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Existence of stable solutions to $(-\Delta)^m u = e^u$ in $\mathbb{R}^N$ with $m \geq 3$ and $N > 2m$

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Abstract

We consider the polyharmonic equation $(-\Delta)^m u = e^u$ in $\mathbb{R}^N$ with $m \geq 3$ and $N > 2m$. We prove the existence of many entire stable solutions. This answers some questions raised by Farina and Ferrero in [7].

MSC: 35J91; 35B08; 35B35

Keywords: Polyharmonic equation; Entire stable solution

1. Introduction

In this paper, we are interested in the existence of entire stable solutions of the polyharmonic equation

$$(-\Delta)^m u = e^u \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

with $m \geq 3$ and $N > 2m$.

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Definition 1. A solution u to (1.1) is said to be stable in $\Omega \subseteq \mathbb{R}^N$ if

$$\begin{cases} \int_\Omega |\nabla(\Delta^{\frac{m-1}{2}}\phi)|^2 dx - \int_\Omega e^u \phi^2 dx \geq 0 \quad \text{for any } \phi \in C_0^\infty(\Omega), & \text{when } m \text{ is odd;} \\ \int_\Omega |\Delta^{\frac{m}{2}}\phi|^2 dx - \int_\Omega e^u \phi^2 dx \geq 0 \quad \text{for any } \phi \in C_0^\infty(\Omega), & \text{when } m \text{ is even.} \end{cases}$$

Moreover, a solution to (1.1) is said to be stable outside a compact set K if it's stable in $\mathbb{R}^N \setminus K$. For simplicity, we say also that u is stable if $\Omega = \mathbb{R}^N$.

For $m = 1$, Farina [6] showed that (1.1) has no stable classical solution in $\mathbb{R}^N$ for $1 \leq N \leq 9$. He also proved that any classical solution which is stable outside a compact set in $\mathbb{R}^2$ verifies $e^u \in L^1(\mathbb{R}^2)$, therefore u is provided by the stereographic projection thanks to Chen–Li's classification result in [3], that is, there exist $\lambda > 0$ and $x_0 \in \mathbb{R}^2$ such that

$$u(x) = \ln\left[\frac{32\lambda^2}{(4 + \lambda^2|x - x_0|^2)^2}\right] \quad \text{for some } \lambda > 0. \tag{1.2}$$

Later on, Dancer and Farina [4] showed that (1.1) admits classical entire solutions which are stable outside a compact set of $\mathbb{R}^N$ if and only if $N \geq 10$.

It is well known that for any $m \geq 1$, $\lambda > 0$ and $x_0 \in \mathbb{R}^{2m}$, the function u defined in (1.2) resolves (1.1) in the conformal dimension $\mathbb{R}^{2m}$, there are the so-called spherical solutions, since they are provided by the stereographic projections.

For $m = 2$, the stability properties of entire solutions to (1.1) were studied in many works, especially the study for radial solutions is complete. Let $u(x) = u(r)$ be a smooth radial solution to (1.1), then u satisfies the following initial value problem

$$\begin{cases} (-\Delta)^m u = e^u, \\ u^{(2k+1)}(0) = 0, & \forall\, 0 \leq k \leq m-1, \\ \Delta^k u(0) = a_k, & \forall\, 0 \leq k \leq m-1. \end{cases} \tag{1.3}$$

Here the Laplacian Δ is seen as $\Delta u = r^{1-N}\left(r^{N-1}u'\right)'$ and a_k are constants in $\mathbb{R}$. Equivalently, let $v_k = (-\Delta)^k u$ for $0 \leq k \leq m-1$, the equation (1.3) can be written as a system

$$-v_k'' - \frac{N-1}{r}v_k' = v_{k+1} \text{ for } 0 \leq k \leq m-2; \quad \text{and } -v_{m-1}'' - \frac{N-1}{r}v_{m-1}' = e^{v_0} \tag{1.4}$$

where $v_k(0) = (-1)^k a_k$ and $v_k'(0) = 0$ for any $0 \leq k \leq m-1$.

