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Trace and inverse trace of Steklov eigenvalues II

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Abstract

This is a continuation of our previous work (Y. Shi and C. Yu, 2016 [21]) on the trace and inverse trace of Steklov eigenvalues. More new inequalities for the trace and inverse trace of Steklov eigenvalues are obtained.

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1. Introduction

For a compact oriented Riemannian manifold (M^n, g) with nonempty boundary, the Dirichlet-to-Neumann map $L^{(p)} : A^p(\partial M) \to A^p(\partial M)$ for differential p-forms maps $\omega \in A^p(\partial M)$ to $i_\nu d\hat{\omega}$ where $\hat{\omega} \in A^p(M)$ is the tangential harmonic extension of ω and ν is the outward unit normal vector on ∂M. This new notion of Dirichlet-to-Neumann map was recently introduced by Raulot and Savo [18]. When $p = 0$, $L^{(0)}$ coincides with the classical Dirichlet-to-Neumann map or Steklov operator essentially introduced by Steklov [22]. The same as the Steklov operator (see [23] for example), $L^{(p)}$ was proved to be a nonnegative self-adjoint first order elliptic

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pseudo-differential operator by Raulot–Savo [18]. Hence, the eigenvalues of $L^{(p)}$ are discrete, and can be listed in ascending order counting multiplicities as follows:

$$0 \le \sigma_1^{(p)} \le \sigma_2^{(p)} \le \cdots \le \sigma_k^{(p)} \le \cdots . \tag{1.1}$$

They are called Steklov eigenvalues for differential p-forms on M.

It is clear that $\sigma_1^{(0)} = 0$ with constant function as the eigenfunction and $\sigma_2^{(0)} > 0$. However, this is not true for $p \ge 1$. Indeed, it is not hard to see that (see [18])

$$\ker L^{(p)} = \mathcal{H}_N^p(M). \tag{1.2}$$

Here

$$\mathcal{H}_N^p(M) = \{\omega \in A^p(M) \mid d\omega = \delta\omega = 0 \text{ and } i_\nu \omega = 0\}. \tag{1.3}$$

By Hodge theory on compact manifolds with nonempty boundary (see [20]),

$$\dim \mathcal{H}_N^p(M) = b_p \tag{1.4}$$

where b_p is the p-th Betti number of M. Hence, the multiplicity of the eigenvalue 0 for $L^{(p)}$ is the same as the p-th Betti number of M.

There has been many works on Steklov eigenvalues (see for example [2–7,9–13,16–19,26,25]) and Dirichlet-to-Neumann map since their importance in mathematical physics (see [15]) and applied mathematics (see [24]). It is really hard to give a complete list for works on estimates of Steklov eigenvalues. One can consult the survey [8] for recent progresses.

In [12], Hersch–Payne–Schiffer proved the following interesting inequality for bounded simply connected planar domain Ω by using harmonic conjugate:

$$\sigma_{p+1}^{(0)} \sigma_{q+1}^{(0)} L(\partial\Omega)^2 \le \begin{cases} (p+q)^2\pi^2 & p+q \text{ is even} \\ (p+q-1)^2\pi^2 & p+q \text{ is odd.} \end{cases} \tag{1.5}$$

Here $L(\partial\Omega)$ means the length of $\partial\Omega$. This result was generalized by Girouard and Polterovich [9] to general surfaces:

$$\sigma_{p+1}^{(0)} \sigma_{q+1}^{(0)} L(\partial M)^2 \le \begin{cases} (\gamma+k)^2(p+q)^2\pi^2 & p+q \text{ is even} \\ (\gamma+k)^2(p+q-1)^2\pi^2 & p+q \text{ is odd.} \end{cases} \tag{1.6}$$

Here M is a compact surface with genus γ and k boundary components. Note that, by setting $p = q$ in (1.5) and (1.6), one can obtain estimates for Steklov eigenvalues that generalized the classical result of Weinstock [26] and a result of Fraser and Schoen [6] respectively.

In [28], Liangwei Yang and the second author generalized (1.5) to higher dimensional case by applying the trick of harmonic conjugate introduced by Hersch–Payne–Schiffer [12] to the new setting of Steklov eigenvalues for differential forms introduced by Raulot–Savo [18]. The result is as follows:

$$\sigma_{1+p}^{(0)} \sigma_{b_{n-2}+q}^{(n-2)} \le \lambda_{b_{n-1}+p+q}(\partial M). \tag{1.7}$$

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Here M is of dimension n and $\lambda_k(\partial M)$ means the k-th eigenvalues of ∂M for the Laplacian operator. This result produces new estimates even in the case of surfaces. Indeed, in [14], Karpukhin proved the following inequality by using (1.7):

$$\sigma_{p+1}^{(0)}\sigma_{q+1}^{(0)}L(\partial M)^2 \le \begin{cases} (p+q+2\gamma+2k-2)^2\pi^2 & p+q \text{ is even} \\ (p+q+2\gamma+2k-1)^2\pi^2 & p+q \text{ is odd.} \end{cases} \tag{1.8}$$

Here M is a compact oriented surface with genus γ and k boundary components. It is clear that inequality (1.8) is sharper than (1.6).

