{\log^\alpha(e+h)}$ is integrable and $\varepsilon^\alpha h^{(1-\varepsilon)q} \to 0$ almost everywhere as $\varepsilon \to 0$, by the Dominated Convergence Theorem, the desired inclusion follows. $\square$

Corollary 2.4. *Let $\Theta(t)=\frac{t^q}{\log^{\frac{1}{q}}(e+t)}$, $q\geqslant 1$, and let $u\in\mathscr{W}^{2,\Theta}(\Omega)$. Then*

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{1}{q}}\int_\Omega \left|D^2u\right|^{(1-\varepsilon)q}=0.$$

We will conclude this section by recalling the following particular case of a result due to T. Iwaniec and C. Sbordone (see Proposition 1 in [12]). It will be a key tool to our proof of Theorem 4.2.

Theorem 2.5. *(See [12].) Let $1\leqslant r_1<r<r_2<\infty$ and let*

$$T:\mathscr{L}^p\left(\Omega,\mathbb{R}^{n\times n}_{sym}\right)\to\mathscr{L}^p\left(\Omega,\mathbb{R}^{n\times n}_{sym}\right) \tag{2.5}$$

be a bounded linear operator for any $p\in[r_1,r_2]$. Then, for any $1-\frac{r}{r_1}\leqslant\varepsilon\leqslant 1-\frac{r}{r_2}$ and any $V\in\mathscr{L}^r(\Omega,\mathbb{R}^{n\times n}_{sym})$ such that $T(V)=0$ it holds

$$\left\|T\left(|V|^{-\varepsilon}V\right)\right\|_{\frac{r}{1-\varepsilon}}\leqslant C_r|\varepsilon|\|V\|_r^{1-\varepsilon} \tag{2.6}$$

where

$$C_r=\frac{2r(r_2-r_1)}{(r-r_1)(r_2-r)}\sup_{r_1\leqslant p\leqslant r_2}\|T\|_p.$$

Here $\|T\|_p$ denotes the norm of the operator (2.5).

Remark 2.6. Under suitable condition on Φ, the estimate (2.6) has counterparts in Orlicz space $\mathscr{L}^\Phi$, namely

$$\left\|T\left(|V|^{-\varepsilon}V\right)\right\|_\Phi\leqslant C_\Phi|\varepsilon|\left\||V|^{-\varepsilon}V\right\|_\Phi. \tag{2.7}$$

3. Hodge decomposition

For any matrix field $V=[V^{ij}]_{i,j=1}^n$ we define

$$\mathrm{DIV}(V)=\sum_{i,j=1}^n\frac{\partial^2}{\partial x_i\partial x_j}V^{ij}.$$

If $V=[V^{ij}]$ belongs to $\mathscr{L}^p(\Omega,\mathbb{R}^{n\times n}_{sym})$, $1<p<\infty$, the bi-Laplacian equation

$$\Delta\Delta v=\mathrm{DIV}(V)$$

can be solved by using the Riesz transforms $R=(R_1,\dots,R_n)$,

$$D^2v=(R\otimes R)\left[\langle R\otimes R|V\rangle\right]=:\mathscr{R}V$$

where $R \otimes R = [R_i \circ R_j]_{i,j=1,\dots,n}$, i.e.

$$v_{x_k x_l} = R_k\left(R_l\left(\sum_{ij=1}^{n} R_i\left(R_j V^{ij}\right)\right)\right), \quad k,l = 1,\dots,n.$$

Hence the Hodge decomposition of V reads as

$$V = D^2 v + H$$

where v is a $\mathscr{W}^{2,p}(\mathbb{R}^n)$ function and $H = [H^{ij}]$ is a symmetric matrix field verifying

$$\sum_{i,j=1}^{n} \frac{\partial^2 H^{ij}}{\partial x_i \partial x_j} = 0. \tag{3.1}$$

Notice that

$$\mathscr{R} : \mathscr{L}^p\left(\mathbb{R}^n, \mathbb{R}^{n\times n}\right) \to \mathscr{L}^p\left(\mathbb{R}^n, \mathbb{R}^{n\times n}\right)$$

and the range of the operator

$$\mathcal{T} = \mathcal{I}d - \mathscr{R} : \mathscr{L}^p\left(\mathbb{R}^n, \mathbb{R}^{n\times n}\right) \to \mathscr{L}^p\left(\mathbb{R}^n, \mathbb{R}^{n\times n}\right)$$

consists of the matrix fields satisfying (3.1).

In other words,

$$V = \mathscr{R}V + \mathcal{T}V.$$

Moreover, from the $\mathscr{L}^p$-boundedness of the Riesz transforms it follows that

$$\|\mathscr{R}V\|_p + \|\mathcal{T}V\|_p \leqslant C(p)\|V\|_p \tag{3.2}$$

where $C(p)$ is a positive constant depending only upon p (see [11] for the independence of the constant C of the dimension n).