Let $m = 2$, $a_0 = u(0) = 0$ (it's always possible by the scaling $u(\lambda x) + 2m\ln\lambda$). Denote by u_β the solution to (1.3) verifying $a_1 = \beta$, it's known from [1,5,11] that:

- There is no global solutions to (1.3) if $N \leq 2$.
- For $N \geq 3$, there exists $\beta_0 < 0$ depending on N such that the solution to (1.3) is globally defined, if and only if $\beta \leq \beta_0$.
- If $N = 3$ or 4, any entire solution u_β is unstable in $\mathbb{R}^N$, but stable outside a compact set.

- If $5 \le N \le 12$, then u_β is stable outside a compact set for every $\beta < \beta_0$ while u_{β_0} is unstable outside every compact set.
- If $5 \le N \le 12$, there exists $\beta_1 < \beta_0$ such that u_β is stable in $\mathbb{R}^N$, if and only if $\beta \le \beta_1$.
- If $N \ge 13$, u_β is stable for every $\beta \le \beta_0$.

Moreover, Dupaigne et al. showed in [5] the examples of non-radial stable solutions for $\Delta^2 u = e^u$ in $\mathbb{R}^N$ with any $N \ge 5$, and Warnault proved in [11] that no stable (radial or not) smooth solution exists for $\Delta^2 u = e^u$ if $N \le 4$.

Recently, Farina and Ferrero [7] studied (1.1) for general $m \ge 3$, they obtained many results about the existence and stability of solutions, especially for the radial solutions. More precisely, they proved that:

- For $N \le 2m$, no stable solution (radial or not) exists;
- For $m \ge 1$ odd and $1 \le N \le 2m - 1$, any radial solution is stable outside a compact set;
- For $m \ge 1$ and $N = 2m$, the spherical solutions, i.e. solutions given by (1.2) are stable outside a compact set;
- For $m \ge 3$ odd, if $(-1)^k a_k \le 0$ for same $1 \le k \le m - 1$, then the radial solution is stable outside a compact set;
- For $m \ge 2$ even and $u(0) = 0$, there exists a function $\Phi : \mathbb{R}^{m-1} \to (-\infty, 0)$ (depending on N) such that the solution to (1.3) is global if and only if $a_{m-1} \le \Phi(a_1, \ldots, a_{m-2})$. Moreover, if $a_{m-1} < \Phi(a_1, \ldots, a_{m-2})$, then the solution is stable outside a compact set.

It is also worthy to mention that for the conformal or critical dimension $N = 2m$ with $m \ge 2$, many existence results were established by prescribing the behavior of u at infinity. See [2,12,5] for $m = 2$ and see [8,9] for $m \ge 3$. Clearly, these results imply the existence of many non-radial solutions which are stable outside a compact set.

However, in the supercritical dimensions $N > 2m$ with $m \ge 3$, less is known for stable solutions. Farina and Ferrero raised then the question (see for instance Problem 4.1 (iii) in [7]) about the existence of stable solutions. In this work, we will provide rich examples of stable solutions. First we consider radial solutions to (1.3) and show that the solution is stable if we allow a_{m-1} to be negative enough.

Theorem 1.1. *Let $m \ge 2$ and $N > 2m$. Given any $(a_k)_{0 \le k \le m-2}$, there exists $\beta \in \mathbb{R}$ such that the solution to equation (1.3) is stable in $\mathbb{R}^N$ for any $a_{m-1} \le \beta$.*

Furthermore, given any $N > 2m$, we prove the existence of non-radial stable solution to (1.1) and the existence of stable radial solutions for the following *borderline* situations.

(i) $N > 2m$, $m \ge 3$ is odd, and $(-1)^k a_k > 0$ for any $1 \le k \le m - 1$;
(ii) $N > 2m$, $m \ge 4$ is even, $u(0) = 0$ and $a_{m-1} = \Phi(a_1, \ldots, a_{m-2})$.

The existence of stable radial solutions on the borderline for $m \ge 4$ even in arbitrary supercritical dimension is a new phenomenon comparing to $m = 2$, where the borderline solutions are not stable out of any compact set if $5 \le N \le 12$.

