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# Approximations of random dispersal operators/equations by nonlocal dispersal operators/equations ☆

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**Abstract**

This paper concerns the approximations of random dispersal operators/equations by nonlocal dispersal operators/equations. It first proves that the solutions of properly rescaled nonlocal dispersal initial–boundary value problems converge to the solutions of the corresponding random dispersal initial–boundary value problems. Next, it proves that the principal spectrum points of nonlocal dispersal operators with properly rescaled kernels converge to the principal eigenvalues of the corresponding random dispersal operators. Finally, it proves that the unique positive time periodic solutions of nonlocal dispersal KPP equations with properly rescaled kernels converge to the unique positive time periodic solutions of the corresponding random dispersal KPP equations.

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## 1. Introduction

Both random dispersal evolution equations (or reaction diffusion equations) and nonlocal dispersal evolution equations (or differential integral equations) are widely used to model diffusive systems in applied sciences. Random dispersal equations of the form

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + F(t,x,u), & x \in D, \\ B_{r,b}u(t,x) = 0, & x \in \partial D \ (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N), \end{cases} \tag{1.1}$$

are usually used to model diffusive systems which exhibit local internal interactions (i.e. the movements of organisms in the systems occur randomly between adjacent spatial locations) and have been extensively studied (see [1–3,6,19,20,24,29,32,42,46], etc.). In (1.1), the domain $D$ is either a bounded smooth domain in $\mathbb{R}^N$ or $D = \mathbb{R}^N$. When $D$ is a bounded domain, either $B_{r,b}u = B_{r,D}u := u$ (in such case, $B_{r,D}u = 0$ on $\partial D$ represents homogeneous Dirichlet boundary condition), or $B_{r,b}u = B_{r,N}u := \frac{\partial u}{\partial \mathbf{n}}$ (in such case, $B_{r,N}u = 0$ on $\partial D$ represents homogeneous Neumann boundary condition), and when $D = \mathbb{R}^N$, it is assumed that $F(t,x,u)$ is periodic in $x_j$ with period $p_j$ and $B_{r,b}u = B_{r,P}u := u(t, x + p_j\mathbf{e_j}) - u(t,x)$ with $\mathbf{e_j} = (\delta_{1j}, \delta_{2j}, \cdots, \delta_{Nj})$ ($\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$) (in such case, $B_{r,P}u = 0$ in $\mathbb{R}^N$ represents periodic boundary condition).

Many applied systems exhibit nonlocal internal interaction (i.e. the movements of organisms in the systems occur between non-adjacent spatial locations). Nonlocal dispersal evolution equations of the form

$$\begin{cases} \partial_t u(t,x) = \nu \int_{D \cup D_b} k(y-x)[u(t,y) - u(t,x)]dy + F(t,x,u), & x \in \bar{D}, \\ B_{n,b}u(t,x) = 0, & x \in D_b \text{ if } D_b \neq \emptyset, \end{cases} \tag{1.2}$$

are often used to model diffusive systems which exhibit nonlocal internal interactions and have been recently studied by many authors (see [4,7–9,12–14,18,21,26,28,30,31,44], etc.). In (1.2), $D$ is either a smooth bounded domain of $\mathbb{R}^N$ or $D = \mathbb{R}^N$; $\nu$ is the dispersal rate; the kernel function $k(\cdot)$ is a smooth and nonnegative function with compact support (the size of the support reflects the dispersal distance) and $\int_{\mathbb{R}^N} k(z)dz = 1$. When $D$ is bounded, either $D_b = D_D := \mathbb{R}^N \backslash \bar{D}$ and $B_{n,b}u = B_{n,D} := u$ (in such case, $u = 0$ on $\mathbb{R}^N \backslash \bar{D}$ represents homogeneous Dirichlet type boundary condition), or $D_b = D_N := \emptyset$ (in such case, nonlocal diffusion takes place only in $\bar{D}$ and hence $D_N = \emptyset$ represents homogeneous Neumann type boundary condition); when $D = \mathbb{R}^N$, it is assumed that $F(t, x + p_j\mathbf{e_j}, u) = F(t,x,u)$, $D_b = D_P := \mathbb{R}^N$, and $B_{n,b}u = B_{n,P}u := u(t, x + p_j\mathbf{e_j}) - u(t,x)$ (hence $B_{n,P}u = 0$ on $\mathbb{R}^N$ represents periodic boundary condition).

Observe that (1.2) with $D_b = D_D$ and $B_{n,b}u = B_{n,D}u$ can be rewritten as

$$\partial_t u(t,x) = \nu \left[ \int_D k(y-x)u(t,y)dy - u(t,x) \right] + F(t,x,u), \quad x \in \bar{D}; \tag{1.3}$$

that (1.2) with $D_b = D_N$ reduces to

$$\partial_t u(t,x) = \nu \int_D k(y-x)\left[u(t,y) - u(t,x)\right]dy + F(t,x,u), \quad x \in \bar{D}; \tag{1.4}$$

and that (1.2) with $D = D_P$, $F(t,x,u)$ being periodic in $x_j$ with period $p_j$, and $B_{n,b}u = B_{n,P}u$ can be written as

$$\begin{cases} \partial_t u(t,x) = \nu \int_{\mathbb{R}^N} k(y-x)\left[u(t,y)-u(t,x)\right]dy + F(t,x,u), & x \in \mathbb{R}^N, \\ u(t,x) = u(t, x+p_j\mathbf{e_j}), & x \in \mathbb{R}^N \end{cases} \tag{1.5}$$

$(j = 1, 2, \cdots N)$.

A huge amount of research has been carried out toward various dynamical aspects of random dispersal evolution equations of the form (1.1). There are also many research works toward various dynamical aspects of nonlocal dispersal evolution equations of the form (1.2). It has been seen that random dispersal evolution equations with Dirichlet, or Neumann, or period boundary condition and nonlocal dispersal evolution equations with the corresponding boundary condition share many similar properties. For example, a comparison principle holds for both equations. There are also many differences between these two types of dispersal evolution equations. For example, solutions of random dispersal evolution equations have smoothness and certain compactness properties, but solutions of nonlocal dispersal evolution equations do not have such properties. Nevertheless, it is expected that nonlocal dispersal evolution equations with Dirichlet, or Neumann, or periodic boundary condition and small dispersal distance possess similar dynamical behaviors as those of random dispersal evolution equations with the corresponding boundary condition and that certain dynamics of random dispersal evolution equations with Dirichlet, or Neumann, or periodic boundary condition can be approximated by the dynamics of nonlocal dispersal evolution equations with the corresponding boundary condition and properly rescaled kernels. It is of great theoretical and practical importance to investigate whether such naturally expected properties actually hold or not.

The objective of the current paper is to investigate how the dynamics of random dispersal operators/equations can be approximated by those of nonlocal dispersal operators/equations from three different perspectives, that is, from initial–boundary value problem point of view, from spectral problem point of view, and from asymptotic behavior point of view. To this end, we assume that $k(\cdot)$ is of the form,

$$k(z) = k_\delta(z) := \frac{1}{\delta^N} k_0\left(\frac{z}{\delta}\right) \tag{1.6}$$

for some $k_0(\cdot)$ satisfying that $k_0(\cdot)$ is a smooth, nonnegative, and symmetric (in the sense that $k_0(z) = k_0(z')$ whenever $|z| = |z'|$) function supported on the unit ball $B(0,1)$ and $\int_{\mathbb{R}^N} k_0(z)dz = 1$, where $\delta(>0)$ is called the dispersal distance. We also assume that

$$\nu = \nu_\delta := \frac{C}{\delta^2}, \tag{1.7}$$

where $C = \left(\frac{1}{2}\int_{\mathbb{R}^N} k_0(z) z_N^2 dz\right)^{-1}$. Throughout the rest of this paper, we will distinguish the three boundary conditions by $i = 1, 2, 3$. Let

$$X_1 = X_2 = \{u(\cdot) \in C(\bar{D}, \mathbb{R})\}$$

with $\|u\|_{X_i} = \max_{x\in\bar{D}} |u(x)| (i = 1, 2)$,

$$X_3 = \{u \in C(\mathbb{R}^N, \mathbb{R}) | u(x + p_j\mathbf{e_j}) = u(x)\},$$

with $\|u\|_{X_3} = \max_{x\in\mathbb{R}^N} |u(x)|$. Let

$$X_i^+ = \{u \in X_i \mid u(x) \geq 0\}$$

($i = 1, 2, 3$). For $u^1(x), u^2(x) \in X_i$, we define

$$u^1 \leq u^2 (u^1 \geq u^2) \text{ if } u^2 - u^1 \in X_i^+ (u^1 - u^2 \in X_i^+)$$

($i = 1, 2, 3$). Note that $X_1 = X_2$ and the introduction of $X_2$ is for convenience.

First, we investigate the approximations of solutions to the initial–boundary value problem associated to (1.1), that is,

$$\begin{cases} \partial_t u(t,x) = \Delta u + F(t,x,u), & x \in D, \\ B_{r,b}(t,x)u = 0, & x \in \partial D \quad (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N), \\ u(s,x) = u_0(x), & x \in \bar{D} \end{cases} \tag{1.8}$$

by solutions to the initial–boundary value problem associated to (1.2) with $k(\cdot) = k_\delta(\cdot)$ and $\nu = \nu_\delta$, that is,

$$\begin{cases} \partial_t u(t,x) = \nu_\delta \int_{D\cup D_b} k_\delta(y-x)[u(t,y) - u(t,x)]dy + F(t,x,u), & x \in \bar{D}, \\ B_{n,b}u(t,x) = 0, & x \in D_b \text{ if } D_b \neq \emptyset, \\ u(s,x) = u_0(x), & x \in \bar{D}, \end{cases} \tag{1.9}$$

where $B_{r,b} = B_{r,D}$ (resp. $B_{n,b} = B_{n,D}$ and $D_b = D_D$), or $B_{r,b} = B_{r,N}$ (resp. $D_b = D_N$ $(= \emptyset)$), or $B_{r,b} = B_{r,P}$ (resp. $B_{n,b} = B_{n,P}$ and $D_b = D_P$). In the rest of this paper, we assume

**(H0)** *$D \subset \mathbb{R}^N$ is either a bounded $C^{2+\alpha}$ domain for some $0 < \alpha < 1$ or $D = \mathbb{R}^N$; $k_\delta(\cdot)$ is as in (1.6) and $\nu_\delta$ is as in (1.7); $F(t,x,u)$ is $C^1$ in $t \in \mathbb{R}$ and $C^3$ in $(x,u) \in \mathbb{R}^N \times \mathbb{R}$, and when $D = \mathbb{R}^N$, $F$ is periodic in $x_j$ with period $p_j$, that is, $F(t, x + p_j\mathbf{e_j}, u) = F(t,x,u)$ for $j = 1, 2, \cdots, N$.*

Note that, by general semigroup theory (see [22,35]), for any $s \in \mathbb{R}$ and any $u_0 \in X_i \cap C^1(\bar{D})$ with $B_{r,b}u_0 = 0$ on $\partial D$, (1.8) with $b = D$ if $i = 1$, $b = N$ if $i = 2$, and $b = P$ if $i = 3$ has a unique (local) solution, denoted by $u_i(t,x;s,u_0)$. Similarly, for any $s \in \mathbb{R}$ and any $u_0 \in X_i$, (1.9) with $b = D$ if $i = 1$, $b = N$ if $i = 2$, and $b = P$ if $i = 3$ has a unique (local) solution, denoted by $u_i^\delta(t,x;s,u_0)$.

Among others, we prove

**Theorem A.** *Assume that for given $1 \leq i \leq 3$, $\delta_0 > 0$, $s \in \mathbb{R}$, $T > 0$, and $u_0 \in X_i \cap C^3(\bar{D})$ with $B_{r,b}u_0 = 0$ if $D$ is bounded ($b = D$ if $i = 1$ and $b = N$ if $i = 2$), $u_i(t,x;s,u_0)$ and $u_i^\delta(t,x;s,u_0)$ exist on $[s, s+T]$ for all $0 < \delta \leq \delta_0$. Assume also that $\sup_{s\leq t\leq s+T, x\in\bar{D}, 0<\delta\leq\delta_0} |u_i(t,x;s,u_0)| < \infty$. Then,*

$$\lim_{\delta\to 0} \sup_{t\in[s,s+T]} \|u_i^\delta(t,\cdot;s,u_0) - u_i(t,\cdot;s,u_0)\|_{X_i} = 0.$$

It should be pointed out that Theorem A is the basis for the study of approximations of various dynamics of random dispersal evolution equations by those of nonlocal dispersal evolution equations. It should also be pointed out that when $F(t,x,u) \equiv 0$ in (1.8) and (1.9), similar results to Theorem A have been proved in [10] and [11] for the Dirichlet and Neumann boundary condition cases, respectively.

