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Singularity formation for one dimensional full Euler equations

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Abstract

We investigate the basic open question on the global existence v.s. finite time blow-up phenomena of classical solutions for the one-dimensional compressible Euler equations of adiabatic flow. For isentropic flows, it is well-known that the solutions develop singularity if and only if initial data contain any compression (the Riemann variables have negative spatial derivative). The situation for non-isentropic flow is not quite clear so far, due to the presence of non-constant entropy. In [4], it is shown that initial weak compressions do not necessarily develop singularity in finite time, unless the compression is strong enough for general data. In this paper, we identify a class of solutions of the full (non-isentropic) Euler equations, developing singularity in finite time even though their initial data do not contain any compression. This is in sharp contrast to the isentropic flow.

MSC: 76N15; 35L65; 35L67

Keywords: Compressible Euler equations; Singularity formation; Global regularity; Large data

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1. Introduction

Compressible Euler equations, introduced by Euler, is a fundamental PDE model for compressible inviscid fluids. In spite of its long history and many celebrated achievements, its mathematical theory is still far from completion, even in one space dimension. In this paper, we will address one of the basic open questions on global existence of classical solutions v.s. finite time blow up for the following Cauchy problem of the full (non-isentropic) Euler equations under Lagrangian coordinates

$$\begin{cases} \tau_t - u_x = 0\,, \\ u_t + p_x = 0\,, \\ s_t = 0\,, \\ (\tau, u, s)(x,0) = (\tau_0, u_0, s_0)(x). \end{cases} \tag{1.1}$$

Here x is the Lagrangian spatial variable, $t \in \mathbb{R}^+$ is the time. $\tau = \rho^{-1}$ denotes the specific volume for the density ρ. p, u and s stand for the pressure, the velocity, and the specific entropy, respectively. We further assume that the flow is polytropic ideal gases, such that

$$p = K\, e^{\frac{s}{c_v}}\, \tau^{-\gamma}\,, \quad 1 < \gamma < 3\,, \tag{1.2}$$

where K and c_v are positive constants, see [8] or [23]. For smooth solutions, we see that $s(x,t) = s_0(x) := s(x)$.

Throughout this paper, we assume that initial data (τ_0, u_0, s_0) satisfy the following conditions:

Assumption 1.1. Assume that $(\tau_0(x), u_0(x)) \in C^1(\mathbb{R})$, $s_0(x) \in C^2(\mathbb{R})$, and there are uniform positive constants M_1 and M_2 such that

$$\|(\tau_0, u_0)(x)\|_{C^1} + \|s_0(x)\|_{C^2} \le M_1, \;\; \tau_0 \ge M_2.$$

Furthermore, assume that there exists a positive constant V such that

$$V = \frac{1}{2c_v} \int_{-\infty}^{+\infty} |s_0'(x)|\, dx < \infty\,. \tag{1.3}$$

It is often convenient to choose some new variables. Define

$$m := e^{\frac{s}{2c_v}} > 0, \quad c := \sqrt{-p_\tau} = \sqrt{K\gamma}\, \tau^{-\frac{\gamma+1}{2}}\, e^{\frac{s}{2c_v}}\,, \tag{1.4}$$

and

$$\eta := \int_\tau^\infty \frac{c}{m}\, d\tau = \tfrac{2\sqrt{K\gamma}}{\gamma-1}\, \tau^{-\frac{\gamma-1}{2}} > 0\,, \tag{1.5}$$

where c is the nonlinear Lagrangian sound speed. Direct calculations show that (cf. [2,5])

$$\begin{aligned} \tau &= K_\tau\, \eta^{-\frac{2}{\gamma-1}}\,, \\ p &= K_p\, m^2\, \eta^{\frac{2\gamma}{\gamma-1}}\,, \\ c &= c(\eta, m) = K_c\, m\, \eta^{\frac{\gamma+1}{\gamma-1}}\,, \end{aligned} \tag{1.6}$$

where K_τ, K_p and K_c are positive constants given by

$$K_\tau := \Big(\frac{2\sqrt{K\gamma}}{\gamma-1}\Big)^{\frac{2}{\gamma-1}}\,, \quad K_p := K\, K_\tau^{-\gamma}\,, \quad \text{and} \quad K_c := \sqrt{K\gamma}\, K_\tau^{-\frac{\gamma+1}{2}}\,. \tag{1.7}$$