Now, let Ω be a subdomain of $\mathbb{R}^n$. In view of the Hodge decomposition of a matrix field $V \in \mathscr{L}^p(\Omega, \mathbb{R}^{n\times n}_{sym})$ we consider the inhomogeneous problem

$$\begin{cases} \Delta\Delta v = f \in \mathscr{W}^{-2,p}(\Omega), & \\ v = 0 & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial \vec{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

When Ω is smooth, problem (3.3) admits a unique solution $v \in \mathscr{W}^{2,p}(\Omega)$ (see for example [2,14,15,21,22]). If f has a *divergence form*, say $f = \mathrm{DIV}(V)$, with $V = [V^{ij}] \in C_0^\infty(\Omega, \mathbb{R}^{n\times n}_{sym})$, then the solution of the system (3.3) takes the form

$$\mathscr{R}_\Omega(V) := D^2 v. \tag{3.4}$$

Denote by $\mathscr{H}^p(\Omega, \mathbb{R}^{n\times n}_{sym})$ the closure of the range of the Hessian operator

$$D^2 : C_0^\infty(\Omega) \to \mathscr{L}^p\big(\Omega, \mathbb{R}^{n\times n}_{sym}\big),$$

$1 < p < \infty$. If Ω is smooth then $\mathscr{R}_\Omega$ has a continuous extension to all $\mathscr{L}^p(\Omega, \mathbb{R}^{n\times n}_{sym})$ spaces, and so (3.4) extends to all $V \in \mathscr{L}^p(\Omega, \mathbb{R}^{n\times n}_{sym})$ and gives a solution v of the problem (3.3) with $D^2 v \in \mathscr{H}^p(\Omega, \mathbb{R}^{n\times n}_{sym})$, $1 < p < \infty$.

Definition 3.1. A subdomain Ω in $\mathbb{R}^n$ is said to be regular if the linear operator $\mathscr{R}_\Omega$ is continuous in $\mathscr{L}^p(\Omega, \mathbb{R}^{n\times n}_{sym})$, for every $p \in]1, \infty[$.

Any C^1 bounded domain is regular (see [2], Theorem 2.1).

Now, let Ω be a regular domain and let us introduce the operator

$$\mathcal{T}_\Omega = \mathcal{I}d - \mathscr{R}_\Omega : \mathscr{L}^p\big(\Omega, \mathbb{R}^{n\times n}_{sym}\big) \to \mathscr{L}^p\big(\Omega, \mathbb{R}^{n\times n}_{sym}\big). \tag{3.5}$$

The range of this operator consists of matrix fields on Ω which satisfy (3.1). Hence we obtain (as before) the Hodge decomposition of a matrix field $V \in \mathscr{L}^p(\Omega, \mathbb{R}^{n\times n}_{sym})$:

$$V = D^2 v + H, \qquad \mathrm{DIV}(H) = 0, \qquad D^2 v \in \mathscr{H}^p\big(\Omega, \mathbb{R}^{n\times n}_{sym}\big)$$

together with the uniform estimate

$$\big\|D^2 v\big\|_{p,\Omega} + \|H\|_{p,\Omega} \leqslant C(p,\Omega)\|V\|_{p,\Omega}$$

Since for any $u \in \mathscr{W}^{2,p}(\Omega)$ it holds $\mathcal{T}_\Omega(D^2 u) = 0$, by Theorem 2.5 with $T = \mathcal{T}_\Omega$, we conclude with the inequality

$$\big\|\mathcal{T}_\Omega\big(\big|D^2 u\big|^{-\varepsilon} D^2 u\big)\big\|_{\frac{p}{1-\varepsilon}} \leqslant C_p|\varepsilon|\big\|D^2 u\big\|_p^{1-\varepsilon}. \tag{3.6}$$

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2 in slightly more general assumption on $D^2 u$ (see Theorem 4.2 below). One of the key ingredients in our proof is Lemma 9.2 in [6].

This lemma depends upon the notion of contact set. If u is an arbitrary continuous function on Ω, the upper contact set of u, denoted Γ^+ or Γ_u^+, is the subset of Ω where the graph of u lies below a support hyperplane in $\mathbb{R}^{n+1}$, that is,

$$\Gamma^+ = \big\{y \in \Omega \;\big|\; u(x) \leqslant u(y) + p\cdot(x-y) \text{ for all } x \in \Omega, \text{ for some } p = p(y) \in \mathbb{R}^n\big\}.$$

Clearly, u is a concave function in Ω if and only if $\Gamma_u^+ = \Omega$.

