



# Existence theorems for non-Abelian Chern–Simons–Higgs vortices with flavor

Shouxin Chen<sup>a</sup>, Xiaosen Han<sup>a,b,\*</sup>, Gustavo Lozano<sup>c</sup>,  
Fidel A. Schaposnik<sup>d</sup>

<sup>a</sup> *Institute of Contemporary Mathematics, School of Mathematics and Statistics, Henan University, Kaifeng, Henan 475004, PR China*

<sup>b</sup> *Taida Institute for Mathematical Sciences, Center for Advanced Study in Theoretical Science, National Taiwan University, Taipei, 10617, Taiwan, ROC*

<sup>c</sup> *Departamento de Física e Instituto de Física de Buenos Aires, FCEyN, Pabellón 1, Ciudad Universitaria, Universidad de Buenos Aires, 1428, Buenos Aires, Argentina*

<sup>d</sup> *Departamento de Física, Universidad Nacional de La Plata, Instituto de Física La Plata, C.C. 67, 1900 La Plata, Argentina*

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## Abstract

In this paper we establish the existence of vortex solutions for a Chern–Simons–Higgs model with gauge group  $SU(N) \times U(1)$  and flavor  $SU(N)$ . These symmetries ensure the existence of genuine non-Abelian vortices through a color–flavor locking. Under a suitable ansatz we reduce the problem to a  $2 \times 2$  system of nonlinear elliptic equations with exponential terms. We study this system over the full plane and over a doubly periodic domain, respectively. For the planar case we use a variational argument to establish the existence result and derive the decay estimates of the solutions. Over the doubly periodic domain we show that the system admits at least two gauge-distinct solutions carrying the same physical energy by using a constrained minimization approach and the mountain-pass theorem. In both cases we get the quantized vortex magnetic fluxes and electric charges.

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\* Corresponding author. Corresponding address: School of Mathematics and Statistics, Henan University, Kaifeng, Henan 475004, PR China.

E-mail addresses: [chensx@henu.edu.cn](mailto:chensx@henu.edu.cn) (S. Chen), [hanxiaosen@henu.edu.cn](mailto:hanxiaosen@henu.edu.cn) (X. Han), [lozano@df.uba.ar](mailto:lozano@df.uba.ar) (G. Lozano), [fidel@fisica.unlp.edu.ar](mailto:fidel@fisica.unlp.edu.ar) (F.A. Schaposnik).

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## 1. Introduction

Magnetic vortex configurations were investigated by Abrikosov [1] more than fifty years ago in the context of Ginzburg–Landau theory of superconductivity. Sixteen years later Nielsen and Olesen stressed the relevance to high-energy physics of vortex-line solutions of the Abelian Higgs model in the context of dual string models [39]. Since then the interest on vortices has continued to grow both in condensed-matter and particle physics.

Very early it was observed that when the Ginzburg–Landau free-energy parameter ratio takes the critical value bordering type-I and type-II superconductivity one can find the BPS (named after the famous works of Bogomol’nyi [8] and Prasad–Sommerfield [42]) equations which are equivalent to the more involved second-order Ginzburg–Landau equations [27]. These BPS equations were rediscovered in the context of high-energy physics in [8,16] with the Ginzburg–Landau parameters identified with the gauge charge and the symmetry breaking potential coupling constant of the Abelian Higgs model. A rigorous study of the self-dual equations with the coupling constant ratio at its critical value was presented in [46] where it was proved that the self-dual vortex solutions are uniquely determined by a set of  $N$  not necessarily distinct points in the plane corresponding to the zeros of the Higgs field. Every set of  $N$  points determines exactly one such solution.

Interestingly enough, as first pointed out in [16], the value of the parameter ratio leading to the BPS equations is precisely the one required to extend the Abelian Higgs model to an  $\mathcal{N} = 2$  supersymmetric model in  $d = 3$  space-time dimensions. This supersymmetry connection opened the way to the computation of some data (like the exact particle spectra in certain gauge theories) at strong coupling even when the full theory is not solvable (see [43] and references therein).

In  $(2 + 1)$  space-time dimensions the usual Maxwell term of the Abelian Higgs model can be replaced by a Chern–Simons (CS) term [14,15] leading to the so-called Abelian Chern–Simons–Higgs theory. Also in this case a specific choice of parameters and of the Higgs potential (a sixth-order one) leads to the BPS equations [13,29,30]. The presence of the CS action drastically changes the vortex solutions which carry in this case both magnetic flux and electric charge. A major interest in the CS theories is closely connected to several problems in planar matter physics. In particular, large scale properties of a Quantum Hall system can be described in terms of a CS theory with the Hall conductivity related to the inverse of the coefficient of the CS action [21]. Also, the central role played by the CS term in bosonization of massive [20] and massless [3,37] fermions has also been revealed.

The existence of self-dual vortex solutions of the Abelian Chern–Simons model with boundary conditions on  $\mathbb{R}^2$  has been established in Refs. [44,48]. The model was also studied in the case of gauge-periodic boundary conditions defined on a periodic cell [28,49] and in this case it has been proved that there is a critical value of the CS coupling parameter above which vortex solutions do not exist [10,45]. The existence of a critical value of the parameter is related to the area of the periodic domain and has been observed also in Abelian and Yang–Mills–Higgs systems in different compact geometries [9,23,35]. Let us point out that periodic field configuration is particularly relevant in the context of condensed matter systems where vortices appear as a lattice array (the Abrikosov lattice) [1].

Topologically stable vortex solutions in  $SU(N)$  gauge theories with both Yang–Mills and Chern–Simons terms were constructed in [17,18] with the scalar fields breaking the symmetry to

$Z_N$  and gauge fields restricted to the Cartan subalgebra. Under the same assumptions self-dual equations for the pure non-Abelian Chern–Simons–Higgs model were studied in [34]. A proof of the existence of vortex solutions with the Cartan restriction was given in [50]. Because in these models the vortex magnetic fluxes turn out to be in the Cartan subalgebra direction the solutions can be seen as the result of an Abelian embedding [43].

It is known that adding flavor to the Yang–Mills–Higgs theory one can arrange the symmetry breaking so that some global diagonal combination of color and flavor groups survives. This pattern of symmetry breaking is known as color–flavor locking procedure (see [43] and the references therein) and leads to genuine non-Abelian vortices with an orientational moduli related to the presence of the surviving symmetry subgroup. Existence and uniqueness theorems for the solutions of this model were presented in [33]. In an independent study [38], Navarro-Lerida and Tchrakian considered genuinely non-Abelian Yang–Mills–Higgs vortices for which the flux fields do not lie in a fixed direction of the Cartan subalgebra.

It is interesting to note that a bi-level Chern–Simons–Higgs theory was recently developed in Aharony–Bergman–Jafferis–Maldacena [4], known as ABJM theory. Various BPS structures for this theory were discovered by the work of Auzzi and Kumar [7] and Kim–Kim–Kwon–Nakajima [32]. On the other hand, the existence and uniqueness theorems for the corresponding BPS equations have been established in [11,26].

Within the color–flavor locking symmetry breaking pattern referred above and an appropriate cylindrically symmetric ansatz, genuine non-Abelian vortex solutions were constructed numerically for the  $SU(N) \times U(1)$  Chern–Simons–Higgs theory with  $SU(N)$  flavor in [34]. It is the purpose of this work to present rigorous existence theorems for this problem which can be reduced to a  $2 \times 2$  system of nonlinear elliptic equations with exponential terms.

It is worth noting that we mainly establish the existence results for the BPS type equations. For the existence results for the non-BPS equations, we mention the work by Chen–Guo–Spirn–Yang [12].

We shall consider the model defined in two types of domains: over the full plane and over a doubly periodic domain. Over the plane, this type of system (with different symmetry groups) was studied in [50] in a general form by using the Cholesky decomposition of a positive definite matrix to find a variational structure. However, for our concrete  $2 \times 2$  system, we find a more explicitly variational structure than that of [50] without using the Cholesky decomposition. Over the doubly periodic domain, we find a sufficient condition on the coupling parameter such that the system admits at least two different solutions as in [41,45], which are obtained by using a constrained minimization approach [10,24,25,41] and the mountain-pass theorem. These two solutions are necessarily gauge inequivalent. Since they both carry the same electric charge and magnetic flux and are of self-dual type, they have the same energy. This phenomenon is in sharp contrast with that in the classical Abelian Higgs model [31,47].

The rest of our paper is organized as follows. In Section 2 we introduce the non-Abelian Chern–Simons model with gauge group  $SU(N) \times U(1)$ , reduce it to a  $2 \times 2$  system of nonlinear elliptic partial differential equations with exponential terms, and state our main results. In Section 3 we establish the existence result for the planar case and derive the decay estimates of the solutions. Section 4 is devoted to the existence result for the doubly periodic domain case.

## 2. The model and main results

In this section we follow [34] to derive the non-Abelian Chern–Simons–Higgs self-dual equations and state our main results. We study the bosonic sector of the  $\mathcal{N} = 2$  SUSY  $SU(N) \times U(1)$  Chern–Simons–Higgs action in  $2 + 1$  dimensions

$$S = \int d^3x \left\{ \frac{\kappa_1}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu}^0 A_\rho^0 + \frac{\kappa_2}{2} \epsilon^{\mu\nu\rho} \left( F_{\mu\nu}^I A_\rho^I - \frac{1}{3} f^{IJK} A_\mu^I A_\nu^J A_\rho^K \right) + (D_\mu \phi^f)^\dagger (D_\mu \phi^f) - V[\phi, \phi^\dagger] \right\}$$

where  $\epsilon^{012} = 1, g^{00} = 1, f^{IJK}$  are the structural constants of  $SU(N), \mu, \nu, \rho = 0, 1, 2$  are Lorentz indices,  $I, J, K = 1, \dots, N^2 - 1$  are the  $SU(N)$  “color” group indices,  $\tau_I$  are the anti-Hermitian generators of  $SU(N)$ . The covariant derivatives and field strengths are defined as

$$\begin{aligned} D_\mu \phi_a^f &= \partial_\mu \phi_a^f + (A_\mu^{SU(N)})_a^b \phi_b^f + (A_\mu^{U(1)})_a^b \phi_b^f, \\ A_\mu^{SU(N)} &= A_\mu^I \tau_I, \quad A_\mu^{U(1)} = A_\mu^0 \tau_0, \\ F_{\mu\nu}^0 &= \partial_{[\mu} A_{\nu]0}, \quad F_{\mu\nu}^I = \partial_{[\mu} A_{\nu]I} + f^{IJK} A_\mu^J A_\nu^K, \end{aligned}$$

where  $\tau^0$  and  $\tau^I$  are the  $U(1)$  and  $SU(N)$  generators.

The potential  $V[\phi, \phi^\dagger]$  is a sixth order polynomial of the form

$$\begin{aligned} V[\phi, \phi^\dagger] &= \frac{1}{16\kappa_1^2 N^2} \phi_f^\dagger \phi^f (\phi_g^\dagger \phi^g - N\xi)^2 + \frac{1}{4\kappa_2^2 N^2} \phi_f^\dagger \tau^I \tau^J \phi^f (\phi_g^\dagger \tau^I \phi^g) (\phi_h^\dagger \tau^I \phi^h) \\ &\quad - \frac{1}{4\kappa_1 \kappa_2 N} (\phi_f^\dagger \tau^I \phi^f)^2 (\phi_g^\dagger \phi^g - N\xi). \end{aligned} \tag{2.1}$$

The scalar complex multiplets have both the gauge index  $a, b, c = 1, \dots, N$  and flavor index  $f, g, h = 1, \dots, N$ , and can be written as an  $N \times N$  matrix. The choice of the sixth order potential is dictated by the aim of getting the BPS equations whose static solutions correspond to a lower bound of the energy. As it is well known, the existence of BPS equations is directly related to the  $\mathcal{N} = 2$  supersymmetry of the theory, with the central charge of the supersymmetry algebra related to the topological charge of the corresponding BPS solutions [43]. It is precisely supersymmetry which requires the particular choice of the potential which, in contrast with the usual forth-order one, has two phases. Indeed, up to gauge transformations, the minima of the potential are given by

$$\begin{aligned} \phi^f &= 0 && \text{symmetric phase} \\ \phi^f \phi_f^\dagger &= \xi \text{diag}\{1, \dots, 1\} && \text{asymmetric phase} \end{aligned}$$

In the asymmetric phase, where the original gauge symmetry is broken, topologically non-trivial solutions can be found (see the discussion below). In what follows we set, without loss of generality,  $\xi = 1$ .

The Euler–Lagrange equations of the theory are

$$\begin{aligned} \kappa_1 \epsilon_\mu^{\alpha\beta} F_{\alpha\beta}^0 &= J_\mu^0 \equiv \phi_f^\dagger \tau^0 D_\mu \phi^f - (D_\mu \phi^f)^\dagger \tau^0 \phi^f, \\ \kappa_2 \epsilon_\mu^{\alpha\beta} F_{\alpha\beta}^I &= J_\mu^I \equiv \phi_f^\dagger \tau^I D_\mu \phi^f - (D_\mu \phi^f)^\dagger \tau^I \phi^f, \\ D_\mu D^\mu \phi^f &= \frac{\partial V}{\partial \phi_f^\dagger}. \end{aligned}$$

Using the Gauss law,

$$\kappa_1 F_{12}^0 = J_0^0 = \phi_f^\dagger \tau^0 D_0 \phi^f - (D_0 \phi^f)^\dagger \tau^0 \phi^f, \tag{2.2}$$

$$\kappa_2 F_{12}^I = J_0^I = \phi_f^\dagger \tau^I D_0 \phi^f - (D_0 \phi^f)^\dagger \tau^I \phi^f, \tag{2.3}$$

one finds that the energy density is given by

$$\mathcal{H} = (D_0 \phi^f)^\dagger (D_0 \phi^f) + (D_i \phi^f)^\dagger (D_i \phi^f) + V[\phi, \phi^\dagger].$$

One can see that the energy can be written as a sum of squares

$$\begin{aligned} H = \int d^2x \mathcal{H} = \int d^2x & \left\{ \left[ D_0 \phi^f \mp i \left( \frac{1}{4\kappa_1 N} (\phi_g^\dagger \phi^g) \phi^f - \frac{1}{2\kappa_2} \phi_g^\dagger \tau^I \phi^f \right) \right]^\dagger \right. \\ & \times \left. \left[ D_0 \phi^f \mp i \left( \frac{1}{4\kappa_1 N} (\phi_g^\dagger \phi^g) \phi^f - \frac{1}{2\kappa_2} \phi_g^\dagger \tau^I \phi^f \right) \right] + (D_\mp \phi^f)^\dagger (D_\mp \phi^f) \pm \sqrt{2N} F_{12}^0 \right\}, \end{aligned}$$

where

$$D_\pm = D_1 \pm iD_2.$$

Energy minima are then obtained by solving the BPS equations

$$D_0 \phi^f \mp i \left( \frac{1}{4\kappa_1 N} (\phi_g^\dagger \phi^g) \phi^f - \frac{1}{2\kappa_2} \phi_g^\dagger \tau^I \phi^f \right) = 0, \tag{2.4}$$

$$D_\mp \phi^f = 0. \tag{2.5}$$

Note that because of the presence of the Chern–Simons term, magnetic vortices are electrically charged and their magnetic flux  $\mathcal{F}$

$$\mathcal{F}^0 \equiv \int F_{12}^0 dx, \quad \mathcal{F}^I \equiv \int F_{12}^I dx$$

and electric charge  $\mathcal{Q}$

$$\mathcal{Q}^0 \equiv \int J_0^0 dx, \quad \mathcal{Q}^I \equiv \int J_0^I dx$$

are related. Indeed, one has from Eqs. (2.2)–(2.3)

$$\mathcal{Q}^0 = \kappa_1 \mathcal{F}^0, \quad \mathcal{Q}^I = \kappa_2 \mathcal{F}^I.$$