Secondly, we investigate the principal eigenvalues of time periodic random dispersal eigenvalue problems of the form

$$\begin{cases} -\partial_t u + \Delta u + a(t,x)u = \lambda u, & x \in D, \\ B_{r,b} u = 0, & x \in \partial D \ (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N), \\ u(t+T,x) = u(t,x), & x \in D, \end{cases} \tag{1.10}$$

and their nonlocal counterparts of the form

$$\begin{cases} -\partial_t u + \nu_\delta \int_{D \cup D_b} k_\delta(y-x)\big[u(t,y) - u(t,x)\big]\,dy + a(t,x)u = \lambda u, & x \in \bar{D}, \\ B_{n,b} u = 0, & x \in D_b \text{ if } D_b \neq \emptyset, \\ u(t+T,x) = u(t,x), & x \in \bar{D}, \end{cases} \tag{1.11}$$

where $a(t+T,x) = a(t,x)$, and when $D = \mathbb{R}^N$, $a(t+T, x+p_j\mathbf{e_j}) = a(t,x)$ for $j = 1,2,\cdots,N$, and $B_{r,b} = B_{r,D}$ (resp. $B_{n,b} = B_{n,D}$ and $D_b = D_D$), or $B_{r,b} = B_{r,N}$ (resp. $D_b = D_N$ $(= \emptyset)$) or $B_{r,b} = B_{r,P}$ (resp. $B_{n,b} = B_{n,P}$ and $D_b = D_P$). We assume that $a(t,x)$ is a $C^1$ function in $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.

The eigenvalue problems of (1.10), in particular, their associated principal eigenvalue problems, are extensively studied and quite well understood (see [15–17,23,25,27,34,38], etc.). For example, with any one of the three boundary conditions, it is known that the largest real part, denoted by $\lambda^r(a)$, of the spectrum set of (1.10) is an isolated algebraically simple eigenvalue with a positive eigenfunction, and for any other $\lambda$ in the spectrum set of (1.10), $\mathrm{Re}\lambda \le \lambda^r(a)$ ($\lambda^r(a)$ is called the *principal eigenvalue* of (1.10) in literature).

The eigenvalue problems (1.11) have also been studied recently by many authors (see [5,12,27,36,38–41], etc.). Let $\lambda^\delta(a)$ be the largest real part of the spectrum set of (1.11) with any one of the three boundary conditions. $\lambda^\delta(a)$ is called the *principal spectrum point* of (1.11). $\lambda^\delta(a)$ is also called the *principal eigenvalue* of (1.11), if it is an isolated algebraically simple eigenvalue with a positive eigenfunction (see Definition 3.1 for detail). Note that $\lambda^\delta(a)$ may not be an eigenvalue of (1.11) (see [12,39] for examples). Hence the principal eigenvalue of (1.11) may not exist. In [41], the authors of the current paper studied the dependence of principal spectrum points or principal eigenvalues (if exist) of nonlocal dispersal operators on underlying parameters ($\delta$, $a(\cdot)$, and $\nu$) in a spatially heterogeneous but temporally homogeneous case. However, the understanding is still little to many interesting questions regarding the principal spectrum points or principal eigenvalues (if exist) of (1.11). In this paper, we show that the principal eigenvalue of (1.10) can be approximated by the principal spectrum point of (1.11). In fact, we show

**Theorem B.** $\lim_{\delta \to 0} \lambda^\delta(a) = \lambda^r(a)$.

We remark that Theorem B is another basis for the study of approximations of various dynamics of random dispersal evolution equations by those of nonlocal dispersal evolution equations. We also remark that some necessary and sufficient conditions are provided in [36] and [37] for

$\lambda_\delta(a)$ to be the principal eigenvalue of (1.11). Among other, it is proved in [36, Theorem A] and [37, Theorem 3.1] that $\lambda^\delta(a)$ is the principal eigenvalue of (1.11) if and only if

$$\lambda^\delta(a) > \max_{x\in\bar{D}} \left\{ -\frac{C}{\delta^2} + \frac{1}{T}\int_0^T a(t,x)dt \right\}.$$

This together with Theorem B implies the following remark.

**Remark 1.1.** $\lambda^\delta(a)$ is the principal eigenvalue of (1.11), provided $\delta \ll 1$.

Thirdly, we explore the asymptotic dynamics of the following time periodic dispersal evolution equations,

$$\begin{cases} \partial_t u = \Delta u + uf(t,x,u), & x \in D, \\ B_{r,b}u = 0, & x \in \partial D \ \ (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N), \end{cases} \tag{1.12}$$

and

$$\begin{cases} \partial_t u = \nu_\delta \int_{D\cup D_b} k_\delta(y-x)[u(t,y)-u(t,x)]dy + uf(t,x,u), & x \in \bar{D}, \\ B_{n,b}u = 0, & x \in D_b \ \text{ if } D_b \neq \emptyset, \end{cases} \tag{1.13}$$

where $D$ is as in **(H0)**. In the rest of this paper, we assume that

**(H1)** *$f$ is $C^1$ in $t \in \mathbb{R}$ and $C^3$ in $(x,u) \in \mathbb{R}^N \times \mathbb{R}$; $f(t,x,u) < 0$ for $u \gg 1$ and $\partial_u f(t,x,u) < 0$ for $u \geq 0$;* $f(t+T,x,u) = f(t,x,u)$; and when $D = \mathbb{R}^N$, $f(t+T,x,u) = f(t,x+p_j\mathbf{e_j},u) = f(t,x,u)$ for $j = 1,2,\cdots,N$.

**(H2)** *For (1.12), $\lambda^r(f(\cdot,\cdot,0)) > 0$, where $\lambda^r(f(\cdot,\cdot,0))$ is the principle eigenvalue of (1.10) with* $a(t,x) = f(t,x,0)$.

**(H2)$_\delta$** *For (1.13), $\lambda^\delta(f(\cdot,\cdot,0)) > 0$, where $\lambda^\delta(f(\cdot,\cdot,0))$ is the principle spectrum point of (1.11) with $a(t,x) = f(t,x,0)$.*

Equations (1.12) and (1.13) are widely used to model population dynamics of species exhibiting random interactions and nonlocal interactions, respectively (see [4,14,33], etc. for (1.12) and [36] for (1.13)). Thanks to the pioneering works of Fisher [20] and Kolmogorov et al. [29] on the following special case of (1.12),

$$\partial_t u = u_{xx} + u(1-u), \quad x \in \mathbb{R},$$

(1.12) and (1.13) are referred to as Fisher type or KPP type equations.

The dynamics of (1.12) and (1.13) have been studied in many papers (see [24,33,45] and references therein for (1.12), and [36] and references therein for (1.13)). With conditions (H1) and (H2), it is proved that (1.12) has exactly two nonnegative time periodic solutions, one is $u \equiv 0$ which is unstable and the other one, denoted by $u^*(t,x)$, is asymptotically stable and strictly positive (see [45, Theorem 3.1], see also [33, Theorems 1.1, 1.3]). Similar results for (1.13) under the assumptions (H1) and $(H2)_\delta$ are proved in [36, Theorem E]. We denote the strictly positive time periodic solution of (1.13) by $u^*_\delta(t,x)$.

Note that, by Theorem B and Remark 1.1, (H2) implies $(H2)_\delta$ when $0<\delta\ll 1$. Hence, we only assume (H2) in the following theorem. In this paper, we show that

**Theorem C.** *If (H1) and (H2) hold, then for any $\epsilon>0$, there exists $\delta_0>0$, such that for all $0<\delta<\delta_0$, we have*

$$\sup_{t\in[0,T]}\|u^*_\delta(t,\cdot)-u^*(t,\cdot)\|_{C(\bar D,\mathbb{R})}\le\epsilon.$$

Theorems A–C above show that many important dynamics of random dispersal equations can be approximated by the corresponding dynamics of nonlocal dispersal equations, which is of both great theoretical and practical importance.

The rest of the paper is organized as follows. In Section 2, we explore the approximation of solutions of random dispersal evolution equations by the solutions of nonlocal dispersal evolution equations and prove Theorem A. In Section 3, we investigate the approximation of principal eigenvalues of time periodic random dispersal operators by the principal spectrum points of time periodic nonlocal dispersal operators and prove Theorem B. We study in Section 4 the approximation of the asymptotic dynamics of time periodic KPP equations with random dispersal by the asymptotic dynamics of time periodic KPP equations with nonlocal dispersal and prove Theorem C.

## 2. Approximation of initial–boundary value problems of random dispersal equations by nonlocal dispersal equations

In this section, we explore the approximation of solutions to (1.8) by the solutions to (1.9). We first present some comparison principle for (1.8) and (1.9). Then we prove Theorem A. Though the ideas of the proofs of Theorem A for different types of boundary conditions are the same, different techniques are needed for different boundary conditions. We hence give proofs of Theorem A for different boundary conditions in different subsections.

### 2.1. Comparison principle for random and nonlocal dispersal evolution equations

In this subsection, we present a comparison principle for random and nonlocal evolution equations, which will be applied in the proof of Theorem A in this section as well as in the proofs of Theorems B and C in Sections 3 and 4.

**Definition 2.1** *(Super- and sub-solutions).* A continuous function $u(t,x)$ on $[s,s+T)\times\mathbb{R}^N$ is called a super-solution (sub-solution) of (1.9) on $(s,s+T)$ if for any $x\in\bar D$, $u(t,x)$ is differentiable on $(s,s+T)$ and satisfies that

$$\begin{cases}\partial_t u(t,x)\ge(\le)\nu_\delta\int_{D\cup D_b}k_\delta(y-x)[u(t,y)-u(t,x)]dy+F(t,x,u), & x\in\bar D,\\ B_{n,b}u(t,x)\ge(\le)0, & x\in D_b \text{ if } D_b\ne\emptyset,\\ u(s,x)\ge(\le)u_0(x), & x\in\bar D,\end{cases}$$

when $b = D$ or $N$, or that

$$\begin{cases} \partial_t u(t,x) \geq (\leq) \nu_\delta \int_{\mathbb{R}^N} k_\delta(y-x)[u(t,y)-u(t,x)]dy + F(t,x,u), & x \in \mathbb{R}^N, \\ B_{n,b}u(t,x) = 0, & x \in \mathbb{R}^N, \\ u(s,x) \geq (\leq) u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

when $b = P$.

Super-solutions and sub-solutions of (1.8) on $(s, s+T)$ are defined in an analogous way.

**Proposition 2.1** *(Comparison principle).*

(1) *Suppose that $u^-(t,x)$ and $u^+(t,x)$ are sub-solution and super-solution of* (1.8) *on $(s, s+T)$, respectively, then*

$$u^-(t,x) \leq u^+(t,x) \quad \forall\, t \in [s, s+T),\ x \in \bar{D}.$$

(2) *Suppose that $u^-(t,x)$ and $u^+(t,x)$ are sub-solution and super-solution of* (1.9) *on $(s, s+T)$, respectively, then*

$$u^-(t,x) \leq u^+(t,x) \quad \forall\, t \in [s, s+T),\ x \in \bar{D}.$$

**Proof.** (1) It follows from comparison principle for parabolic equations.

(2) It follows from [36, Proposition 3.1]. □

### 2.2. *Proof of Theorem A in the Dirichlet boundary condition case*

In this subsection, we prove Theorem A in the Dirichlet boundary case. Throughout this subsection, we assume (H0), and $B_{r,b}u = B_{r,D}u$ in (1.8), and $D_b = D_D$ $(= \mathbb{R}^N \backslash \bar{D})$ and $B_{n,b}u = B_{n,D}u$ in (1.9). Note that $D \cup D_b = \mathbb{R}^N$ in this case. Without loss of generality, we assume $s = 0$.