Theorem 1.2. *For $m \ge 3$ odd and $N > 2m$, there exists entire stable solution u of (1.3) satisfying $sign(a_k) = (-1)^k$ for all $1 \le k \le m - 1$.*

Theorem 1.3. *For any $m \geq 3$ and $N > 2m$, there exist non-radial stable solutions to (1.1). Moreover, when $m \geq 4$ is even, there are radial stable solutions on the borderline hypersurface of existence, i.e. when $a_{m-1} = \Phi(a_k)$.*

The proof of Theorem 1.3 is based on the following result, which is inspired by [5], where we construct some stable solutions to (1.1) by super-sub-solution method.

Proposition 1.4. *For any $m \geq 2$ and $N > 2m$, let $P(x)$ be a polynomial verifying*

$$\lim_{|x|\to\infty} \frac{P(x)}{\ln|x|} = \infty \quad \text{and} \quad \deg(P) \leq 2m-2.$$

Then there exists $C_P \in \mathbb{R}$ such that for any $C \geq C_P$, we have a solution u of (1.1) verifying

$$-P(x) - C \leq u(x) \leq -P(x) - C + (1+|x|^2)^{m-\frac{N}{2}} \quad \text{in } \mathbb{R}^N.$$

Consequently, there exists $\widetilde{C}_P \in \mathbb{R}$ such that the above solution u is stable in $\mathbb{R}^N$ for any $C \geq \widetilde{C}_P$.

It will be interesting to know if all radial solutions are stable in high dimensions as for $m = 2$ and $N \geq 13$. Unfortunately, we are not able to answer this question completely, but we can prove that for $m \geq 3$ odd, and a wide class of initial data (a_k), the corresponding radial solutions are always effectively stable in large dimensions.

Theorem 1.5. *Let $m \geq 3$ be odd, then there exists N_0 depending only on m such that for any $N \geq N_0$, the radial solution to (1.3) with $a_k \leq 0$ for $1 \leq k \leq m-1$ is stable in $\mathbb{R}^N$.*

The following Hardy inequalities will play an important role in our study of stability, see Theorem 3.3 in [10]. Let $m \geq 2$ and $N > 2m$. If m is odd, then

$$\lambda_{N,m} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^{2m}} dx \leq \int_{\mathbb{R}^N} |\nabla(\Delta^{\frac{m-1}{2}}\varphi)|^2 dx \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^N),$$

where

$$\lambda_{N,m} := \frac{(N-2)^2}{16^{\frac{m}{2}}} \prod_{i=1}^{\frac{m-1}{2}} (N-4i-2)^2(N+4i-2)^2. \tag{1.5}$$

If m is even, then

$$\mu_{N,m} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^{2m}} dx \leq \int_{\mathbb{R}^N} |\Delta^{\frac{m}{2}}\varphi|^2 dx \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^N),$$

where

$$\mu_{N,m} := \frac{1}{16^{\frac{m}{2}}} \prod_{i=0}^{\frac{m-2}{2}} (N+4i)^2(N-4i-4)^2. \tag{1.6}$$

Theorem 1.1 is proved in Section 2. More examples of stable solutions will be given in Section 3, including the proofs of Theorems 1.2, 1.3 and 1.5.

2. A first existence result

Here we prove Theorem 1.1. We will make use of a well-known comparison result (see for instance Proposition 13.2 in [7]).

Lemma 2.1. *Let $u, v \in C^{2m}([0, R))$ be two radial functions such that $\Delta^m u - e^u \geq \Delta^m v - e^v$ in $[0, R)$ and*

$$\Delta^k u(0) \geq \Delta^k v(0),\ (\Delta^k u)'(0) \geq (\Delta^k v)'(0), \quad \forall\, 0 \leq k \leq m-1. \tag{2.1}$$

Then for any $r \in [0, R)$ we have

$$\Delta^k u(r) \geq \Delta^k v(r), \quad \textit{for all}\ 0 \leq k \leq m-1.$$

Now, we consider radial solutions to the initial value problem (1.3). Denote

$$c_k = \Delta^k(r^{2k}) = \prod_{i=1}^{k} 2i(N-2+2i) \quad \text{for any } k \geq 1. \tag{2.2}$$

Case 1: $m \geq 3$ is odd.