Since the BPS equations (2.4)–(2.5) are difficult to deal with directly, we make the following ansatz, which coincides with that in [34] except that cylindrical symmetry is not assumed

$$\Phi = \text{diag}\{\phi, \dots, \phi, \phi_N\}, \tag{2.6}$$

$$A_0^0 = \frac{1}{\sqrt{2N}} f_0, \quad A_i^0 = -\sqrt{\frac{2}{N}} f_i, \tag{2.7}$$

$$A_0^I = 0, \quad A_i^I = 0, \quad I = 1, \dots, N^2 - 2, \tag{2.8}$$

$$A_0^{N^2-1} = \sqrt{\frac{N-1}{2N}} f_0^{N^2-1}, \quad A_i^{N^2-1} = \sqrt{\frac{2(N-1)}{N}} f_i^{N^2-1}, \tag{2.9}$$

where  $\phi$  and  $\phi_N$  are complex-valued functions,  $f_0, f_i, f_0^{N^2-1}, f_i^{N^2-1}$  are real functions and we have chosen

$$\tau^0 = \frac{i}{\sqrt{2N}} \text{diag}\{1, \dots, 1\}, \quad \tau^{N^2-1} = \frac{i}{\sqrt{2N(N-1)}} \text{diag}\{1, \dots, 1, 1 - N\}.$$

In (2.6) we have written the Higgs fields in terms of an  $N \times N$  matrix  $\Phi$  with entries  $\Phi_{af} = \phi_a^f$  where  $a$  runs over the gauge group indices and  $f$  over the flavor indices. The gauge and flavor groups act on  $\Phi$  according to  $\Phi \rightarrow U\Phi V$  with  $U$  an element of the gauge group and  $V$  an element of the flavor group. The choice of ansatz for  $\Phi$  produces, with appropriate boundary conditions for  $\phi$  and  $\phi_N$ , the spontaneous breaking of both gauge and flavor symmetries with a surviving diagonal global  $SU(N)_{C+F}$  in what is known as a color–flavor locking in the vacuum [43] that will ensure topological stable solutions. Indeed, for the asymmetric phase the first and third terms in the potential (2.1) force  $\Phi$  to develop a vacuum expectation value while the second one forces it to be diagonal. Such vacuum expectation value is preserved only for transformations in which  $U = V^{-1}$ , which corresponds to performing a global gauge transformation and a related (inverse) global flavor transformation. The relevant homotopy group is then  $\Pi_1(SU(N) \times U(1)/Z_N)$ , leading to  $Z_N$  non-Abelian vortices. Let us finally note that the choice of a different non-trivial  $A_i^I$  component together with the corresponding column permutation in  $\Phi$  in ansatz (2.6)–(2.9) leads to other  $Z_N$  vortex solutions.

With the above ansatz, the Gauss law (2.2)–(2.3) and the BPS equations (2.4)–(2.5) can be simplified as

$$\kappa_1 F_{12}^0 = -\frac{1}{N\sqrt{2N}} \left( [N-1]|\phi|^2 + |\phi_N|^2 \right) f_0 + [N-1] \left[ |\phi|^2 - |\phi_N|^2 \right] f_0^{N^2-1}, \tag{2.10}$$

$$\kappa_2 F_{12}^{N^2-1} = -\frac{\sqrt{N-1}}{N\sqrt{2N}} \left( [|\phi|^2 - |\phi_N|^2] f_0 + [|\phi|^2 + [N-1]|\phi_N|^2] f_0^{N^2-1} \right), \tag{2.11}$$

$$(\partial_1 - i\partial_2)\phi = \frac{i}{N} \left( [f_1 - f_1^{N^2-1}] - i[f_2 - f_2^{N^2-1}] \right) \phi, \tag{2.12}$$

$$(\partial_1 - i\partial_2)\phi_N = \frac{i}{N} \left( [f_1 + [N-1]f_1^{N^2-1}] - i[f_2 + [N-1]f_2^{N^2-1}] \right) \phi_N, \tag{2.13}$$

$$f_0 = \frac{1}{2\kappa_1} \left( [N-1]|\phi|^2 + |\phi_N|^2 - N \right), \tag{2.14}$$

$$f_0^{N^2-1} = \frac{1}{2\kappa_2} \left( |\phi|^2 - |\phi_N|^2 \right). \tag{2.15}$$

Without loss of generality we have chosen the upper sign in parameter  $\epsilon = \pm 1$  introduced when we wrote the energy density as a sum of squares. The lower sign will just correspond to vortex magnetic fluxes with opposite sign.

From Eqs. (2.12)–(2.13) of  $\phi$  and  $\phi_N$ , we see that the zeros of them are isolated. For topological solutions ( $\phi, \phi_N \rightarrow 1$  at infinity), the zeros are finite. We denote the zero sets of  $\phi$  and  $\phi_N$  by

$$Z_i = \{p_{is}, s = 1, \dots, n_i\}, \quad i = 1, 2.$$

Let

$$\partial \equiv \frac{1}{2}(\partial_1 - i\partial_2),$$

and note

$$\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2), \quad \partial\bar{\partial} = \bar{\partial}\partial = \frac{1}{4}\Delta.$$

By a direct computation, we obtain

$$\Delta \ln |\phi|^2 = -\sqrt{\frac{2}{N}} F_{12}^0 - \sqrt{\frac{2}{N(N-1)}} F_{12}^{N^2-1}, \tag{2.16}$$

$$\Delta \ln |\phi_N|^2 = -\sqrt{\frac{2}{N}} F_{12}^0 + \sqrt{\frac{2(N-1)}{N}} F_{12}^{N^2-1}. \tag{2.17}$$

Then, from (2.10)–(2.11), (2.14)–(2.15) and (2.16)–(2.17) we have, away from the zeros of the Higgs fields

$$\begin{aligned} \Delta \ln |\phi|^2 &= \frac{1}{4N^2} \left\{ \frac{1}{\kappa_1^2} \left( [N-1]|\phi|^2 + |\phi_N|^2 \right) \left( [N-1]|\phi|^2 + |\phi_N|^2 - N \right) \right. \\ &\quad + \frac{N-1}{\kappa_1\kappa_2} \left( |\phi|^2 - |\phi_N|^2 \right)^2 + \frac{1}{\kappa_1\kappa_2} \left( [N-1]|\phi|^2 + |\phi_N|^2 - N \right) \left( |\phi|^2 - |\phi_N|^2 \right) \\ &\quad \left. + \frac{1}{\kappa_2^2} \left( |\phi|^2 + [N-1]|\phi_N|^2 \right) \left( |\phi|^2 - |\phi_N|^2 \right) \right\}, \tag{2.18} \end{aligned}$$

$$\begin{aligned} \Delta \ln |\phi_N|^2 &= \frac{1}{4N^2} \left\{ \frac{1}{\kappa_1^2} \left( [N-1]|\phi|^2 + |\phi_N|^2 \right) \left( [N-1]|\phi|^2 + |\phi_N|^2 - N \right) \right. \\ &\quad + \frac{N-1}{\kappa_1\kappa_2} \left( |\phi|^2 - |\phi_N|^2 \right)^2 - \frac{N-1}{\kappa_1\kappa_2} \left( [N-1]|\phi|^2 + |\phi_N|^2 - N \right) \left( |\phi|^2 - |\phi_N|^2 \right) \\ &\quad \left. - \frac{N-1}{\kappa_2^2} \left( |\phi|^2 + [N-1]|\phi_N|^2 \right) \left( |\phi|^2 - |\phi_N|^2 \right) \right\}. \tag{2.19} \end{aligned}$$

Let

$$u_1 = \ln |\phi|^2, \quad u_2 = \ln |\phi_N|^2, \quad \lambda \equiv \frac{1}{4\kappa_1^2}, \quad \kappa \equiv \frac{\kappa_1}{\kappa_2}.$$

We can rewrite the above equations (2.18)–(2.19) as

$$\begin{aligned} \Delta u_1 = \lambda & \left\{ \frac{1}{N^2} \left( [N - 1 + \kappa]^2 e^{2u_1} - [\kappa - 1](N - [N - 2][\kappa - 1]) e^{u_1 + u_2} \right. \right. \\ & \left. \left. + [1 - \kappa](1 + [N - 1]\kappa) e^{2u_2} \right) - \frac{1}{N} \left( [N - 1 + \kappa] e^{u_1} + [1 - \kappa] e^{u_2} \right) \right\} \\ & + 4\pi \sum_{s=1}^{n_1} \delta_{p_{1s}}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} \Delta u_2 = \lambda & \left\{ \frac{1}{N^2} \left( [N - 1][1 - \kappa][N - 1 + \kappa] e^{2u_1} - [N - 1][\kappa - 1](2 + [N - 2]\kappa) e^{u_1 + u_2} \right. \right. \\ & \left. \left. + (1 + [N - 1]\kappa)^2 e^{2u_2} \right) - \frac{1}{N} \left( [N - 1][1 - \kappa] e^{u_1} + (1 + [N - 1]\kappa) e^{u_2} \right) \right\} \\ & + 4\pi \sum_{s=1}^{n_2} \delta_{p_{2s}}. \end{aligned} \tag{2.21}$$

We are interested in the existence of solutions of (2.20)–(2.21) for two cases. In the first case, we consider the system (2.20)–(2.21) over the plane with the topological boundary condition

$$u_1 \rightarrow 0, \quad u_2 \rightarrow 0, \quad |x| \rightarrow +\infty. \tag{2.22}$$

In the second case we study the equations over a doubly periodic domain  $\Omega$ , governing multiple vortices hosted in  $\Omega$  such that the field configurations are subject to the 't Hooft boundary condition [28,49,51] under which periodicity is achieved modulo gauge transformations.

Defining the matrix  $K$  as

$$K \equiv \frac{1}{N} \begin{pmatrix} N - 1 + \kappa & 1 - \kappa \\ (N - 1)(1 - \kappa) & 1 + (N - 1)\kappa \end{pmatrix}, \tag{2.23}$$

the system (2.20)–(2.21) can be rewritten in a compact form as

$$\Delta u_i = \lambda \left( \sum_{j=1}^2 \sum_{k=1}^2 e^{u_j} K_{ij} e^{u_k} K_{jk} - \sum_{j=1}^2 K_{ij} e^{u_j} \right) + 4\pi \sum_{s=1}^{n_i} \delta_{p_{is}}, \quad i = 1, 2. \tag{2.24}$$

Thus, our model has equations of motion with the same structure as those studied by Yang [50] in connection to the Non-Abelian Chern–Simons–Higgs model with matter in the *adjoint* representation. Indeed, setting  $N = 2$  and  $\kappa = 3$  in (2.23) gives the same equations as those arising in the  $SU(3)$  model studied in [40,50,51],

$$\Delta u_1 = \lambda \left( 4e^{2u_1} - e^{u_1+u_2} - 2e^{2u_2} - 2e^{u_1} + e^{u_2} \right) + 4\pi \sum_{s=1}^{n_1} \delta_{p_{1s}}, \tag{2.25}$$

$$\Delta u_2 = \lambda \left( -2e^{2u_1} - e^{u_1+u_2} + 4e^{2u_2} + e^{u_1} - 2e^{u_2} \right) + 4\pi \sum_{s=1}^{n_2} \delta_{p_{2s}}. \tag{2.26}$$

As already pointed out in [50], an existence theorem for the system (2.24) in  $\mathbb{R}^2$  can be established for more general matrices  $K$  not necessarily connected to a specific  $SU(N)$  model with adjoint matter. Our equations provide an explicit realization of this idea. Notice though, that even if the existence of solutions for the system (2.24) in  $\mathbb{R}^2$  can be established as a result of the theorem shown in [50], our matrix  $K$  does not satisfy the hypothesis used in [50] to derive the decay estimates (the symmetrization of  $K$  is not positive definite). It should be stressed that this decay estimates are relevant to make the connection between the topological charge (the magnetic flux) and the number of zeros of the components of the Higgs fields. We will then present in this paper a direct variational argument by using an explicit variational structure to get the solutions. More importantly, we will be able to get the decay estimates of the solutions and quantized fluxes. In addition, it is interesting to get an existence theory for the problem (2.20)–(2.21) over a doubly periodic domain. We modify the approach in [10,24,25,41] to deal with this case. Although the abstract approach [25] may be applied for this case, we present in this work more simplified estimates than that in [25], which is of independent interest. This motivates us to give a complete analysis of the nonlinear elliptic system (2.20)–(2.21) in both cases.

Our main results read as follows.

**Theorem 2.1.** *Consider Eqs. (2.20)–(2.21) over the full plane subject to the topological boundary condition (2.22). For any distribution of points  $p_{i1}, \dots, p_{in_i} \in \mathbb{R}^2, i = 1, 2, \kappa > 0, \lambda > 0$ , there exists a solution  $(u_1, u_2)$  for Eqs. (2.20)–(2.21) realizing the boundary condition (2.22). Moreover, there hold the following decay estimates: for any small  $\varepsilon \in (0, 1)$ , the solution satisfies*

$$|(N - 1)u_1 + u_2|^2 + |u_1 - u_2|^2 \leq C(\varepsilon)e^{-\sigma_0\sqrt{2\lambda}(1-\varepsilon)|x|}, \tag{2.27}$$

$$|\nabla([N - 1]u_1 + u_2)|^2 + |\nabla(u_1 - u_2)|^2 \leq C(\varepsilon)e^{-\sigma_0\sqrt{2\lambda}(1-\varepsilon)|x|} \tag{2.28}$$

as  $|x|$  is sufficiently large, where  $C(\varepsilon)$  is a positive constant depending only on  $\varepsilon, \sigma_0 = \min\{1, \kappa\}$ .

**Theorem 2.2.** *Consider Eqs. (2.20)–(2.21) over a doubly periodic domain  $\Omega$  in  $\mathbb{R}^2$ . For any given points  $p_{i1}, \dots, p_{in_i} \in \Omega, i = 1, 2$ , which need not to be distinct,  $\lambda > 0$ , and  $\kappa > 1$ , we have the following conclusion:*

1. Every solution  $(u_1, u_2)$  of (2.20)–(2.21) satisfies

$$u_i < 0, \quad i = 1, 2. \tag{2.29}$$

2. If Eqs. (2.20)–(2.21) admit a solution, then

$$\lambda > \lambda_0 \equiv \frac{16\pi([N - 1]n_1 + n_2)}{N|\Omega|}. \tag{2.30}$$

3. There exists a positive constant  $\lambda_1 > \lambda_0$  such that when  $\lambda > \lambda_1$  Eqs. (2.20)–(2.21) admit at least two distinct solutions over  $\Omega$ , one of which satisfies the behavior

$$e^{u_i} \rightarrow 1, \quad \text{as } \lambda \rightarrow +\infty, \quad i = 1, 2 \tag{2.31}$$

pointwise a.e. in  $\Omega$  and strongly in  $L^p(\Omega)$  for any  $p \geq 1$ .

**Theorem 2.3.** In both planar and doubly periodic cases, for the solutions  $(u_1, u_2)$  obtained above, the vortex magnetic fluxes take the quantized form

$$\mathcal{F}^{U(1)} \equiv \int F_{12}^0 dx = \frac{4\pi}{\sqrt{2N}} ([N - 1]n_1 + n_2), \tag{2.32}$$

$$\mathcal{F}^{SU(N)} \equiv \int F_{12}^{N^2-1} dx = 4\pi \sqrt{\frac{N-1}{2N}} (n_1 - n_2). \tag{2.33}$$

### 3. Planar case

In this section we aim to find solutions of (2.20)–(2.21) under the topological boundary condition (2.22) and establish the decay estimates of the solutions, which allow us to get the quantized fluxes stated in Theorem 2.3 for the planar case.