For C^1 solutions, the problem (1.1) is equivalent to (cf. [9,23])

$$\begin{cases} \eta_t + \frac{c}{m}\, u_x = 0\,, \\ u_t + m\, c\, \eta_x + 2\frac{p}{m}\, m_x = 0\,, \\ m_t = 0\,, \\ (\eta, u, m)(x,0) = (\eta_0, u_0, m_0)(x) = (\eta(\tau_0(x)), u_0(x), m(s_0(x))). \end{cases} \tag{1.8}$$

In the regime of smooth solutions, m is independent of time, we thus fix $m = m(x) = m_0(x)$ in the rest of this paper. Therefore, formally, one can still treat (1.8) as a system of two (significant) equations, with fluxes (pressure) depending on x explicitly. Like in the case of isentropic flows, two truly nonlinear characteristic fields are

$$\frac{dx^+}{dt} = c \quad \text{and} \quad \frac{dx^-}{dt} = -c\,, \tag{1.9}$$

and we denote the corresponding directional derivatives along these by

$$\partial_+ := \frac{\partial}{\partial t} + c\,\frac{\partial}{\partial x} \quad \text{and} \quad \partial_- := \frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}\,,$$

respectively. We further introduce the following Riemann variables

$$w := u - m\,\eta\,, \qquad z := u + m\,\eta\,, \tag{1.10}$$

which vary along characteristics

$$\partial_+ z = \frac{1}{2\gamma}\,\frac{c\, m_x}{m}\,(z-w)\,, \tag{1.11}$$

$$\partial_- w = \frac{1}{2\gamma}\,\frac{c\, m_x}{m}\,(z-w)\,. \tag{1.12}$$

Following the wisdoms of many previous works, cf. [2,4,15], a good choice of derivative variables is

$$\begin{aligned} y &:= m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}\, \eta^{\frac{\gamma+1}{2(\gamma-1)}}\, \big(z_x - \tfrac{2}{3\gamma-1}\, m_x\, \eta\big), \\ q &:= m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}\, \eta^{\frac{\gamma+1}{2(\gamma-1)}}\, \big(w_x + \tfrac{2}{3\gamma-1}\, m_x\, \eta\big)\,, \end{aligned} \tag{1.13}$$

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which satisfy

$$\partial_+ y = a - b\, y^2,$$
$$\partial_- q = a - b\, q^2, \tag{1.14}$$

where

$$a := \frac{K_c}{\gamma}\Big[\frac{\gamma-1}{3\gamma-1}\, m\, m_{xx} - \frac{(3\gamma+1)(\gamma-1)}{(3\gamma-1)^2}\, m_x^2\Big] m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}\, \eta^{\frac{3(\gamma+1)}{2(\gamma-1)}+1},$$
$$b := K_c\, \frac{\gamma+1}{2(\gamma-1)}\, m^{\frac{3(3-\gamma)}{2(3\gamma-1)}}\, \eta^{\frac{3-\gamma}{2(\gamma-1)}}. \tag{1.15}$$

Clearly, if $s_0(x)$ (thus $m(x)$) is a constant, then $a = 0$, and (1.1) reduces to the corresponding Cauchy problem for the well-known p-system for isentropic flow, (1.14) are of Riccati type. The question on the global regularity v.s. finite time blow up for (1.1) turns out to have a clear answer: For isentropic flow, under the Assumption 1.1, the Cauchy problem (1.1) has a unique global-in-time classical solution if and only if

$$y(x,0) \ge 0 \quad \text{and} \quad q(x,0) \ge 0, \quad \text{for all} \quad x \in \mathbb{R}. \tag{1.16}$$

We refer the readers to [12,11,18,14] for small initial data and to [4] for general large initial data, where the density lower bound was obtained.