In [12], Hersch, Payne and Schiffer also obtained some sharp estimates on the inverse trace of Steklov eigenvalues on bounded simply connected planar domain Ω with smooth boundary. Their result is:

$$\sum_{i=1}^{2n} \frac{1}{\sigma_{1+i}^{(0)}} \ge \frac{L(\partial\Omega)}{\pi} \sum_{i=1}^{n} \frac{1}{i} \tag{1.9}$$

for any positive integer n. It is not hard to see that the inequality is sharp on the unit disk. This estimate was generalized to general surfaces by the authors in [21].

The equality (1.9) can be reformulated in majorization relations. Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Rearrange its components in descending order as $x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$. Let $y \in \mathbb{R}^n$ be another vector. We say that x is weakly majorized by y, denoted by $x \prec_w y$, if

$$\sum_{i=1}^{m} x_{[i]} \le \sum_{i=1}^{m} y_{[i]}, \tag{1.10}$$

for any $m = 1, 2, \cdots, n$. Furthermore, we say that x is majorized by y, denoted as $x \prec y$, if $x \prec_w y$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Now, using the majorization relations, (1.9) can be reformulated as

$$\frac{L(\partial\Omega)}{\pi}\left(1, \frac{1}{2}, \cdots, \frac{1}{n}\right) \prec_w \left(\frac{1}{\sigma_2^{(0)}} + \frac{1}{\sigma_3^{(0)}}, \frac{1}{\sigma_4^{(0)}} + \frac{1}{\sigma_5^{(0)}}, \cdots, \frac{1}{\sigma_{2n}^{(0)}} + \frac{1}{\sigma_{2n+1}^{(0)}}\right). \tag{1.11}$$

The following two basic majorization principles are useful for producing new inequalities:

(1) if $x \prec_w y$ and f is an increasing convex function, then

$$(f(x_1), f(x_2), \cdots, f(x_n)) \prec_w (f(y_1), f(y_2), \cdots, f(y_n));$$

(2) if $x \prec y$ and f is a convex function, then

$$(f(x_1), f(x_2), \cdots, f(x_n)) \prec_w (f(y_1), f(y_2), \cdots, f(y_n)).$$

By applying the basic majorization principle with $f(t) = t^2$ to (1.11), one has

$$\left(\frac{1}{\sigma_2^{(0)}}\right)^2 + \left(\frac{1}{\sigma_3^{(0)}}\right)^2 + \cdots + \left(\frac{1}{\sigma_{2n+1}^{(0)}}\right)^2 \ge \frac{L(\partial\Omega)^2}{2\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right). \tag{1.12}$$

As mentioned in [21], a weaker version of this inequality can be found in [4]. In fact, the general estimates in [21] can also be obtained in this way.

In [12], the authors also posed the following interesting question: is the following inequality true for bounded simply connected planar domain Ω with smooth boundary:

$$\frac{1}{\sigma_2^{(0)}\sigma_3^{(0)}}+\frac{1}{\sigma_3^{(0)}\sigma_4^{(0)}}+\cdots+\frac{1}{\sigma_{2n}^{(0)}\sigma_{2n+1}^{(0)}}\geq\frac{L(\partial\Omega)^2}{4\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\right)? \tag{1.13}$$

According to the knowledge of the authors, no answer has been found. (1.12) can be also viewed as a weaker version of the inequality. The question of Hersch–Payne–Schiffer is the motivation of the study of this paper.

Furthermore, note that the inequalities (1.7) and results in [21] have a similar feature. The left hand sides of the inequalities depend on the geometry of M by definition while the right hand sides of the inequalities depend only on the intrinsic geometry of ∂M. This may in some sense relate to the interesting problem of determining the geometry of M from the Steklov spectrum or the Steklov operator. In this paper, we obtain new inequalities in a similar feature on the trace or inverse trace of Steklov eigenvalues by combining the tricks in [27] and [28].

The first main result of this paper is the following inequality mixing up trace and inverse trace of Steklov eigenvalues with weights.

Theorem 1.1. *Let (M^n,g) be a compact oriented Riemannian manifold with nonempty boundary. Then, for any positive integer r, s and m,*

$$\left(\sum_{i=1}^{m}\left(a_i\sigma_{b_{n-2}+s+i-1}^{(n-2)}\right)^{\frac{1}{p}}\right)\left(\sum_{i=1}^{m}\left(\frac{c_i}{\sigma_{r+i}^{(0)}}\right)^{\frac{q}{p}}\right)^{-\frac{1}{q}}\leq\left(\sum_{i=1}^{m}\left(\frac{a_i\lambda_{b_{n-1}+r+s+i-1}}{c_i}\right)^{\frac{q^*}{p}}\right)^{\frac{1}{q^*}} \tag{1.14}$$

where $q\geq p\geq 1$, $q>1$, $\frac{1}{q^*}+\frac{1}{q}=1$,

$$a_1\geq a_2\geq\cdots\geq a_m\geq 0 \text{ and} \tag{1.15}$$

$$c_1\geq c_2\geq\cdots\geq c_m>0. \tag{1.16}$$

Here λ_k means the k-th eigenvalue of the Laplacian operator on ∂M.

When $m=1$, Theorem 1.1 gives us (1.7). The weights $a_1,a_2,\cdots a_m$ and $c_1,c_2,\cdots,c_m$ can be used to make the inequality sharper. For example, when M is a simply connected surface, we can choose suitable weights to make (1.14) sharp on the unit disk.