Fix $\Delta^k u(0) = a_k$ for $0 \leq k \leq m-2$. Consider the solution $u_{(a_k)}$ to (1.3) associated to the initial values a_k, $0 \leq k \leq m-1$. We know that the solution is globally defined in $\mathbb{R}^N$ for any (a_k), see [7]. Clearly, the polynomial

$$\Psi(r) = a_0 + \sum_{1\leq k\leq m-1} \frac{a_k}{c_k} r^{2k} \quad \text{with } c_k \text{ given by (2.2)}$$

verifies $\Delta^m \Psi \equiv 0$ in $\mathbb{R}^N$ and $\Delta^k \Psi(0) = a_k$ for all $0 \leq k \leq m-1$.

As $\Delta^m(u_{(a_k)} - \Psi) = -e^{u_{(a_k)}} < 0$, it's easy to check that $u_{(a_k)}(r) < \Psi(r)$ for any $r > 0$. We claim that: $u_{(a_k)}$ is stable when a_{m-1} is small enough. In fact, we need only to get the following estimate:

$$e^{\Psi(r)} \leq \frac{\lambda_{N,m}}{r^{2m}} \quad \text{in } \mathbb{R}^N, \tag{2.3}$$

where $\lambda_{N,m} > 0$ is given by (1.5). Let

$$h(r) = c_{m-1} r^{2-2m} \left[a_0 + \sum_{1\leq k\leq m-2} \frac{a_k}{c_k} r^{2k} + 2m \ln r - \ln \lambda_{N,m} \right].$$

Obviously $\lim_{r\to+\infty} h(r) = 0$ and $\lim_{r\to 0} h(r) = -\infty$. So $H_0 = \sup_{(0,\infty)} h(r) < \infty$ exists and (2.3) holds if $-a_{m-1} \geq H_0$. We conclude that if $a_{m-1} \leq -H_0$,

$$\int_{\mathbb{R}^N} |\nabla(\Delta^{\frac{m-1}{2}}\phi)|^2 dx - \int_{\mathbb{R}^N} e^{u_{(a_k)}}\phi^2 dx \ge \int_{\mathbb{R}^N} |\nabla(\Delta^{\frac{m-1}{2}}\phi)|^2 dx - \int_{\mathbb{R}^N} e^{\Psi}\phi^2 dx$$
$$\ge \int_{\mathbb{R}^N} |\nabla(\Delta^{\frac{m-1}{2}}\phi)|^2 dx - \lambda_{N,m}\int_{\mathbb{R}^N} \frac{\phi^2}{|x|^{2m}} dx \ge 0,$$

i.e. $u_{(a_k)}$ is stable in $\mathbb{R}^N$.

Case 2: m is even.

Let $\Delta^k u(0) = a_k$ for $0 \le k \le m-2$ be fixed. We can check that the scaling $u(\lambda x) + 2m\ln\lambda$ does not affect the stability of the solution, so we can assume that $a_0 = 0$ without loss of generality. By Theorem 2.2 in [7], the solution to (1.4) is global if and only if $a_{m-1} \le \beta_0 = \Phi(a_k)$. For any $a_{m-1} < \beta_0$, consider

$$\Psi(r) = u_{\beta_0}(r) + \frac{(a_{m-1} - \beta_0)r^{2m-2}}{c_{m-1}},$$

then $\Delta^m\Psi = \Delta^m u_{\beta_0} = e^{u_{\beta_0}} \ge e^{\Psi}$. Using Lemma 2.1, we have $u_{(a_k)} \le \Psi$ in $\mathbb{R}^N$ as $\Delta^k\Psi(0) = \Delta^k u_{(a_k)}(0)$ for any $0 \le k \le m-1$. As above, if there holds

$$e^{u_{\beta_0}} e^{\frac{(a_{m-1}-\beta_0)r^{2m-2}}{c_{m-1}}} \le \frac{\mu_{N,m}}{r^{2m}} \quad \text{in } \mathbb{R}^N, \tag{2.4}$$

with $\mu_{N,m}$ given by (1.6), then $u_{(a_k)}$ is stable in $\mathbb{R}^N$. Let

$$g(r) = c_{m-1}r^{2-2m}\left[u_{\beta_0}(r) - \ln\frac{\mu_{N,m}}{r^{2m}}\right] - \beta_0.$$