Lemma 4.1. *(See [6], Lemma 9.2.) For any $u \in C^2(\Omega) \cap C(\overline{\Omega})$ we have*

$$\sup_{\Omega} u \leqslant \sup_{\partial\Omega} u^+ + \frac{d}{\omega_n^{1/n}} \left(\int_{\Gamma^+} \left|\det D^2 u\right| \right)^{1/n} \tag{4.1}$$

where $d = \operatorname{diam} \Omega$ and ω_n denotes the measure of the unit sphere $\partial B(0,1)$ of $\mathbb{R}^n$.

In this paper, using Lemma 4.1, we shall prove the following

Theorem 4.2. *Let $u \in C(\overline{\Omega}) \cap \mathscr{W}^{2,1}(\Omega)$ be a concave function (i.e. $-D^2 u \geqslant 0$) such that $|D^2 u| \in \Sigma^n_{\frac{1}{n}}(\Omega)$. Then*

$$\sup_{\Omega} u \leqslant \sup_{\partial\Omega} u^+ + \frac{d}{\omega_n^{1/n}} \left\| \frac{Lu}{\mathscr{D}^{1/n}} \right\|_{\mathscr{L}^n(\Omega)}. \tag{4.2}$$

Proof. Let $u \in C(\overline{\Omega}) \cap \mathscr{W}^{2,1}(\Omega)$ be a concave function such that $|D^2 u| \in \Sigma^n_{\frac{1}{n}}(\Omega)$, i.e.

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{1}{n}} \int_{\Omega} \left|D^2 u\right|^{(1-\varepsilon)n} = 0 \tag{4.3}$$

and let $0 < \varepsilon \leqslant 1$. Obviously,

$$\left\langle \mathscr{A}(x) \;\middle|\; D^2 u \right\rangle = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} = Lu \tag{4.4}$$

in $\Omega \subset \mathbb{R}^n$. Multiplying (4.4) by $|D^2 u|^{-\varepsilon}$ we obtain

$$\left\langle \mathscr{A}(x) \;\middle|\; \left|D^2 u\right|^{-\varepsilon} D^2 u \right\rangle = \left|D^2 u\right|^{-\varepsilon} Lu. \tag{4.5}$$

Since by (4.3) $|D^2 u| \in \mathscr{L}^{(1-\varepsilon)n}(\Omega)$, then $|D^2 u|^{-\varepsilon} D^2 u$ belongs to $\mathscr{L}^n(\Omega)$. Hence, we can consider (see Section 3) the Hodge decomposition:

$$\left|D^2 u\right|^{-\varepsilon} D^2 u = D^2 v_\varepsilon + H_\varepsilon \tag{4.6}$$

where $v_\varepsilon \in \mathscr{W}^{2,n}(\Omega)$.

Moreover, using the same notation as in Section 3,

$$D^2 v_\varepsilon = \mathscr{R}\left(\left|D^2 u\right|^{-\varepsilon} D^2 u\right),$$

and

$$H_\varepsilon = \mathcal{T}_\Omega\left(\left|D^2 u\right|^{-\varepsilon} D^2 u\right)$$

by (3.6) we have

$$\|H_\varepsilon\|_n \leqslant C_n\varepsilon \cdot \big\| \left|D^2u\right|^{-\varepsilon} D^2u\big\|_n. \tag{4.7}$$

By (4.6), Eq. (4.5) becomes

$$\left\langle \mathscr{A}(x) \middle| D^2 v_\varepsilon \right\rangle + \left\langle \mathscr{A}(x) \middle| H_\varepsilon \right\rangle = \left|D^2u\right|^{-\varepsilon} Lu.$$

Let now $\Omega' \subset\subset \Omega$, applying Lemma 4.1 to $v_\varepsilon \in \mathscr{W}^{2,n}(\Omega)$, we obtain

$$\sup_{\Omega'} v_\varepsilon \leqslant \sup_{\partial\Omega'} v_\varepsilon^+ + \frac{d}{\omega_n^{1/n}} \left(\int_{\Omega'} \left|\det D^2 v_\varepsilon\right| \right)^{\frac{1}{n}} \tag{4.8}$$

where $d = \operatorname{diam} \Omega'$. Let $\varepsilon \to 0$, our aim is to show that

$$\sup_{\Omega'} u \leqslant \sup_{\partial\Omega'} u^+ + \frac{d}{\omega_n^{1/n}} \left(\int_{\Omega'} \left|\det D^2 u\right| \right)^{\frac{1}{n}}. \tag{4.9}$$