Corollary 1.1. *Let M^2 be a compact oriented simply connected surface. Then, for any positive integer n,*

$$\sum_{i=1}^{n}\frac{\sigma_{2i}^{(0)}+\sigma_{2i+1}^{(0)}}{i^3}\leq\frac{4\sqrt{2}\pi^2}{L(\partial M)^2}\left(\sum_{i=1}^{2n}\left(\frac{1}{\sigma_{1+i}^{(0)}}\right)^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)^{\frac{1}{2}}. \tag{1.17}$$

The equality holds when M is a disk. Letting $n \to \infty$, one has

$$\sum_{i=1}^{\infty} \frac{\sigma_{2i}^{(0)} + \sigma_{2i+1}^{(0)}}{i^3} \leq \frac{4\sqrt{3}\pi^3}{3L(\partial M)^2} \left(\sum_{i=1}^{\infty} \left(\frac{1}{\sigma_{1+i}^{(0)}} \right)^2 \right)^{\frac{1}{2}}. \tag{1.18}$$

Proof. Note that, in this case, $b_0 = 1$ and $b_1 = 0$. Let $p = 1$, $q = 2$, $r = s = 1$, $m = 2n$

$$a_{2i-1} = a_{2i} = \left(\frac{2i\pi}{L(\partial M)} \right)^{-3}, \tag{1.19}$$

and $c_{2i-1} = c_{2i} = 1$ for $i = 1, 2, \cdots, n$ in (1.14). Then, the conclusion follows by noting that

$$\lambda_{2i}(\partial M) = \lambda_{2i+1}(\partial M) = \left(\frac{2i\pi}{L(\partial M)} \right)^2 \tag{1.20}$$

for $i = 1, 2, \cdots$. □

One can produce many inequalities of a similar form with (1.17) which is sharp on the unit disk by choosing suitable weights.

The second main result of this paper is an inequality mixing up different types of inverse traces for Steklov eigenvalues as follows.

Theorem 1.2. *Let (M^n, g) be a compact Riemannian manifold with nonempty boundary. Then, for any positive integer r, s, m and $k = 1, 2, \cdots, m$,*

$$\begin{aligned} &\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} \left(\frac{1}{\sigma_{b_{n-2}+s+i_1-1}^{(n-2)} \sigma_{b_{n-2}+s+i_2-1}^{(n-2)} \cdots \sigma_{b_{n-2}+s+i_k-1}^{(n-2)}} \right)^p + \mu C_{m-1}^{k-1} \sum_{i=1}^{m} \left(\frac{1}{\sigma_{r+i}^{(0)}} \right)^q \\ &\geq \frac{kp+q}{pq} p^{\frac{q}{kp+q}} q^{\frac{kp}{kp+q}} \mu^{\frac{kp}{kp+q}} \times \\ &\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} \left(\frac{1}{\lambda_{b_{n-1}+r+s+i_1-1} \lambda_{b_{n-1}+r+s+i_2-1} \cdots \lambda_{b_{n-1}+r+s+i_k-1}} \right)^{\frac{pq}{kp+q}} \end{aligned} \tag{1.21}$$

where $p > 0$, $q \geq 1$ and $\mu > 0$.

When $p = q$ and $k = \mu = 1$ in (1.21), we have

$$\sum_{i=1}^{m} \left(\frac{1}{\sigma_{r+i}^{(0)}} \right)^q + \sum_{i=1}^{m} \left(\frac{1}{\sigma_{b_{n-2}+r+i}^{(n-2)}} \right)^q \geq 2 \sum_{i=1}^{m} \left(\frac{1}{\lambda_{b_{n-1}+r+s+i-1}} \right)^{\frac{q}{2}}. \tag{1.22}$$

This is a special case of inequality (1.11) in [21]. The weight μ can be used to make the inequality sharper. For example, when M is a simply connected surface, we have the following inequality.

Corollary 1.2. *Let M^2 be compact oriented simply connected surface. Then*

$$\frac{1}{\sigma_2^{(0)}\sigma_3^{(0)}\cdots\sigma_{2n}^{(0)}\sigma_{2n+1}^{(0)}}+\frac{1}{2n}\sum_{i=1}^{2n}\left(\frac{1}{\sigma_{1+i}^{(0)}}\right)^{2n}\geq\frac{L(\partial M)^{2n}}{2^{2n-1}\pi^{2n}(n!)^2}. \tag{1.23}$$

Proof. Let $m=k=2n$, $r=s=1$, $p=1$, $\mu=\frac{1}{2n}$ and $q=2n$ in (1.21). Then, the conclusion follows by noting that

$$\lambda_{2i}=\lambda_{2i+1}=\left(\frac{2i\pi}{L(\partial M)}\right)^2 \tag{1.24}$$

for $i=1,2,\cdots$. □

Note that (1.23) is sharp when $n=1$ and M is the unit disk and is not sharp when $n\geq 2$ on the unit disk.

The strategy to prove Theorem 1.1 and Theorem 1.2 in this paper is similar with that in [21]. The main difference is that we do not apply Courant–Fischer's min–max principle directly to obtain eigenvalue comparison (see Lemma 2.1). This is also a generalization of the key lemma in [28].