**Proof of Theorem A in the Dirichlet boundary condition case.** Let $u_0 \in C^3(\bar{D})$ with $u_0(x) = 0$ for $x \in \partial D$. Let $u_1^\delta(t,x)$ be the solution of (1.9) with $s = 0$ and $u_1(t,x)$ be the solution of (1.8) with $s = 0$. Suppose that $u_1(t,x)$ and $u_1^\delta(t,x)$ exist on $[0,T]$. By regularity of solutions for parabolic equations, $u_1 \in C^{1+\frac{\alpha}{2},2+\alpha}((0,T] \times \bar{D}) \cap C^{0,2+\alpha}([0,T] \times \bar{D})$. Let $\tilde{u}_1$ be an extension of $u_1$ to $[0,T] \times \mathbb{R}^N$ satisfying that $\tilde{u}_1 \in C^{0,2+\alpha}([0,T] \times \mathbb{R}^N)$. Define

$$L_\delta(z)(t,x) = \nu_\delta \int_{\mathbb{R}^N} k_\delta(y-x)[z(t,y)-z(t,x)]dy.$$

Let $G(t,x) = \tilde{u}_1(t,x)$ for $(t,x) \in [0,T] \times \mathbb{R}^N \backslash \bar{D}$. Then $\tilde{u}_1$ verifies

$$\begin{cases} \partial_t \tilde{u}_1(t,x) = L_\delta(\tilde{u}_1)(t,x) + F_\delta(t,x) + F(t,x,\tilde{u}_1(t,x)), & x \in \bar{D}, \quad t \in (0,T], \\ \tilde{u}_1(t,x) = G(t,x), & x \in \mathbb{R}^N \backslash \bar{D}, t \in [0,T], \\ \tilde{u}_1(0,x) = u_0(x), & x \in \bar{D}, \end{cases}$$

where

$$F_\delta(t,x) = \Delta\tilde{u}_1(t,x) - L_\delta(\tilde{u}_1)(t,x)$$
$$= \Delta\tilde{u}_1(t,x) - \nu_\delta \int\limits_{\mathbb{R}^N} k_\delta(y-x)(\tilde{u}_1(t,y) - \tilde{u}_1(t,x))dy.$$

Let $w_1^\delta = \tilde{u}_1 - u_1^\delta$. We then have

$$\begin{cases} \partial_t w_1^\delta(t,x) = L_\delta(w_1^\delta)(t,x) + F_\delta(t,x) + a_1^\delta(t,x)w_1^\delta(t,x), & x \in \bar{D}, \quad t \in (0,T], \\ w_1^\delta(t,x) = G(t,x), & x \in \mathbb{R}^N \backslash \bar{D},\ t \in [0,T], \\ w_1^\delta(0,x) = 0, & x \in \bar{D}, \end{cases} \tag{2.1}$$

where $a_1^\delta(t,x) = \int_0^1 F_u[t,x,u_1^\delta(t,x) + \theta(\tilde{u}_1(t,x) - u_1^\delta(t,x))]d\theta$.

We claim that

$$\begin{cases} \sup_{t\in[0,T]} \|F_\delta(t,\cdot)\|_{X_1} = O(\delta^\alpha), \\ \sup_{t\in[0,T],x\in\mathbb{R}^N\backslash\bar{D},\mathrm{dist}(x,\partial D)\le\delta} |G(t,x)| = O(\delta). \end{cases} \tag{2.2}$$

In fact,

$$\Delta\tilde{u}_1(t,x) - \nu_\delta \int\limits_{\mathbb{R}^N} k_\delta(y-x)(\tilde{u}_1(t,y) - \tilde{u}_1(t,x))dy$$
$$= \Delta\tilde{u}_1(t,x) - \nu_\delta \int\limits_{\mathbb{R}^N} \frac{1}{\delta^N} k_0\left(\frac{y-x}{\delta}\right)(\tilde{u}_1(t,y) - \tilde{u}_1(t,x))dy$$
$$= \Delta\tilde{u}_1(t,x) - \nu_\delta \int\limits_{\mathbb{R}^N} k_0(z)(\tilde{u}_1(t,x+\delta z) - \tilde{u}_1(t,x))dz$$
$$= \Delta\tilde{u}_1(t,x) - \nu_\delta \int\limits_{\mathbb{R}^N} k_0(z)\left[\frac{\delta^2 z_N^2}{2!}\Delta\tilde{u}_1(t,x) + O(\delta^{2+\alpha})\right]dz$$
$$= \Delta\tilde{u}_1(t,x) - \left[\nu_\delta\delta^2 \int\limits_{\mathbb{R}^N} k_0(z)\frac{z_N^2}{2}dz\right]\Delta\tilde{u}_1(t,x) + O(\delta^\alpha)$$
$$= \Delta\tilde{u}_1(t,x) - \Delta\tilde{u}_1(t,x) + O(\delta^\alpha)$$
$$= O(\delta^\alpha) \quad \forall\, x \in \bar{D},$$

and

$$|G(t,x)| = |\tilde{u}_1(t,x)|$$
$$\le \sup_{t\in[0,T],x\in\mathbb{R}^N\backslash D,z\in\partial D,\mathrm{dist}(x,z)\le\delta} |\tilde{u}_1(t,x) - u_1(t,z)|$$
$$= O(\delta) \quad \forall\, x \in \mathbb{R}^N\backslash\bar{D},\ \mathrm{dist}(x,\partial D) \le \delta.$$

Therefore, (2.2) holds.

Next, let $\bar{w}$ be given by

$$\bar{w}(t,x) = e^{At}(K_1\delta^\alpha t) + K_2\delta,$$

where $A = \max_{t\in[0,T],x\in\bar{D},0<\delta\le\delta_0} a_1^\delta(t,x)$. By direct calculation, we have

$$\begin{cases} \partial_t \bar{w}(t,x) = L_\delta(\bar{w}) + a_1^\delta(t,x)\bar{w} + \bar{F}_\delta(t,x) & x\in\bar{D}, \quad t\in(0,T], \\ \bar{w}(t,x) = e^{At}(K_1\delta^\alpha t) + K_2\delta, & x\in\mathbb{R}^N\backslash\bar{D}, \quad t\in[0,T], \\ \bar{w}(0,x) = K_2\delta, & x\in\bar{D}, \end{cases} \tag{2.3}$$

where

$$\bar{F}_\delta(t,x) = e^{At}K_1\delta^\alpha + [A - a_1^\delta(t,x)]e^{At}K_1\delta^\alpha t - a_1^\delta(t,x)K_2\delta.$$

By (2.2), there are $0 < \tilde{\delta}_0 \le \delta_0$ and $K_1, K_2 > 0$ such that

$$\begin{cases} F_\delta(t,x) \le \bar{F}_\delta(t,x), & x\in\bar{D}, \quad t\in[0,T], \\ G(t,x) \le e^{At}(K_1\delta^\alpha t) + K_2\delta, & x\in\mathbb{R}^N\backslash\bar{D}, \ \operatorname{dist}(x,\partial D)\le\delta, \ t\in[0,T], \end{cases} \tag{2.4}$$

when $0<\delta<\tilde{\delta}_0$. By (2.1), (2.3), (2.4), and Proposition 2.1, we obtain

$$w^\delta(t,x) \le \bar{w}(t,x) = e^{At}(K_1\delta^\alpha t) + K_2\delta \quad \forall x\in\bar{D}, \ t\in[0,T] \tag{2.5}$$

for $0<\delta<\tilde{\delta}_0$.

Similarly, let $\underline{w}(t,x) = e^{At}(-K_1\delta^\alpha t) - K_2\delta$. We can prove that for $0<\delta<\tilde{\delta}_0$ (by reducing $\tilde{\delta}_0$ if necessary),

$$w^\delta(t,x) \ge \underline{w}(t,x) = -e^{At}(K_1\delta^\alpha t) - K_2\delta \quad \forall\, x\in\bar{D}, \ t\in[0,T]. \tag{2.6}$$

By (2.5) and (2.6) we have

$$|w^\delta(t,x)| \le e^{At}K_1\delta^\alpha t + K_2\delta \quad \forall\, x\in\bar{D}, \ t\in[0,T],$$

which implies that there is $C(T)>0$ such that for any $0<\delta<\tilde{\delta}_0$,

$$\sup_{t\in[0,T]} \|u_1(\cdot,t) - u_1^\delta(\cdot,t)\|_{X_1} \le C(T)\delta^\alpha.$$

Theorem A in the Dirichlet boundary condition case then follows. □

**Remark 2.1.** If the homogeneous Dirichlet boundary conditions $B_{r,D}u = u = 0$ on $\partial D$ and $B_{n,D}u = u = 0$ on $\mathbb{R}^N\backslash\bar{D}$ are changed to nonhomogeneous Dirichlet boundary conditions $B_{r,D}u = u = g(t,x)$ on $\partial D$ and $B_{n,D}u = u = g(t,x)$ on $\mathbb{R}^N\backslash\bar{D}$, Theorem A also holds, which can be proved by the similar arguments as above.

### 2.3. Proof of Theorem A in the Neumann boundary condition case

In this subsection, we prove Theorem A in the Neumann boundary condition case. Throughout this subsection, we assume (H0), and $B_{r,b}u = B_{r,N}u$ in (1.8), and $D_b = D_N = \emptyset$ in (1.9). Without loss of generality, we assume $s = 0$.

We first introduce two lemmas. To this end, for given $\delta > 0$ and $d_0 > 0$, let $D_\delta = \{z \in D \mid \mathrm{dist}(z, \partial D) < d_0\delta\}$.

**Lemma 2.1.** *Let $\theta \in C^{1+\frac{\alpha}{2},2+\alpha}((0,T] \times \times \mathbb{R}^N) \cap C^{0,2+\alpha}([0,T] \times \mathbb{R}^N)$ and $\frac{\partial\theta}{\partial \mathbf{n}} = h$ on $\partial D$, then for $x \in D_\delta$ and $\delta$ small,*

$$
\begin{aligned}
&\frac{1}{\delta^2}\int_{\mathbb{R}^N\setminus D} k_\delta(y-x)(\theta(t,y)-\theta(t,x))dy \\
&= \frac{1}{\delta}\int_{\mathbb{R}^N\setminus D} k_\delta(y-x)\mathbf{n}(\bar{x}) \cdot \frac{y-x}{\delta} h(\bar{x},t)dy \\
&+ \int_{\mathbb{R}^N\setminus D} k_\delta(y-x) \sum_{|\beta|=2} \frac{D^\beta\theta}{2}(\bar{x},t)\left[\left(\frac{y-\bar{x}}{\delta}\right)^\beta - \left(\frac{x-\bar{x}}{\delta}\right)^\beta\right]dy + O(\delta^\alpha),
\end{aligned}
$$

*where $\bar{x}$ is the orthogonal projection of $x$ on the boundary of $D$ so that $\|\bar{x} - y\| \le 2d_0\delta$ and $\mathbf{n}(\bar{x})$ is the exterior unit normal vector of $\partial D$ at $\bar{x}$.*

**Proof.** See [10, Lemma 3]. □

**Lemma 2.2.** *There exist $K > 0$ and $\bar{\delta} > 0$ such that for $\delta < \bar{\delta}$,*

$$
\int_{\mathbb{R}^N\setminus D} k_\delta(y-x)\mathbf{n}(\bar{x})\frac{y-x}{\delta}dy \ge K \int_{\mathbb{R}^N\setminus D} k_\delta(y-x)dy.
$$

**Proof.** See [10, Lemma 4]. □

**Proof of Theorem A in the Neumann boundary condition case.** Suppose that $u_0 \in C^3(\bar{D})$. Let $u_2^\delta(t,x)$ be the solution to (1.9) with $s = 0$ and $u_2(t,x)$ be the solution to (1.8) with $s = 0$. Assume that $u_2(t,x)$ and $u_2^\delta(t,x)$ exist on $[0,T]$. Then $u_2 \in C^{1+\frac{\alpha}{2},2+\alpha}((0,T] \times \bar{D})$. Let $\tilde{u}_2$ be an extension of $u_2$ to $[0,T] \times \mathbb{R}^N$ satisfying that $\tilde{u}_2 \in C^{1+\frac{\alpha}{2},2+\alpha}((0,T] \times \mathbb{R}^N) \cap C^{0,2+\alpha}([0,T] \times \mathbb{R}^N)$. Define

$$
L_\delta(z)(t,x) = \nu_\delta \int_D k_\delta(y-x)(z(t,y)-z(t,x))dy,
$$

and

$$
\tilde{L}_\delta(z)(t,x) = \nu_\delta \int_{\mathbb{R}^N} k_\delta(y-x)(z(t,y)-z(t,x))dy.
$$