Indeed, for any matrix $M \in \mathbb{R}^{n\times n}$, let $|M|$ the usual Euclidean norm of M. We have the following inequality

$$\det(A+B) \leqslant |\det A| + C_n \sum_{k=1}^{n-1} |A|^k |B|^{n-k} + |\det B|, \tag{4.10}$$

for $A, B \in \mathbb{R}^{n\times n}$. Then by Young inequality

$$|A|^k |B|^{n-k} \leqslant \frac{k}{n} \varepsilon^{\frac{1}{n}} |A|^n + \frac{n-k}{n} \varepsilon^{-\frac{k}{n(n-k)}} |B|^n, \quad k = 1, \dots, n-1. \tag{4.11}$$

Combining (4.6) and (4.8) this yields

$$\sup_{\Omega'} v_\varepsilon \leqslant \sup_{\partial\Omega'} v_\varepsilon^+ + \frac{d'}{\omega_n^{1/n}} \int_{\Omega'} \left(\left|\det\left(\left|D^2u\right|^{-\varepsilon} D^2u - H_\varepsilon\right)\right| \right)^{\frac{1}{n}}. \tag{4.12}$$

We apply (4.10) and (4.11) with $A = A_\varepsilon = |D^2u|^{-\varepsilon} D^2u$ and $B = B_\varepsilon = -H_\varepsilon$ by (4.10) and (4.11) to obtain

$$\begin{aligned}
\sup_{\Omega'} v_\varepsilon &\leqslant \sup_{\partial\Omega'} v_\varepsilon^+ + \frac{d'}{\omega_n^{1/n}} \left(\int_{\Omega'} |\det A| + C_n \sum_{k=1}^{n-1} |A|^k |B|^{n-k} + |\det B| \right)^{\frac{1}{n}} \\
&\leqslant \sup_{\partial\Omega'} v_\varepsilon^+ + \frac{d'}{\omega_n^{1/n}} \left(\int_{\Omega'} |\det A| \right)^{\frac{1}{n}} + C(n,d) \left[\left(\varepsilon^{\frac{1}{n}} \sum_{k=1}^{n-1} \frac{k}{n} \int_{\Omega'} |A|^n \right)^{\frac{1}{n}} \right.
\end{aligned}$$

$$+\varepsilon^{-\frac{n-1}{n^2}}\left(\int_{\Omega'}|B|^n\right)^{\frac{1}{n}}+\left(\int_{\Omega'}|\det B|\right)^{\frac{1}{n}}\Bigg]$$

$$=\sup_{\partial\Omega'} v_\varepsilon^+ + \frac{d'}{\omega_n^{1/n}}\left(\int_{\Omega'}|\det A|\right)^{\frac{1}{n}} + C(n,d)[J_1+J_2+J_3], \tag{4.13}$$

where the terms J_1, J_2 and J_3 stands for the integrals between rectangular brackets, respectively. Next, we estimate the quantities J_1, J_2 and J_3. First observe that

$$|\det B| \leqslant c_n |H_\varepsilon|^n,$$

so by (4.7), we have

$$J_1 \leqslant C(n)\left(\varepsilon^{\frac{1}{n}}\int_{\Omega}\left|D^2u\right|^{(1-\varepsilon)n}\right)^{\frac{1}{n}},$$

$$J_2 \leqslant C(n)\varepsilon^{1-\frac{n-1}{n^2}}\left(\int_{\Omega}\left|D^2u\right|^{(1-\varepsilon)n}\right)^{\frac{1}{n}},$$

$$J_3 \leqslant C(n)\left(\int_{\Omega}|H_\varepsilon|^n\right)^{\frac{1}{n}} \leqslant C(n)\varepsilon\left(\int_{\Omega}\left|D^2u\right|^{(1-\varepsilon)n}\right)^{\frac{1}{n}}.$$

Corollary 2.4 shows that

$$J_1, J_2, J_3 \to 0, \quad \text{as } \varepsilon \to 0. \tag{4.14}$$

Let us consider

$$\begin{aligned}\int_{\Omega'}|\det A| &= \int_{\Omega'}\left|\det\left|D^2u\right|^{-\varepsilon}D^2u\right| \\ &\leqslant \int_{\Omega'}\left|D^2u\right|^{-n\varepsilon}\left|\det D^2u\right| \\ &= \int_{\Omega'\cap\{|D^2u|\geqslant 1\}}\left|D^2u\right|^{-n\varepsilon}\left|\det D^2u\right| + \int_{\Omega'\cap\{|D^2u|<1\}}\left|D^2u\right|^{-n\varepsilon}\left|\det D^2u\right| \\ &\leqslant \int_{\Omega'\cap\{|D^2u|\geqslant 1\}}\left|\det D^2u\right| + \int_{\Omega'\cap\{|D^2u|<1\}} c_n\left|D^2u\right|^{n(1-\varepsilon)}.\end{aligned}$$

Since $\int|\det D^2u| < \infty$ (otherwise estimate (4.9) is obvious), we can apply the Dominated Convergence Theorem to deduce that

$$\left(\int_{\Omega'} |\det A|\right)^{\frac{1}{n}} \to \left(\int_{\Omega'} \left|\det D^2 u\right|\right)^{\frac{1}{n}} \quad \text{as } \varepsilon \to 0.$$

Observe that

$$v_\varepsilon \to u \quad \big(\text{uniformly in } \Omega'\big), \tag{4.15}$$

then letting $\varepsilon \to 0$ in (4.13), we obtain (4.9).