By [7], the borderline entire solution $u_{\beta_0}(r) = o(r^{2m-2})$ as $r \to \infty$. So $\lim_{r\to+\infty} g(r) = -\beta_0$, $\lim_{r\to 0} g(r) = -\infty$, and (2.4) holds if we take $-a_{m-1} \ge \sup_{(0,\infty)} g$. $\square$

3. More stable solutions

Here we show more examples of stable solutions by proving Theorems 1.2, 1.3 and 1.5.

3.1. *Proof of Theorem 1.2*

Consider u^ε, solution of (1.3) with the initial conditions $a_k = (-1)^k\varepsilon$ for $0 \le k \le m-3$; $a_{m-2} = -\beta$ with $\beta > 0$ and $a_{m-1} = \varepsilon$. Here $\varepsilon \in (0, 1]$ is a small parameter, for simplicity, we will omit the exponent ε in the following. Let

$$\Psi(r) := -\frac{\beta}{c_{m-2}}r^{2m-4} + \varepsilon H(r),$$

where

$$H(r) := 1 + \sum_{k=1}^{m-3}\frac{(-1)^k}{c_k}r^{2k} + \frac{r^{2m-2}}{c_{m-1}} \quad \text{with } c_k \text{ given by (2.2).}$$

Therefore $(-\Delta)^m \Psi \equiv 0$ and $\Delta^k \Psi(0) = \Delta^k u(0)$ for any $0 \le k \le m-1$. Denote also

$$H_+(r) := 1 + \sum_{k=1}^{m-3} \frac{r^{2k}}{c_k} + \frac{r^{2m-2}}{c_{m-1}}.$$

As we have

$$u \le \Psi \le -\frac{\beta}{c_{m-2}} r^{2m-4} + \varepsilon H_+(r) \quad \text{in } [0, \infty),$$

there holds $u(r) \le \varepsilon H_+(1)$ in $[0, 1]$. Denote $\gamma_0 := e^{H_+(1)}$ and consider $v := u - \Psi + \frac{\gamma_0}{c_m} r^{2m}$. Then $\Delta^m v = \Delta^m u + \gamma_0 = -e^u + \gamma_0 \ge 0$ for any $\varepsilon \le 1$ and $r \in [0, 1]$. Since $\Delta^k v(0) = 0$ for any $0 \le k \le m-1$, we get $v \ge 0$ in $[0, 1]$, hence

$$\begin{aligned} u(r) &\ge \varepsilon H(r) - \frac{\beta}{c_{m-2}} r^{2m-4} - \frac{\gamma_0}{c_m} r^{2m} \\ &> -H_+(1) - \frac{\beta}{c_{m-2}} - \frac{\gamma_0}{c_m} =: \xi_0, \quad \forall\, r \in [0, 1],\ \varepsilon \le 1. \end{aligned}$$

Inversely, consider $w := u - \Psi + \frac{e^{\xi_0}}{c_m} r^{2m}$ in $[0, 1]$, there holds $\Delta^m w = e^{\xi_0} - e^u \le 0$ in $[0, 1]$. By Lemma 2.1, we have then $\Delta^k w(r) \le 0$ in $[0, 1]$ for any $0 \le k \le m$, so that for $r \in [0, 1]$,

$$\Delta^{m-1} u(r) \le \varepsilon - \frac{e^{\xi_0}}{2N} r^2, \quad \Delta^{m-2} u(r) \le -\beta + \frac{\varepsilon}{2N} r^2 - \frac{e^{\xi_0}}{8N(N+2)} r^4.$$