The outline of the remaining parts of this paper is as follows. In Section 2, we recall some preliminaries including harmonic conjugate, eigenvalue comparison and matrix inequalities that will be used in Section 3. In Section 3, we prove Theorem 1.1 and Theorem 1.2.

2. Preliminaries

We first prove an eigenvalue comparison in the same spirit with Courant–Fischer's min–max principle. The result generalizes the key lemma in [28]. The proof is similar with that in [28].

Lemma 2.1. *Let (M^n,g) be a compact oriented Riemannian manifold with nonempty boundary and*

$$\epsilon_1,\epsilon_2,\cdots,\epsilon_k,\cdots$$

be a complete orthonormal system of positive Steklov eigenvalues for p-forms according to eigenvalues listed in ascending order. Let V be a finite dimensional subspace of

$$\left\{\omega\in A^p(M)\,\middle|\,\begin{array}{l}\Delta\omega=0,\quad i_\nu\omega=0,\ \omega\perp_{L^2(\partial M)}\mathcal{H}_N^p(M),\\ \text{and }\omega\perp_{L^2(\partial M)}\epsilon_1,\epsilon_2,\cdots,\epsilon_{s-1}\end{array}\right\}.$$

Here $p=0,1,2,\cdots,n-1$, s is a positive integer, ν is the unit outward normal vector on ∂M,

$$\mathcal{H}_N^p(M)=\{\omega\in A^p(M)\mid d\omega=\delta\omega=0\text{ and }i_\nu\omega=0\}, \tag{2.1}$$

and Δ is Hodge–Laplacian operator. Suppose that $\dim V=m$. *Then*

$$\sigma_{b_p+s+k-1}^{(p)}\leq\lambda_k(A) \tag{2.2}$$

for $k = 1, 2, \cdots, m$. Here

$$\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_m(A)$$

are the eigenvalues of the linear transform $A : V \to V$ defined by

$$\int_M (\langle d(A\alpha), d\beta\rangle + \langle \delta(A\alpha), \delta\beta\rangle)\, dV_M = \int_{\partial M} \langle i_\nu d\alpha, i_\nu d\beta\rangle dV_{\partial M} \tag{2.3}$$

for any $\alpha, \beta \in V$, and b_p is the p-th Betti number of M.

Proof. Let $\alpha_1, \alpha_2, \cdots, \alpha_m \in V$ be eigenforms of A for $\lambda_1(A), \lambda_2(A), \cdots, \lambda_m(A)$ respectively. By linear algebra, we can also assume that

$$\int_M \left(\left\langle d\alpha_i, d\alpha_j\right\rangle + \left\langle \delta\alpha_i, \delta\alpha_j\right\rangle\right) dV_M = \delta_{ij}. \tag{2.4}$$

Then

$$\begin{aligned} \int_{\partial M} \left\langle i_\nu d\alpha_i, i_\nu d\alpha_j\right\rangle dV_{\partial M} &= \lambda_i(A) \int_M \left(\left\langle d\alpha_i, d\alpha_j\right\rangle + \left\langle \delta\alpha_i, \delta\alpha_j\right\rangle\right) dV_M \\ &= \lambda_i(A)\delta_{ij}. \end{aligned} \tag{2.5}$$

Let $E_k = \mathrm{span}\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$. Then,

$$E_k \cap \overline{\mathrm{span}\{\epsilon_{s+k-1}, \epsilon_{s+k}, \cdots\}} \ne 0$$

by dimension reasons.

Let $\omega \in E_k \cap \overline{\mathrm{span}\{\epsilon_{s+k-1}, \epsilon_{s+k}, \cdots\}}$ be nonzero. Suppose

$$\omega = \sum_{i=s+k-1}^{\infty} c_i \epsilon_i.$$

Then

$$\begin{aligned} \frac{\int_{\partial M} \langle i_\nu d\omega, i_\nu d\omega\rangle dV_{\partial M}}{\int_M \langle d\omega, d\omega\rangle + \langle \delta\omega, \delta\omega\rangle dV_M} &= \frac{\int_{\partial M} \langle i_\nu d\omega, i_\nu d\omega\rangle dV_{\partial M}}{\int_{\partial M} \langle i_\nu d\omega, \omega\rangle dV_M} \\ &= \frac{\sum_{i=s+k-1}^{\infty} {\sigma_{b_p+i}^{(p)}}^2 c_i^2}{\sum_{i=s+k-1}^{\infty} \sigma_{b_p+i}^{(p)} c_i^2} \\ &\ge \sigma_{b_p+s+k-1}^{(p)}. \end{aligned} \tag{2.6}$$

On the other hand, suppose $\omega = \sum_{i=1}^{k} c_i\alpha_i$, by (2.4) and (2.5),

$$\frac{\int_{\partial M} \langle i_\nu d\omega, i_\nu d\omega\rangle dV_{\partial M}}{\int_M \langle d\omega, d\omega\rangle + \langle \delta\omega, \delta\omega\rangle dV_M} = \frac{\sum_{i=1}^{k} \lambda_i(A)c_i^2}{\sum_{i=1}^{k} c_i^2} \le \lambda_k(A). \tag{2.7}$$

Combining the above two inequalities, we obtain the conclusion. □

Secondly, recall the following result about harmonic conjugate of harmonic functions for higher dimensional manifolds in [28,21].