#### 3.1. Existence of solutions

Following [31], we introduce at this point the background functions

$$u_i^0 = - \sum_{s=1}^{n_i} \ln(1 + \mu|x - p_{is}|^{-2}), \quad \mu > 0, \quad i = 1, 2,$$

which satisfy

$$\Delta u_i^0 = 4\pi \sum_{s=1}^{n_i} \delta_{p_{is}} - h_i, \tag{3.1}$$

where

$$h_i = \sum_{s=1}^{n_i} \frac{4\mu}{(\mu + |x - p_{is}|^2)^2}, \quad i = 1, 2. \tag{3.2}$$

Writing  $u_i = u_i^0 + v_i$ , we then recast (2.20)–(2.21) into

$$\begin{aligned} \Delta v_1 = \lambda \left\{ \frac{1}{N^2} \left( [N - 1 + \kappa]^2 e^{2u_1^0 + 2v_1} - [\kappa - 1](N - [N - 2][\kappa - 1]) e^{u_1^0 + u_2^0 + v_1 + v_2} \right. \right. \\ \left. \left. + [1 - \kappa](1 + [N - 1]\kappa) e^{2u_2^0 + 2v_2} \right) - \frac{1}{N} \left( [N - 1 + \kappa] e^{u_1^0 + v_1} + [1 - \kappa] e^{u_2^0 + v_2} \right) \right\} \\ + h_1, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \Delta v_2 = \lambda \left\{ \frac{[N-1][1-\kappa]}{N^2} \left( [N-1+\kappa]e^{2u_1^0+2v_1} + (2+[N-2]\kappa)e^{u_1^0+u_2^0+v_1+v_2} \right. \right. \\ \left. \left. + \frac{(1+[N-1]\kappa)^2}{N^2} e^{2u_2^0+2v_2} \right) - \frac{1}{N} \left( [N-1][1-\kappa]e^{u_1^0+v_1} + (1+[N-1]\kappa)e^{u_2^0+v_2} \right) \right\} \\ + h_2. \end{aligned} \tag{3.4}$$

The boundary condition (2.22) now reads as

$$v_1 \rightarrow 0, \quad v_2 \rightarrow 0, \quad |x| \rightarrow +\infty. \tag{3.5}$$

To see the variational structure of (3.3)–(3.4) clearly, it is convenient to rewrite them equivalently as

$$\begin{aligned} \left( N-1 + \frac{1}{\kappa} \right) \Delta v_1 + \left( 1 - \frac{1}{\kappa} \right) \Delta v_2 = \lambda \left( [N-1+\kappa]e^{2u_1^0+2v_1} \right. \\ \left. - Ne^{u_1^0+v_1} - [\kappa-1]e^{u_1^0+u_2^0+v_1+v_2} \right) + \left( N-1 + \frac{1}{\kappa} \right) h_1 + \left( 1 - \frac{1}{\kappa} \right) h_2, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \left( 1 - \frac{1}{\kappa} \right) \Delta v_1 + \left( \frac{1}{N-1} + \frac{1}{\kappa} \right) \Delta v_2 = \lambda \left( \left[ \frac{1}{N-1} + \kappa \right] e^{2u_2^0+2v_2} \right. \\ \left. - \frac{N}{N-1} e^{u_2^0+v_2} - [\kappa-1]e^{u_1^0+u_2^0+v_1+v_2} \right) + \left( 1 - \frac{1}{\kappa} \right) h_1 + \left( \frac{1}{N-1} + \frac{1}{\kappa} \right) h_2. \end{aligned} \tag{3.7}$$

We will work on the space  $W^{1,2}(\mathbb{R}^2) \times W^{1,2}(\mathbb{R}^2)$ . Let us define

$$A(N, \kappa) \equiv \begin{pmatrix} N-1 + \frac{1}{\kappa} & 1 - \frac{1}{\kappa} \\ 1 - \frac{1}{\kappa} & \frac{1}{N-1} + \frac{1}{\kappa} \end{pmatrix} \tag{3.8}$$

and introduce the notation

$$\mathbf{v}^\tau \equiv (v_1, v_2), \mathbf{U}^\tau \equiv (e^{u_1^0+v_1}, e^{u_2^0+v_2}), \mathbf{h}^\tau \equiv (h_1, h_2), \mathbf{1}^\tau \equiv (1, 1). \tag{3.9}$$

Then, it is straightforward to see that Eqs. (3.6)–(3.7) are the Euler–Lagrange equations of the following functional

$$I(v_1, v_2) \equiv \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_{i=1}^2 \partial_i \mathbf{v}^\tau A(N, \kappa) \partial_i \mathbf{v} + \frac{\lambda}{2} (\mathbf{U} - \mathbf{1})^\tau A(N, \kappa^{-1}) (\mathbf{U} - \mathbf{1}) + \mathbf{h}^\tau A(N, \kappa) \mathbf{v} \right\} dx. \tag{3.10}$$

Then, to solve Eqs. (3.6)–(3.7) (or equivalently (3.3)–(3.4)), we just need to find the critical points of the functional  $I$  defined in (3.10).

To seek the critical points of the functional  $I$ , we first show that it is coercive over  $W^{1,2}(\mathbb{R}^2) \times W^{1,2}(\mathbb{R}^2)$ .

It is easy to see that  $A(N, \kappa)$ , as defined in (3.8), is positive definite, and its smaller eigenvalue is

$$\alpha_0(\kappa) \equiv \frac{1}{2} \left( \frac{(N-1)^2 + 1}{N-1} + \frac{2}{\kappa} - \sqrt{\frac{N^2(N-2)^2}{(N-1)^2} + 4 \left(1 - \frac{1}{\kappa}\right)^2} \right) > 0 \tag{3.11}$$

for any  $N \geq 2, \kappa > 0$ .

Then

$$\begin{aligned} I(v_1, v_2) &\geq \frac{\alpha_0(\kappa)}{2} \left( \|\nabla v_1\|_2^2 + \|\nabla v_2\|_2^2 \right) + \frac{\lambda}{2} \alpha_0(\kappa^{-1}) \int_{\mathbb{R}^2} \left\{ \left( e^{u_1^0 + v_1} - 1 \right)^2 + \left( e^{u_2^0 + v_2} - 1 \right)^2 \right\} dx \\ &\quad + \int_{\mathbb{R}^2} \mathbf{h}^t A(N, \kappa) \mathbf{v} dx. \end{aligned} \tag{3.12}$$

From the expression of  $h_i$  (3.2), we see that

$$\|h_i\|_2 \leq \frac{C}{\sqrt{\mu}}, \quad i = 1, 2, \tag{3.13}$$

where and in the following we use  $C$  to denote a generic positive constant independent of  $\mu$ . Then it follows from the Hölder inequality and (3.13) that

$$\int_{\mathbb{R}^2} \left( \left[ N - 1 + \frac{1}{\kappa} \right] h_1 + \left[ 1 - \frac{1}{\kappa} \right] h_2 \right) v_1 dx \geq -\frac{C}{\sqrt{\mu}} \|v_1\|_2, \tag{3.14}$$

$$\int_{\mathbb{R}^2} \left( \left[ 1 - \frac{1}{\kappa} \right] h_1 + \left[ \frac{1}{N-1} + \frac{1}{\kappa} \right] h_2 \right) v_2 dx \geq -\frac{C}{\sqrt{\mu}} \|v_2\|_2. \tag{3.15}$$

Next we need to estimate the  $L^2$ -terms in (3.12). Noting the fact  $e^{u_i^0} - 1 \in L^2(\mathbb{R}^2)$  and using the elementary inequality  $|e^t - 1| \geq \frac{|t|}{1+|t|}, t \in \mathbb{R}$ , we have

$$\begin{aligned} \int_{\mathbb{R}^2} \left( e^{u_i^0 + v_i} - 1 \right)^2 dx &= \int_{\mathbb{R}^2} \left( e^{u_i^0} [e^{v_i} - 1] + e^{u_i^0} - 1 \right)^2 dx \\ &\geq \frac{1}{2} \int_{\mathbb{R}^2} e^{2u_i^0} (e^{v_i} - 1)^2 dx - \int_{\mathbb{R}^2} \left( e^{u_i^0} - 1 \right)^2 dx \\ &\geq \frac{1}{2} \int_{\mathbb{R}^2} e^{2u_i^0} \frac{|v_i|^2}{(1+|v_i|)^2} dx - C, \quad i = 1, 2. \end{aligned} \tag{3.16}$$

From the definition of  $u_i^0$ , we see that  $e^{2u_i^0}$  satisfies  $0 \leq e^{2u_i^0} < 1$  and vanishes at the points  $p_{i1}, \dots, p_{in_i}, i = 1, 2$ . To proceed further, we use a decomposition of  $\mathbb{R}^2$  as in [31]

$$\mathbb{R}^2 = \Omega_1^i \cup \Omega_2^i, \quad i = 1, 2, \tag{3.17}$$

where

$$\Omega_1^i = \left\{ x \in \mathbb{R}^2 \mid e^{2u_i^0} \leq \frac{1}{2} \right\}, \quad \Omega_2^i = \left\{ x \in \mathbb{R}^2 \mid e^{2u_i^0} \geq \frac{1}{2} \right\}, \quad i = 1, 2. \tag{3.18}$$

Next we need the inverse Hölder inequality (see [2]).

**Lemma 3.1.** *For any measurable functions  $g_1, g_2$  over  $\Omega$ , there holds*

$$\int_{\Omega} |g_1 g_2| dx \geq \left( \int_{\Omega} |g_1|^q dx \right)^{\frac{1}{q}} \left( \int_{\Omega} |g_2|^{q'} dx \right)^{\frac{1}{q'}}, \tag{3.19}$$

where  $q, q' \in \mathbb{R}, 0 < q < 1, q' < 0$  with  $\frac{1}{q} + \frac{1}{q'} = 1$ .

On  $\Omega_1^i$ , we have  $0 \leq e^{2u_i^0} \leq \frac{1}{2}$  and  $e^{2u_i^0}$  approaches 0 at most  $4n_i$  order near the vortex points  $p_{i,s}, s = 1, \dots, n_i, i = 1, 2$ . We choose  $q_i'$  to satisfy  $-\frac{1}{2n_i} < q_i' < 0$ . Then the integrals

$$\int_{\Omega_1^i} e^{2q_i' u_i^0} dx, \quad i = 1, 2$$

exist and are positive constants. In view of the inverse Hölder inequality (3.19), we obtain

$$\begin{aligned} \int_{\Omega_1^i} e^{2u_i^0} \frac{|v_i|^2}{(1 + |v_i|)^2} dx &\geq \left( \int_{\Omega_1^i} \frac{|v_i|^{2q_i}}{(1 + |v_i|)^{2q_i}} dx \right)^{\frac{1}{q_i}} \left( \int_{\Omega_1^i} e^{2q_i' u_i^0} dx \right)^{\frac{1}{q_i'}} \\ &\geq C \left( \int_{\Omega_1^i} \frac{|v_i|^{2q_i}}{(1 + |v_i|)^{2q_i}} dx \right)^{\frac{1}{q_i}}, \quad i = 1, 2, \end{aligned} \tag{3.20}$$

where  $0 < q_i < \frac{1}{1+2n_i}, i = 1, 2$ .

Since  $0 < \frac{|v_i|}{1+|v_i|} < 1$ , using the Young inequality, we have

$$\begin{aligned} \left( \int_{\Omega_1^i} \frac{|v_i|^{2q_i}}{(1 + |v_i|)^{2q_i}} dx \right)^{\frac{1}{q_i}} &\geq \left( \int_{\Omega_1^i} \frac{|v_i|^2}{(1 + |v_i|)^2} dx \right)^{\frac{1}{q_i}} \\ &\geq \frac{1}{2} \int_{\Omega_1^i} \frac{|v_i|^2}{(1 + |v_i|)^2} dx - C. \end{aligned} \tag{3.21}$$

Inserting (3.21) into (3.20), we find that

$$\int_{\Omega_1^i} e^{2u_i^0} \frac{|v_i|^2}{(1 + |v_i|)^2} dx \geq C \int_{\Omega_1^i} \frac{|v_i|^2}{(1 + |v_i|)^2} dx - C, \quad i = 1, 2. \tag{3.22}$$

On  $\Omega_2^i$ , we have

$$\int_{\Omega_2^i} e^{2u_i^0} \frac{|v_i|^2}{(1 + |v_i|)^2} dx \geq \frac{1}{2} \int_{\Omega_2^i} \frac{|v_i|^2}{(1 + |v_i|)^2} dx, \quad i = 1, 2. \tag{3.23}$$

Combining (3.16), (3.22) and (3.23), we finally obtain

$$\int_{\mathbb{R}^2} \left( e^{u_i^0 + v_i} - 1 \right)^2 dx \geq C \int_{\mathbb{R}^2} \frac{|v_i|^2}{(1 + |v_i|)^2} dx - C, \quad i = 1, 2. \tag{3.24}$$

Let us now analyze  $\|\nabla v_i\|_2^2$ . Noting the interpolation inequality over  $W^{1,2}(\mathbb{R}^2)$ :

$$\int_{\mathbb{R}^2} w^4 dx \leq 2 \int_{\mathbb{R}^2} w^2 dx \int_{\mathbb{R}^2} |\nabla w|^2 dx, \quad \forall w \in W^{1,2}(\mathbb{R}^2), \tag{3.25}$$

we obtain

$$\begin{aligned} & \left( \int_{\mathbb{R}^2} |v_i|^2 dx \right)^2 \\ &= \left( \int_{\mathbb{R}^2} \frac{|v_i|}{1 + |v_i|} [1 + |v_i|] |v_i| dx \right)^2 \\ &\leq \int_{\mathbb{R}^2} \frac{|v_i|^2}{(1 + |v_i|)^2} dx \int_{\mathbb{R}^2} (|v_i| + |v_i|^2)^2 dx \\ &\leq 4 \int_{\mathbb{R}^2} \frac{|v_i|^2}{(1 + |v_i|)^2} dx \int_{\mathbb{R}^2} |v_i|^2 dx \left( \int_{\mathbb{R}^2} |\nabla v_i|^2 + 1 \right) \\ &\leq \frac{1}{2} \left( \int_{\mathbb{R}^2} |v_i|^2 dx \right)^2 + C \left( \left[ \int_{\mathbb{R}^2} \frac{|v_i|^2}{[1 + |v_i|]^2} dx \right]^4 + \left[ \int_{\mathbb{R}^2} |\nabla v_i|^2 dx \right]^4 + 1 \right), \end{aligned} \tag{3.26}$$

which implies

$$\|v_i\|_2 \leq C \left( \int_{\mathbb{R}^2} \frac{|v_i|^2}{[1 + |v_i|]^2} dx + \int_{\mathbb{R}^2} |\nabla v_i|^2 dx + 1 \right), \quad i = 1, 2. \tag{3.27}$$

Then from (3.12), (3.14), (3.15), and (3.24), we conclude that

$$\begin{aligned} I(v_1, v_2) &\geq \frac{\alpha_0(\kappa)}{2} \left( \|\nabla v_2\|_2^2 + \|\nabla v_1\|_2^2 \right) + C \left( \int_{\mathbb{R}^2} \frac{|v_1|^2}{[1 + |v_1|]^2} dx + \int_{\mathbb{R}^2} \frac{|v_2|^2}{[1 + |v_2|]^2} dx \right) \\ &\quad - \frac{C}{\sqrt{\mu}} (\|v_1\|_2 + \|v_2\|_2) - C. \end{aligned} \tag{3.28}$$

At this point, taking  $\mu$  sufficiently large in (3.28) and using (3.27), we have

$$I(v_1, v_2) \geq C \left( \left( \|\nabla v_1\|_2^2 + \|\nabla v_2\|_2^2 \right) + \int_{\mathbb{R}^2} \frac{|v_1|^2}{[1 + |v_1|]^2} dx + \int_{\mathbb{R}^2} \frac{|v_2|^2}{[1 + |v_2|]^2} dx \right) - C. \tag{3.29}$$

Thus, from (3.29) and (3.27), we get

$$I(v_1, v_2) \geq C (\|v_1\|_{W^{1,2}(\mathbb{R}^2)} + \|v_2\|_{W^{1,2}(\mathbb{R}^2)}) - C, \tag{3.30}$$

which gives the coerciveness of the functional  $I$  over  $W^{1,2}(\mathbb{R}^2) \times W^{1,2}(\mathbb{R}^2)$ .