Set $w_2^\delta = u_2^\delta - \tilde{u}_2$. Then

$$\begin{aligned}\partial_t w_2^\delta(t,x) &= \partial_t u_2^\delta(t,x) - \partial_t \tilde{u}_2(t,x)\\ &= [L_\delta(u_2^\delta)(t,x) + F(t,x,u_2^\delta)] - [\Delta \tilde{u}_2(t,x) + F(t,x,\tilde{u}_2)]\\ &= L_\delta(w_2^\delta)(t,x) + a_2^\delta(t,x) w_2^\delta(t,x) + F_\delta(t,x),\end{aligned}$$

where $a_2^\delta(t,x) = \int_0^1 F_u(t,x,\tilde{u}_2(t,x) + \theta(u_2^\delta(t,x) - \tilde{u}_2(t,x)))d\theta$ and

$$F_\delta(t,x) = \tilde{L}_\delta(\tilde{u}_2)(t,x) - \Delta\tilde{u}_2(t,x) - \nu_\delta \int\limits_{\mathbb{R}^N\setminus D} k_\delta(y-x)(\tilde{u}_2(t,y) - \tilde{u}_2(t,x))dy.$$

Hence $w_2^\delta$ verifies

$$\begin{cases} \partial_t w_2^\delta(t,x) = L_\delta(w_2^\delta)(t,x) + a_2^\delta(t,x) w_2^\delta(t,x) + F_\delta(t,x), & x \in \bar{D},\\ w_2^\delta(0,x) = 0, & x \in \bar{D}. \end{cases} \tag{2.7}$$

To prove the theorem, let us pick an auxiliary function $v$ as a solution to

$$\begin{cases} \partial_t v(t,x) = \Delta v(t,x) + a_2^\delta(t,x) v(t,x) + h(t,x), & x \in D, \quad t \in (0,T],\\ \frac{\partial v}{\partial \mathbf{n}}(t,x) = g(t,x), & x \in \partial D,\ t \in [0,T],\\ v(0,x) = v_0(x), & x \in D \end{cases}$$

for some smooth functions $h(t,x) \geq 1$, $g(t,x) \geq 1$ and $v_0(x) \geq 0$ such that $v(t,x)$ has an extension $\tilde{v}(t,x) \in C^{1+\frac{\alpha}{2},2+\alpha}((0,T]\times\mathbb{R}^N) \cap C^{0,2+\alpha}([0,T]\times\mathbb{R}^N)$. Then $v$ is a solution to

$$\begin{cases} \partial_t v(t,x) = L_\delta(v)(t,x) + a_2^\delta(t,x) v(t,x) + H(t,x,\delta), & x \in \bar{D}, t \in (0,T],\\ v(0,x) = v_0(x), & x \in \bar{D}, t \in [0,T], \end{cases} \tag{2.8}$$

where

$$H(t,x,\delta) = \Delta\tilde{v}(t,x) - \tilde{L}_\delta(v)(t,x) + \nu_\delta \int\limits_{\mathbb{R}^N\setminus D} k_\delta(y-x)(\tilde{v}(t,y) - \tilde{v}(t,x))dy + h(t,x).$$

By Lemma 2.1 and the first estimate in (2.2), we have the following estimate for $H(x,t,\delta)$:

$$\begin{aligned} H(t,x,\delta) &= \Delta\tilde{v}(t,x) - \tilde{L}_\delta(v)(t,x) + \frac{C}{\delta^2} \int\limits_{\mathbb{R}^N\setminus D} k_\delta(y-x)(\tilde{v}(t,y) - \tilde{v}(t,x))dy + h(t,x)\\ &\geq \frac{C}{\delta} \int\limits_{\mathbb{R}^N\setminus D} k_\delta(y-x)\mathbf{n}(\bar{x})\frac{y-x}{\delta} g(\bar{x},t)dy\\ &\quad + C \int\limits_{\mathbb{R}^N\setminus D} k_\delta(y-x) \sum_{|\beta|=2} \frac{D^\beta \tilde{v}}{2}(\bar{x},t)\left[\left(\frac{y-\bar{x}}{\delta}\right)^\beta - \left(\frac{x-\bar{x}}{\delta}\right)^\beta\right] dy + 1 - C_1\delta^\alpha \end{aligned}$$

$$\geq \frac{C}{\delta} g(\bar{x},t) \int\limits_{\mathbb{R}^N \setminus D} k_\delta(y-x)\mathbf{n}(\bar{x})\frac{y-x}{\delta}dy - D_1 C \int\limits_{\mathbb{R}^N \setminus D} k_\delta(y-x)dy + \frac{1}{2} \tag{2.9}$$

for some constants $D_1$ and $C_1$ and $\delta$ sufficiently small such that $C_1\delta^\alpha \leq \frac{1}{2}$. Then Lemma 2.2 implies that there exist $C' > 0$ and $\delta'$ such that

$$\frac{1}{\delta} \int\limits_{\mathbb{R}^N \setminus D} k_\delta(y-x)\mathbf{n}(\bar{x})\frac{y-x}{\delta}dy \geq \frac{C'}{\delta} \int\limits_{\mathbb{R}^N \setminus D} k_\delta(y-x)dy,$$

if $\delta < \delta'$. This implies that

$$H(x,t,\delta) \geq \left[\frac{CC'g(\bar{x},t)}{\delta} - D_1\right] \int\limits_{\mathbb{R}^N \setminus D} k_\delta(y-x)dy + \frac{1}{2}, \tag{2.10}$$

if $\delta < \delta'$.

We now estimate $F_\delta(t,x)$. By Lemmas 2.1, 2.2, the first estimate in (2.2), and the fact that $\frac{\partial \tilde{u}_2}{\partial \mathbf{n}} = 0$ on $\partial D$, we have

$$\begin{aligned}
F_\delta(t,x) &= O(\delta^\alpha) + \nu_\delta \int\limits_{\mathbb{R}^N \setminus D} k_\delta(y-x)(\tilde{u}_2(t,y) - \tilde{u}_2(t,x))dy \\
&= O(\delta^\alpha) + C \int\limits_{\mathbb{R}^N \setminus D} k_\delta(y-x) \sum_{|\beta|=2} \frac{D^\beta \theta}{2}(\bar{x},t)\left[\left(\frac{y-\bar{x}}{\delta}\right)^\beta - \left(\frac{x-\bar{x}}{\delta}\right)^\beta\right]dy \\
&\leq C_2\delta^\alpha + D_1 C \int\limits_{\mathbb{R}^N \setminus D} k_\delta(y-x)dy \\
&= C_2\delta^\alpha + D_2 \int\limits_{\mathbb{R}^N \setminus D} k_\delta(y-x)dy
\end{aligned} \tag{2.11}$$

for some $C_2 > 0$ and $D_2 > 0$. Given $\epsilon > 0$, let $v_\epsilon = \epsilon v$. By (2.8), $v_\epsilon$ satisfies

$$\begin{cases} \partial_t v_\epsilon(t,x) - L_\delta(v_\epsilon)(t,x) - a_2^\delta(t,x)v_\epsilon(t,x) = \epsilon H(t,x,\delta), & x \in \bar{D}, \\ v_\epsilon(0,x) = \epsilon v_0(x), & x \in \bar{D}. \end{cases} \tag{2.12}$$

By (2.10) and (2.11), there exist $C_3 > 0$ and $0 < \tilde{\delta}_0 < \delta_0$ such that for $0 < \delta \leq \tilde{\delta}_0$,

$$\begin{aligned}
F_\delta(t,x) &\leq C\delta^\alpha + D_2 \int\limits_{\mathbb{R}^N \setminus D} k_\delta(y-x)dy \\
&\leq \frac{\epsilon}{2} + \frac{C_3\epsilon}{\delta} \int\limits_{\mathbb{R}^N \setminus D} k_\delta(y-x)dy = \epsilon H(x,t,\delta) \quad \forall x \in \bar{D},\ t \in [0,T].
\end{aligned} \tag{2.13}$$

Then by (2.7), (2.12), (2.13), and Proposition 2.1, we have

$$-M\epsilon \le -v_\epsilon \le w_2^\delta \le v_\epsilon \le M\epsilon \quad \forall\, \delta \le \tilde{\delta}_0,$$

where $M = \max_{t\in[0,T],x\in\bar{D}} v(t,x)$. This implies

$$\sup_{t\in[0,T]} \|u_2^\delta(t,\cdot) - u_2(t,\cdot)\|_{X_2} \to 0, \quad \text{as } \delta \to 0.$$

Theorem A in the Neumann boundary condition is thus proved. □

### 2.4. *Proof of Theorem A in the periodic boundary condition case*

In this subsection, we prove Theorem A in the periodic boundary condition case. Throughout this subsection, we assume (H0), $B_{r,b}u = B_{r,P}u$ in (1.8), and $B_{n,b}u = B_{n,P}u$ in (1.9). Without loss of generality again, we assume $s = 0$.

**Proof of Theorem A in the periodic boundary case.** Suppose that $u_0 \in X_3 \cap C^3(\mathbb{R}^N)$. Let $u_3^\delta(t,x)$ be the solution to (1.9) with $s = 0$ and $u_3(t,x)$ be the solution to (1.8) with $s = 0$. Suppose that $u_3(t,x)$ and $u_3^\delta(t,x)$ exist on $[0,T]$. Set $w_3^\delta = u_3^\delta - u_3$. Then $w_3^\delta$ satisfies

$$\begin{cases} \partial_t w_3^\delta(t,x) = \nu_\delta \int_{\mathbb{R}^N} k_\delta(y-x)(w_3^\delta(t,y) - w_3^\delta(t,x))dy + a_3^\delta(t,x)w_3^\delta(t,x) + F_\delta(t,x), \\ \quad x \in \mathbb{R}^N,\ t \in (0,T], \\ w_3^\delta(t,x) = w_3^\delta(t, x + p_j\mathbf{e}_j), \quad x \in \mathbb{R}^N,\ t \in [0,T], \\ w_3^\delta(0,x) = 0, \qquad\qquad\qquad\quad x \in \mathbb{R}^N, \end{cases} \tag{2.14}$$

where $a_3^\delta(t,x) = \int_0^1 F_u(t,x,u_3(t,x) + \theta(u_3^\delta(t,x) - u_3(t,x)))d\theta$ and $F_\delta(t,x) = \nu_\delta \int_{\mathbb{R}^N} k_\delta(y - x)[u_3(t,y) - u_3(t,x)]dy - \Delta u_3$. Let

$$\bar{w}(t,x) = e^{At}(K_1\delta^\alpha t) + K_2\delta,$$

where $A = \max_{t\in[0,T],x\in\mathbb{R}^N,0<\delta\le\delta_0} a_3^\delta(t,x)$. Applying the similar approach as in the Dirichlet boundary condition case, we can show that there are $K_1 > 0$, $K_2 > 0$, and $\delta_0 > 0$ such that for $0 < \delta < \delta_0$,

$$-\bar{w}(t,x) \le w_3^\delta(t,x) \le \bar{w}(t,x) \quad \forall\, x \in \mathbb{R}^N,\ t \in [0,T].$$

Theorem A in the periodic boundary condition case then follows. □

## 3. Approximation of principal eigenvalues of time periodic random dispersal operators by nonlocal dispersal operators

In this section, we investigate the approximation of principal eigenvalues of time periodic random dispersal operators by the principal spectrum points of time periodic nonlocal dispersal operators. We first recall some basic properties of principal eigenvalues of time periodic random dispersal or parabolic operators, and basic properties of principal spectrum points of time periodic nonlocal dispersal operators. We then prove Theorem B.

*3.1. Basic properties*

In this subsection, we present basic properties of principal eigenvalues of time periodic parabolic operators and basic properties of principal spectrum points of time periodic nonlocal dispersal operators.