Lemma 2.2. *Let (M^n, g) be a compact oriented Riemannian manifold with nonempty boundary and u be a harmonic function on M. Suppose that*

$$*du \perp_{L^2(M)} \mathcal{H}_N^{(n-1)}(M). \tag{2.8}$$

Then, there is a unique $\omega \in A^{n-2}(M)$ such that

(1) $d\omega = *du$;
(2) $\delta\omega = 0$;
(3) $i_\nu\omega = 0$ *and*
(4) $\omega \perp_{L^2(\partial M)} \mathcal{H}_N^{n-2}(M)$.

Here

$$\mathcal{H}_N^p = \{\gamma \in A^p(M) \mid d\gamma = \delta\gamma = 0 \text{ and } i_\nu\gamma = 0\}. \tag{2.9}$$

ω is called the harmonic conjugate of u.

Next, recall some matrix inequalities that will be used in the next section. The inequalities are simple and may be well known for experts. However, since we can not find direct reference for them, proofs of them are also given.

Lemma 2.3. *Let A be a $m \times m$ matrix that is positive definite and*

$$\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_m(A) \tag{2.10}$$

be its eigenvalues. Then,

(1) *for any $0 \le p \le 1$ and $a_1 \ge a_2 \ge \cdots \ge a_m \ge 0$,*

$$\sum_{i=1}^{m} (a_i\lambda_i(A))^p \le \sum_{i=1}^{m} (a_i A(i,i))^p; \tag{2.11}$$

(2) *for* $p \geq 1$ *or* $p \leq 0$, *and* $0 < a_1 \leq a_2 \leq \cdots \leq a_m$,

$$\sum_{i=1}^{m} (a_i \lambda_i(A))^p \geq \sum_{i=1}^{m} (a_i A(i,i))^p; \tag{2.12}$$

(3) *for any* $p \leq 0$ *and* $k = 1, 2, \cdots, m$

$$\begin{aligned} &\sum_{1\leq i_1<i_2<\cdots<i_k\leq m} \big(\lambda_{i_1}(A)\lambda_{i_2}(A)\cdots\lambda_{i_k}(A)\big)^p \\ \geq &\sum_{1\leq i_1<i_2<\cdots<i_k\leq m} (A(i_1,i_1)A(i_2,i_2)\cdots A(i_k,i_k))^p . \end{aligned} \tag{2.13}$$

Here $A(i,j)$ *means the* (i,j)*-entry of* A.

Proof.

(1) By Schur's Theorem (see [1]),

$$\{A(1,1), A(2,2), \cdots, A(m,m)\} \prec \{\lambda_1(A), \lambda_2(A), \cdots, \lambda_m(A)\}. \tag{2.14}$$

Note that, $f(t) = -t^p$ is convex for $0 \leq p \leq 1$. By basic majorization principles, we have

$$\begin{aligned} &\{-A(1,1)^p, -A(2,2)^p, \cdots, -A(m,m)^p\} \\ &\prec_w \{-\lambda_1(A)^p, -\lambda_2(A)^p, \cdots, -\lambda_m(A)^p\}. \end{aligned} \tag{2.15}$$

Let $\sigma : \{1, 2, \cdots, m\} \to \{1, 2, \cdots, m\}$ be a permutation such that

$$A(\sigma(1),\sigma(1)) \leq A(\sigma(2),\sigma(2)) \leq \cdots \leq A(\sigma(m),\sigma(m)).$$

Then, by (2.15) and rearrangement inequality,

$$\begin{aligned} &-a_1^p\lambda_1^p - a_2^p\lambda_2^p - \cdots - a_m^p\lambda_m^p \\ &= -(a_1^p - a_2^p)\lambda_1^p - (a_2^p - a_3^p)(\lambda_1^p + \lambda_2^p) - \cdots - (a_{m-1}^p - a_m^p)(\lambda_1^p + \lambda_2^p + \cdots + \lambda_{m-1}^p) \\ &\quad - a_m^p(\lambda_1^p + \lambda_2^p + \cdots + \lambda_m^p) \\ &\geq -(a_1^p - a_2^p)A(\sigma(1),\sigma(1))^p - (a_2^p - a_3^p)(A(\sigma(1),\sigma(1))^p + A(\sigma(2),\sigma(2))^p) - \cdots \\ &\quad - (a_{m-1}^p - a_m^p)(A(\sigma(1),\sigma(1))^p + A(\sigma(2),\sigma(2))^p + \cdots + A(\sigma(m-1),\sigma(m-1))^p) \\ &\quad - a_m^p(A(\sigma(1),\sigma(1))^p + A(\sigma(2),\sigma(2))^p + \cdots + A(\sigma(m),\sigma(m))^p) \\ &= -a_1^p A(\sigma(1),\sigma(1))^p - a_2^p A(\sigma(2),\sigma(2))^p - \cdots - a_m^p A(\sigma(m),\sigma(m))^p \\ &\geq -a_1^p A(1,1)^p - a_2^p A(2,2)^p - \cdots - a_m^p A(m,m)^p . \end{aligned} \tag{2.16}$$

This completes the proof of (1).

(2) Note that $f(t) = t^p$ is convex for $p \geq 1$ or $p \leq 0$. So, by basic majorization principles,

$$\begin{aligned}&\{A(1,1)^p, A(2,2)^p, \cdots, A(m,m)^p\}\\ \prec_w\ &\{\lambda_1(A)^p, \lambda_2(A)^p, \cdots, \lambda_m(A)^p\}.\end{aligned}$$

Then, a similar argument as in (2.16) will give us (2).