Moreover, as $\Delta^{m-1} u$ is decreasing, we have $\Delta^{m-1} u(r) \le \Delta^{m-1} u(1) \le \varepsilon - \frac{e^{\xi_0}}{2N}$ in $(1, \infty)$. Consequently, for $r > 1$,

$$\begin{aligned} \Delta^{m-2} u(r) &= \Delta^{m-2} u(1) + \int_1^r \rho^{1-N} \int_0^\rho s^{N-1} \Delta^{m-1} u(s) ds d\rho \\ &\le -\beta + \frac{\varepsilon}{2N} - \frac{e^{\xi_0}}{8N(N+2)} + \int_1^r \rho^{1-N} \int_0^\rho \left[\varepsilon - e^{\xi_0} \frac{\min(1, s)^2}{2N} \right] s^{N-1} ds d\rho \\ &= -\beta + \varepsilon \frac{r^2}{2N} - e^{\xi_0} \left[\frac{1}{8N(N+2)} + \frac{1}{2N^2} \int_1^r \left(\rho - \frac{2}{N+2} \rho^{1-N} \right) d\rho \right] \\ &= -\beta + \frac{e^{\xi_0}}{8N(N-2)} + \left(\varepsilon - \frac{e^{\xi_0}}{4N^2} \right) r^2 - \frac{e^{\xi_0}}{N^2(N^2-4)} r^{2-N}. \end{aligned}$$

Combining the above estimates, we conclude that if $0 < \varepsilon \le \varepsilon_1 := \min(1, \frac{e^{\xi_0}}{4N^2})$,

$$\Delta^{m-2} u(r) \le -\beta + \frac{e^{\xi_0}}{2N} =: h(\beta) \quad \text{for any } r \in [0, \infty).$$

This yields then for $\varepsilon \le \varepsilon_1$, by Young's inequality,

$$u(r) \le \varepsilon + \varepsilon \sum_{k=1}^{m-3} \frac{(-1)^k}{c_k} r^{2k} + h(\beta)\frac{r^{2m-4}}{c_{m-2}} \le 2\varepsilon_1 + \Big[C_1 + h(\beta)\Big]\frac{r^{2m-4}}{c_{m-2}}, \quad \forall r > 0.$$

As $\lim_{\beta\to\infty} h(\beta) = -\infty$, there exists β_1 large such that $u(r) \le \ln \lambda_{N,m} - 2m \ln r$ in $(0, \infty)$ if $\beta \ge \beta_1$. This means that u is stable for any $0 < \varepsilon \le \varepsilon_1$ and $\beta \ge \beta_1$. $\square$

3.2. *Proof of Proposition 1.4 and Theorem 1.3*

As already mentioned, Theorem 1.3 is a direct consequence of Proposition 1.4. So we will consider firstly Proposition 1.4.

Let P be s polynomial in $\mathbb{R}^N$ with $\deg(P) \le 2m - 2$ such that $\ln |x| = o(P(x))$ as $|x|$ goes to infinity. We are looking for a solution u of the form $u(x) = -P(x) - C + z(x)$ with

$$(-\Delta)^m z(x) = e^{-P(x)-C+z(x)} \text{ in } \mathbb{R}^N \quad \text{and} \quad z(x) = O(|x|^{2m-N}) \text{ as } |x| \to \infty. \tag{3.5}$$

Equivalently, we will resolve the following system:

$$\begin{cases} -\Delta z = (N-2m)(2m-2)v_1 & \text{in } \mathbb{R}^N, \\ -\Delta v_k = (N-2m+2k)(2m-2k-2)v_{k+1} & \text{in } \mathbb{R}^N, \ 1 \le k \le m-2 \\ -\Delta v_{m-1} = d_m e^{-P(x)-C} e^z & \text{in } \mathbb{R}^N. \end{cases} \tag{3.6}$$

Here

$$\frac{1}{d_m} = \prod_{i=1}^{m-1} 2i(N-2i-2).$$

Set $W_j := (1+|x|^2)^{j-\frac{N}{2}}$ for $j \in \mathbb{Z}$, the straightforward calculations yield that

$$-\Delta W_j = (N-2j)(2j-2)W_{j-1} + (N-2j)(N-2j+2)W_{j-2} \quad \text{for any } j \in \mathbb{Z}.$$

Therefore, for $2 \le j < \frac{N}{2}$, we have $-\Delta W_j \ge (N-2j)(2j-2)W_{j-1}$.