Estimate (1.4) follows, by the following lemma.

Lemma 4.3. *Let $u \in C(\overline{\Omega}) \cap \mathscr{W}^{2,1}(\Omega)$ be concave. Then*

$$\left|\det D^2 u\right| \leqslant \frac{(Lu)^n}{n^n \mathscr{D}}.$$

Proof. The above estimate is immediate from the following matrix inequality:

$$\det M \det N \leqslant \left(\frac{\operatorname{trace} MN}{n}\right)^n, \qquad M, N \text{ symmetric} \geqslant 0. \tag{4.16}$$

Indeed, taking $M = -D^2 u$ and $N = [a_{ij}]$ we have

$$\left|\det D^2 u\right| = \det\big(-D^2 u\big) \leqslant \frac{1}{\mathscr{D}}\left(\frac{\operatorname{Tr}[\mathscr{A}(x) D^2 u]}{n}\right)^n = \left(\frac{Lu}{n\mathscr{D}^{\frac{1}{n}}}\right)^n. \qquad \square$$

Combining Lemma 4.3 with (4.9) the thesis (1.5) follows. $\square$

5. Proof of Theorem 1.3

Proof of Theorem 1.3. Let $u \in C(\overline{\Omega}) \cap \mathscr{W}^{2,1}(\Omega)$ verifying (1.7) and let $0 < \varepsilon \leqslant 1$. Using the Hodge decomposition in $\mathscr{L}^n \log \mathscr{L}$ (see Remark 2.6 and [8]) we write

$$\left|D^2 u\right|^{-\varepsilon} D^2 u = D^2 v_\varepsilon + H_\varepsilon$$

where $D^2 v_\varepsilon \in \mathscr{L}^n \log \mathscr{L}(\Omega)$ and

$$\|H_\varepsilon\|_{\mathscr{L}^n \log \mathscr{L}} \leqslant C_n \varepsilon \cdot \left\| \left|D^2 u\right|^{-\varepsilon} D^2 u \right\|_{\mathscr{L}^n \log \mathscr{L}}. \tag{5.1}$$

Using again Lemma 4.1 we have

$$\sup_{\Omega'} v_\varepsilon \leqslant \sup_{\partial \Omega'} v_\varepsilon^+ + \frac{d}{\omega_n^{1/n}} \left(\int_{\Gamma_\varepsilon^+} \left|\det D^2 v_\varepsilon\right|\right)^{\frac{1}{n}} \tag{5.2}$$

Now, via the matrix inequality (4.16) taking $M = -D^2 v_\varepsilon$ and $N = [a_{ij}]$, we have on Γ_ε^+

$$\left|\det D^2 v_\varepsilon\right| \leqslant \frac{1}{\mathscr{D}}\left(\frac{\operatorname{Tr}[\mathscr{A}(x) D^2 v_\varepsilon]}{n}\right)^n. \tag{5.3}$$

Hence, combining (5.2) and (5.3) we have

$$\begin{aligned}
\sup_{\Omega'} v_\varepsilon &\leqslant \sup_{\partial\Omega'} v_\varepsilon^+ + \frac{d}{\omega_n^{1/n}}\left(\int_{\Gamma_\varepsilon^+} \frac{1}{\mathscr{D}}\left(\frac{\operatorname{Tr}[\mathscr{A}(x) D^2 v_\varepsilon]}{n}\right)^n\right)^{\frac{1}{n}} \\
&\leqslant \sup_{\partial\Omega'} v_\varepsilon^+ + \frac{d}{n\omega_n^{1/n}}\left(\int_{\Gamma_\varepsilon^+} \frac{1}{\mathscr{D}}\left(\operatorname{Tr}\left[\mathscr{A}(x)\left|D^2 u\right|^{-\varepsilon} D^2 u\right] - \operatorname{Tr}\left[\mathscr{A}(x) H_\varepsilon(x)\right]\right)^n\right)^{\frac{1}{n}} \\
&\leqslant \sup_{\partial\Omega'} v_\varepsilon^+ + \frac{c(n)d}{n\omega_n^{1/n}}\left(\int_{\Omega'} \frac{1}{\mathscr{D}}\left|D^2 u\right|^{-\varepsilon n}|Lu|^n\right)^{\frac{1}{n}} \\
&\quad + c(n,\Omega)\left(\int_{\Omega'} \frac{1}{\mathscr{D}}\left|\mathscr{A}(x)\right|^n |H_\varepsilon|^n\right)^{\frac{1}{n}}.
\end{aligned} \tag{5.4}$$