(3) Applying (2) to the exterior power $\wedge^k A$ of A (see [1]) with all a_i's being 1, we have

$$\sum_{1\leq i_1<i_2<\cdots<i_r\leq m} \left(\lambda_{i_1}(A)\lambda_{i_2}(A)\cdots\lambda_{i_r}(A)\right)^p \geq \sum_{1\leq i_1<i_2<\cdots<i_r\leq m} \left(A\begin{bmatrix} i_1, i_2, \cdots, i_r \\ i_1, i_2, \cdots, i_r\end{bmatrix}\right)^p. \tag{2.17}$$

Here

$$A\begin{bmatrix} i_1, i_2, \cdots, i_r \\ i_1, i_2, \cdots, i_r\end{bmatrix} = \begin{vmatrix} A(i_1,i_1) & A(i_1,i_2) & \cdots & A(i_1,i_r) \\ A(i_2,i_1) & A(i_2,i_2) & \cdots & A(i_2,i_r) \\ \vdots & \vdots & \cdots & \vdots \\ A(i_r,i_1) & A(i_r,i_2) & \cdots & A(i_r,i_r)\end{vmatrix}. \tag{2.18}$$

By Hadamard's inequality,

$$A\begin{bmatrix} i_1, i_2, \cdots, i_r \\ i_1, i_2, \cdots, i_r\end{bmatrix} \leq A(i_1,i_1)A(i_2,i_2)\cdots A(i_r,i_r). \tag{2.19}$$

So, we have (3). □

Finally, recall the following elementary inequality that will be used in the next section. For completeness, we also give a proof.

Lemma 2.4. *Let $x_1, x_2, \cdots, x_m$ and $p_1, p_2, \cdots, p_m$ be positive numbers. Then*

$$x_1^{p_1} + x_2^{p_2} + \cdots + x_m^{p_m} \geq \frac{1}{p}\left(p_1^{\frac{1}{p_1}} p_2^{\frac{1}{p_2}} \cdots p_m^{\frac{1}{p_m}}\right)^p (x_1x_2\cdots x_m)^p. \tag{2.20}$$

Here

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}. \tag{2.21}$$

Proof. Let $q_i = \frac{p_i}{p}$ for $i = 1, 2, \cdots, m$. Then

$$\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} = 1. \tag{2.22}$$

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By Young's inequality,

$$
\begin{aligned}
& x_1^{p_1}+x_2^{p_2}+\cdots+x_m^{p_m} \\
& =\frac{1}{q_1}\left(q_1^{\frac{1}{q_1}} x_1^p\right)^{q_1}+\frac{1}{q_2}\left(q_2^{\frac{1}{q_2}} x_2^p\right)^{q_2}+\cdots \frac{1}{q_m}\left(q_m^{\frac{1}{q_m}} x_m^p\right)^{q_m} \\
& \geq q_1^{\frac{1}{q_1}} q_2^{\frac{1}{q_2}} \cdots q_m^{\frac{1}{q_m}}\left(x_1 x_2 \cdots x_m\right)^p \\
& =\frac{1}{p}\left(p_1^{\frac{1}{p_1}} p_2^{\frac{1}{p_2}} \cdots p_m^{\frac{1}{p_m}}\right)^p\left(x_1 x_2 \cdots x_m\right)^p .
\end{aligned}
\tag{2.23}
$$

This completes the proof of the inequality. □

3. Proof of the main theorems

In this section, we prove Theorem 1.1 and Theorem 1.2. First, by using Lemma 2.1 and Lemma 2.2, we have the following comparison of eigenvalues.

Lemma 3.1. *Let (M^n, g) be a compact oriented Riemannian manifold with nonempty boundary. Then, for any positive integers r, s and m, there are two $m \times m$ matrices A and B that are both positive definite such that*

(1) $\sigma_{r+i}^{(0)} \leq \lambda_i(A)$;
(2) $\sigma_{b_{n-2}+s+i-1}^{(n-2)} \leq \lambda_i(B)$ *and*
(3) $B(i,i) \leq A^{-1}(i,i)\lambda_{b_{n-1}+r+s+i-1}$,

for $i = 1, 2, \cdots, m$. Here $A^{-1}(i,j)$ and $B(i,j)$ mean the (i,j)-entry of A^{-1} and B respectively, λ_k means the k-th eigenvalue for the Laplacian operator on ∂M, and b_k means the k-th Betti number of M.

Proof. Let

$$\phi_1 = \frac{1}{\sqrt{A(\partial M)}}, \phi_2, \cdots, \phi_k, \cdots$$

be a complete orthonormal system for eigenvalues of the Laplacian operator of ∂M according to eigenvalues

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots .$$

Here $A(\partial M)$ means the area of ∂M. Moreover, let

$$\psi_1, \psi_2, \cdots, \psi_k, \cdots$$

and

$$\epsilon_1, \epsilon_2, \cdots, \epsilon_k, \cdots$$

be complete orthonormal systems for positive Steklov eigenvalues of functions and $(n-2)$-forms respectively, according to eigenvalues listed in ascending order.