It is easy to see that the functional  $I$  is continuous, differentiable, and weakly lower semi-continuous on  $W^{1,2}(\mathbb{R}^2) \times W^{1,2}(\mathbb{R}^2)$ . Then by (3.30), we infer that the functional  $I$  admits a critical point  $(v_1, v_2) \in W^{1,2}(\mathbb{R}^2) \times W^{1,2}(\mathbb{R}^2)$ , which is a weak solution of Eqs. (3.6)–(3.7).

Using the well-known inequality

$$\|e^w - 1\|_2 \leq C \exp(C \|w\|_{W^{1,2}(\mathbb{R}^2)}^2), \quad \forall w \in W^{1,2}(\mathbb{R}^2),$$

we see that the right hand sides of Eqs. (3.6)–(3.7) belong to  $L^2(\mathbb{R}^2)$ . Hence using the  $L^2$ -elliptic estimates we conclude that  $(v_1, v_2) \in W^{2,2}(\mathbb{R}^2) \times W^{2,2}(\mathbb{R}^2)$ , which gives the desired boundary condition (3.5) at infinity. Similarly, we obtain that  $\partial_i v_1, \partial_i v_2 \rightarrow 0$  ( $i = 1, 2$ ) when  $|x| \rightarrow \infty$ .  $\square$

### 3.2. Decay estimates and quantized fluxes

In this subsection our purpose is to derive the decay estimates of  $(N - 1)u_1 + u_2, u_1 - u_2$  and their derivatives when  $|x| \rightarrow \infty$ . As an application of the decay estimates we may calculate the quantized fluxes stated in Theorem 2.3 for the planar case.

To establish the decay estimates, it is convenient to write the equations in a vector form. We will use the notation (3.9) and

$$\mathbf{u} = (u_1, u_2)^\tau, \quad \mathbf{U} = \text{diag}\{e^{u_1}, e^{u_2}\}. \tag{3.31}$$

Then away from the vortex points Eqs. (2.20)–(2.21) can be rewritten as

$$\Delta \mathbf{u} = \lambda KUK(\mathbf{U} - \mathbf{1}), \tag{3.32}$$

where  $K$  is defined by (2.23).

Let

$$O \equiv \begin{pmatrix} N-1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{w} = (w_1, w_2)^\tau \equiv O\mathbf{u}. \tag{3.33}$$

When  $|x| > R$  with  $R > 0$  sufficiently large such that  $R > |p_{is}|$  for  $s = 1, \dots, n_i$  and  $i = 1, 2$ , we have

$$\Delta \mathbf{u} = \lambda KUKU_\xi \mathbf{u} = \lambda K^2 \mathbf{u} + \lambda(KUKU_\xi \mathbf{u} - K^2 \mathbf{u}), \tag{3.34}$$

where  $U_\xi \equiv \text{diag}\{e^{u_1^\xi}, e^{u_2^\xi}\}$ , and  $u_i^\xi$  lies between 0 and  $u_i$  for  $i = 1, 2$ . From (3.34), we see that when  $|x| > R$

$$\begin{aligned} \Delta \mathbf{w} &= \lambda OK^2 O^{-1} \mathbf{w} + \lambda(OKUKU_\xi O^{-1} \mathbf{w} - OK^2 O^{-1} \mathbf{w}) \\ &= \lambda D\mathbf{w} + \lambda(OKUKU_\xi O^{-1} \mathbf{w} - D\mathbf{w}), \end{aligned} \tag{3.35}$$

where  $D \equiv \text{diag}\{1, \kappa^2\}$ .

Then as  $|x| > R$  we have

$$\begin{aligned} \Delta |\mathbf{w}|^2 &\geq 2\mathbf{w}^\tau \Delta \mathbf{w} = 2\lambda \mathbf{w}^\tau D\mathbf{w} + \lambda \mathbf{w}^\tau (OKUKU_\xi O^{-1} \mathbf{w} - D\mathbf{w}) \\ &\geq 2\lambda \sigma_0^2 |\mathbf{w}|^2 - f(x) |\mathbf{w}|^2, \end{aligned} \tag{3.36}$$

where  $\sigma_0 = \min\{1, \kappa\}$ ,  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Therefore, for any sufficiently small  $\varepsilon \in (0, 1)$ , there exists an  $R_\varepsilon > R$  such that

$$\Delta |\mathbf{w}|^2 \geq 2\lambda \sigma_0^2 \left(1 - \frac{\varepsilon}{2}\right) |\mathbf{w}|^2, \quad |x| > R_\varepsilon. \tag{3.37}$$

Noting that  $|\mathbf{w}|^2 = 0$  at infinity, we conclude from (3.37) that there exists a positive constant  $C(\varepsilon)$  such that

$$|\mathbf{w}|^2 \leq C(\varepsilon) e^{-\sigma_0 \sqrt{2\lambda}(1-\varepsilon)|x|}, \quad |x| > R_\varepsilon, \tag{3.38}$$

which gives the desired estimate (2.27).

To get the decay estimate for the derivatives of  $w_1$  and  $w_2$ , we follow the same procedure. Let  $\partial$  be any one of the two partial derivatives  $\partial_1$  and  $\partial_2$ . Then from (3.32) we get

$$\begin{aligned} \Delta(\partial \mathbf{u}) &= \lambda KUVK(\mathbf{U} - \mathbf{1}) + \lambda KUKU\partial \mathbf{u} \\ &= \lambda K^2 \partial \mathbf{u} + \lambda(KUVK(\mathbf{U} - \mathbf{1}) + KUKU\partial \mathbf{u} - K^2 \partial \mathbf{u}), \end{aligned} \tag{3.39}$$

where  $V \equiv \text{diag}\{\partial u_1, \partial u_2\}$ .

Hence we have

$$\begin{aligned} \Delta(\partial \mathbf{w}) &= \lambda OK^2 O^{-1} \partial \mathbf{w} + \lambda(OKUVK(\mathbf{U} - 1) + OKUKUO^{-1} \partial \mathbf{w} - OK^2 O^{-1} \partial \mathbf{w}) \\ &= \lambda D \partial \mathbf{w} + \lambda(OKUVK(\mathbf{U} - 1) + OKUKUO^{-1} \partial \mathbf{w} - D \partial \mathbf{w}). \end{aligned} \tag{3.40}$$

Then as  $|x|$  is sufficiently large we get

$$\begin{aligned} \Delta|\partial \mathbf{w}|^2 &\geq 2(\partial \mathbf{w})^\tau \Delta(\partial \mathbf{w}) \\ &= \lambda(\partial \mathbf{w})^\tau D \partial \mathbf{w} + \lambda(\partial \mathbf{w})^\tau (OKUVK(\mathbf{U} - 1) + OKUKUO^{-1} \partial \mathbf{w} - D \partial \mathbf{w}). \end{aligned} \tag{3.41}$$

Hence, similar to (3.37), for any sufficiently small  $\varepsilon \in (0, 1)$ , there exists an  $R_\varepsilon > R$  such that

$$\Delta|\partial \mathbf{w}|^2 \geq 2\lambda\sigma_0^2 \left(1 - \frac{\varepsilon}{2}\right) |\partial \mathbf{w}|^2, \quad |x| > R_\varepsilon. \tag{3.42}$$

Since we have shown that  $|\partial \mathbf{w}|^2 \rightarrow 0$  as  $|x| \rightarrow \infty$ , from (3.42) we get that there exists a positive constant  $C(\varepsilon)$  such that

$$|\partial \mathbf{w}|^2 \leq C(\varepsilon)e^{-\sigma_0\sqrt{2\lambda}(1-\varepsilon)|x|}, \quad |x| > R_\varepsilon, \tag{3.43}$$

which gives (2.28).

Using the decay estimates one can now calculate the magnetic fluxes. Indeed, from Eqs. (2.16)–(2.17) one has

$$F_{12}^0 = -\frac{1}{\sqrt{2N}} \Delta \left( [N - 1] \ln |\phi|^2 + |\ln \phi_N|^2 \right), \tag{3.44}$$

$$F_{12}^{N^2-1} = -\sqrt{\frac{N-1}{2N}} \Delta \left( \ln |\phi|^2 - \ln |\phi_N|^2 \right), \tag{3.45}$$

so that

$$\mathcal{F}^{U(1)} = -\frac{1}{\sqrt{2N}} \int_{\mathbb{R}^2} \Delta \left( [N - 1] \ln |\phi|^2 + |\ln \phi_N|^2 \right) dx, \tag{3.46}$$

$$\mathcal{F}^{SU(N)} = -\sqrt{\frac{N-1}{2N}} \int_{\mathbb{R}^2} \Delta \left( \ln |\phi|^2 - |\ln \phi_N|^2 \right) dx, \tag{3.47}$$

and direct integration leads to (2.32)–(2.33) which show that the vortex magnetic fluxes are completely determined by the numbers of zeros of the Higgs scalars.

Now we have completed the proof of Theorem 2.2 and Theorem 2.3 for the planar case.  $\square$

### 4. Doubly periodic case

In this section we establish the existence of doubly periodic solutions for (2.20)–(2.21). We will use a constrained minimization approach, developed in [10] and later refined by [24,25,41, 45], to establish the existence of the first solution.

Consider Eqs. (2.20)–(2.21) over a doubly periodic domain  $\Omega$ . We first derive a priori estimates for the solutions to (2.20)–(2.21).

**Lemma 4.1.** Any solution  $(u_1, u_2)$  to (2.20)–(2.21) satisfies

$$u_i < 0 \quad \text{in } \Omega, \quad i = 1, 2.$$

**Proof.** To prove this lemma, it is convenient to rewrite (2.20)–(2.21) equivalently as

$$\begin{aligned} \Delta u_1 = & \lambda \left\{ \frac{\kappa - 1}{N^2} ([N - 1 + \kappa]e^{u_1} + [N - 1][\kappa - 1]e^{u_2})(e^{u_1} - e^{u_2}) \right. \\ & \left. + \frac{1}{N} ([N - 1 + \kappa]e^{u_1}[e^{u_1} - 1] + [\kappa - 1]e^{u_2}[1 - e^{u_2}]) \right\} + 4\pi \sum_{s=1}^{n_1} \delta_{p_{1s}}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \Delta u_2 = & \lambda \left\{ \frac{(N - 1)(\kappa - 1)}{N^2} ([\kappa - 1]e^{u_1} + [1 + [N - 1]\kappa]e^{u_2})(e^{u_2} - e^{u_1}) \right. \\ & \left. + \frac{1}{N} ([1 + [N - 1]\kappa]e^{u_2}[e^{u_2} - 1] + [N - 1][\kappa - 1]e^{u_1}[1 - e^{u_1}]) \right\} + 4\pi \sum_{s=1}^{n_2} \delta_{p_{2s}}. \end{aligned} \tag{4.2}$$

It is easy to see that  $u_i$  may achieve its maximum value at some point  $\tilde{x}_i \in \Omega \setminus \{p_{i1}, \dots, p_{in_i}\}$ ,  $i = 1, 2$ . Denote  $\tilde{u}_i \equiv \max_{x \in \Omega} u_i = u_i(\tilde{x}_i)$ ,  $i = 1, 2$ .

We first show that  $\tilde{u}_i \leq 0$ ,  $i = 1, 2$ . If  $\tilde{u}_1 \geq \tilde{u}_2$ , using (4.1), we have

$$\begin{aligned} 0 \geq \Delta u_1(\tilde{x}_1) = & \lambda \left\{ \frac{\kappa - 1}{N^2} ([N - 1 + \kappa]e^{\tilde{u}_1} + [N - 1][\kappa - 1]e^{u_2(\tilde{x}_1)})(e^{\tilde{u}_1} - e^{u_2(\tilde{x}_1)}) \right. \\ & \left. + \frac{1}{N} ([N - 1 + \kappa]e^{\tilde{u}_1}[e^{\tilde{u}_1} - 1] + [\kappa - 1]e^{u_2(\tilde{x}_1)}[1 - e^{u_2(\tilde{x}_1})]) \right\} \\ \geq & \frac{\lambda}{N} ([N - 1 + \kappa]e^{\tilde{u}_1} - [\kappa - 1]e^{u_2(\tilde{x}_1)})(e^{\tilde{u}_1} - 1). \end{aligned}$$

Then from maximum principle we see that  $\tilde{u}_1 \leq 0$ . Then in this case both  $\tilde{u}_1$  and  $\tilde{u}_2$  are nonpositive.

If  $\tilde{u}_2 \geq \tilde{u}_1$ , using (4.2), we obtain

$$\begin{aligned} 0 \geq \Delta u_2(\tilde{x}_2) = & \lambda \left\{ \frac{(N - 1)(\kappa - 1)}{N^2} ([\kappa - 1]e^{u_1(\tilde{x}_2)} + [1 + [N - 1]\kappa]e^{\tilde{u}_2})(e^{\tilde{u}_2} - e^{u_1(\tilde{x}_2)}) \right. \\ & \left. + \frac{1}{N} ([1 + [N - 1]\kappa]e^{\tilde{u}_2}[e^{\tilde{u}_2} - 1] + [N - 1][\kappa - 1]e^{u_1(\tilde{x}_2)}[1 - e^{u_1(\tilde{x}_2})]) \right\} \end{aligned}$$

$$\geq \frac{\lambda}{N} \left( [1 + [N - 1]\kappa]e^{\tilde{u}_2} - [N - 1][\kappa - 1]e^{u_1(\tilde{x}_2)} \right) (e^{\tilde{u}_2} - 1).$$

Using maximum principle again, we get  $\tilde{u}_2 \leq 0$ . Therefore we have  $u_i \leq 0, i = 1, 2$ .

To prove the strict inequality, we can apply the strong maximum principle. In fact, it is sufficient to note that

$$\Delta u_1 + \alpha_1(x)u_1 = \frac{\lambda}{N^2}(\kappa - 1) \left( [N - 1 + \kappa]e^{u_1} + (1 + [N - 1]\kappa)e^{u_2} \right) (1 - e^{u_2}) \geq 0,$$

$$\Delta u_2 + \alpha_2(x)u_2 = \frac{\lambda}{N^2}(N - 1)(\kappa - 1) \left( [N - 1 + \kappa]e^{u_1} + (1 + [N - 1]\kappa)e^{u_2} \right) (1 - e^{u_1}) \geq 0,$$

where

$$\alpha_1(x) = \frac{\lambda}{N^2} \left( [N - 1 + \kappa]^2 e^{u_1} + [N - 1][\kappa - 1]^2 e^{u_2} \right) \frac{1 - e^{u_1}}{u_1},$$

$$\alpha_2(x) = \frac{\lambda}{N^2} \left( [N - 1][\kappa - 1]^2 e^{u_1} + [1 + [N - 1][\kappa - 1]]^2 e^{u_2} \right) \frac{1 - e^{u_2}}{u_2}.$$

Then Lemma 4.1 follows from the strong maximum principle.  $\square$

From Lemma 4.1, we get the first part of Theorem 2.2.