It remains to prove that the second integral in the last inequality goes to zero as $\varepsilon \to 0$. By the duality pairing between the spaces $\operatorname{Exp}(\Omega')$ and the Zygmund space $\mathscr{L}\log\mathscr{L}(\Omega')$ we have

$$\int_{\Omega'} \frac{1}{\mathscr{D}}\left|\mathscr{A}(x)\right|^n |H_\varepsilon|^n \leqslant \left\|\frac{\mathscr{A}(x)^n}{\mathscr{D}}\right\|_{\operatorname{Exp}(\Omega')} \cdot \left\||H_\varepsilon|^n\right\|_{\mathscr{L}\log\mathscr{L}(\Omega')}. \tag{5.5}$$

Moreover, by (5.1) and Proposition 2.1 it holds

$$\left\||H_\varepsilon|^n\right\|_{\mathscr{L}\log\mathscr{L}} \leqslant c(n)\varepsilon^n \left\|D^2 u\right\|^{(1-\varepsilon)n}_{\mathscr{L}^{(1-\varepsilon)n}\log\mathscr{L}}. \tag{5.6}$$

Hence, combining (5.5) (5.6), (1.6) and (1.7) we obtain

$$\left(\int_{\Omega'} \frac{1}{\mathscr{D}}\left|\mathscr{A}(x)\right|^n |H_\varepsilon|^n\right)^{\frac{1}{n}} \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.7}$$

Finally we observe that

$$\begin{aligned}
\int_{\Omega'} \frac{1}{\mathscr{D}}\left|D^2 u\right|^{-\varepsilon n}|Lu|^n &\leqslant \int_{\Omega'\cap\{|D^2 u|\geqslant 1\}} \frac{|Lu|^n}{\mathscr{D}} + \int_{\Omega'\cap\{|D^2 u|<1\}} \left|D^2 u\right|^{-\varepsilon n}\frac{|Lu|^n}{\mathscr{D}} \\
&= \int_{\Omega'\cap\{|D^2 u|\geqslant 1\}} \frac{|Lu|^n}{\mathscr{D}} + \int_{\Omega'\cap\{|D^2 u|<1\}} \frac{|\langle\mathscr{A}(x)|D^2 u|D^2 u|^{-\varepsilon}\rangle|^n}{\mathscr{D}}.
\end{aligned}$$

Then, since

$$\frac{|\langle \mathscr{A}(x)|D^2u|D^2u|^{-\varepsilon}\rangle|^n}{\mathscr{D}} \leqslant \frac{|\mathscr{A}|^n}{\mathscr{D}}\left|D^2u\right|^{(1-\varepsilon)n},$$

we can apply the Dominated Convergence Theorem to conclude

$$\left(\int_{\Omega'} \frac{1}{\mathscr{D}}\left|D^2u\right|^{-\varepsilon n}|Lu|^n\right)^{\frac{1}{n}} \to \left(\int_{\Omega'} \frac{1}{\mathscr{D}}|Lu|^n\right)^{\frac{1}{n}} \quad \text{as } \varepsilon \to 0. \tag{5.8}$$

Combining (5.4), (5.7) and (5.8) we arrive at the desired inequality. □

We conclude this section by observing that in dimension $n = 2$, under assumption (1.6) a continuity result is obtained in [19].

6. The uniformly elliptic case

Proof of Theorem 1.4. Let $u \in C(\overline{\Omega}) \cap \mathscr{W}^{2,n)}(\Omega)$. Using the same arguments of the proof of Theorem 4.2, we obtain

$$\left\langle \mathscr{A}(x) \,\middle|\, \left|D^2u\right|^{-\varepsilon} D^2u\right\rangle = \left|D^2u\right|^{-\varepsilon} Lu. \tag{6.1}$$

Then, we consider the Hodge decomposition

$$\left|D^2u\right|^{-\varepsilon} D^2u = D^2v_\varepsilon + H_\varepsilon \tag{6.2}$$

where $v_\varepsilon \in \mathscr{W}^{2,n}(\Omega)$ and

$$\|H_\varepsilon\|_n \leqslant C_n\varepsilon \left\|\left|D^2u\right|^{-\varepsilon} D^2u\right\|_n. \tag{6.3}$$