For general adiabatic flows, a is not constant zero, (1.14) are not of Riccati type, and these different ODE structures lead to different behaviors of solutions. We further remark that, although b is positive as long as vacuum does not appear, a usually will change sign. This makes the analysis even more difficult. When initial data are uniformly small perturbation near a constant equilibrium, T. Liu [18] proved that singularity must develop in finite time if there is some compression (y, or q has negative x derivative) initially. We also refer the readers to [22,19,20,1,13,6,7] for beautiful singularity formation theories in multidimension. In a recent paper [4], Chen, Pan and Zhu focus on the general data without smallness assumption of the solutions. They found that initial weak compressions do not necessarily develop singularity in finite time unless the compression is stronger than a critical strength identified in [4]. Furthermore they find that this critical maximum strength of compression can be attained by this class of initial data admitting global smooth solutions. However, if there is an initial point where the compression is stronger than this critical value, [4] proved that the singularity must develop in finite time. In a recent paper, Zheng [25] extends this result of singularity formation of [4] to the case with general pressure law. The other side of story on the global regularity is much less satisfactory. Many mathematicians tried to find a general initial condition to preserve global regularity without much success, even for small initial data, see [24,17,16] for some unsuccessful attempts. Therefore, it remains as a significant open problem even for small initial data.

One natural question is whether such type of initial data (1.16) can prevent finite time singularity formation from the classical solutions of (1.1). As mentioned earlier, this is an open problem even for small initial data except for some special class of solutions such as those constructed in [4,10,21,26]. In this paper, we will provide a negative answer to this question. Indeed, we shall prove that even under the following stronger condition,

$$y(x,0) > 0 \quad \text{and} \quad q(x,0) > 0, \ \text{for all} \quad x \in \mathbb{R}, \tag{1.17}$$

there are broad class of initial data for (1.1) developing singularity in finite time.

In the next section, we will first present an example to illustrate the (new) major mechanism of singularity formation. Then we shall identify a (broad) class of initial data generating such mechanism for finite time blow up of smooth solutions. In the last section, we will present some further discussion on the global existence of smooth solutions.

2. Main results

We first summarize some useful knowledge about (1.1), mainly cited from [4,5]. From Assumption 1.1, it is clear that there are positive constants M_L, M_U, M_w and M_z such that

$$0 < M_L < m(x) < M_U\,, \quad |z_0(x)| < M_z, \quad |w_0(x)| < M_w\,. \tag{2.1}$$

For $\overline{V} = \frac{V}{2\gamma}$, we now denote N_1, N_2 by

$$N_1 := M_z + \overline{V}\,M_w + \overline{V}\,(\overline{V}\,M_z + \overline{V}^2\,M_w)\,e^{\overline{V}^2},$$
$$N_2 := M_w + \overline{V}\,M_z + \overline{V}\,(\overline{V}\,M_w + \overline{V}^2\,M_z)\,e^{\overline{V}^2}.$$

As known by [5] (using a non-trivial characteristic method), (1.12) gives the following uniform estimates

$$|w| \le N_2,\ |z| \le N_1, \tag{2.2}$$

and the quantity η is bounded by

$$\eta(x,t) \le \frac{N_1+N_2}{2} M_L{}^{\frac{1}{2\gamma}-1} := E_U, \tag{2.3}$$

i.e., the density has an upper bound M_ρ. Therefore, there exists a constant M_c such that

$$c(x,t) = K_c\,m\,\eta^{\frac{\gamma+1}{\gamma-1}} \le M_c = K_c M_U (E_U)^{\frac{\gamma+1}{\gamma-1}}. \tag{2.4}$$

Besides, we define

$$N := \sqrt{\frac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)} M_3}\; E_U^{\frac{3\gamma-1}{2(\gamma-1)}}\; M_L^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}, \tag{2.5}$$

where M_3 is a (best possible) positive constant such that

$$|m\,m_{xx} - \tfrac{3\gamma+1}{3\gamma-1}\,m_x^2| \le M_3. \tag{2.6}$$

Indeed, such N is chosen to be the upper bound of $|\frac{a}{b}|^{\frac{1}{2}}$, see [4],

$$|\frac{a}{b}| \le N^2. \tag{2.7}$$

Thus, combining (2.7) with ODE comparison theorem, [4] obtained the uniform upper bounds for quantities y and q, i.e.,

$$y(x,t) \le \max\left\{N,\ \sup_x\{y(x,0)\}\right\} =: \bar{Y}\,,$$

$$q(x,t) \le \max\left\{N,\ \sup_x\{q(x,0)\}\right\} =: \bar{Q}\,.$$

One of the significant contributions of [4] is the following density lower bound estimate.