Let  $u_i^0$  be the solution of the following problem (see [6])

$$\Delta u_i^0 = 4\pi \sum_{s=1}^{n_i} \delta_{p_{is}} - \frac{4\pi n_i}{|\Omega|}, \quad \int_{\Omega} u_i^0 dx = 0, \quad i = 1, 2,$$

and  $u_i = u_i^0 + v_i, i = 1, 2$ . Then over  $\Omega$  Eqs. (2.20)–(2.21) can be reduced as

$$\begin{aligned} \Delta v_1 = \lambda \left\{ \frac{1}{N^2} \left( [N - 1 + \kappa]^2 e^{2u_1^0 + 2v_1} - [\kappa - 1](N - [N - 2][\kappa - 1])e^{u_1^0 + u_2^0 + v_1 + v_2} \right. \right. \\ \left. \left. + [1 - \kappa](1 + [N - 1]\kappa)e^{2u_2^0 + 2v_2} \right) - \frac{1}{N} \left( [N - 1 + \kappa]e^{u_1^0 + v_1} + [1 - \kappa]e^{u_2^0 + v_2} \right) \right\} \\ + \frac{4\pi n_1}{|\Omega|}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \Delta v_2 = \lambda \left\{ \frac{1}{N^2} \left( [N - 1][1 - \kappa][N - 1 + \kappa]e^{2u_1^0 + 2v_1} \right. \right. \\ \left. \left. - [N - 1][\kappa - 1](2 + [N - 2]\kappa)e^{u_1^0 + u_2^0 + v_1 + v_2} + (1 + [N - 1]\kappa)^2 e^{2u_2^0 + 2v_2} \right) \right. \\ \left. - \frac{1}{N} \left( [N - 1][1 - \kappa]e^{u_1^0 + v_1} + (1 + [N - 1]\kappa)e^{u_2^0 + v_2} \right) \right\} + \frac{4\pi n_2}{|\Omega|}. \end{aligned} \tag{4.4}$$

To find a variational principle for the problem (4.3)–(4.4), as in the full plane case we rewrite Eqs. (4.3)–(4.4) equivalently as

$$\left(N - 1 + \frac{1}{\kappa}\right) \Delta v_1 + \left(1 - \frac{1}{\kappa}\right) \Delta v_2 = \lambda \left( [N - 1 + \kappa] e^{2u_1^0 + 2v_1} - N e^{u_1^0 + v_1} - [\kappa - 1] e^{u_1^0 + u_2^0 + v_1 + v_2} \right) + \frac{b_1}{|\Omega|}, \tag{4.5}$$

$$\left(1 - \frac{1}{\kappa}\right) \Delta v_1 + \left(\frac{1}{N - 1} + \frac{1}{\kappa}\right) \Delta v_2 = \lambda \left( \left[\frac{1}{N - 1} + \kappa\right] e^{2u_2^0 + 2v_2} - \frac{N}{N - 1} e^{u_2^0 + v_2} - [\kappa - 1] e^{u_1^0 + u_2^0 + v_1 + v_2} \right) + \frac{b_2}{|\Omega|}, \tag{4.6}$$

where the notation

$$b_1 \equiv \frac{4\pi ([1 + [N - 1]\kappa]n_1 + [\kappa - 1]n_2)}{\kappa}, \quad b_2 \equiv \frac{4\pi ([N - 1][\kappa - 1]n_1 + [N - 1 + \kappa]n_2)}{(N - 1)\kappa} \tag{4.7}$$

will be used throughout this paper.

We will work on the space  $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ , where  $W^{1,2}(\Omega)$  is the set of  $\Omega$ -periodic  $L^2$ -functions whose derivatives also belong to  $L^2(\Omega)$ . We denote the usual norm on  $W^{1,2}(\Omega)$  by  $\|\cdot\|$  as given by  $\|w\|^2 = \|w\|_2^2 + \|\nabla w\|_2^2 = \int_{\Omega} w^2 dx + \int_{\Omega} |\nabla w|^2 dx$ .

Then, as in the planar case, we readily see that Eqs. (4.5)–(4.6) are the Euler–Lagrange equations of the functional

$$I(v_1, v_2) = \int_{\Omega} \left\{ \frac{1}{2} \sum_{i=1}^2 \partial_i \mathbf{v}^t A(N, \kappa) \partial_i \mathbf{v} + \frac{\lambda}{2} (\mathbf{U} - \mathbf{1})^t A(N, \kappa^{-1}) (\mathbf{U} - \mathbf{1}) + \frac{\mathbf{b}^t \cdot \mathbf{v}}{|\Omega|} \right\} dx, \tag{4.8}$$

where the notation (3.9) and  $\mathbf{b}^t \equiv (b_1, b_2)$  are used. Hence in the following subsections we concentrate on finding the critical points of the functional  $I$ .

#### 4.1. Constrained minimization procedure

To find a first critical point of the functional  $I$ , we carry out a constrained minimization procedure.

For any solution  $(v_1, v_2)$  of (4.5)–(4.6), integrating over  $\Omega$  gives the following constraints

$$\begin{aligned} (N - 1 + \kappa) \int_{\Omega} e^{2u_1^0 + 2v_1} dx - N \int_{\Omega} e^{u_1^0 + v_1} dx - (\kappa - 1) \int_{\Omega} e^{u_1^0 + u_2^0 + v_1 + v_2} dx + \frac{b_1}{\lambda} &= 0, \tag{4.9} \\ \left(\frac{1}{N - 1} + \kappa\right) \int_{\Omega} e^{2u_2^0 + 2v_2} - \frac{N}{N - 1} \int_{\Omega} e^{u_2^0 + v_2} - (\kappa - 1) \int_{\Omega} e^{u_1^0 + u_2^0 + v_1 + v_2} dx + \frac{b_2}{\lambda} &= 0. \end{aligned} \tag{4.10}$$

We first establish the necessary condition stated in Theorem 2.2 for the existence of solutions to (2.20)–(2.21). To this end, for any solution  $(v_1, v_2)$  of (4.5)–(4.6), from (4.9)–(4.10) we observe that

$$\int_{\Omega} \left( \mathbf{U}^{\tau} A(N, \kappa^{-1}) \mathbf{U} - N e^{u_1^0 + v_1} - \frac{N}{N-1} e^{u_2^0 + v_2} \right) dx = -\frac{4\pi([N-1]n_1 + n_2)}{(N-1)\lambda}, \tag{4.11}$$

which can be rewritten as

$$\int_{\Omega} \left\{ \left( \mathbf{U} - \frac{\mathbf{1}}{2} \right)^{\tau} A(N, \kappa^{-1}) \left( \mathbf{U} - \frac{\mathbf{1}}{2} \right) \right\} dx = \frac{N|\Omega|}{4(N-1)} - \frac{4\pi([N-1]n_1 + n_2)}{(N-1)\lambda}. \tag{4.12}$$

Since the matrix  $A(N, \kappa^{-1})$  is positive definite, (4.12) implies the Bradlow bound [36]

$$\lambda > \frac{16\pi([N-1]n_1 + n_2)}{N|\Omega|}.$$

Then we get the necessary condition (2.30) stated in Theorem 2.2.

Using (4.9)–(4.10), we also find

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} (\mathbf{U} - \mathbf{1})^{\tau} A(N, \kappa^{-1}) (\mathbf{U} - \mathbf{1}) dx \\ &= \frac{1}{2} \int_{\Omega} \left( N [1 - e^{u_1^0 + v_1}] + \frac{N}{N-1} [1 - e^{u_2^0 + v_2}] \right) dx - \frac{2\pi N}{\lambda} \left( n_1 + \frac{n_2}{N-1} \right). \end{aligned} \tag{4.13}$$

It is well-known that the space  $W^{1,2}(\Omega)$  can be decomposed as

$$W^{1,2}(\Omega) = \mathbb{R} \oplus \dot{W}^{1,2}(\Omega),$$

where

$$\dot{W}^{1,2}(\Omega) = \left\{ w \in W^{1,2}(\Omega) \mid \int_{\Omega} w dx = 0 \right\}$$

is a closed subspace of  $W^{1,2}(\Omega)$ .

Then, for  $v_i \in W^{1,2}(\Omega)$  we have the decomposition

$$v_i = c_i + w_i,$$

where

$$\int_{\Omega} w_i dx = 0, \quad c_i = \frac{1}{|\Omega|} \int_{\Omega} v_i dx, \quad i = 1, 2.$$

If  $(v_1, v_2) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$  satisfies (4.9)–(4.10), we obtain

$$(N - 1 + \kappa)e^{2c_1} a_{11} - e^{c_1} P_1(w_1, w_2, e^{c_2}) + \frac{b_1}{\lambda} = 0, \tag{4.14}$$

$$\left(\frac{1}{N - 1} + \kappa\right) e^{2c_2} a_{22} - e^{c_2} P_2(w_1, w_2, e^{c_1}) + \frac{b_2}{\lambda} = 0, \tag{4.15}$$

where

$$a_{ij} \equiv \int_{\Omega} e^{u_i^0 + u_j^0 + w_i + w_j} dx, \quad a_i \equiv \int_{\Omega} e^{u_i^0 + w_i} dx, \quad i, j = 1, 2,$$

$$P_1(w_1, w_2, e^{c_2}) \equiv Na_1 + (\kappa - 1)e^{c_2} a_{12}, \tag{4.16}$$

$$P_2(w_1, w_2, e^{c_1}) \equiv \frac{N}{N - 1} a_2 + (\kappa - 1)e^{c_1} a_{12}. \tag{4.17}$$

Then (4.14)–(4.15) are solvable with respect to  $c_1, c_2$  if and only if

$$P_1^2(w_1, w_2, e^{c_2}) \geq \frac{4b_1}{\lambda} (N - 1 + \kappa) a_{11}, \tag{4.18}$$

$$P_2^2(w_1, w_2, e^{c_1}) \geq \frac{4b_2}{\lambda} \left(\frac{1}{N - 1} + \kappa\right) a_{22}. \tag{4.19}$$

In view of (4.18)–(4.19), we choose the following inequality-type constraints

$$a_1^2 \geq \frac{4(N - 1 + \kappa)b_1 a_{11}}{N^2 \lambda}, \tag{4.20}$$

$$a_2^2 \geq \frac{4(N - 1)(1 + [N - 1]\kappa)b_2 a_{22}}{N^2 \lambda}. \tag{4.21}$$

We introduce the admissible set as

$$\mathcal{A} = \left\{ \mathbf{w}^T \equiv (w_1, w_2) \in \dot{W}^{1,2}(\Omega) \times \dot{W}^{1,2}(\Omega) \text{ satisfies (4.20), (4.21)} \right\}. \tag{4.22}$$

Then, for any  $(w_1, w_2) \in \mathcal{A}$ , we can obtain a solution of (4.14)–(4.15) with respect to  $c_1$  and  $c_2$  by solving the following equations

$$e^{c_1} = \frac{P_1(w_1, w_2, e^{c_2}) + \sqrt{P_1^2(w_1, w_2, e^{c_2}) - \frac{4(N-1+\kappa)b_1}{\lambda} a_{11}}}{2(N - 1 + \kappa) a_{11}}$$

$$\equiv f_1(e^{c_2}), \tag{4.23}$$

$$e^{c_2} = \frac{P_2(w_1, w_2, e^{c_1}) + \sqrt{P_2^2(w_1, w_2, e^{c_1}) - \frac{4(1+[N-1]\kappa)b_2}{(N-1)\lambda} a_{22}}}{2\left(\frac{1}{N-1} + \kappa\right) a_{22}}$$

$$\equiv f_2(e^{c_1}). \tag{4.24}$$

To solve (4.23)–(4.24), we just need to find the zeros of the function

$$f(t) \equiv t - f_1(f_2(t)), \quad t \geq 0.$$

In order to do that, it is sufficient to prove the following proposition.

**Proposition 4.1.** *For any  $(w_1, w_2) \in \mathcal{A}$ , the equation*

$$f(t) = t - f_1(f_2(t)) = 0$$

*admits a unique positive solution  $t_0$ .*

For any  $(w_1, w_2) \in \mathcal{A}$ , using this proposition, we can get a solution of (4.14)–(4.15) with respect to  $c_1, c_2$ .

**Proof of Proposition 4.1.** We see from (4.23) and (4.24) that

$$f_i(t) > 0, \quad \forall t \geq 0, \quad i = 1, 2. \tag{4.25}$$

Hence  $f(0) = -f_1(f_2(0)) < 0$ . It is easy to check that

$$\frac{df_1(t)}{dt} = \frac{(\kappa - 1)f_1(t)a_{12}}{\sqrt{P_1^2(w_1, w_2, t) - \frac{4(N-1+\kappa)b_1}{\lambda}a_{11}}}, \tag{4.26}$$

$$\frac{df_2(t)}{dt} = \frac{(\kappa - 1)f_2(t)a_{12}}{\sqrt{P_2^2(w_1, w_2, t) - \frac{4(1+[N-1]\kappa)b_2}{(N-1)\lambda}a_{22}}}, \tag{4.27}$$

which are all positive since  $\kappa > 1$ . Thus, the functions  $f_i(t)$ , ( $i = 1, 2$ ) are strictly increasing for all  $t > 0$ .

A direct computation gives

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{f_1(t)}{t} &= \frac{(\kappa - 1)a_{12}}{(N - 1 + \kappa)a_{11}}, \\ \lim_{t \rightarrow +\infty} \frac{f_2(t)}{t} &= \frac{(\kappa - 1)a_{12}}{\left(\frac{1}{N-1} + \kappa\right)a_{22}}, \end{aligned}$$

which implies

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{f(t)}{t} &= 1 - \frac{(\kappa - 1)^2 a_{12}^2}{(N - 1 + \kappa) \left(\frac{1}{N-1} + \kappa\right) a_{11} a_{22}} \\ &\geq 1 - \frac{(\kappa - 1)^2}{(N - 1 + \kappa) \left(\frac{1}{N-1} + \kappa\right)} \\ &= \frac{N^2 \kappa}{(N - 1)(\kappa - 1)^2 + N^2 \kappa} > 0 \end{aligned}$$

Hence we conclude that

$$\lim_{t \rightarrow +\infty} f(t) = +\infty.$$

In view of the fact  $f(0) < 0$ , then we infer that the function  $f(\cdot)$  has at least one zero point  $t_0 > 0$ .

Now we prove that the zero point of  $f(\cdot)$  is also unique. From (4.26)–(4.27) and (4.23)–(4.24) we obtain

$$\begin{aligned} \frac{df(t)}{dt} &= 1 - \frac{(\kappa - 1)^2 f_1(f_2(t)) f_2(t) a_{12}^2}{\sqrt{P_1^2(w_1, w_2, t) - \frac{4(N-1+\kappa)b_1}{\lambda} a_{11}}} \frac{1}{\sqrt{P_2^2(w_1, w_2, t) - \frac{4(1+[N-1]\kappa)b_2}{(N-1)\lambda} a_{22}}} \\ &> 1 - \frac{f_1(f_2(t))}{t} = \frac{f(t)}{t}, \end{aligned}$$

which gives

$$\left(\frac{f(t)}{t}\right)' > 0.$$

Thus we see that  $\frac{f(t)}{t}$  is strictly increasing for  $t > 0$ . Consequently,  $f(t)$  is strictly increasing for  $t > 0$ , which implies  $f(t)$  has a unique zero point. Then the proof of Proposition 4.1 is complete.  $\square$

From the above discussion we conclude that, for any  $(w_1, w_2) \in \mathcal{A}$ , there is a pair  $(c_1(w_1, w_2), c_2(w_1, w_2))$  given by (4.23)–(4.24), which solves (4.14)–(4.15), such that  $(v_1, v_2)$  defined by

$$v_i = w_i + c_i(w_1, w_2), \quad i = 1, 2$$

satisfies (4.9)–(4.10).

Thus, to seek the critical points of  $I$ , we may consider the functional

$$J(w_1, w_2) \equiv I(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2)), \quad (w_1, w_2) \in \mathcal{A}. \tag{4.28}$$

In view of (4.13), we may write the functional  $J$  as

$$\begin{aligned} J(w_1, w_2) &= \frac{1}{2} \sum_{i=1}^2 \int_{\Omega} \partial_i \mathbf{w}^t A(N, \kappa) \partial_i \mathbf{w} dx \\ &\quad + \frac{\lambda}{2} \left( N \int_{\Omega} [1 - e^{c_1} e^{u_1^0 + w_1}] dx + \frac{N}{N-1} \int_{\Omega} [1 - e^{c_2} e^{u_2^0 + w_2}] dx \right) \\ &\quad - 2\pi N \left( n_1 + \frac{n_2}{N-1} \right) + b_1 c_1 + b_2 c_2, \end{aligned} \tag{4.29}$$

where the notations (3.4) and (4.7) are used.