By the same argument as in the proof of Theorem 1.1 in [21], there are nonconstant harmonic functions $u_1, u_2, \cdots, u_m$ such that

(1) $*du_i \perp_{L^2(M)} \mathcal{H}_N^{n-1}(M)$;
(2) $u_i \perp_{L^2(\partial M)} \psi_1, \psi_2, \cdots, \psi_{r-1}$;
(3) $\omega_i \perp_{L^2(\partial M)} \epsilon_1, \epsilon_2, \cdots, \epsilon_{s-1}$ where ω_i is the harmonic conjugate of u_i as in Lemma 2.2;
(4) $u_i \in \operatorname{span}\{\hat{\phi}_2, \hat{\phi}_3, \cdots, \hat{\phi}_{b_{n-1}+r+s+i-1}\}$ where $\hat{\phi}_i$ means the harmonic extension of ϕ_i;
(5) $\int_M \langle du_i, du_j\rangle dV_M = \delta_{ij}$

for $i, j = 1, 2, \cdots, m$. For making the argument more self-contained, we sketch the construction of $u_1, u_2, \cdots, u_m$ in the following. Suppose that $u_1, u_2, \cdots, u_{k-1}$ satisfying (1), (2), (3), (4) and

$$\int_M \langle du_i, du_j\rangle dV_M = \delta_{ij} \text{ for } i, j = 1, 2, \cdots, k-1, \tag{3.1}$$

have been constructed. Suppose that

$$u_k = c_2\hat{\phi}_2 + c_3\hat{\phi}_3 + \cdots + c_{b_{n-1}+r+s+k-1}\hat{\phi}_{b_{n-1}+r+s+k-1} \tag{3.2}$$

with c_i's constants to be determined. Note that (1), (2), (3) and

$$\int_M \langle du_k, du_i\rangle dV_M = 0 \text{ for } i = 1, 2, \cdots, k-1 \tag{3.3}$$

make

$$b_{n-1} + (r-1) + s - 1 + k - 1 = b_{n-1} + r + s + k - 3$$

homogeneous linear restrictions on the $b_{n-1}+r+s+k-2$ unknowns $c_2, c_3, \cdots, c_{b_{n-1}+r+s+k-1}$. Because the number of unknowns is greater than the number of homogeneous linear restrictions, there is a nonconstant u_k satisfying (1), (2), (3), (4) and (3.3). By re-scaling u_k, we can suppose that

$$\int_M \langle du_k, du_k\rangle dV_M = 1. \tag{3.4}$$

This gives us the construction of $u_1, u_2, \cdots, u_m$.

Note that

$$\int_M \langle d\omega_i, d\omega_j\rangle dV_M = \int_M \langle du_i, du_j\rangle dV_M = \delta_{ij} \tag{3.5}$$

for $i, j = 1, 2, \cdots, m$.

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Let $V = \text{span}\{u_1, u_2, \cdots, u_m\}$ and $W = \text{span}\{\omega_1, \omega_2, \cdots, \omega_m\}$. Let $A : V \to V$ and $B : W \to W$ be linear transformations on V and W such that

$$\int_M \langle du, dv\rangle dV_M = \int_{\partial M} \langle Au, v\rangle dV_{\partial M} \tag{3.6}$$

for any $u, v \in V$ and

$$\int_M \langle dB\alpha, d\beta\rangle = \int_{\partial M} \langle i_\nu\alpha, i_\nu\beta\rangle dV_{\partial M} \tag{3.7}$$

for any $\alpha, \beta \in W$ respectively. Then, by Courant–Fischer's min–max principle,

$$\sigma^{(0)}_{r+i} \leq \lambda_i(A), \tag{3.8}$$

and by Lemma 2.1,

$$\sigma^{(n-2)}_{b_{n-2}+s+i-1} \leq \lambda_i(B) \tag{3.9}$$

for $i = 1, 2, \cdots, m$.

Denote the matrix of A and B under the bases $\{u_1, u_2, \cdots, u_m\}$ and $\{\omega_1, \omega_2, \cdots, \omega_m\}$ as A and B respectively. Then, by (3.6) and (3.7),

$$A^{-1}(i, j) = \int_{\partial M} \langle u_i, u_j\rangle dV_{\partial M} \text{ and } B(i, j) = \int_{\partial M} \langle i_\nu\omega_i, i_\nu\omega_j\rangle dV_{\partial M}, \tag{3.10}$$

for $i, j = 1, 2, \cdots, m$. Moreover,

$$\begin{aligned}\frac{B(i,i)}{A^{-1}(i,i)} &= \frac{\int_{\partial M} \langle i_\nu\omega_i, i_\nu\omega_i\rangle dV_{\partial M}}{\int_{\partial M} \langle u_i, u_i\rangle dV_{\partial M}}\\ &= \frac{\int_{\partial M} \langle i_\nu * du_i, i_\nu * du_i\rangle dV_{\partial M}}{\int_{\partial M} \langle u_i, u_i\rangle dV_{\partial M}}\\ &= \frac{\int_{\partial M} \langle du_i, du_i\rangle dV_{\partial M}}{\int_{\partial M} \langle u_i, u_i\rangle dV_{\partial M}}\\ &\leq \lambda_{b_{n-1}+r+s+i-1}.\end{aligned} \tag{3.11}$$

This completes the proof of Lemma 3.1. □

Remark 3.1. (3.8) can also be shown by similar arguments as in the proof of (2.1).