Lemma 2.1. *([4]) For $1 < \gamma < 3$, as long as the solutions $(\tau, u, s)(x,t)$ of (1.1) is C^1 on $\mathbb{R} \times [0,t]$, there is a positive constant K_1, such that*

$$\rho(x,t) \ge \left[\tau_0^{\frac{3-\gamma}{4}}(x) + K_1(\bar{Y} + \bar{Q})t\right]^{-\frac{4}{3-\gamma}}, \tag{2.8}$$

where K_1 only depends on γ.

Remark 2.2. In [3], G. Chen improved the density lower bound to the order of $(1+t)^{-1+\delta}$ for any small $\delta > 0$. In order to keep the presentation simple, we did not adopt this new estimate here.

This Lemma implies that there is a positive constant K_0 depending only on γ such that

$$b \ge K_0 M_L^{\frac{3(3-\gamma)}{2(3\gamma-1)}} \left(M_1^{\frac{3-\gamma}{4}} + K_1(\bar{Y} + \bar{Q})t\right)^{-1}. \tag{2.9}$$

Indeed, with the help of this Lemma 2.1 and (2.9), Chen, Pan, and Zhu proved in [4] that under Assumption 1.1, the classical solution of (1.1) must develop singularity in finite time if

$$\inf_x\{y(x,0), q(x,0)\} < -N,$$

where N is defined in (2.5).

The aim of this section is to show that strictly rarefactive condition on initial data, i.e. (1.17), is not sufficient to offer global regularity. In the rest of this section, we first show an explicit example to illustrate the idea, followed by the main theorem of this paper.

2.1. An example

Consider the Cauchy problem of equations (1.1) with the following initial data

$$s_0(x) = arctan(x), \quad \eta_0(x) = m_0^{-\frac{3\gamma-3}{3\gamma-1}} \quad \text{and} \quad u_0 = \epsilon\, arctan(x). \tag{2.10}$$

Here $0 < \epsilon < 1$ is small to be chosen later. Therefore, Assumption 1.1 is satisfied.

Direct computations give

$$y(x,0) = q(x,0) = \frac{\epsilon}{1+x^2}\left(e^{\frac{arctan(x)}{2c_v}}\right)^{-\frac{6}{3\gamma-1}} > 0, \quad \forall x \in \mathbb{R},$$

which checks (1.17).

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We now look at the evolution of y along the following characteristic curve defined by

$$\begin{cases} \frac{dX(t)}{dt} = c, \\ X(0) = 1. \end{cases}$$

Thus, $X(t) > 1$ is an increasing function. Using (2.4), we know for $t > 0$ that

$$1 \le X(t) = 1 + \int_0^t c(X(\sigma), \sigma)\, d\sigma \le 1 + M_c t. \tag{2.11}$$

Recall that y satisfies

$$\begin{cases} \partial_+ y = a - by^2, \\ y(0) = y_0 = y(1,0) > 0, \end{cases} \tag{2.12}$$

and $b > 0$, hence we have, as long as C^1 solution exists up to time $t > 0$,

$$y(X(t), t) < y_0 + \int_0^t a(X(\sigma), \sigma)\, d\sigma. \tag{2.13}$$

We now claim that $a(X(\sigma), \sigma) < 0$ for any $\sigma \ge 0$. Indeed, if we define