Let

$$\mathcal{X}_1 = \mathcal{X}_2 = \{u \in C(\mathbb{R} \times \bar{D}, \mathbb{R}) | u(t+T, x) = u(t, x)\}$$

with norm $\|u\|_{\mathcal{X}_i} = \sup_{t\in[0,T]} \|u(t,\cdot)\|_{X_i} (i = 1, 2)$,

$$\mathcal{X}_3 = \{u \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}) | u(t+T, x) = u(t, x + p_j\mathbf{e_j}) = u(t, x)\}$$

with norm $\|u\|_{\mathcal{X}_3} = \sup_{t\in[0,T]} \|u(t,\cdot)\|_{X_3}$, and

$$\mathcal{X}_i^+ = \{u \in \mathcal{X}_i | u(t, x) \geq 0\}$$

$(i = 1, 2, 3)$. And for $u^1, u^2 \in \mathcal{X}_i$, we define

$$u^1 \leq u^2 (u^1 \geq u^2) \text{ if } u^2 - u^1 \in \mathcal{X}_i^+ \ (u_1 - u_2 \in \mathcal{X}_i^+)$$

$(i = 1, 2, 3)$. For given $a(\cdot,\cdot) \in \mathcal{X}_i \cap C^1(\mathbb{R} \times \mathbb{R}^N)$, let $L_i^\delta(a) : \mathcal{D}(L_i^\delta(a)) \subset \mathcal{X}_i \to \mathcal{X}_i$ be defined as follows,

$$\begin{aligned}(L_1^\delta(a)u)(t,x) = & -\partial_t u(t,x) + \nu_\delta \left[ \int_D k_\delta(y-x)u(t,y)dy - u(t,x) \right] \\ & + a(t,x)u(t,x), \quad (t,x) \in \mathbb{R} \times \bar{D}, \end{aligned} \tag{3.1}$$

$$\begin{aligned}(L_2^\delta(a)u)(t,x) = & -\partial_t u(t,x) + \nu_\delta \int_D k_\delta(y-x)[u(t,y) - u(t,x)]dy \\ & + a(t,x)u(t,x), \quad (t,x) \in \mathbb{R} \times \bar{D}, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}(L_3^\delta(a)u)(t,x) = & -\partial_t u(t,x) + \nu_\delta \int_{\mathbb{R}^N} k_\delta(y-x)[u(t,y) - u(t,x)]dy \\ & + a(t,x)u(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N. \end{aligned} \tag{3.3}$$

We first recall the definition of principal spectrum points/eigenvalues of time periodic nonlocal dispersal operators.

**Definition 3.1.** Let

$$\lambda_i^\delta(a) = \sup\{\mathrm{Re}\lambda | \lambda \in \sigma(L_i^\delta(a))\}$$

for $i = 1, 2, 3$.

(1) $\lambda_i^\delta(a)$ is called the principal spectrum point of $L_i^\delta(a)$.
(2) If $\lambda_i^\delta(a)$ is an isolated algebraically simple eigenvalue of $L_i^\delta(a)$ with a positive eigenfunction, then $\lambda_i^\delta(a)$ is called the principal eigenvalue of $L_i^\delta(a)$ or it is said that $L_i^\delta(a)$ has a principal eigenvalue.

For the time periodic random dispersal operators, let $a(\cdot,\cdot) \in \mathcal{X}_i \cap C^1(\mathbb{R}\times\mathbb{R}^N)$, and $L_i(a) : \mathcal{D}(L_i(a)) \subset \mathcal{X}_i \to \mathcal{X}_i$ be defined as follows,

$$(L_i(a)u)(t,x) = -\partial_t u(t,x) + \Delta u(t,x) + a(t,x)u(t,x)$$

for $i = 1, 2, 3$. Note that for $u \in \mathcal{D}(L_1(a))$, $B_{r,D}u = u = 0$ on $\partial D$ and for $u \in \mathcal{D}(L_2(a))$, $B_{r,N}u = \frac{\partial u}{\partial \mathbf{n}} = 0$ on $\partial D$. Let

$$\lambda_i^r(a) = \sup\{\mathrm{Re}\lambda | \lambda \in \sigma(L_i(a))\}.$$

It is well known that $\lambda_i^r(a)$ is an isolated algebraically simple eigenvalue of $L_i(a)$ with a positive eigenfunction (see [23]) and $\lambda_i^r(a)$ is called the *principal eigenvalue* of $L_i(a)$.

Next we derive some properties of the principal spectrum points of nonlocal dispersal operators by using the spectral radius of the solution operators of the associated evolution equations. To this end, for $i = 1, 2, 3$, define $\Phi_i^\delta(t,s;a) : X_i \to X_i$ by

$$(\Phi_i^\delta(t,s;a)u_0)(\cdot) = u_i(t,\cdot;s,u_0,a), \quad u_0 \in X_i,$$

where $u_1(t,\cdot;s,u_0,a)$ is the solution to

$$\partial_t u(t,x) = \nu_\delta \left[ \int_D k_\delta(y-x)u(t,y)dy - u(t,x) \right] + a(t,x)u(t,x), \quad x \in \bar{D} \tag{3.4}$$

with $u_1(s,\cdot;s,u_0,a) = u_0(\cdot) \in X_1$, $u_2(t,\cdot;s,u_0,a)$ is the solution to

$$\partial_t u(t,x) = \nu_\delta \int_D k_\delta(y-x)[u(t,y)-u(t,x)]dy + a(t,x)u(t,x), \quad x \in \bar{D} \tag{3.5}$$

with $u_2(s,\cdot;s,u_0,a) = u_0(\cdot) \in X_2$, and $u_3(t,\cdot;s,u_0,a)$ is the solution to

$$\partial_t u(t,x) = \nu_\delta \left[ \int_{\mathbb{R}^N} k_\delta(y-x)u(t,y)dy - u(t,x) \right] + a(t,x)u(t,x), \quad x \in \mathbb{R}^N \tag{3.6}$$

with $u_3(s,\cdot;s,u_0,a) = u_0(\cdot) \in X_3$. By general semigroup property, $\Phi_i^\delta(t,s;a)$ $(i = 1, 2, 3)$ is well defined.

Let $A_1$ be $-\Delta$ with Dirichlet boundary condition acting on $X_1 \cap C_0(D)$. Let

$$X_1^r = \mathcal{D}(A_1^\alpha) \tag{3.7}$$

for some $0 < \alpha < 1$ such that $C^1(\bar{D}) \subset X_1^r$ with $\|u\|_{X_1^r} = \|A_1^\alpha u\|_{X_1}$. Similarly, let $A_2$ be $-\Delta$ with Neumann boundary condition acting on $X_2$. Let

$$X_2^r = X_2 \tag{3.8}$$

with $\|u\|_{X_2^r} = \|u\|_{X_2}$, and

$$X_3^r = X_3 \tag{3.9}$$

with $\|u\|_{X_3^r} = \|u\|_{X_3}$. Let

$$X_i^{r,+} = \{u \in X_i^r | u(x) \geq 0\}$$

($i = 1, 2, 3$). Similarly, for $i = 1, 2, 3$, define $\Phi_i(t, s; a) : X_i^r \to X_i^r$ by

$$(\Phi_i(t, s; a)u_0)(\cdot) = u_i(t, \cdot; s, u_0, a), \quad u_0 \in X_i^r,$$

where $u_1(t, \cdot; s, u_0, a)$ is the solution to

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + a(t,x)u(t,x), & x \in D, \\ u(t,x) = 0, & x \in \partial D \end{cases} \tag{3.10}$$

with $u_1(s, \cdot; s, u_0, a) = u_0(\cdot) \in X_1^r$, $u_2(t, \cdot; s, u_0, a)$ is the solution to

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + a(t,x)u(t,x), & x \in D, \\ \frac{\partial u}{\partial \mathbf{n}}(t,x) = 0, & x \in \partial D \end{cases} \tag{3.11}$$

with $u_2(s, \cdot; s, u_0, a) = u_0(\cdot) \in X_2^r$, and $u_3(t, \cdot; s, u_0, a)$ is the solution to

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + a(t,x)u(t,x), & x \in \mathbb{R}^N, \\ u(t, x + p_j \mathbf{e_j}) = u(t,x), & x \in \mathbb{R}^N \end{cases} \tag{3.12}$$

with $u_3(s, \cdot; s, u_3, a) = u_0(\cdot) \in X_3^r$.

Let $r(\Phi_i^\delta(T, 0; a))$ be the spectral radius of $\Phi_i^\delta(T, 0; a)$ and $\lambda_i^\delta(a)$ be the principal spectrum point of $L_i^\delta(a)$. We have the following proposition.

**Proposition 3.1.** *Let $1 \leq i \leq 3$ be given. Then*

$$r(\Phi_i^\delta(T, 0; a)) = e^{\lambda_i^\delta(a)T}.$$

**Proof.** See [41, Proposition 3.3]. □

Similarly, let $r(\Phi_i(T, 0; a))$ be the spectral radius of $\Phi_i(T, 0; a)$ and $\lambda_i^r(a)$ be the principal eigenvalue of $L_i(a)$. Note that $X_i^r$ is a strongly ordered Banach space with the positive cone $C = \{u \in X_i^r \,|\, u(x) \geq 0\}$ and by the regularity, *a priori* estimates, and comparison principle for parabolic equations, $\Phi_i(T, 0; a) : X_i^r \to X_i^r$ is strongly positive and compact. Then by the Kreĭn–Rutman Theorem (see [43]), $r(\Phi_i(T, 0; a))$ is an isolated algebraically simple eigenvalue of $\Phi_i(T,0;a)$ with a positive eigenfunction $u_i^r(\cdot)$ and for any $\mu \in \sigma(\Phi_i(T,0;a)) \setminus \{r(\Phi_i(T,0;a))\}$,

$$\mathrm{Re}\mu < r(\Phi_i(T,0;a)).$$

The following propositions then follow.

**Proposition 3.2.** *Let $1 \le i \le 3$ be given. Then*

$$r(\Phi_i(T,0;a)) = e^{\lambda_i^r(a)T}.$$

*Moreover, there is a codimension one subspace $Z_i$ of $X_i^r$ such that*

$$X_i^r = Y_i \oplus Z_i,$$

*where $Y_i = \mathrm{span}\{u_i^r(\cdot)\}$, and there are $M > 0$ and $\gamma > 0$ such that for any $w_i \in Z_i$, there holds*

$$\frac{\|\Phi_i(nT,0;a)w_i\|_{X_i^r}}{\|\Phi_i(nT,0;a)u_i^r\|_{X_i^r}} \le Me^{-\gamma nT}.$$

**Proposition 3.3.** *For given $1 \le i \le 3$ and $a_1, a_2 \in \mathcal{X}_i \cap C^1(\mathbb{R} \times \mathbb{R}^N)$,*

$$|\lambda_i^\delta(a_1) - \lambda_i^\delta(a_2)| \le \max_{t\in[0,T],x\in\bar{D}} |a_1(t,x) - a_2(t,x)|, \tag{3.13}$$

*and*

$$|\lambda_i^r(a_1) - \lambda_i^r(a_2)| \le \max_{t\in[0,T],x\in\bar{D}} |a_1(t,x) - a_2(t,x)|. \tag{3.14}$$

**Proof.** Let $a_0 = \max_{t\in[0,T],x\in\bar{D}} |a_1(t,x) - a_2(t,x)|$ and

$$a_1^\pm(t,x) = a_1(t,x) \pm a_0.$$

It is not difficult to see that

$$\Phi_i^\delta(t,s;a_1^\pm) = e^{\pm a_0(t-s)}\Phi_i^\delta(t,s;a_1).$$

It then follows that

$$r(\Phi_i^\delta(T,0;a_1^\pm)) = e^{(\lambda_i^\delta(a_1)\pm a_0)T}. \tag{3.15}$$

Observe that by Proposition 2.1, for any $u_0 \in X_i^+$,

$$\Phi_i^\delta(T,0;a_1^-)u_0 \le \Phi_i^\delta(T,0;a_2)u_0 \le \Phi_i^\delta(T,0;a_1^+)u_0.$$

This implies that

$$r(\Phi_i^\delta(T,0;a_1^-)) \le r(\Phi_i^\delta(T,0;a_2)) \le r(\Phi_i^\delta(T,0;a_1^+)).$$

This together with (3.15) implies that

$$\lambda_i^\delta(a_1) - a_0 \le \lambda_i^\delta(a_2) \le \lambda_i^\delta(a_1) + a_0, \tag{3.16}$$

that is, (3.13) holds.