Let $N > 2m$,

$$Z(x) := W_m(x) > 0, \quad V_k := W_{m-k}(x) > 0 \text{ for } 1 \le k \le m-1.$$

So $-\Delta Z \ge (N-2m)(2m-2)V_1$, $-\Delta V_k \ge (N-2m+2k)(2m-2k-2)V_{k+1}$ for $1 \le k \le m-2$ and

$$-\Delta V_{m-1} = N(N-2)W_{-1} = N(N-2)(1+|x|^2)^{-1-\frac{N}{2}}.$$

Consider

$$f(x) := -P(x) + \frac{N+2}{2}\ln(1+|x|^2) + \ln d_m - \ln[N(N-2)] + (1+|x|^2)^{m-\frac{N}{2}},$$

by our assumption on P and $m < \frac{N}{2}$, readily $\max_{\mathbb{R}^N} f(x) = C_P < \infty$ exists. For any $C \ge C_P$, we have

$$-\Delta V_{m-1} \ge d_m e^{-P(x)-C_P} e^Z \ge d_m e^{-P(x)-C} e^Z \quad \text{in } \mathbb{R}^N.$$

In other words, $(Z, V_1, \dots, V_{m-1})$ is a super-solution in $\mathbb{R}^N$ to the system (3.6) for $C \ge C_P$.

Since the system (3.6) is cooperative, $(0, 0, \dots, 0)$ and $(Z, V_1, \dots, V_{m-1})$ form a pair of ordered sub- and super-solutions, we obtain the existence of a solution to (3.6), hence a solution of (3.5). Moreover, the solution u satisfies $-P(x) - C \le u(x) \le -P(x) - C + Z(x)$ in $\mathbb{R}^N$.

To ensure the stability of u, it's sufficient to choose C such that

$$e^{u(x)} \le e^{-P(x)-C+Z(x)} \le e^{-P(x)-C+1} \le \frac{\gamma_{N,m}}{|x|^{2m}} \quad \text{in } \mathbb{R}^N, \tag{3.7}$$

where $\gamma_{N,m} = \lambda_{N,m}$ in (1.5) if m is odd and $\gamma_{N,m} = \mu_{N,m}$ given by (1.6) if m is even. Let $g(x) = 1 - \ln \gamma_{N,m} - P(x) + 2m \ln |x|$, clearly $C'_P = \max_{\mathbb{R}^N \setminus \{0\}} g(x) < \infty$ exists since

$$\lim_{|x| \to 0} g(x) = \lim_{|x| \to \infty} g(x) = -\infty.$$

Therefore, if we take $\widetilde{C}_p = \max(C_P, C'_P)$, u is a stable solution in $\mathbb{R}^N$ if $C \ge \widetilde{C}_P$. The proof of Proposition 1.4 is complete. □

Remark 3.1. We do not know if the assumption $\lim_{|x| \to \infty} \frac{P(x)}{\ln |x|} = \infty$ is equivalent or not to the apparently weaker condition $\lim_{|x| \to \infty} P(x) = \infty$.

Proof of Theorem 1.3. Indeed, if P is non-radial in Proposition 1.4, the solution u constructed is clearly non-radial. On the other hand, if P is radial, as our super- and sub-solutions are radial, we can work in the subclass of radial functions to get a radial solution u. So for $m \ge 4$ even, if we consider polynomials $P(r) = \sum_{0 \le k \le j} b_k r^{2k}$ with $b_j > 0$ and $1 \le j \le m - 2$, we obtain radial stable solutions u satisfying $u(r) = o(r^{2m-2})$ at infinity. By [7], such radial solutions must be on the borderline hypersurface $a_{m-1} = \Phi(a_k)$. □

Remark 3.2. For $m \ge 3$ odd, if we take $P(x) = P(r) = b_1 r^2$ with $b_1 > 0$, the radial stable solutions obtained verify that $(-\Delta)^k u(0) > 0$, i.e. $\text{sign}(a_k) = (-1)^k$ for $1 \le k \le m - 1$, since otherwise $u(r) \le -Cr^4$ at infinity, see [7]. The solutions obtained in the proof of Theorem 1.2 are different, because they satisfy $\lim_{r \to \infty} \Delta^{m-1} u < 0$.