We easily see that the functional  $J$  is Fréchet differentiable in the interior of  $\mathcal{A}$ . If we find a critical point  $(w_1, w_2)$  of  $J$ , which lies in the interior of  $\mathcal{A}$ , then  $(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2))$  is a critical point of  $I$ . Therefore in what follows we just need to find the critical points for the functional  $J$ .

In the rest of this work we use the notation: we write  $(\alpha_1, \alpha_2)^\tau < (\leq)(\beta_1, \beta_2)^\tau$  if  $\alpha_i < (\leq)\beta_i$ ,  $i = 1, 2$ ; for two matrices  $H = (h_{ij})_{2 \times 2}, M = (m_{ij})_{2 \times 2}$ , we write  $H < (\leq)M$  if  $h_{ij} < (\leq)m_{ij}$  for  $i, j = 1, 2$ . We first establish the following lemma.

**Lemma 4.2.** *For any  $(w_1, w_2) \in \mathcal{A}$ , there holds*

$$e^{c_i} \int_{\Omega} e^{u_i^0 + w_i} \leq |\Omega|, \quad i = 1, 2. \tag{4.30}$$

$$e^{c_i} \leq 1, \quad i = 1, 2. \tag{4.31}$$

**Proof.** For any  $(w_1, w_2) \in \mathcal{A}$ , from (4.23)–(4.24) we obtain

$$e^{c_1} \leq \frac{Na_1 + (\kappa - 1)e^{c_2}a_{12}}{(N - 1 + \kappa)a_{11}}, \tag{4.32}$$

$$e^{c_2} \leq \frac{Na_2 + (N - 1)(\kappa - 1)e^{c_1}a_{12}}{(1 + [N - 1]\kappa)a_{22}}. \tag{4.33}$$

Let

$$\tilde{K} \equiv \tilde{K}(w_1, w_2) \equiv \begin{pmatrix} (N - 1 + \kappa)a_{11} & (1 - \kappa)a_{12} \\ (N - 1)(1 - \kappa)a_{12} & (1 + [N - 1]\kappa)a_{22} \end{pmatrix}. \tag{4.34}$$

Then the inequalities (4.32)–(4.33) can be rewritten as

$$\tilde{K}(e^{c_1}, e^{c_2})^\tau \leq N(a_1, a_2)^\tau. \tag{4.35}$$

Using the Hölder inequality we have

$$\text{diag} \left\{ a_{11}^{\frac{1}{2}}, a_{22}^{\frac{1}{2}} \right\} N K \text{diag} \left\{ a_{11}^{\frac{1}{2}}, a_{22}^{\frac{1}{2}} \right\} (e^{c_1}, e^{c_2})^\tau \leq \tilde{K}(e^{c_1}, e^{c_2})^\tau. \tag{4.36}$$

Note the inverse of  $K$  is given by

$$K^{-1} = \frac{1}{N\kappa} \begin{pmatrix} 1 + [N - 1]\kappa & \kappa - 1 \\ (N - 1)(\kappa - 1) & N - 1 + \kappa \end{pmatrix}, \tag{4.37}$$

whose entries are positive since  $N \geq 2$  and  $\kappa > 1$ . Hence from (4.35)–(4.36), we have

$$(e^{c_1}, e^{c_2})^\tau \leq \text{diag} \left\{ a_{11}^{-\frac{1}{2}}, a_{22}^{-\frac{1}{2}} \right\} K^{-1} \text{diag} \left\{ a_{11}^{-\frac{1}{2}}, a_{22}^{-\frac{1}{2}} \right\} (a_1, a_2)^\tau, \tag{4.38}$$

which with Hölder’s inequality yields

$$(a_1 e^{c_1}, a_2 e^{c_2})^\tau \leq \text{diag} \left\{ a_1 a_{11}^{-\frac{1}{2}}, a_2 a_{22}^{-\frac{1}{2}} \right\} K^{-1} \text{diag} \left\{ a_1 a_{11}^{-\frac{1}{2}}, a_2 a_{22}^{-\frac{1}{2}} \right\} \mathbf{1} \leq |\Omega| \mathbf{1}. \tag{4.39}$$

Then we obtain (4.30), which implies (4.31) by using Jensen’s inequality. The proof of Lemma 4.2 is complete.  $\square$

Now we need the following inequalities.

**Lemma 4.3.** *For any  $(w_1, w_2) \in \mathcal{A}$  and  $s \in (0, 1)$ , there holds*

$$\int_{\Omega} e^{w_1^0 + w_1} dx \leq \left( \frac{N^2 \lambda}{4[N - 1 + \kappa] b_1} \right)^{\frac{1-s}{s}} \left( \int_{\Omega} e^{s w_1^0 + s w_1} dx \right)^{\frac{1}{s}}, \tag{4.40}$$

$$\int_{\Omega} e^{w_2^0 + w_2} dx \leq \left( \frac{N^2 \lambda}{4[N - 1](1 + [N - 1]\kappa) b_2} \right)^{\frac{1-s}{s}} \left( \int_{\Omega} e^{s w_2^0 + s w_2} dx \right)^{\frac{1}{s}}. \tag{4.41}$$

The proof of such type inequalities can be found in [40,41].

With the help of Lemma 4.3 and the Moser–Trudinger inequality (see [6,19])

$$\int_{\Omega} e^w dx \leq C_1 \exp \left( \frac{1}{16\pi} \|\nabla w\|_2^2 \right), \quad \forall w \in \dot{W}^{1,2}(\Omega), \tag{4.42}$$

where  $C_1$  is a positive constant depending on  $\Omega$  only, we may show the coerciveness of the functional  $J$ .

**Lemma 4.4.** *For any  $(w_1, w_2) \in \mathcal{A}$ , there exists a suitable positive constant  $C$ , independent of  $\lambda$ , such that*

$$J(w_1, w_2) \geq \frac{\alpha_0(\kappa)}{4} \left( \|\nabla w_1\|_2^2 + \|\nabla w_2\|_2^2 \right) - C(\ln \lambda + 1). \tag{4.43}$$

**Proof.** Noting that the matrices  $A(N, \kappa)$  and  $A(N, \kappa^{-1})$  defined by (3.8) are both positive definite, then using (4.13), (4.29), we have

$$J(w_1, w_2) \geq \frac{\alpha_0(\kappa)}{2} \left( \|\nabla w_1\|_2^2 + \|\nabla w_2\|_2^2 \right) + b_1 c_1 + b_2 c_2, \tag{4.44}$$

where  $\alpha_0(\kappa)$  is a positive constant defined by (3.11).

Next we estimate  $c_1, c_2$  in (4.44). We see from (4.23)–(4.24) that

$$e^{c_1} \geq \frac{N a_1}{2(N - 1 + \kappa) a_{11}},$$

$$e^{c_2} \geq \frac{N a_2}{2(1 + [N - 1]\kappa) a_{22}}.$$

Then, using the constraints (4.20)–(4.21), we obtain

$$e^{c_1} \geq \frac{2b_1}{N\lambda a_1}, \quad e^{c_2} \geq \frac{2(N-1)b_2}{N\lambda a_2},$$

which give

$$c_1 \geq \ln \frac{2b_1}{N} - \ln \lambda - \ln a_1, \tag{4.45}$$

$$c_2 \geq \ln \frac{2(N-1)b_2}{N} - \ln \lambda - \ln a_2. \tag{4.46}$$

For any  $s \in (0, 1)$ , in view of Lemma 4.3 and the Trudinger–Moser inequality (4.42), we have

$$\begin{aligned} \ln \int_{\Omega} e^{u_1^0 + w_1} dx &\leq \frac{1-s}{s} \left( \ln \lambda + \ln \frac{N^2}{4[N-1+\kappa]b_1} \right) + \frac{1}{s} \ln \int_{\Omega} e^{su_1^0 + sw_1} dx \\ &\leq \frac{s}{16\pi} \|\nabla w_1\|_2^2 + \frac{1}{s} \ln C_1 + \max_{\Omega} u_1^0 \\ &\quad + \frac{1-s}{s} \left( \ln \lambda + \ln \frac{N^2}{4[N-1+\kappa]b_1} \right). \end{aligned} \tag{4.47}$$

Similarly, we obtain

$$\begin{aligned} \ln \int_{\Omega} e^{u_2^0 + w_2} dx &\leq \frac{s}{16\pi} \|\nabla w_2\|_2^2 + \frac{1}{s} \ln C_1 + \max_{\Omega} u_2^0 \\ &\quad + \frac{1-s}{s} \left( \ln \lambda + \ln \frac{N^2}{4b_2[N-1](1+[N-1]\kappa)} \right). \end{aligned} \tag{4.48}$$

Plugging (4.47) and (4.48) into (4.45) and (4.46), respectively, gives

$$\begin{aligned} c_1 &\geq -\frac{s}{16\pi} \|\nabla w_1\|_2^2 + \ln \frac{N}{2(N-1+\kappa)} - \max_{\Omega} u_1^0 \\ &\quad - \frac{1}{s} \left( \ln \lambda + \ln C_1 + \ln \frac{N^2}{4(N-1+\kappa)b_1} \right), \end{aligned} \tag{4.49}$$

$$\begin{aligned} c_2 &\geq -\frac{s}{16\pi} \|\nabla w_2\|_2^2 + \ln \frac{N}{2(1+[N-1]\kappa)} - \max_{\Omega} u_2^0 \\ &\quad - \frac{1}{s} \left( \ln \lambda + \ln C_1 + \ln \frac{N^2}{4(N-1)(1+[N-1]\kappa)b_2} \right). \end{aligned} \tag{4.50}$$

Hence, inserting (4.49)–(4.50) into (4.44), we obtain

$$\begin{aligned}
 J(w_1, w_2) \geq & \left( \frac{\alpha_0(\kappa)}{2} - \frac{b_1 s}{16\pi} \right) \|\nabla w_1\|_2^2 + \left( \frac{\alpha_0(\kappa)}{2} - \frac{b_2 s}{16\pi} \right) \|\nabla w_2\|_2^2 \\
 & - \frac{b_1 + b_2}{s} \ln \lambda + b_1 \ln \frac{N}{2(N-1+\kappa)} + b_2 \ln \frac{N}{2(1+[N-1]\kappa)} \\
 & - b_1 \max_{\Omega} u_1^0 - b_2 \max_{\Omega} u_2^0 - \frac{b_1 + b_2}{s} \ln C_1 \\
 & - \frac{1}{s} \left( b_1 \ln \frac{N^2}{4(N-1+\kappa)b_1} + b_2 \ln \frac{N^2}{4(N-1)(1+[N-1]\kappa)b_2} \right). \tag{4.51}
 \end{aligned}$$

Thus, we get (4.43) by taking  $s > 0$  sufficiently small in (4.51). Then the proof of Lemma 4.4 is complete.  $\square$

Noting that the functional  $J$  is weakly lower semi-continuous in  $\mathcal{A}$ , and using Lemma 4.4, we infer that  $J$  admits a minimizer in  $\mathcal{A}$ . In the sequel, we show that the minimizer of  $J$  lies in the interior of  $\mathcal{A}$  when  $\lambda$  is sufficiently large.

**Lemma 4.5.** *On the boundary of  $\mathcal{A}$ , there exists a constant  $C > 0$  independent of  $\lambda$  such that*

$$\inf_{(w_1, w_2) \in \partial \mathcal{A}} J(w_1, w_2) \geq \frac{N|\Omega|}{2(N-1)} \lambda - C(\ln \lambda + \sqrt{\lambda} + 1). \tag{4.52}$$

**Proof.** On the boundary of  $\mathcal{A}$  we have

$$a_1^2 = \frac{4b_1(N-1+\kappa)}{N^2\lambda} a_{11}, \tag{4.53}$$

or

$$a_2^2 = \frac{4b_2(N-1)(1+[N-1]\kappa)}{N^2\lambda} a_{22}. \tag{4.54}$$

If (4.53) holds, in view of (4.38) and the Hölder inequality, we have

$$\begin{aligned}
 e^{c_1} a_1 & \leq \frac{(1+[N-1]\kappa)a_1^2}{N\kappa a_{11}} + \frac{(\kappa-1)a_1 a_2 a_{12}}{N\kappa a_{11} a_{22}} \\
 & \leq \frac{4(N-1+\kappa)(1+[N-1]\kappa)b_1}{N^3\kappa\lambda} + \frac{2(\kappa-1)}{N^2\kappa} \sqrt{\frac{(N-1+\kappa)b_1|\Omega|}{\lambda}}. \tag{4.55}
 \end{aligned}$$

Hence, from Lemma 4.2 and (4.55), we have

$$\frac{\lambda}{2} \left( N \int_{\Omega} [1 - e^{u_1^0+v_1}] dx + \frac{N}{N-1} \int_{\Omega} [1 - e^{u_2^0+v_2}] dx \right) \geq \frac{N|\Omega|}{2} \lambda - C(\sqrt{\lambda} + 1), \tag{4.56}$$

where  $C$  is a positive constant independent of  $\lambda$ .

Similarly, if (4.54) holds, we can conclude that

$$\frac{\lambda}{2} \left( N \int_{\Omega} [1 - e^{u_1^0+v_1}] dx + \frac{N}{N-1} \int_{\Omega} [1 - e^{u_2^0+v_2}] dx \right) \geq \frac{N|\Omega|}{2(N-1)} \lambda - C(\sqrt{\lambda} + 1), \tag{4.57}$$

where  $C$  is a positive constant independent of  $\lambda$ .

Now, using (4.56)–(4.57), and estimating  $c_1, c_2$  as that in Lemma 4.4, we can obtain (4.52). Then Lemma 4.5 follows.  $\square$

To estimate the value of  $J$  in the interior of  $\mathcal{A}$ , we use the approach of [41] to find the test functions.

It was shown in [45] that, for  $\mu > 0$  sufficiently large, the problem

$$\Delta v = \mu e^{u_i^0+v} (e^{u_i^0+v} - 1) + \frac{4\pi n_i}{|\Omega|} \quad \text{in } \Omega$$

admits solutions  $v_i^\mu, i = 1, 2$  such that  $u_i^0 + v_i^\mu < 0$  in  $\Omega, c_i^\mu = \frac{1}{|\Omega|} \int_{\Omega} v_i^\mu dx \rightarrow 0$  and  $w_i^\mu = v_i^\mu - c_i^\mu \rightarrow -u_i^0$  pointwise and a.e. as  $\mu \rightarrow +\infty$ .

Since  $e^{u_i^0+w_i^\mu} \in L^\infty(\Omega), i = 1, 2$  we have

$$e^{u_i^0+w_i^\mu} \rightarrow 1 \quad \text{strongly in } L^p(\Omega) \quad \text{for any } p \geq 1$$

as  $\mu \rightarrow +\infty$ . In particular, we have

$$\lim_{\mu \rightarrow \infty} \int_{\Omega} e^{u_i^0+u_j^0+w_i^\mu+w_j^\mu} dx = |\Omega|, \quad i, j = 1, 2. \tag{4.58}$$

Hence we obtain

$$\lim_{\mu \rightarrow \infty} \tilde{K}(w_1^\mu, w_2^\mu) = |\Omega|NK, \tag{4.59}$$

where  $K$  is defined by (2.23).