Similarly, we can prove that (3.14) holds. □

### *3.2. Proof of Theorem B in the Dirichlet boundary condition case*

In this subsection, we prove Theorem B in the Dirichlet boundary condition case. Throughout this subsection, we assume $B_{r,b}u = B_{r,D}u$ in (1.10), and $D_b = D_D$ $(= \mathbb{R}^N \setminus \bar{D})$ and $B_{n,b}u = B_{n,D}u$ in (1.11). Note that for any $a \in \mathcal{X}_1 \cap C^1(\mathbb{R} \times \mathbb{R}^N)$, there are $a_n \in \mathcal{X}_1 \cap C^3(\mathbb{R} \times \mathbb{R}^N)$ such that $\sup_{t \in [0,T]} \|a_n(t,\cdot) - a(t,\cdot)\|_{X_1} \to 0$ as $n \to \infty$. By Proposition 3.3, without loss of generality, we may assume that $a \in \mathcal{X}_1 \cap C^3(\mathbb{R} \times \mathbb{R}^N)$.

**Proof of Theorem B in the Dirichlet boundary condition case.** First of all, for the simplicity in notation, we put

$$\Phi(T,0) = \Phi_1(T,0;a), \qquad \lambda_1^r = \lambda_1^r(a),$$

and

$$\Phi^\delta(T,0) = \Phi_1^\delta(T,0;a), \quad \lambda_1^\delta = \lambda_1^\delta(a).$$

Let $u^r(\cdot)$ be a positive eigenfunction of $\Phi(T,0)$ corresponding to $r(\Phi(T,0))$. Without loss of generality, we assume that $\|u^r\|_{X_1^r} = 1$.

We first show that for any $\epsilon > 0$, there is $\delta_1 > 0$ such that for $0 < \delta < \delta_1$,

$$\lambda_1^\delta \ge \lambda_1^r - \epsilon. \tag{3.17}$$

In order to do so, choose $D_0 \subset\subset D$ and $u_0 \in X_1^r \cap C^3(\bar{D})$ such that $u_0(x) = 0$ for $x \in D \backslash D_0$, and $u_0(x) > 0$ for $x \in \mathrm{Int} D_0$. By Proposition 3.2, there exist $\alpha > 0$, $M > 0$, and $u' \in Z_1$, such that

$$u_0(x) = \alpha u^r(x) + u'(x), \tag{3.18}$$

and

$$\|\Phi(nT,0)u'\|_{X_1^r} \le Me^{-\gamma nT} e^{\lambda_1^r nT}. \tag{3.19}$$

By Theorem A, there is $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, there hold

$$\big(\Phi^\delta(nT,0)u^r\big)(x) \ge \big(\Phi(nT,0)u^r\big)(x) - C^1(nT,\delta), \tag{3.20}$$

and

$$\big(\Phi^\delta(nT,0)u'\big)(x) \le \big(\Phi(nT,0)u'\big)(x) + C^2(nT,\delta), \tag{3.21}$$

where $C^i(nT,\delta) \to 0$ as $\delta \to 0$ $(i = 1, 2)$. Hence for $0 < \delta < \delta_0$,

$$\begin{aligned}\big(\Phi^\delta(nT,0)u_0\big)(x) &= \alpha\big(\Phi^\delta(nT,0)u^r\big)(x) + \big(\Phi^\delta(nT,0)u'\big)(x)\\ &\geq \alpha\big(\Phi(nT,0)u^r\big)(x) - \alpha C^1(nT,\delta) - C^2(nT,\delta) - \|\Phi(nT,0)u'\|_{X_1^r}\\ &\geq \alpha e^{\lambda_1^r nT}u^r(x) - \alpha C^1(nT,\delta) - C^2(nT,\delta) - Me^{-\gamma nT}e^{\lambda_1^r nT}\\ &= e^{(\lambda_1^r-\epsilon)nT}e^{\epsilon nT}(\alpha u^r(x) - Me^{-\gamma nT}) - \alpha C^1(nT,\delta) - C^2(nT,\delta). \end{aligned} \tag{3.22}$$

Note that there exists $m > 0$ such that

$$u^r(x) \geq m > 0 \quad \text{ for } x \in \bar{D}_0.$$

Hence for any $0 < \epsilon < \gamma$, there is $n_1 > 0$ such that for $n \geq n_1$,

$$e^{\epsilon nT}(\alpha u^r(x) - Me^{-\gamma nT}) \geq u_0(x) + 1 \quad \text{for} \quad x \in \bar{D}_0, \tag{3.23}$$

and there is $\delta_1 \leq \delta_0$ such that for $0 < \delta < \delta_1$,

$$C^1(n_1T,\delta) + C^2(n_1T,\delta) \leq e^{(\lambda_1^r-\epsilon)n_1T}. \tag{3.24}$$

Note that $u_0(x) = 0$ for $x \in D\backslash D_0$ and $\big(\Phi^\delta(n_1T,0)u_0\big)(x) \geq 0$ for all $x \in \bar{D}$. This together with (3.22)–(3.24) implies that for $\delta < \delta_1$,

$$\big(\Phi^\delta(n_1T,0)u_0\big)(x) \geq e^{(\lambda_1^r-\epsilon)n_1T}u_0(x), \quad x \in \bar{D}. \tag{3.25}$$

By (3.25) and Proposition 2.1, for any $0 < \delta < \delta_1$ and $n \geq 1$,

$$(\Phi^\delta(nn_1T,0)u_0)(\cdot) \geq e^{(\lambda_1^r-\epsilon)nn_1T}u_0(\cdot).$$

This together with Proposition 3.1 implies that for $0 < \delta < \delta_1$,

$$e^{\lambda_1^\delta T} = r(\Phi^\delta(T,0)) \geq e^{(\lambda_1^r-\epsilon)T}.$$

Hence (3.17) holds.

Next, we prove that for any $\epsilon > 0$, there is $\delta_2 > 0$ such that for $0 < \delta < \delta_2$,

$$\lambda_1^\delta \leq \lambda_1^r + \epsilon. \tag{3.26}$$

To this end, first, choose a sequence of smooth domains $\{D_m\}$ such that $D_1 \supset D_2 \supset D_3 \cdots \supset D_m \supset \cdots \supset \bar{D}$, and $\cap_{m=1}^\infty D_m = \bar{D}$. Consider the following evolution equation

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + a(t,x)u(t,x), & x \in D_m,\\ u(t,x) = 0, & x \in \partial D_m. \end{cases} \tag{3.27}$$

Let

$$X_{1,m} = \{u \in C(\bar{D}_m, \mathbb{R})\},$$

and

$$X_{1,m}^r = \mathcal{D}(A_{1,m}^\alpha),$$

where $A_{1,m}$ is $-\Delta$ with Dirichlet boundary condition acting on $X_{1,m} \cap C_0(D_m)$ and $0 < \alpha < 1$. We denote the solution of (3.27) by $u_m(t, \cdot; s, u_0) = (\Phi_m(t, s)u_0)(\cdot)$ with $u(s, \cdot; s, u_0) = u_0(\cdot) \in X_{1,m}^r$. By Proposition 3.2, we have

$$r(\Phi_m(T, 0)) = e^{\lambda_{1,m}^r T},$$

where $\lambda_{1,m}^r$ is the principal eigenvalue of the following eigenvalue problem,

$$\begin{cases} -\partial_t u + \Delta u + a(t, x)u = \lambda u, & x \in D_m, \\ u(t + T, x) = u(t, x), & x \in D_m, \\ u(t, x) = 0, & x \in \partial D_m. \end{cases}$$

By the dependence of the principle eigenvalue on the domain perturbation (see [15]), for any $\epsilon > 0$, there exists $m_1$ such that

$$\lambda_{1,m_1}^r \le \lambda_1^r + \frac{\epsilon}{2}. \tag{3.28}$$

Secondly, let $u_{m_1}^r(\cdot)$ be a positive eigenfunction of $\Phi_{m_1}(T, 0)$ corresponding to $r(\Phi_{m_1}(T, 0))$. By regularity for parabolic equations, $u_{m_1}^r \in C^3(\bar{D}_{m_1})$. Let $(\Phi_{m_1}^\delta(t, 0)u_{m_1}^r)(x)$ be the solution to

$$\begin{cases} u_t = \nu_\delta \left[\int_{D_{m_1}} k_\delta(y - x)u(t, y)dy - u(t, x)\right] + a(t, x)u(t, x), & x \in \bar{D}_{m_1}, \\ u(0, x) = u_{m_1}^r(x). \end{cases} \tag{3.29}$$

Then by Theorem A,

$$\left(\Phi_{m_1}^\delta(nT, 0)u_{m_1}^r\right)(x) \le \left(\Phi_{m_1}(nT, 0)u_{m_1}^r\right)(x) + C(nT, \delta) \quad \forall\, x \in \bar{D}_{m_1},$$

where $C(nT, \delta) \to 0$ as $\delta \to 0$. By Proposition 2.1,

$$\left(\Phi^\delta(nT, 0)u_{m_1}^r|_{\bar{D}}\right)(x) \le \left(\Phi_{m_1}^\delta(nT, 0)u_{m_1}^r\right)(x) \quad \forall\, x \in \bar{D}.$$

It then follows that for $x \in \bar{D}$,

$$\begin{aligned} \left(\Phi^\delta(nT, 0)u_{m_1}^r|_{\bar{D}}\right)(x) &\le \left(\Phi_{m_1}(nT, 0)u_{m_1}^r\right)(x) + C(nT, \delta) \\ &= e^{\lambda_{m_1}^r nT} u_{m_1}^r(x) + C(nT, \delta) \\ &\le e^{(\lambda_1^r + \frac{\epsilon}{2})nT} u_{m_1}^r(x) + C(nT, \delta) \\ &= e^{(\lambda_1^r + \epsilon)nT} e^{-\frac{\epsilon}{2}nT} u_{m_1}^r(x) + C(nT, \delta). \end{aligned} \tag{3.30}$$

Note that

$$\min_{x \in \bar{D}} u_{m_1}^r(x) > 0.$$

Hence for any $\epsilon > 0$, there is $n_2 \geq 1$ such that

$$e^{-\frac{\epsilon}{2}n_2 T} \leq \frac{1}{2}, \tag{3.31}$$

and there is $\delta_2 > 0$ such that for $0 < \delta < \delta_2$,

$$C(n_2 T, \delta) \leq \frac{1}{2} e^{(\lambda_1^r + \epsilon)n_2 T} u_{m_1}^r(x) \quad \forall x \in \bar{D}. \tag{3.32}$$

By (3.30)–(3.32),

$$\big(\Phi^\delta(n_2 T, 0) u_{m_1}^r|_{\bar{D}}\big)(x) \leq e^{(\lambda_1^r + \epsilon)n_2 T} u_{m_1}^r(x) \quad \forall\, x \in \bar{D}.$$

This together with Proposition 2.1 implies that for $0 < \delta < \delta_2$,

$$\big(\Phi^\delta(n n_2 T, 0) u_{m_1}^r|_{\bar{D}}\big)(x) \leq e^{(\lambda_1^r + \epsilon)n n_2 T} u_{m_1}^r(x) \quad \forall\, x \in \bar{D}. \tag{3.33}$$

This together with Proposition 3.1 implies that

$$\lambda_1^\delta \leq \lambda_1^r + \epsilon$$

for $0 < \delta < \delta_2$, that is, (3.26) holds.

Theorem B in the Dirichlet boundary condition case then follows from (3.17) and (3.26). □

### *3.3. Proofs of Theorem B in the Neumann and periodic boundary condition cases*

**Proof of Theorem B in the Neumann boundary condition case.** We assume $B_{r,b}u = B_{r,N}u$ in (1.10), and $D_b = D_N$ $(= \emptyset)$ in (1.11). The proof in the Neumann boundary condition case is similar to the arguments in the Dirichlet boundary condition case (it is simpler). For the completeness, we give a proof in the following. Without loss of generality, we may also assume that $a \in \mathcal{X}_2 \cap C^3(\mathbb{R} \times \mathbb{R}^N)$.