By (6.2), Eq. (6.1) takes the form

$$\left\langle \mathscr{A}(x) \,\middle|\, D^2v_\varepsilon\right\rangle = \left|D^2u\right|^{-\varepsilon} Lu - \left\langle \mathscr{A}(x) \,\middle|\, H_\varepsilon\right\rangle.$$

Applying Theorem 1.1 to v_ε we obtain, for any $\Omega' \subset\subset \Omega$,

$$\begin{aligned}
\sup_{\Omega'} v_\varepsilon &\leqslant \sup_{\partial\Omega'} v_\varepsilon^+ + \frac{d'}{\omega_n^{1/n}}\left(\int_{\Omega'} \frac{||D^2u|^{-\varepsilon}Lu - \langle \mathscr{A}(x)|H_\varepsilon\rangle|^n}{\mathscr{D}}\right)^{\frac{1}{n}} \\
&\leqslant \sup_{\partial\Omega'} v_\varepsilon^+ + c_n\frac{d'}{\omega_n^{1/n}}\left[\left(\int_{\Omega'} \frac{||D^2u|^{-\varepsilon}Lu|^n}{\mathscr{D}}\right)^{\frac{1}{n}} + \left(\int_{\Omega'} \frac{|\langle \mathscr{A}(x)|H_\varepsilon\rangle|^n}{\mathscr{D}}\right)^{\frac{1}{n}}\right]
\end{aligned} \tag{6.4}$$

where $d' = \operatorname{diam} \Omega'$.

We are going to prove that

$$\left[\left(\int_{\Omega'} \frac{||D^2u|^{-\varepsilon}Lu|^n}{\mathscr{D}}\right)^{\frac{1}{n}} + \left(\int_{\Omega'} \frac{|\langle \mathscr{A}(x)|H_\varepsilon\rangle|^n}{\mathscr{D}}\right)^{\frac{1}{n}}\right] \longrightarrow \left(\int_{\Omega'} \frac{|Lu|^n}{\mathscr{D}}\right)^{\frac{1}{n}}, \tag{6.5}$$

as $\varepsilon \to 0$. To this aim, let us preliminary observe that by the Schwarz inequality it holds

$$\int_{\Omega'} \frac{\langle \mathscr{A}(x)|H_\varepsilon\rangle|^n}{\mathscr{D}} \leqslant \int_{\Omega'} \frac{|\mathscr{A}|^n|H_\varepsilon|^n}{\mathscr{D}} \leqslant c_n \int_{\Omega'} \frac{\Lambda^n(x)}{\lambda^n(x)}|H_\varepsilon|^n. \tag{6.6}$$

By assumption $K = \frac{\Lambda(x)}{\lambda(x)} \in \mathscr{L}^\infty$ and by (6.3)

$$\begin{aligned}
\left(\int_{\Omega'} \frac{|\langle \mathscr{A}(x)|H_\varepsilon\rangle|^n}{\mathscr{D}}\right)^{\frac{1}{n}} &\leqslant c_n\|K\|_\infty\|H_\varepsilon\|_n \\
&\leqslant c_n\|K\|_\infty\varepsilon \cdot \left\|\left|D^2u\right|^{-\varepsilon}D^2u\right\|_n \\
&\leqslant c_n\|K\|_\infty\varepsilon^{1-\frac{1}{n}}\left(\varepsilon\int_{\Omega'}\left|D^2u\right|^{(1-\varepsilon)n}\right)^{\frac{1}{n}}.
\end{aligned} \tag{6.7}$$

As $\varepsilon \to 0$, last expression tends to zero. On the other hand,

$$\begin{aligned}
\int_{\Omega'} \frac{||D^2u|^{-\varepsilon}Lu|^n}{\mathscr{D}} &= \int_{\Omega'\cap\{|D^2u|\geqslant 1\}} \frac{||D^2u|^{-\varepsilon}Lu|^n}{\mathscr{D}} + \int_{\Omega'\cap\{|D^2u|<1\}} \frac{||D^2u|^{-\varepsilon}Lu|^n}{\mathscr{D}} \\
&\leqslant \int_{\Omega'\cap\{|D^2u|\geqslant 1\}} \frac{|Lu|^n}{\mathscr{D}} + \int_{\Omega'\cap\{|D^2u|<1\}} \frac{|\langle \mathscr{A}(x)|D^2u|D^2u|^{-\varepsilon}\rangle|^n}{\mathscr{D}}.
\end{aligned}$$