$$g(x) = s_{xx} - \frac{1}{c_v(3\gamma - 1)} s_x^2, \tag{2.14}$$

where $s = s_0(x) = \arctan(x)$, it holds that

$$g(x) \le s_{xx} = -\frac{2x}{(1+x^2)^2} < 0, \ \forall\, x \in [1, +\infty),$$

which implies that

$$\begin{aligned} a &= \frac{K_c}{\gamma}\Big[\frac{\gamma-1}{3\gamma-1} m m_{xx} - \frac{(3\gamma+1)(\gamma-1)}{(3\gamma-1)^2} m_x^2\Big] m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}} \eta^{\frac{3(\gamma+1)}{2(\gamma-1)}+1} \\ &= \frac{K_c(\gamma-1)}{\gamma(3\gamma-1)}\Big[m m_{xx} - \frac{(3\gamma+1)}{(3\gamma-1)} m_x^2\Big] m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}} \eta^{\frac{3(\gamma+1)}{2(\gamma-1)}+1} \\ &= \frac{K_c(\gamma-1)}{2c_v\gamma(3\gamma-1)} m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}+2} \eta^{\frac{3(\gamma+1)}{2(\gamma-1)}+1} g(x) < 0. \end{aligned} \tag{2.15}$$

Using (2.10) and Lemma 2.1, we have

$$
\begin{aligned}
&e^{-\frac{\pi}{4c_v}} \leq m \leq e^{\frac{\pi}{4c_v}}, \ K_\tau e^{-\frac{3\pi}{2c_v(3\gamma-1)}} \leq \tau_0 \leq K_\tau e^{\frac{3\pi}{2c_v(3\gamma-1)}} =: \tau_1, \\
&\eta^{\frac{3(\gamma+1)}{2(\gamma-1)}+1} \geq (\frac{2\sqrt{K\gamma}}{\gamma-1})^{\frac{3(\gamma+1)}{2(\gamma-1)}+1}\big[\tau_1^{\frac{3-\gamma}{4}} + K_1(\bar{Y}+\bar{Q})t\big]^{-\frac{5\gamma+1}{3-\gamma}}.
\end{aligned}
\tag{2.16}
$$

On the other hand, simple calculations show that

$$
g'(x) \geq \frac{4x^2}{(1+x^2)^3} > 0, \ \forall\, x \in [1,\infty).
$$

Denoting

$$
K_a = \frac{K_c(\gamma-1)}{2c_v\gamma(3\gamma-1)} e^{-\frac{(15\gamma-13)\pi}{8c_v(3\gamma-1)}} (\frac{2\sqrt{K\gamma}}{\gamma-1})^{\frac{3(\gamma+1)}{2(\gamma-1)}+1},
$$

we thus conclude that, for all $t > 0$, it holds that

$$
\begin{aligned}
a(X(t),t) &\leq K_a g(X(t))\big[\tau_1^{\frac{3-\gamma}{4}} + K_1(\bar{Y}+\bar{Q})t\big]^{-\frac{5\gamma+1}{3-\gamma}} \\
&\leq -K_a \frac{2(1+M_ct)}{(1+(1+M_ct)^2)^2}\big[\tau_1^{\frac{3-\gamma}{4}} + K_1(\bar{Y}+\bar{Q})t\big]^{-\frac{5\gamma+1}{3-\gamma}} \\
&\leq -K_a K_2 \frac{2(1+M_ct)}{(1+(1+M_ct)^2)^2}(1+t)^{-\frac{5\gamma+1}{3-\gamma}},
\end{aligned}
\tag{2.17}
$$

where

$$
K_2 = \big[\tau_1^{\frac{3-\gamma}{4}} + K_1(\bar{Y}+\bar{Q})\big]^{-\frac{5\gamma+1}{3-\gamma}}.
$$

Therefore, there is a positive constant K_3 such that

$$
a(X(t),t) \leq -K_3(1+t)^{-\frac{5\gamma+1}{3-\gamma}-3},
$$

and thus

$$
\int_0^\infty a(X(t),t)\, dt \leq -K_4,
$$

for some positive constant

$$
K_4 = \int_0^\infty K_3(1+t)^{-\frac{5\gamma+1}{3-\gamma}-3}\, dt.
$$

Note that

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$$y_0 = y(1,0) \le \frac{\epsilon}{2}.$$

If we choose

$$\epsilon < K_4,$$

there must exist a $T > 0$, such that

$$y(X(T),T) \le -\frac{1}{2}K_4 < 0. \tag{2.18}$$

We recall (2.12), and the fact that $a < 0$ for $t > 0$. We have

$$\begin{cases} \partial_+ y \le -by^2, \\ y(T) = y(X(T),T) \le -\frac{1}{2}K_4. \end{cases} \tag{2.19}$$

Using Lemma 2.1, and the estimate on b in (2.9), it is clear that the solution for the following ODE

$$\begin{cases} \dfrac{dZ(t)}{dt} = -bZ^2(t), \\ Z(T) = -\frac{1}{2}K_4 < 0, \end{cases}$$

blow up in finite time, say, T^*, and $Z(t)$ tends to $-\infty$ as $t \to T^*-$. Using comparison principle, we know that the solution of (2.19) blows up before T^*, so does the solution of (2.12).