For the simplicity in notation, put

$$\Phi(nT, 0) = \Phi_2(nT, 0; a), \qquad \lambda_2^r = \lambda_2^r(a),$$

and

$$\Phi^\delta(nT, 0) = \Phi_2^\delta(nT, 0; a), \quad \lambda_2^\delta = \lambda_2^\delta(a).$$

By Propositions 3.1 and 3.2,

$$r(\Phi(T, 0)) = e^{\lambda_2^r T}, \tag{3.34}$$

and

$$r(\Phi^\delta(T, 0)) = e^{\lambda_2^\delta T}. \tag{3.35}$$

Let $u^r(\cdot)$ be a positive eigenfunction of $\Phi(T,0)$ corresponding to $r(\Phi(T,0))$. By regularity for parabolic equations, $u^r \in C^3(\bar{D})$. By Theorem A, we have

$$\|\Phi^\delta(nT,0)u^r - \Phi(nT,0)u^r\|_{X_2} \le C(nT,\delta),$$

where $C(nT,\delta) \to 0$ as $\delta \to 0$. This implies that for all $x \in \bar{D}$,

$$\begin{aligned}\big(\Phi^\delta(nT,0)u^r\big)(x) &\ge \big(\Phi(nT,0)u^r\big)(x) - C(nT,\delta)\\ &= e^{\lambda_2^r nT}u^r(x) - C(nT,\delta)\\ &= e^{(\lambda_2^r-\epsilon)nT}e^{\epsilon nT}u^r(x) - C(nT,\delta),\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}\big(\Phi^\delta(nT,0)u^r\big)(x) &\le \big(\Phi(nT,0)u^r\big)(x) + C(nT,\delta)\\ &= e^{\lambda_2^r nT}u^r(x) + C(nT,\delta)\\ &= e^{(\lambda_2^r+\epsilon)nT}e^{-\epsilon nT}u^r(x) + C(nT,\delta).\end{aligned} \tag{3.37}$$

Note that

$$\min_{x\in\bar{D}} u^r(x) > 0. \tag{3.38}$$

Hence for any $\epsilon > 0$, there is $n_1 > 1$ such that

$$\begin{cases} e^{\epsilon n_1 T}u^r(x) \ge \frac{3}{2}u^r(x) & \forall x \in \bar{D},\\ e^{-\epsilon n_1 T}u^r(x) \le \frac{1}{2}u^r(x) & \forall x \in \bar{D},\end{cases} \tag{3.39}$$

and there is $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$,

$$C(n_1T)\delta < \frac{1}{2}e^{(\lambda_2^r-\epsilon)n_1T}u^r(x) \quad \forall x \in \bar{D}. \tag{3.40}$$

By (3.36)–(3.40), we have that for any $0 < \delta < \delta_0$,

$$e^{(\lambda_2^r-\epsilon)n_1T}u^r(x) \le \big(\Phi^\delta(n_1T,0)u^r\big)(x) \le e^{(\lambda_2^r+\epsilon)n_1T}u^r(x) \quad \forall x \in \bar{D}.$$

This together with Proposition 2.1 implies that for all $n \ge 1$,

$$e^{(\lambda_2^r-\epsilon)n_1nT}u^r(x) \le \big(\Phi^\delta(n_1nT,0)u^r\big)(x) \le e^{(\lambda_2^r+\epsilon)n_1nT}u^r(x) \quad \forall x \in \bar{D}.$$

It then follows that for any $0 < \delta < \delta_0$,

$$e^{(\lambda_2^r-\epsilon)T} \le r(\Phi^\delta(T,0)) \le e^{(\lambda_2^r+\epsilon)T}.$$

By Proposition 3.1, we have

$$|\lambda_2^\delta - \lambda_2^r| < \epsilon \quad \forall 0 < \delta < \delta_0.$$

Theorem B in the Neumann boundary condition case is thus proved. $\square$

**Proof of Theorem B in the periodic boundary condition case.** We assume $D = \mathbb{R}^N$, and $B_{r,b}u = B_{r,P}u$ in (1.10), and $B_{n,b}u = B_{n,P}u$ in (1.11). It can be proved by the same arguments as in the Neumann boundary condition case. □

## 4. Approximation of time periodic positive solutions of random dispersal KPP equations by nonlocal dispersal KPP equations

In this section, we study the approximation of the asymptotic dynamics of time periodic KPP equations with random dispersal by those of time periodic KPP equations with nonlocal dispersal. We first recall the existing results about time periodic positive solutions of KPP equations with random as well as nonlocal dispersal. Then we prove Theorem C. Throughout this section, we assume that $D$ is as in (H0), and (H1), (H2) and (H2)$_\delta$ hold. Recall that, (H2) implies (H2)$_\delta$ for $\delta$ sufficiently small by Theorem B.

### *4.1. Basic properties*

In this subsection, we present some basic known results for (1.12) and (1.13). Let $X_1^r$, $X_2^r$, and $X_3^r$ be defined as in (3.7), (3.8), and (3.9), respectively. For $u_0 \in X_i^r$, let $(U(t,0)u_0)(\cdot) = u(t,\cdot;u_0)$, where $u(t,\cdot;u_0)$ is the solution to (1.12) with $u(0,\cdot;u_0) = u_0(\cdot)$ and $B_{r,b}u = B_{r,D}u$ when $i = 1$, $B_{r,b}u = B_{r,N}u$ when $i = 2$, and $B_{r,b}u = B_{r,P}u$ when $i = 3$. Similarly, for $u_0 \in X_i$, let $(U^\delta(t,0)u_0)(\cdot) = u^\delta(t,\cdot;u_0)$, where $u^\delta(t,\cdot;u_0)$ is the solution to (1.13) with $u^\delta(0,\cdot;u_0) = u_0(\cdot)$ and $D_b = D_D$ $(= \mathbb{R}^N \backslash \bar{D})$, $B_{n,b}u = B_{n,D}u$ when $i = 1$, $D_b = D_N$ $(= \emptyset)$ when $i = 2$, and $B_{n,b}u = B_{n,P}u$ and $D_b = D_p$ $(= \mathbb{R}^N)$ when $i = 3$.

**Proposition 4.1.**

(1) *If* $u_0 \geq 0$, *solution* $u(t,\cdot;u_0)$ *to* (1.12) *with* $u(0,\cdot;u_0) = u_0(\cdot)$ *exists for all* $t \geq 0$ *and* $u(t,\cdot;u_0) \geq 0$ *for all* $t \geq 0$.
(2) *If* $u_0 \geq 0$, *solution* $u(t,\cdot;u_0)$ *to* (1.13) *with* $u(0,\cdot;u_0) = u_0(\cdot)$ *exists for all* $t \geq 0$ *and* $u(t,\cdot;u_0) \geq 0$ *for all* $t \geq 0$.

**Proof.** (1) Note that $u(\cdot) \equiv 0$ is a sub-solution of (1.12) and $u(\cdot) \equiv M$ is a super-solution of (1.12) for $M \gg 1$. Then by Proposition 2.1, there is $M \gg 1$ such that

$$0 \leq u(t,x;u_0) \leq M \quad \forall\, x \in \bar{D},\ t \in (0,t_{\max}),$$

where $(0,t_{\max})$ is the interval of existence of $u(t,\cdot;u_0)$. This implies that we must have $t_{\max} = \infty$ and hence (1) holds.

(2) It can be proved by similar arguments as in (1). □

**Proposition 4.2.**

(1) (1.12) *has a unique globally stable positive time periodic solution* $u^*(t,x)$.
(2) (1.13) *has a unique globally stable time periodic positive solution* $u_\delta^*(t,x)$.

**Proof.** (1) See [45, Theorem 3.1] (see also [33, Theorems 1.1, 1.3]).

(2) See [36, Theorem E]. □

**Remark 4.1.** By Proposition 4.2(2), if there is $u_0 \in X_i^+ \setminus \{0\}$ such that $(U^\delta(nT,0)u_0)(\cdot) \ge u_0(\cdot)$ for some $n \ge 1$, then we must have $\lim_{n\to\infty}(U^\delta(nT,0)u_0)(\cdot) = u_\delta^*(0,\cdot)$ and hence

$$(U^\delta(nT,0)u_0)(\cdot) \le u_\delta^*(0,\cdot).$$

Similarly, if there is $u_0 \in X_i^+ \setminus \{0\}$ such that $(U^\delta(nT,0)u_0)(\cdot) \le u_0(\cdot)$ for some $n \ge 1$, then

$$(U^\delta(nT,0)u_0)(\cdot) \ge u_\delta^*(0,\cdot).$$

### 4.2. *Proof of Theorem C in the Dirichlet boundary condition case*

In this subsection, we prove Theorem C in the Dirichlet boundary condition case. Throughout this subsection, we assume that $B_{r,b}u = B_{r,D}u$ in (1.12), and $D_b = D_D$ and $B_{n,b}u = B_{n,D}u$ in (1.13).

**Proof of Theorem C in the Dirichlet boundary condition case.** First of all, note that it suffices to prove that for any $\epsilon > 0$, there is $\delta_0 > 0$ such that for $0 < \delta < \delta_0$,

$$u_\delta^*(t,x) - \epsilon \le u^*(t,x) \le u_\delta^*(t,x) + \epsilon \quad \forall\, t \in [0,T],\ x \in \bar{D}.$$

We first show that for any $\epsilon > 0$, there is $\delta_1 > 0$ such that for $0 < \delta < \delta_1$,

$$u^*(t,x) \le u_\delta^*(t,x) + \epsilon \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{4.1}$$

To this end, choose a smooth function $\phi_0 \in C_0^\infty(D)$ satisfying that $\phi_0(x) \ge 0$ for $x \in D$ and $\phi_0(\cdot) \not\equiv 0$. Let $0 < \eta \ll 1$ be such that

$$u_-(x) := \eta\phi_0(x) < u^*(0,x) \quad \text{for} \quad x \in \bar{D}.$$

Then there is $\epsilon_0 > 0$ such that

$$u^*(0,x) \ge u_-(x) + \epsilon_0 \quad \text{for} \quad x \in \operatorname{supp}(\phi_0). \tag{4.2}$$

By Proposition 4.2, there is $N \gg 1$ such that

$$\big(U(NT,0)u_-\big)(x) \ge u^*(NT,x) - \epsilon_0/2 = u^*(0,x) - \epsilon_0/2 \quad \forall\, x \in \bar{D}.$$

By Theorem A, there is $\bar{\delta}_1 > 0$ such that for $0 < \delta < \bar{\delta}_1$, we have

$$\big(U^\delta(NT,0)u_-\big)(x) \ge \big(U(NT,0)u_-\big)(x) - \epsilon_0/2 \quad \forall\, x \in \bar{D}.$$

Hence for $0 < \delta < \bar{\delta}_1$,

$$\big(U^\delta(NT,0)u_-\big)(x) \ge u^*(0,x) - \epsilon_0 \quad \forall\, x \in \bar{D}. \tag{4.3}$$

Note that

$$\big(U^\delta(NT,0)u_-\big)(x) \ge 0 \quad \forall\, x \in \bar{D}.$$

It then follows from (4.2) and (4.3) that for $0 < \delta < \bar{\delta}_1$,

$$\big(U^\delta(NT,0)u_-\big)(x) \ge u_-(x) \quad \forall\, x \in \bar{D}.$$

This together with Proposition 4.2 (2) implies that

$$\big(U^\delta(NT,0)u_-\big)(x) \le u_\delta^*(0,x) \quad \forall\, x \in \bar{D} \tag{4.4}$$

(see Remark 4.1).

By Proposition 4.2 (1) again, for $n \gg 1$,

$$u^*(t,x) \le (U(nNT+t,0)u_-)(x) + \epsilon/2 \;\; \forall\, t \in [0,T],\;\; x \in \bar{D}. \tag{4.5}$$

Fix an $n \gg 1$ such that (4.5) holds. By Theorem A, there is $0 < \tilde{\delta}_1 \le \bar{\delta}_1$ such that for $0 < \delta < \tilde{\delta}_1$,

$$(U(nNT+t,0)u_-)(x) \le (U^\delta(nNT+t,0)u_-)(x) + C_1(\delta), \tag{4.6}$$

where $C_1(\delta) \to 0$ as $\delta \to 0$. By (4.4), Proposition 2.1, and Proposition 4.2 (2),

$$\big(U^\delta(nNT+t,0)u_-\big)(x) \le \big(U^\delta(t,0)u_\delta^*(0,\cdot)\big)(x) = u_\delta^*(t,x) \tag{4.7}$$

for $t \in [0,T]$ and $x \in \bar{D}$. Let $0 < \delta_1 \le \tilde{\delta}_1$ be such that

$$C_1(\delta) < \epsilon/2 \quad \forall\, 0 < \delta < \delta_1. \tag{4.8}$$

(4.1) then follows from (4.5)–(4.8).