Note that in $\Omega' \cap \{|D^2u| < 1\}$ it holds

$$\begin{aligned}
\frac{|\langle \mathscr{A}(x)|D^2u|D^2u|^{-\varepsilon}\rangle|^n}{\mathscr{D}} &\leqslant \frac{|\mathscr{A}|^n|D^2u|^{(1-\varepsilon)n}}{\mathscr{D}} \\
&\leqslant c_n\frac{\Lambda^n(x)}{\lambda^n(x)}\left|D^2u\right|^{(1-\varepsilon)n} \\
&\leqslant c_nK^n(x).
\end{aligned}$$

Using the Dominated Convergence Theorem we obtain

$$\left(\int_{\Omega'} \frac{||D^2u|^{-\varepsilon}Lu|^n}{\mathscr{D}}\right)^{\frac{1}{n}} \longrightarrow \left(\int_{\Omega'} \frac{|Lu|^n}{\mathscr{D}}\right)^{\frac{1}{n}}. \tag{6.8}$$

Combining (6.7) and (6.8), (6.5) follows.

Letting $\varepsilon \to 0$ in (6.4), thanks to (6.5) and (6.7) we complete the proof. $\square$

References

[1] R. Adams, Sobolev Spaces, Pure Appl. Math., vol. 65, Academic Press, New York, London, 1975.
[2] V. Adolfsson, J. Pipher, The inhomogeneous Dirichlet problem for Δ^2 in Lipschitz domains, J. Funct. Anal. 159 (1998) 137–190.
[3] A.D. Alexandrov, Certain estimate for the Dirichlet problem, Dokl. Akad Nauk. SSSR 134 (1960) 1001–1004 (in Russian); English Translation in Sov. Math. Dokl. 1 (1960) 1151–1154.
[4] A.D. Alexandrov, The impossibility of general estimates for solutions and of uniqueness for linear equations with norms weaker than in $\mathscr{L}^n$, Vestnik Leningrad. Univ. 21 (1996) 5–10, Amer. Math. Soc. Transl. Ser. 2 68 (1968) 162–168.
[5] K. Astala, T. Iwaniec, G. Martin, Pucci's conjecture and the Alexandrov inequality for elliptic PDEs in the plane, J. Reine Angew. Math. 591 (2006) 49–74.
[6] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1983.
[7] L. Greco, A remark on the equality $\det Df = \operatorname{Det} Df$, Differential Integral Equations 6 (5) (1993) 1089–1100.
[8] L. Greco, T. Iwaniec, G. Moscariello, Limits of the improved integrability of the volume forms, Indiana Univ. Math. J. 44 (2) (1995) 305–339.
[9] L. Greco, G. Moscariello, T. Radice, Nondivergence elliptic equations with unbounded coefficients, Discrete Contin. Dyn. Syst. Ser. B 11 (1) (2009) 131–143.
[10] T. Iwaniec, G. Martin, Geometric Function Theory and Non-Linear Analysis, Oxford Math. Monogr., The Clarendon Press, Oxford University Press, New York, 2001.
[11] T. Iwaniec, G. Martin, Riesz transforms and related singular integrals, J. Reine Angew. Math. 473 (1996) 25–57.
[12] T. Iwaniec, C. Sbordone, Weak minima of variational integrals, J. Reine Angew. Math. 454 (1994) 143–161.
[13] T. Iwaniec, C. Sbordone, On the integrability of the Jacobian under minimal hypothesis, Arch. Ration. Mech. Anal. 119 (1992) 129–143.
[14] D. Jerison, C. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains, J. Funct. Anal. 130 (1995) 161–219.
[15] J. Pipher, G. Verchota, The Dirichlet problem in L^p for the biharmonic equation on Lipschitz domains, Amer. J. Math. 114 (1992) 923–972.
[16] C. Pucci, Limitazioni per soluzioni di equazioni ellittiche, Ann. Mat. Pura Appl. (4) 72 (1966) 15–30.
[17] C. Pucci, Operatori ellittici estremanti, Ann. Mat. Pura Appl. (4) 74 (1966) 15–30.
[18] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Marcel Dekker, Inc., New York, 1991.
[19] T. Ricciardi, G. Zecca, On the continuity of solutions to degenerate elliptic equations in two dimensions, Potential Anal. 37 (2) (2012) 171–185.
[20] G. Talenti, Sopra una classe di equazioni ellittiche a coefficienti misurabili, Ann. Mat. Pura Appl. (4) 69 (1965) 285–304.
[21] G. Verchota, The Dirichlet problem for the biharmonic equations in C^1 domains, Indiana Math. J. 36 (1987) 867–895.
[22] G. Verchota, The Dirichlet problem for the polyharmonic equation in Lipschitz domains, Indiana Math. J. 39 (1990) 671–702.
[23] A. Verde, G. Zecca, Lower semicontinuity of certain quasiconvex functionals in Orlicz–Sobolev spaces, Nonlinear Anal. Theory Methods Appl. 71 (10) (2009) 4515–4524.