Now, we are ready to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let A, B be the matrices in Lemma 3.1. Then, by Lemma 3.1, (2.11) and (2.12), we have

$$
\begin{aligned}
\sum_{i=1}^{m}\left(a_i\sigma^{(n-2)}_{b_{n-2}+s+i-1}\right)^{\frac{1}{p}} &\le \sum_{i=1}^{m}(a_i\lambda_i(B))^{\frac{1}{p}} \\
&\le \sum_{i=1}^{m}(a_iB(i,i))^{\frac{1}{p}} \\
&\le \sum_{i=1}^{m}A^{-1}(i,i)^{\frac{1}{p}}\left(a_i\lambda_{b_{n-1}+r+s+i-1}\right)^{\frac{1}{p}} \\
&= \sum_{i=1}^{m}\left(c_iA^{-1}(i,i)\right)^{\frac{1}{p}}\left(a_ic_i^{-1}\lambda_{b_{n-1}+r+s+i-1}\right)^{\frac{1}{p}} \\
&\le \left(\sum_{i=1}^{m}\left(c_iA^{-1}(i,i)\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\left(\sum_{i=1}^{m}\left(\frac{a_i\lambda_{b_{n-1}+r+s+i-1}}{c_i}\right)^{\frac{q^*}{p}}\right)^{\frac{1}{q^*}} \\
&\le \left(\sum_{i=1}^{m}\left(\frac{c_i}{\lambda_i(A)}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\left(\sum_{i=1}^{m}\left(\frac{a_i\lambda_{b_{n-1}+r+s+i-1}}{c_i}\right)^{\frac{q^*}{p}}\right)^{\frac{1}{q^*}} \\
&\le \left(\sum_{i=1}^{m}\left(\frac{c_i}{\sigma^{(0)}_{r+i}}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\left(\sum_{i=1}^{m}\left(\frac{a_i\lambda_{b_{n-1}+r+s+i-1}}{c_i}\right)^{\frac{q^*}{p}}\right)^{\frac{1}{q^*}}.
\end{aligned}
\tag{3.12}
$$

This completes the proof of Theorem 1.1. □

Similarly as in the proof of Theorem 1.1, by using Lemma 3.1 and Lemma 2.3, we can prove Theorem 1.2.

Proof of Theorem 1.2. Let A and B be the matrices in Lemma 3.1. Then, by Lemma 3.1, (2.13), (2.12) and Lemma 2.4,

$$
\begin{aligned}
&\sum_{1\le i_1<i_2<\cdots<i_k\le m}\left(\frac{1}{\sigma^{(n-2)}_{b_{n-2}+s+i_1-1}\sigma^{(n-2)}_{b_{n-2}+s+i_2-1}\cdots\sigma^{(n-2)}_{b_{n-2}+s+i_k-1}}\right)^{p}+\mu C^{k-1}_{m-1}\sum_{i=1}^{m}\left(\frac{1}{\sigma^{(0)}_{r+i}}\right)^{q} \\
\ge&\sum_{1\le i_1<i_2<\cdots<i_k\le m}\left(\frac{1}{\lambda_{i_1}(B)\lambda_{i_2}(B)\cdots\lambda_{i_k}(B)}\right)^{p}+\mu C^{k-1}_{m-1}\sum_{i=1}^{m}\left(\frac{1}{\lambda_i(A)}\right)^{q} \\
\ge&\sum_{1\le i_1<i_2<\cdots<i_k\le m}\left(\frac{1}{B(i_1,i_1)B(i_2,i_2)\cdots B(i_k,i_k)}\right)^{p}+\mu C^{k-1}_{m-1}\sum_{i=1}^{m}\left(A^{-1}(i,i)\right)^{q} \\
=&\sum_{1\le i_1<i_2<\cdots<i_k\le m}\Bigg[\left(\frac{1}{B(i_1,i_1)B(i_2,i_2)\cdots B(i_k,i_k)}\right)^{p} \\
&+\mu\left(A^{-1}(i_1,i_1)\right)^{q}+\mu\left(A^{-1}(i_2,i_2)\right)^{q}+\cdots+\mu\left(A^{-1}(i_k,i_k)\right)^{q}\Bigg]
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
&\geq \frac{kp+q}{pq} p^{\frac{q}{kp+q}} q^{\frac{kp}{kp+q}} \mu^{\frac{kp}{k+q}} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} \left(\frac{A^{-1}(i_1,i_1)A^{-1}(i_2,i_2)\cdots A^{-1}(i_k,i_k)}{B(i_1,i_1)B(i_2,i_2)\cdots B(i_k,i_k)} \right)^{\frac{pq}{kp+q}} \\
&\geq \frac{kp+q}{pq} p^{\frac{q}{kp+q}} q^{\frac{kp}{kp+q}} \mu^{\frac{kp}{kp+q}} \times \\
&\quad \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} \left(\frac{1}{\lambda_{b_{n-1}+r+s+i_1-1}\lambda_{b_{n-1}+r+s+i_2-1}\cdots\lambda_{b_{n-1}+r+s+i_k-1}} \right)^{\frac{pq}{kp+q}}.
\end{aligned}
$$

This completes the proof of Theorem 1.2. □

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