Next, we need to show for any $\epsilon > 0$, there is $\delta_2 > 0$ such that for $0 < \delta < \delta_2$,

$$u^*(t,x) \ge u_\delta^*(t,x) - \epsilon \quad \forall\, t \in [0,T],\;\; x \in \bar{D}. \tag{4.9}$$

To this end, choose a sequence of open sets $\{D_m\}$ with smooth boundaries such that $D_1 \supset D_2 \supset D_3 \cdots \supset D_m \supset \cdots \supset \bar{D}$, and $\bar{D} = \cap_{m=1}^\infty D_m$. According to Corollary 5.11 in [17], $D_m \to D$ regularly and all assertions of Theorem 5.5 in [17] hold.

Consider

$$\begin{cases} \partial_t u = \Delta u + u f(t,x,u), & x \in D_m, \\ u(t,x) = 0, & x \in \partial D_m. \end{cases} \tag{4.10}$$

Let $U_m(t,0)u_0 = u(t,\cdot;u_0)$, where $u(t,\cdot;u_0)$ is the solution to (4.10) with $u(0,\cdot;u_0) = u_0(\cdot)$. By Proposition 4.2, (4.10) has a unique time periodic positive solution $u_m^*(t,x)$. We first claim that

$$\lim_{m\to\infty} u_m^*(t,x) \to u^*(t,x) \;\text{ uniformly in } t \in [0,T] \text{ and } x \in \bar{D}. \tag{4.11}$$

In fact, it is clear that $u^* \in C(\mathbb{R} \times \bar{D}, \mathbb{R})$ and $u_m^* \in C(\mathbb{R} \times \bar{D}_m, \mathbb{R})$. By [15, Theorem 7.1],

$$\sup_{t\in\mathbb{R}} \|u_m^*(t,\cdot) - u^*(t,\cdot)\|_{L^q(D)} \to 0 \quad \text{as} \quad m \to \infty$$

for $1 \le q < \infty$. Let $a(t,x) = f(t,x,u^*(t,x))$ and $a_m(t,x) = f(t,x,u_m^*(t,x))$. Then $u^*(t,x)$ and $u_m^*(t,x)$ are time periodic solutions to the following linear parabolic equations,

$$\begin{cases} u_t = \Delta u + a(t,x)u, & x \in D, \\ u(t,x) = 0, & x \in \partial D, \end{cases} \tag{4.12}$$

and

$$\begin{cases} u_t = \Delta u + a_m(t,x)u, & x \in D_m, \\ u(t,x) = 0, & x \in \partial D_m, \end{cases} \tag{4.13}$$

respectively. Observe that there is $M > 0$, such that

$$\|a\|_{L^\infty([T,2T]\times D)} < M,\ \|a_m\|_{L^\infty([T,2T]\times D_m)} < M,\ \|u^*(0,\cdot)\|_{L^\infty(D)} < M,\ \text{and}$$
$$\|u_m^*(0,\cdot)\|_{L^\infty(D_m)} < M.$$

By [1, Theorem D(1)], $\{u_m^*(t,x)\}$ is equi-continuous on $[T,2T] \times \bar{D}$. Without loss of generality, we may then assume that $u_m^*(t,x)$ converges uniformly on $[T,2T] \times \bar{D}$. But $u_m^*(t,\cdot) \to u^*(t,\cdot)$ in $L^q(D)$ uniformly in $t$. We then must have

$$u_m^*(t,x) \to u^*(t,x) \quad \text{as} \quad n \to \infty$$

uniformly in $(t,x) \in [T,2T] \times \bar{D}$. This together with the time periodicity of $u_m^*$ shows that (4.11) holds.

Next, for any $\epsilon > 0$, fix $m \gg 1$ such that

$$u^*(t,x) \ge u_m^*(t,x) - \epsilon/3 \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{4.14}$$

Choose $M \gg 1$ such that for $0 < \delta \le 1$,

$$M u_m^*(t,x) \ge u_\delta^*(t,x)\ \ \forall\, t \in [0,T],\ x \in \bar{D}. \tag{4.15}$$

Let

$$u_m^+(x) = M u_m^*(0,x), \quad u^+(x) = u_m^+(x)|_{\bar{D}}.$$

By Proposition 4.2, for fixed $m$ and $\epsilon$, there exists $N \gg 1$, such that

$$u_m^*(t,x) \ge \big(U_m(NT+t,0)u_m^+\big)(x) - \epsilon/3 \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{4.16}$$

By Theorem A, there is $0 < \tilde{\delta}_2 < 1$ such that for $0 < \delta < \tilde{\delta}_2$,

$$(U_m(NT+t,0)u_m^+)(x) \ge (U_m^\delta(NT+t,0)u_m^+)(x) - C_2(\delta) \quad \forall t \in [0,T],\ x \in D_m, \tag{4.17}$$

where $C_2(\delta) \to 0$ as $\delta \to 0$ and $(U_m^\delta(t,0)u_0)(\cdot) = u(t,\cdot;u_0)$ is the solution to

$$\begin{cases} u_t(t,x) = \nu_\delta \left[\int_{D_m} k_\delta(y-x)u(t,y)dy - u(t,x)\right] + u(t,x)f(t,x,u(t,x)), & x \in \bar{D}_m \\ u(0,x) = u_0(x), & x \in \bar{D}_m. \end{cases}$$

Let $0 < \delta_2 < \tilde{\delta}_2$ be such that for $0 < \delta < \delta_2$,

$$C_2(\delta) < \epsilon/3. \tag{4.18}$$

By Proposition 2.1, for $x \in \bar{D}$ we have

$$(U_m^\delta(NT+t,0)u_m^+)(x) \geq (U^\delta(NT+t,0)u^+)(x),$$

and

$$(U^\delta(NT+t,0)u^+)(x) = (U^\delta(t,0)U^\delta(NT,0)u^+)(x) \geq (U^\delta(t,0)u_\delta^*(0,\cdot))(x) = u_\delta^*(t,x).$$

This together with (4.14), (4.16), (4.17), and (4.18) implies (4.9).

So, for any $\epsilon > 0$, there exists $\delta_0 = \min\{\delta_1, \delta_2\}$, such that for any $\delta < \delta_0$, we have

$$|u^*(t,x) - u_\delta^*(t,x)| \leq \epsilon \quad \text{uniform in } t > 0 \text{ and } x \in \bar{D}. \qquad \square$$

### *4.3. Proofs of Theorem C in the Neumann and periodic boundary condition cases*

In this subsection, we prove Theorem C in the Neumann and periodic boundary condition cases.

**Proof of Theorem C in the Neumann boundary condition case.** We assume $B_{r,b}u = B_{r,N}u$ in (1.10), and $D_b = D_N$ $(= \emptyset)$ in (1.11). The proof in the Neumann boundary condition case is similar to the arguments in the Dirichlet boundary condition case (it is indeed simpler). For completeness, we provide a proof.

First, we show that for any $\epsilon > 0$, there is $\delta_1 > 0$ such that

$$u^*(t,x) \leq u_\delta^*(t,x) + \epsilon \quad \forall\, t \in [0,T],\ x \in \bar{D}, \tag{4.19}$$

if $0 < \delta < \delta_1$. Choose a smooth function $u_- \in C^\infty(\bar{D})$ with $u_-(\cdot) \geq 0$ and $u_-(\cdot) \not\equiv 0$ such that

$$u_-(x) < u^*(0,x) \quad \forall\, x \in \bar{D}.$$

Then there is $\epsilon_0 > 0$ such that

$$u^*(0,x) \geq u_-(x) + \epsilon_0 \quad \forall\, x \in \bar{D}. \tag{4.20}$$

By Proposition 4.2 (1), there is $N \gg 1$ such that

$$\big(U(NT,0)u_-\big)(x) \geq u^*(0,x) - \epsilon_0/2 \quad \forall\, x \in \bar{D}. \tag{4.21}$$

By Theorem A, there is $\bar{\delta}_1 > 0$ such that for $0 < \delta < \bar{\delta}_1$,

$$(U^\delta(NT,0)u_-)(x) \geq (U(NT,0)u_-)(x) - \epsilon_0/2 \quad \forall\, x \in \bar{D}. \tag{4.22}$$

By (4.20), (4.21) and (4.22),

$$\big(U^\delta(NT,0)u_-\big)(x) \geq u_-(x) \quad \forall\, x \in \bar{D},$$

and then by Proposition 4.2 (2),

$$\big(U^\delta(NT,0)u_-\big)(x) \le u_\delta^*(0,x) \quad \forall\, x \in \bar{D}. \tag{4.23}$$

By Proposition 4.2 (1) again, for any given $\epsilon > 0$, $n \gg 1$, and $0 < \delta < \bar{\delta}_1$,

$$u^*(t,x) \le (U(nNT+t,0)u_-)(x) + \epsilon/2 \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{4.24}$$

By Theorem A, there is $0 < \delta_1 \le \bar{\delta}_1$ such that for $\delta < \delta_1$,

$$(U(nNT+t,0)u_-)(x) \le (U^\delta(nNT+t,0)u_-)(x) + \frac{\epsilon}{2} \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{4.25}$$

By Proposition 2.1 and (4.23), we have

$$\begin{aligned}(U^\delta(nNT+t,0)u_-)(x) &= (U^\delta(t,0)U^\delta(nNT,0)u_-)(x) \le (U^\delta(t,0)u_\delta^*(t,\cdot))(x) \\ &= u_\delta^*(t,x)\end{aligned} \tag{4.26}$$

for $t \in [0,T]$ and $x \in \bar{D}$. (4.19) then follows from (4.24)–(4.26).

Next, we show that for any $\epsilon > 0$, there is $\delta_2 > 0$ such that for $0 < \delta < \delta_2$,

$$u^*(t,x) \ge u_\delta^*(t,x) - \epsilon \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{4.27}$$

Choose $M \gg 1$ such that $f(t,x,M) < 0$ for $t \in \mathbb{R}$ and $x \in \bar{D}$. Put

$$u^+(x) = M \quad \forall\, x \in \bar{D}.$$

Then for all $\delta > 0$,

$$u_\delta^*(0,x) \le u^+(x) \quad \forall\, x \in \bar{D}. \tag{4.28}$$

By Proposition 4.2, there is $N \gg 1$ such that

$$u^*(t,x) \ge (U(NT+t,0)u^+)(x) - \epsilon/2 \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{4.29}$$

By Theorem A, there is $\delta_2 > 0$ such that for $0 < \delta < \delta_2$,

$$(U(NT+t,0)u^+)(x) \ge (U^\delta(NT+t,0)u^+)(x) - \frac{\epsilon}{2} \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{4.30}$$

By (4.28),

$$\begin{aligned}(U^\delta(NT+t,0)u^+)(x) &= (U^\delta(t,0)U^\delta(NT,0)u^+)(x) \ge (U^\delta(t,0)u_\delta^*(t,\cdot))(x) \\ &= u_\delta^*(t,x)\end{aligned} \tag{4.31}$$

for $t \in [0,T]$ and $x \in \bar{D}$. (4.27) then follows from (4.29)–(4.31).

So, for any $\epsilon > 0$, there exists $\delta_0 = \min\{\delta_1, \delta_2\}$, such that for any $0 < \delta < \delta_0$, we have

$$|u^*(t,x) - u_\delta^*(t,x)| \le \epsilon \ \text{ uniform in } t > 0 \text{ and } x \in \bar{D}. \qquad \square$$

**Proof of Theorem C in the periodic boundary condition case.** We assume $D = \mathbb{R}^N$, and $B_{r,b}u = B_{r,P}u$ in (1.10), and $B_{n,b}u = B_{n,P}u$ in (1.11). It can be proved by the similar arguments as in the Neumann boundary condition case. □

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