This example shows that (1.17) is not enough to prevent the singularity formation from smooth initial data. In the next subsection, we will extend this example for a class of initial data.

2.2. *Main theorem*

Inspired by Example in the last subsection with initial data (2.10), we can find a class of initial data that the solutions of system (1.1) will develop singularity in finite time even when condition (1.17) is imposed. For this purpose, we give the following definitions.

Definition 2.3. The initial data $(\tau_0, u_0, s_0)(x)$ is said to satisfy the Y-condition, if there exist a point $x^* \in \mathbb{R}$ and a uniformly bounded function $f_1(x) \ge 0$ decreasing on $[x^*, +\infty)$ such that

- (Y1): For all $x \ge x^*$, $g(x) < 0$,
- (Y2): For all $x \ge x^*$, $|g(x)| \ge f_1(x)$,
- (Y3): $y(x^*,0) < \bar{y}_0 = K_5 \int_0^\infty f_1(x^* + M_c\sigma)\big[M_1^{\frac{3-\gamma}{4}} + K_1(\bar{Y} + \bar{Q})\sigma\big]^{-\frac{5\gamma+1}{3-\gamma}} d\sigma$, where $K_5 = \dfrac{K_c(\gamma-1)}{2c_v\gamma(3\gamma-1)} M_L^{\frac{15\gamma-13}{2(3\gamma-1)}} (\dfrac{2\sqrt{K\gamma}}{\gamma-1})^{\frac{3(\gamma+1)}{2(\gamma-1)}+1}$

Definition 2.4. The initial data $(\tau_0, u_0, s_0)(x)$ is said to satisfy the Q-condition, if there exist a point $x_* \in \mathbb{R}$ and a uniformly bounded function $f_2(x) \ge 0$ increasing on $(-\infty, x_*]$ such that

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- (Q1): For all $x \leq x_*$, $g(x) < 0$,
- (Q2): For all $x \leq x_*$, $|g(x)| \geq f_2(x)$,
- (Q3): $q(x_*, 0) < \bar{q}_0 = K_5 \int_0^\infty f_2(x_* - M_c\sigma)\big[M_1^{\frac{3-\gamma}{4}} + K_1(\bar{Y} + \bar{Q})\sigma\big]^{-\frac{5\gamma+1}{3-\gamma}} \, d\sigma$.

Theorem 2.5. *For $1 < \gamma < 3$, under Assumption 1.1, if the initial data $(\tau_0, u_0, s_0)(x)$ satisfies the Y-condition or Q-condition, then the solutions of Cauchy problem (1.1) must blow up in finite time.*

Proof. We only show the proof of y part with Y-condition here. The case for q with Q-condition is similar. In the same spirit of the last subsection, for $X(t)$ defined by

$$\begin{cases} \dfrac{dX(t)}{dt} = c, \\ X(0) = x^*, \end{cases}$$

it is enough to prove that there exists a finite time T such that $y(X(T), T) < 0$, since we can then compare our y equation with the Riccati-type equation by dropping a term with the initial data $y(X(T), T)$ starting from time T.

It is clear that

$$x^* \leq X(t) \leq x^* + M_c t.$$

From (2.12), we have

$$y(X(t), t) < y_0(x^*) + \int_0^t a(X(\sigma), \sigma)) \, d\sigma.$$

Using Lemma 2.1, (Y1) and (Y2) in Y-condition, we have

$$\int_0^\infty a(X(\sigma), \sigma))d\sigma < -\bar{y}_0, \tag{2.20}$$

for $\bar{y}_0$ defined in (Y3) of Y-condition. Therefore, there exists a finite positive time T, such that $y(X(T), T) < 0$. This completes the proof. □

3. Further discussion

In this section, we will discuss the difficulties on the global existence of classical solution for (1.1).