3.3. Proof of Theorem 1.5

Our argument is based on the following estimate.

Lemma 3.3. *Let ξ be a radial function in $C^2(\mathbb{R}^N)$. Suppose that $\Delta \xi \ge r^\ell g(r)$ with $\ell > -1$ and g nonincreasing in r, then*

$$\xi(r) \ge \xi(0) + \frac{r^{\ell+2}}{(N+\ell)(\ell+2)} g(r), \quad \forall\, r \ge 0.$$

In fact, we have

$$\xi'(r) \geq r^{1-N} \int_0^r g(s)s^{N-1}s^\ell ds \geq r^{1-N} g(r) \int_0^r s^{N+\ell-1} ds = \frac{r^{\ell+1}}{N+\ell} g(r). \tag{3.8}$$

Integrating again, we get

$$\xi(r) \geq \xi(0) + g(r)\frac{r^{\ell+2}}{(N+\ell)(\ell+2)}.$$

Now consider m odd. Let u be the solution to (1.3) with $a_k \leq 0$ for all $1 \leq k \leq m-1$. Denote $w_k := \Delta^k u$ for $1 \leq k \leq m-1$. As $\Delta^{m-1} w_1 = -e^u < 0$ and $\Delta^k w_1(0) = a_{k+1} \leq 0$ for all $0 \leq k \leq m-2$, we get $w_1 \leq 0$ in $\mathbb{R}^N$, hence u is decreasing in r. By Lemma 3.3, as $-\Delta w_{m-1} = e^u$,

$$-w_{m-1}(r) \geq -a_{m-1}(0) + \frac{r^2}{2N}e^{u(r)} \geq \frac{r^2}{2N}e^{u(r)},$$

so we have

$$-\Delta w_{m-2}(r) = -w_{m-1}(r) \geq \frac{r^2}{2N}e^{u(r)}, \quad \forall\, r > 0.$$

Applying again Lemma 3.3, we obtain

$$-w_{m-2}(r) \geq -a_{m-2} + \frac{r^4}{8N(N+2)}e^{u(r)} \geq \frac{r^4}{8N(N+2)}e^{u(r)}.$$

By induction, for all $1 \leq k \leq m-1$,

$$-w_{m-k}(r) \geq \frac{r^{2k}}{P_k(N)}e^{u(r)} \quad \text{for any } r > 0,$$

where

$$P_k(N) = 2^k k! \prod_{\ell=0}^{k-1}(N+2\ell).$$

In particular, there holds

$$-\Delta u(r) = -w_1(r) \geq \frac{r^{2m-2}}{P_{m-1}(N)}e^{u(r)}, \quad \forall\, r > 0.$$

Using (3.8), we get

$$-u'(r) \geq \frac{r^{2m-1}}{(N+2m-2)P_{m-1}(N)}e^{u(r)}, \quad \forall\, r > 0.$$

Therefore

$$e^{-u(r)} \geq e^{-u(0)} + \int_0^r \frac{s^{2m-1}}{(N+2m-2)P_{m-1}(N)} ds \geq \frac{r^{2m}}{P_m(N)},$$

hence

$$e^{u(r)} \leq \frac{P_m(N)}{r^{2m}} \quad \text{for any } r > 0.$$

As polynomial in N, $\deg(P_m) = m$ while $\deg(\lambda_{N,m}) = 2m$, so there exists N_0 such that for $N \geq N_0$, $P_m(N) \leq \lambda_{N,m}$, then $e^u \leq \frac{P_m(N)}{r^{2m}} \leq \frac{\lambda_{N,m}}{r^{2m}}$ i.e. the solution u is stable in $\mathbb{R}^N$. $\square$

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