Since  $\tilde{K}$  is invertible, one has

$$\lim_{\mu \rightarrow \infty} \tilde{K}^{-1}(w_1^\mu, w_2^\mu) = \frac{1}{|\Omega|N} K^{-1}. \tag{4.60}$$

Then, by the definition of  $\mathcal{A}$ , (4.58) and (4.60), we see that, for some fixed  $\lambda_2 > 0$  sufficiently large and any  $\varepsilon \in (0, 1)$ , there is a  $\mu_\varepsilon$  sufficiently large such that

$$(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon}) \in \text{int}\mathcal{A} \tag{4.61}$$

for all  $\lambda > \lambda_2$  and

$$a_{ij}(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon}) < (1 + \varepsilon)|\Omega|, \quad i, j = 1, 2, \tag{4.62}$$

$$\frac{(1 - \varepsilon)}{N|\Omega|} K^{-1} < \tilde{K}^{-1}(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon}) < \frac{(1 + \varepsilon)}{N|\Omega|} K^{-1} < \frac{2}{N|\Omega|} K^{-1}. \tag{4.63}$$

**Lemma 4.6.** As  $\lambda > 0$  is sufficiently large, there holds

$$J(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon}) - \inf_{(w_1, w_2) \in \partial \mathcal{A}} J(w_1, w_2) < -1. \tag{4.64}$$

**Proof.** Using Jensen’s inequality, (4.31), and (4.23)–(4.24), we obtain

$$\begin{aligned} e^{c_1(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})} &\geq \frac{P_1(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon}, e^{c_2(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})})}{2(N - 1 + \kappa)a_{11}} \\ &\quad \times \left( 1 + \sqrt{1 - \frac{4(N - 1 + \kappa)b_1a_{11}}{\lambda P_1^2(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon}, e^{c_2(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})})}} \right) \\ &\geq \frac{P_1(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon}, e^{c_2(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})})}{(N - 1 + \kappa)a_{11}} - \frac{2b_1}{\lambda P_1(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon}, e^{c_2(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})})} \\ &\geq \frac{(N|\Omega| + [\kappa - 1]a_{12}e^{c_2(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})})}{(N - 1 + \kappa)a_{11}} - \frac{2b_1}{N\lambda|\Omega|}. \end{aligned} \tag{4.65}$$

Here and in the sequel we simply write  $a_{ij}(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})$  as  $a_{ij}$ . Analogously, we have

$$e^{c_2(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})} \geq \frac{(N|\Omega| + [N - 1][\kappa - 1]a_{12}e^{c_1(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})})}{(1 + [N - 1]\kappa)a_{22}} - \frac{2(N - 1)b_2}{N\lambda|\Omega|}. \tag{4.66}$$

Then by (4.65)–(4.66) and (4.62), we get

$$\begin{aligned} &\tilde{K} \left( e^{c_1(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})}, e^{c_2(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})} \right)^\tau \\ &\geq N|\Omega|\mathbf{1} - \frac{2}{N|\Omega|\lambda} ((N - 1 + \kappa)b_1a_{11}, (1 + [N - 1]\kappa)b_2a_{22})^\tau \\ &\geq N|\Omega|\mathbf{1} - \frac{4}{N\lambda} ((N - 1 + \kappa)b_1, (1 + [N - 1]\kappa)b_2)^\tau. \end{aligned} \tag{4.67}$$

Hence, in view that the entries of  $K^{-1}$  are positive, it follows from (4.63) and (4.67) that

$$\begin{aligned} &\left( e^{c_1(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})}, e^{c_2(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})} \right)^\tau \\ &\geq (1 - \varepsilon)K^{-1}\mathbf{1} - \frac{8}{N^2\lambda|\Omega|} K^{-1} ((N - 1 + \kappa)b_1, (1 + [N - 1]\kappa)b_2)^\tau \\ &= (1 - \varepsilon)\mathbf{1} - \frac{8}{N^2\lambda|\Omega|} K^{-1} ((N - 1 + \kappa)b_1, (1 + [N - 1]\kappa)b_2)^\tau. \end{aligned} \tag{4.68}$$

Consequently, we have

$$\int_{\Omega} \left( 1 - e^{c_1(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})} e^{u_1^0 + w_1^{\mu_\varepsilon}} \right) \leq |\Omega|\varepsilon - \frac{2(1 + [N - 1]\kappa) ([N - 1 + \kappa]b_1 + [N - 1][\kappa - 1]b_2)}{N^3\lambda\kappa}, \tag{4.69}$$

$$\int_{\Omega} \left( 1 - e^{c_2(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon})} e^{u_2^0 + w_2^{\mu_\varepsilon}} \right) \leq |\Omega|\varepsilon - \frac{2(N - 1 + \kappa) \{ (1 + [N - 1]\kappa)b_1 + [N - 1][\kappa - 1]b_2 \}}{N^3\lambda\kappa}. \tag{4.70}$$

Using (4.31), (4.69) and (4.70), we infer that, for any small  $\varepsilon > 0$ , there exists a positive constant  $C_\varepsilon$  such that

$$J(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon}) \leq \frac{N^2\lambda|\Omega|\varepsilon}{2(N - 1)} + C_\varepsilon. \tag{4.71}$$

Thus, in view of Lemma 4.5, we have

$$J(w_1^{\mu_\varepsilon}, w_2^{\mu_\varepsilon}) - \inf_{(w_1, w_2) \in \partial\mathcal{A}} J(w_1, w_2) \leq \frac{N|\Omega|\lambda}{2(N - 1)} (N\varepsilon - 1) + C(\ln\lambda + \sqrt{\lambda} + 1), \tag{4.72}$$

where  $C$  is a positive constant independent of  $\lambda$ .

Then, taking  $\varepsilon = \frac{1}{2N}$ , and  $\lambda$  sufficiently large in (4.72), we conclude with (4.64).  $\square$

Now from Lemma 4.4 and Lemma 4.6 we infer that, there exists  $\tilde{\lambda} > 0$  such that, for every  $\lambda > \tilde{\lambda}$ , the functional  $J$  achieves its minimum at a point  $(w_{1,\lambda}, w_{2,\lambda})$ , which belongs to the interior of  $\mathcal{A}$ . Moreover,  $(v_{1,\lambda}, v_{2,\lambda})$ , defined by

$$v_{i,\lambda} = w_{i,\lambda} + c_i(w_{1,\lambda}, w_{2,\lambda}), \quad i = 1, 2, \tag{4.73}$$

is a critical point of the functional  $I$  in  $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ , which gives a weak solution of (4.3)–(4.4).

Next we study the behavior of the solution established above.

**Lemma 4.7.** *Let  $(v_{1,\lambda}, v_{2,\lambda})$  be the solution of (4.3)–(4.4) given by (4.73). There holds*

$$e^{u_i^0 + v_{i,\lambda}} \rightarrow 1 \quad \text{as } \lambda \rightarrow +\infty, \quad i = 1, 2, \tag{4.74}$$

pointwise a.e. in  $\Omega$  and in  $L^p(\Omega)$  for any  $p \geq 1$ . Moreover,  $(v_{1,\lambda}, v_{2,\lambda})$  is a local minimizer of the functional  $I$  in  $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ .

**Proof.** Using (4.13) and similar estimates as in Lemma 4.4, for any  $\lambda > \tilde{\lambda}$ , we infer that there exists a positive constant  $C$  independent of  $\lambda$  such that

$$J(w_{1,\lambda}, w_{2,\lambda}) \geq \frac{\alpha_0(\kappa^{-1})\lambda}{2} \left\{ \int_{\Omega} \left( e^{u_1^0 + v_{1,\lambda}} - 1 \right)^2 dx + \int_{\Omega} \left( e^{u_2^0 + v_{2,\lambda}} - 1 \right)^2 dx \right\} - C(\ln\lambda + 1), \tag{4.75}$$

where  $\alpha_0(\kappa^{-1})$  is a positive constant defined by (3.11). Hence, it follows from (4.75) and (4.71) that

$$\int_{\Omega} \left( e^{u_i^0 + v_{i,\lambda}} - 1 \right)^2 dx \rightarrow 0, \quad \text{as } \lambda \rightarrow +\infty, \quad i = 1, 2. \tag{4.76}$$

In view of Lemma 4.1, we have  $e^{u_i^0 + v_{i,\lambda}} < 1, i = 1, 2$ . Then, we conclude (4.74) by the dominated convergence theorem.

Next, we show that  $(v_{1,\lambda}, v_{2,\lambda})$  is a local minimizer of the functional  $I$  in  $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ .

By a direct computation, for any  $(w_1, w_2) \in \mathcal{A}$  and the corresponding  $(c_1, c_2)$  given by (4.23)–(4.24), we obtain

$$\partial_{c_1} I(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2)) = 0 = \partial_{c_2} I(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2))$$

and

$$\begin{aligned} & \partial_{c_1}^2 I(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2)) \\ &= \lambda \left( 2[N - 1 + \kappa] e^{2c_1} a_{11} - e^{c_1} P_1(w_1, w_2, e^{c_1}) \right) \\ &= \lambda \left\{ \left( N \int_{\Omega} e^{u_1^0 + v_1} dx + [\kappa - 1] \int_{\Omega} e^{u_1^0 + u_2^0 + v_1 + v_2} dx \right)^2 \right. \\ & \quad \left. - \frac{4(N - 1 + \kappa)b_1}{\lambda} \int_{\Omega} e^{2u_1^0 + 2v_1} dx \right\}^{\frac{1}{2}}, \end{aligned} \tag{4.77}$$

$$\begin{aligned} & \partial_{c_2}^2 I(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2)) \\ &= \lambda \left( 2 \left[ \frac{1}{N - 1} + \kappa \right] e^{2c_2} a_{22} - e^{c_2} P_2(w_1, w_2, e^{c_2}) \right) \\ &= \lambda \left\{ \left( \frac{N}{N - 1} \int_{\Omega} e^{u_2^0 + v_2} dx + [\kappa - 1] \int_{\Omega} e^{u_1^0 + u_2^0 + v_1 + v_2} dx \right)^2 \right. \\ & \quad \left. - \frac{4(1 + [N - 1]\kappa)b_2}{(N - 1)\lambda} \int_{\Omega} e^{2u_2^0 + 2v_2} dx \right\}^{\frac{1}{2}}, \end{aligned} \tag{4.78}$$

$$\begin{aligned} & \partial_{c_1 c_2}^2 I(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2)) \\ &= \lambda(1 - \kappa) e^{c_1} e^{c_2} a_{12} \\ &= \lambda(1 - \kappa) \int_{\Omega} e^{u_1^0 + u_2^0 + v_1 + v_2} dx. \end{aligned} \tag{4.79}$$

If  $(w_1, w_2)$  belongs to the interior of  $\mathcal{A}$ , then we can use strict inequalities in the constraints (4.20)–(4.21) to get

$$\begin{aligned} \partial_{c_1}^2 I(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2)) &> \lambda(\kappa - 1) \int_{\Omega} e^{u_1^0 + u_2^0 + v_1 + v_2} dx, \\ \partial_{c_2}^2 I(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2)) &> \lambda(\kappa - 1) \int_{\Omega} e^{u_1^0 + u_2^0 + v_1 + v_2} dx. \end{aligned}$$

Thus, we conclude that, if  $(w_1, w_2)$  is an interior point of  $\mathcal{A}$  then the Hessian matrix of  $I(w_1 + c_1, w_2 + c_2)$  with respect to  $(c_1, c_2)$  is strictly positive definite at  $(c_1(w_1, w_2), c_2(w_1, w_2))$ . We apply such property, near the critical point  $(v_{1,\lambda}, v_{2,\lambda})$ . Indeed, by continuity, for  $\delta > 0$  sufficiently small, we can ensure that, if  $(v_1, v_2) = (w_1 + c_1, w_2 + c_2)$  satisfies:

$$\|v_1 - v_{1,\lambda}\| + \|v_2 - v_{2,\lambda}\| \leq \delta,$$

then  $(w_1, w_2)$  belongs to the interior of  $\mathcal{A}$  and

$$\begin{aligned} I(v_1, v_2) = I(w_1 + c_1, w_2 + c_2) &\geq I(w_{1,\lambda} + c_1(w_{1,\lambda}, w_{2,\lambda}), w_{2,\lambda} + c_2(w_{1,\lambda}, w_{2,\lambda})) \\ &= J(w_{1,\lambda}, w_{2,\lambda}) = I(v_{1,\lambda}, v_{2,\lambda}). \end{aligned}$$

Hence,  $(v_{1,\lambda}, v_{2,\lambda})$  is a local minimizer for  $I$  in  $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ . Then the proof of Lemma 4.7 is complete.  $\square$

#### 4.2. A second solution

In this subsection, via the mountain-pass theorem, we find a second critical point of the functional  $I$ , which gives a second solution of (4.3)–(4.4).

For this purpose, we show that the functional  $I$  satisfies the P-S condition.

**Lemma 4.8.** *Every sequence  $(v_{1,n}, v_{2,n}) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$  satisfies*

$$I(v_{1,n}, v_{2,n}) \rightarrow a_0 \quad \text{as } n \rightarrow +\infty, \tag{4.80}$$

$$\|I'(v_{1,n}, v_{2,n})\|_* \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \tag{4.81}$$

*admits a strongly convergent subsequence in  $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ , where  $a_0$  is a constant and  $\|\cdot\|_*$  denotes the norm of the dual space of  $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ .*

**Proof.** Denote  $\varepsilon_n = \|I'(v_{1,n}, v_{2,n})\|_*$ , we have  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow +\infty$ . For any  $(\psi_1, \psi_2) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ , we obtain

$$\begin{aligned} &(I'(v_{1,n}, v_{2,n}))(\psi_1, \psi_2) \\ &= \left(N - 1 + \frac{1}{\kappa}\right) \int_{\Omega} \nabla v_{1,n} \cdot \nabla \psi_1 dx + \left(\frac{1}{N-1} + \frac{1}{\kappa}\right) \int_{\Omega} \nabla v_{2,n} \nabla \psi_2 dx \end{aligned}$$

$$\begin{aligned}
 &+ \left(1 - \frac{1}{\kappa}\right) \int_{\Omega} (\nabla v_{2,n} \cdot \nabla \psi_1 + \nabla v_{1,n} \cdot \nabla \psi_2) dx \\
 &+ \lambda \int_{\Omega} \left( [N - 1 + \kappa] e^{u_1^0 + v_{1,n}} [e^{u_1^0 + v_{1,n}} - 1] + (1 - \kappa) e^{u_1^0 + v_{1,n}} [e^{u_2^0 + v_{2,n}} - 1] \right) \psi_1 dx \\
 &+ \lambda \int_{\Omega} \left( \left[ \frac{1}{N - 1} + \kappa \right] e^{u_2^0 + v_{2,n}} [e^{u_2^0 + v_{2,n}} - 1] + (1 - \kappa) e^{u_2^0 + v_{2,n}} [e^{u_1^0 + v_{1,n}} - 1] \right) \psi_2 dx \\
 &+ \frac{b_1}{|\Omega|} \int_{\Omega} \psi_1 dx + \frac{b_2}{|\Omega|} \int_{\Omega} \psi_2 dx \tag{4.82}
 \end{aligned}$$

and

$$|(I'(v_{1,n}, v_{2,n}))(\psi_1, \psi_2)| \leq \varepsilon_n (\|\psi_1\| + \|\psi_2\|) \tag{4.83}$$

Taking  $(\psi_1, \psi_2) = (1, 1)$  in (4.82), we find

$$\begin{aligned}
 &(I'(v_{1,n}, v_{2,n}))(1, 1) \\
 &= \lambda \int_{\Omega} \left( [N - 1 + \kappa] e^{u_1^0 + v_{1,n}} [e^{u_1^0 + v_{1,n}} - 1] + (1 - \kappa) e^{u_1^0 + v_{1,n}} [e^{u_2^0 + v_{2,n}} - 1] \right) dx \\
 &+ \lambda \int_{\Omega} \left( \left[ \frac{1}{N - 1} + \kappa \right] e^{u_2^0 + v_{2,n}} [e^{u_2^0 + v_{2,n}} - 1] + (1 - \kappa) e^{u_2^0 + v_{2,n}} [e^{u_1^0 + v_{1,n}} - 1] \right) dx \\
 &+ b_1 + b_2 \\
 &= \lambda \left\{ \int_{\Omega} \left( [N - 1 + \kappa] [e^{u_1^0 + v_{1,n}} - 1]^2 + \left[ \frac{1}{N - 1} + \kappa \right] [e^{u_2^0 + v_{2,n}} - 1]^2 \right) dx \right. \\
 &+ 2(1 - \kappa) \int_{\Omega} (e^{u_1^0 + v_{1,n}} - 1)(e^{u_2^0 + v_{2,n}} - 1) dx \\
 &\left. + \int_{\Omega} \left( N [e^{u_1^0 + v_{1,n}} - 1] + \frac{N}{N - 1} [e^{u_2^0 + v_{2,n}} - 1] \right) dx \right\} + b_1 + b_2. \tag{4.84}
 \end{aligned}$$

Noting that the matrix  $A(N, \kappa^{-1})$  defined by (3.8) is positive definite, then from (4.84) and (4.83) we infer that

$$\int_{\Omega} (e^{u_1^0 + v_{1,n}} - 1)^2 dx + \int_{\Omega} (e^{u_2^0 + v_{2,n}} - 1)^2 dx \leq C$$

for some positive constant  $C$ , which implies

$$\int_{\Omega} e^{2u_1^0+2v_{1,n}} dx + \int_{\Omega} e^{2u_2^0+2v_{2,n}} dx \leq C. \tag{4.85}$$

Here and in what follows we use  $C$  to denote a generic positive constant independent of  $n$ . Since  $v_{i,n} \in W^{1,2}(\Omega)$ , we have the following decomposition

$$v_{i,n} = w_{i,n} + c_{i,n}, \quad w_{i,n} \in \dot{W}^{1,2}(\Omega), \quad c_{i,n} \in \mathbb{R}.$$

From (4.85) and the Jensen inequality we conclude that  $c_{i,n}$  is bounded from above.