From the last section, cf. Theorem 2.5, the negative part of quantity a, equivalently $g(x)$, is extremely harmful to the regularity of solutions. Indeed, mathematically, if $a \geq 0$ for all $x \in \mathbb{R}$, then there are certain set of rarefactive initial data offering global classical solutions. However, physically, we can not assume $a \geq 0$ for all x, unless s_0 is a constant (for isentropic flow).

Actually, from the relation (2.15), one finds that $a \geq 0$ is equivalent to $m\, m_{xx} - \frac{3\gamma+1}{3\gamma-1} m_x^2 \geq 0$, which is again equivalent to

$$\left(m^{-\frac{2}{3\gamma-1}}\right)_{xx} \leq 0.$$

Therefore, $m^{-\frac{2}{3\gamma-1}}$ is a concave function over $\mathbb{R}$ if $a \geq 0$ for all x. This contradicts to the physical fact that $0 < M_L \leq m(x) \leq M_U$, unless $m(x)$ is a constant.

In order to see what the possibility is for the global regularity when $a < 0$, let's consider the following ODE model (comparing to the equations of y and q when $a < 0$):

$$\begin{cases} \dfrac{dZ(t)}{dt} = -e^{-2t} - Z^2(t), \\ Z(0) = Z_0. \end{cases} \tag{3.1}$$

If the initial data $Z_0 < \dfrac{1}{2} = \displaystyle\int_0^\infty e^{-2t}\, dt$, one can use the same idea as in the proof of Theorem 2.5 to show that $Z(t)$ must blow up in finite time. However, if $Z_0 \geq 1$, (3.1) admits a global solution. To show this, we first consider the case $Z_0 = 1$. Since $Z(t)$ is decreasing, we know that $Z(t) < 1$ for $t > 0$. It is clear that for a short time period, $0 < Z(t) < 1$. We assume that t_1 is the maximum time for $Z(t) > 0$ such that

$$Z(t) > 0, \forall t \in [0, t_1); \text{ and } \lim_{t \to t_1 -} Z(t) = 0.$$

For any $t \in (0, t_1)$, we have

$$\begin{aligned} Z_t = -e^{-2t} - Z^2 &\geq -e^{-2t} - Z \\ (e^t Z)_t &\geq -e^{-t} \\ e^t Z &\geq 1 + e^{-t} - 1 \\ Z(t) &\geq e^{-2t} > 0. \end{aligned}$$

Therefore, $t_1 = \infty$, and (3.1) admits a global solution $0 < Z(t) \leq 1$ if $Z_0 = 1$. Now, if $Z_0 > 1$, again, we have $Z(t) \leq Z_0$ since it is strictly decreasing. If $Z(t) > 1$ for all the time, we have a global solution $Z(t)$ taking value in $[1, Z_0]$. Now, if for some time $t_2 > 0$, $Z(t_2) = 1$. We now consider

$$\begin{cases} \dfrac{dZ(t)}{dt} = -e^{-2t} - Z^2(t), \\ Z(t_2) = 1, \end{cases} \tag{3.2}$$

for $t \geq t_2$. Similar argument like the case of $Z_0 = 1$ gives the following estimate

$$Z(t) \geq (e^{t_2} - e^{-t_2})e^{-t} + e^{-2t} \geq e^{-2t},\ \forall\, t \geq t_2.$$

Therefore, (3.1) has a global solution $Z(t)$ taking values in $(0, Z_0]$ if $Z_0 \geq 1$.

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Inspired by the example (3.1), one may want to ask for strong rarefaction condition, such as

$$y(x,0) > \delta \quad \text{and} \quad q(x,0) > \delta, \ \forall\, x \in \mathbb{R}, \tag{3.3}$$

for some positive number δ, with the hope of global regularity. Unfortunately, this condition contradicts to the Assumption 1.1 of initial data, which implies that $y(x,0)$ and $q(x,0)$ tend to zero when x goes to infinity. Therefore, under Assumption 1.1, it seems very complicated to find a relatively simple set of conditions for the global existence of classical solution for (1.1). For some discussion based on the linear analysis, we also refer the readers to [16].

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