Noting that the matrix  $A(N, \kappa)$  and  $A(N, \kappa^{-1})$  defined by (3.8) are both positive definite, we estimate  $I(v_{1,n}, v_{2,n})$  as

$$\begin{aligned} I(v_{1,n}, v_{2,n}) &\geq \frac{\alpha_0(\kappa)}{2} \left( \|\nabla w_{1,n}\|_2^2 + \|\nabla w_{2,n}\|_2^2 \right) \\ &\quad + \frac{\alpha_0(\kappa^{-1})\lambda}{2} \left( \int_{\Omega} \left[ e^{u_1^0+v_{1,n}} - 1 \right]^2 dx + \int_{\Omega} \left[ e^{u_2^0+v_{2,n}} - 1 \right]^2 dx \right) + b_1 c_{1,n} + b_2 c_{2,n}, \end{aligned} \tag{4.86}$$

where  $\alpha_0(\kappa)$  and  $\alpha_0(\kappa^{-1})$  are a positive constants defined by (3.11).

Let  $\varphi_n \equiv Nw_{1,n} + \frac{N}{N-1}w_{2,n}$  and  $\varphi_n^+ \equiv \max\{\varphi_n, 0\}$ . Then, taking  $(\psi_1, \psi_2) = (\varphi_n^+, \varphi_n^+)$  in (4.82), we get

$$\begin{aligned} &\|\nabla \varphi_n^+\|_2^2 + \lambda \int_{\Omega} \left( \sqrt{N-1 + \kappa} e^{u_1^0+v_{1,n}} - \sqrt{\frac{1}{N-1} + \kappa} e^{u_2^0+v_{2,n}} \right)^2 \varphi_n^+ dx \\ &\quad + 2\lambda \left( \sqrt{(\kappa-1)^2 + \frac{N^2\kappa}{N-1}} - [\kappa-1] \right) \int_{\Omega} e^{u_1^0+u_2^0+v_{1,n}+v_{2,n}} \varphi_n^+ dx \\ &\leq C(\|\varphi_n^+\|_2 + \varepsilon_n \|\varphi_n^+\|). \end{aligned} \tag{4.87}$$

Therefore, using the Poincaré inequality in (4.87), we find that

$$\int_{\Omega} e^{u_1^0+u_2^0+v_{1,n}+v_{2,n}} \varphi_n^+ dx \leq C(\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2). \tag{4.88}$$

Let  $(\psi_1, \psi_2) = (w_{1,n}, w_{2,n})$  in (4.82), and noting that the matrix  $A(N, \kappa)$  defined by (3.8) is positive definite, we have

$$\begin{aligned} &(I'(v_{1,n}, v_{2,n}))(w_{1,n}, w_{2,n}) \\ &= \left( N - 1 + \frac{1}{\kappa} \right) \|\nabla w_{1,n}\|_2^2 + \left( \frac{1}{N-1} + \frac{1}{\kappa} \right) \|\nabla w_{2,n}\|_2^2 + 2 \left( 1 - \frac{1}{\kappa} \right) \int_{\Omega} \nabla w_{1,n} \cdot \nabla w_{2,n} dx \end{aligned}$$

$$\begin{aligned}
 & + \lambda \left\{ [N - 1 + \kappa] \int_{\Omega} e^{2u_1^0 + 2v_{1,n}} w_{1,n} dx + \left[ \frac{1}{N - 1} + \kappa \right] \int_{\Omega} e^{2u_2^0 + v_{2,n}} w_{2,n} dx \right. \\
 & + (1 - \kappa) \int_{\Omega} e^{u_1^0 + 2v_{1,n} + u_2^0 + v_{2,n}} (w_{1,n} + w_{2,n}) dx \\
 & \left. - N \int_{\Omega} e^{u_1^0 + v_{1,n}} w_{1,n} dx - \frac{N}{N - 1} \int_{\Omega} e^{2u_2^0 + 2v_{2,n}} w_{2,n} dx \right\} \\
 & \geq \alpha_0(\kappa) \left( \|\nabla w_{1,n}\|_2^2 + \|\nabla w_{2,n}\|_2^2 \right) \\
 & + \lambda \left\{ [N - 1 + \kappa] \int_{\Omega} e^{2u_1^0 + 2v_{1,n}} w_{1,n} dx + \left[ \frac{1}{N - 1} + \kappa \right] \int_{\Omega} e^{2u_2^0 + v_{2,n}} w_{2,n} dx \right. \\
 & + (1 - \kappa) \int_{\Omega} e^{u_1^0 + 2v_{1,n} + u_2^0 + v_{2,n}} (w_{1,n} + w_{2,n}) dx \\
 & \left. - N \int_{\Omega} e^{u_1^0 + v_{1,n}} w_{1,n} dx - \frac{N}{N - 1} \int_{\Omega} e^{2u_2^0 + 2v_{2,n}} w_{2,n} dx \right\}, \tag{4.89}
 \end{aligned}$$

where  $\alpha_0(\kappa)$  is a positive constant defined by (3.11). In view of (4.85), we see that

$$\left| \int_{\Omega} e^{u_i^0 + v_{i,n}} w_{i,n} dx \right| \leq C \|w_{i,n}\|_2, \quad i = 1, 2. \tag{4.90}$$

Since we have shown that  $c_{i,n}$  is bounded from above, there holds

$$\begin{aligned}
 \int_{\Omega} e^{2u_i^0 + 2v_{i,n}} w_{i,n} dx & = \int_{\Omega} e^{2u_i^0 + 2c_{i,n}} (e^{2w_{i,n}} - 1) w_{i,n} dx + \int_{\Omega} e^{2u_i^0 + 2c_{i,n}} w_{i,n} dx \\
 & \geq -C \|w_{i,n}\|_2.
 \end{aligned} \tag{4.91}$$

It easily follows that

$$\begin{aligned}
 & \int_{\Omega} e^{u_1^0 + v_{1,n} + u_2^0 + v_{2,n}} (w_{1,n} + w_{2,n}) dx \\
 & \leq \int_{\Omega} e^{u_1^0 + v_{1,n} + u_2^0 + v_{2,n}} (w_{1,n} + w_{2,n})_+ dx \\
 & = \int_{\{w_{1,n} \leq 0 \leq w_{2,n}\}} e^{u_1^0 + c_{1,n}} (e^{w_{1,n}} - 1) e^{u_2^0 + v_{2,n}} (w_{1,n} + w_{2,n})_+ dx
 \end{aligned}$$

$$\begin{aligned}
 &+ \int_{\{w_{1,n} \leq 0 \leq w_{2,n}\}} e^{u_1^0 + c_{1,n}} e^{u_2^0 + v_{2,n}} (w_{1,n} + w_{2,n})_+ dx \\
 &+ \int_{\{w_{2,n} \leq 0 \leq w_{1,n}\}} e^{u_2^0 + c_{2,n}} (e^{w_{2,n}} - 1) e^{u_1^0 + v_{1,n}} (w_{1,n} + w_{2,n})_+ dx \\
 &+ \int_{\{w_{2,n} \leq 0 \leq w_{1,n}\}} e^{u_2^0 + c_{2,n}} e^{u_1^0 + v_{1,n}} (w_{1,n} + w_{2,n})_+ dx \\
 &+ \int_{\{w_{1,n} > 0\} \cap \{w_{2,n} > 0\}} e^{u_1^0 + v_{1,n} + u_2^0 + v_{2,n}} (w_{1,n} + w_{2,n})_+ dx \\
 &\leq C (\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2) + \int_{\Omega} e^{u_1^0 + v_{1,n} + u_2^0 + v_{2,n}} \varphi_n^+ dx,
 \end{aligned}$$

which together with (4.88) imply

$$\int_{\Omega} e^{u_1^0 + v_{1,n} + u_2^0 + v_{2,n}} (w_{1,n} + w_{2,n}) dx \leq C (\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2). \tag{4.92}$$

Now from (4.89)–(4.92), we see that

$$\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2 \leq C. \tag{4.93}$$

Noting that we have shown that  $\{c_{i,n}\}$  is bounded from above, by (4.93), (4.80) and (4.86), we infer that  $c_{i,n}$  is also bounded from below,  $i = 1, 2$ . Hence, using (4.93) again, we conclude that  $\{v_{i,n}\}$  is uniformly bounded in  $W^{1,2}(\Omega)$ ,  $i = 1, 2$ .

Therefore, up to a subsequence, there exists  $v_i \in W^{1,2}(\Omega)$ , such that  $v_{i,n} \rightharpoonup v_i$  weakly in  $W^{1,2}(\Omega)$ , strongly in  $L^p(\Omega)$  for any  $p \geq 1$ , pointwise a.e. in  $\Omega$ , and  $e^{u_i^0 + v_{i,n}} \rightarrow e^{u_i^0 + v_i}$  in  $L^p(\Omega)$  for any  $p \geq 1$ , as  $n \rightarrow +\infty$ ,  $i = 1, 2$ .

Hence we see that  $(v_1, v_2)$  is a critical point for the functional  $I$ . From the above convergence results we obtain

$$\begin{aligned}
 &\alpha_0(\kappa) \left( \|\nabla(v_{1,n} - v_1)\|_2^2 + \|\nabla(v_{2,n} - v_2)\|_2^2 \right) \\
 &\leq \left( N - 1 + \frac{1}{\kappa} \right) \|\nabla(v_{1,n} - v_1)\|_2^2 + \left( \frac{1}{N - 1} + \frac{1}{\kappa} \right) \|\nabla(v_{2,n} - v_2)\|_2^2 \\
 &\quad + 2 \left( 1 - \frac{1}{\kappa} \right) \int_{\Omega} \nabla(v_{1,n} - v_1) \cdot \nabla(v_{2,n} - v_2) dx \\
 &= (I'(v_{1,n}, v_{2,n}) - I'(v_1, v_2))(v_{1,n} - v_1, v_{2,n} - v_2) + o(1) \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \tag{4.94}
 \end{aligned}$$

where  $\alpha_0(\kappa)$  is a positive constant defined by (3.11).

Then we conclude from the estimate (4.94) that  $(v_{1,n}, v_{2,n}) \rightarrow (v_1, v_2)$  strongly in  $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$  as  $n \rightarrow +\infty$ . Then Lemma 4.8 follows.  $\square$

To find a second solution of (4.3)–(4.4), noting that we have proved that  $(v_{1,\lambda}, v_{2,\lambda})$  given in (4.73) is a local minimizer of the functional  $I$ , we only need to consider the following two cases.

Case 1.  $(v_{1,\lambda}, v_{2,\lambda})$  is a degenerate minimum. In other words, for any sufficiently small  $\delta > 0$ ,

$$\inf_{\|v_1 - v_{1,\lambda}\| + \|v_2 - v_{2,\lambda}\| = \delta} I(v_1, v_2) = I(v_{1,\lambda}, v_{2,\lambda}).$$

Thus, we conclude from Corollary 1.6 of [22] that there is a one parameter family of degenerate local minimizer of the functional  $I$ . Automatically, a second solution of (4.3)–(4.4) for this case can be obtained.

Case 2.  $(v_{1,\lambda}, v_{2,\lambda})$  is a strict local minimum. That is, for any sufficiently small  $\delta > 0$ , there holds

$$I(v_{1,\lambda}, v_{2,\lambda}) < \inf_{\|v_1 - v_{1,\lambda}\| + \|v_2 - v_{2,\lambda}\| = \delta} I(v_1, v_2) \equiv \gamma_0. \tag{4.95}$$

We observe that

$$I(v_{1,\lambda} - \xi, v_{2,\lambda} - \xi) \rightarrow -\infty \quad \text{as} \quad \xi \rightarrow +\infty.$$

Hence, for a sufficiently large  $\xi_0 > 1$ , letting

$$\tilde{v}_i = v_{i,\lambda} - \xi_0, \quad i = 1, 2,$$

we can obtain

$$\|\tilde{v}_1 - v_{1,\lambda}\| + \|\tilde{v}_2 - v_{2,\lambda}\| > \delta \tag{4.96}$$

and

$$I(\tilde{v}_1, \tilde{v}_2) < I(v_{1,\lambda}, v_{2,\lambda}) - 1. \tag{4.97}$$

Now we introduce the paths

$$\mathcal{P} = \left\{ \Gamma(t) \mid \Gamma \in C([0, 1], W^{1,2}(\Omega) \times W^{1,2}(\Omega)), \quad \Gamma(0) = (v_{1,\lambda}, v_{2,\lambda}), \quad \Gamma(1) = (\tilde{v}_1, \tilde{v}_2) \right\}$$

and define

$$\theta_0 = \inf_{\Gamma \in \mathcal{P}} \sup_{t \in [0,1]} I(\Gamma(t)).$$

Then we obtain

$$\theta_0 > I(\tilde{v}_1, \tilde{v}_2). \tag{4.98}$$

At last, noting Lemma 4.8, (4.95)–(4.97), we can use the mountain-pass theorem of Ambrosetti–Rabinowitz [5] to conclude that  $\theta_0$  is also a critical value of the functional  $I$ , which gives another critical point of  $I$ . In view of (4.98), we obtain a second solution of (4.3)–(4.4),

which is different from  $(v_{1,\lambda}, v_{2,\lambda})$  defined by (4.73). Then the proof of Theorem 2.2 is complete.  $\square$

### 4.3. Quantized fluxes over $\Omega$

In this short subsection we calculate the quantized fluxes stated in Theorem 2.3 for the doubly periodic domain case. In fact, as in the planar case we obtain from (2.16)–(2.17) that

$$F_{12}^0 = -\frac{1}{\sqrt{2N}} \Delta \left( [N-1] \ln |\phi|^2 + |\ln \phi_N|^2 \right),$$

$$F_{12}^{N^2-1} = -\sqrt{\frac{N-1}{2N}} \Delta \left( \ln |\phi|^2 - \ln \phi_N^2 \right),$$

which gives

$$\mathcal{F}^{U(1)} = -\frac{1}{\sqrt{2N}} \int_{\Omega} \Delta \left( [N-1] \ln |\phi|^2 + |\ln \phi_N|^2 \right) dx,$$

$$\mathcal{F}^{SU(N)} = -\sqrt{\frac{N-1}{2N}} \int_{\Omega} \Delta \left( \ln |\phi|^2 - |\ln \phi_N|^2 \right) dx.$$

Then, using Eqs. (2.20)–(2.21) and a direct integration, we get the desired quantized fluxes (2.32)–(2.33).  $\square$

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