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Reducibility for wave equations of finitely smooth potential with periodic boundary conditions

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Abstract

In the present paper, the reducibility is derived for the wave equations with finitely smooth and time-quasi-periodic potential subject to periodic boundary conditions. More exactly, the linear wave equation $u_{tt} - u_{xx} + Mu + \varepsilon(V_0(\omega t)u_{xx} + V(\omega t, x)u) = 0,\ x \in \mathbb{R}/2\pi\mathbb{Z}$ can be reduced to a linear Hamiltonian system with a constant coefficient operator which is of pure imaginary point spectrum set, where V is finitely smooth in (t, x), quasi-periodic in time t with Diophantine frequency $\omega \in \mathbb{R}^n$, and V_0 is finitely smooth and quasi-periodic in time t with Diophantine frequency $\omega \in \mathbb{R}^n$. Moreover, it is proved that the corresponding wave operator possesses the property of pure point spectra and zero Lyapunov exponent.

MSC: 35P05; 37K55; 81Q15

Keywords: KAM theory; Reducibility; Quasi-periodic wave operator; Finitely smooth potential; Periodic boundary conditions; Pure-point spectrum

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1. Introduction

In the present paper, we investigate the reducibility of

$$u_{tt}-u_{xx}+Mu+\varepsilon(V_0(\omega t)u_{xx}+V(\omega t,x)u)=0,\ \ x\in\mathbb{R}/2\pi\mathbb{Z}. \tag{1.1}$$

To that end, we need the following conditions:

Assumption A. *Assume $M>0$ is a constant, and V_0, V_1 are C^N-smooth and quasi-periodic in time t with frequency $\omega\in\mathbb{R}^n$, which means, there are hull functions $\mathscr{V}_0(\theta)\in C^N(\mathbb{T}^n;\mathbb{R})$, $\mathscr{V}(\theta,x)\in C^N(\mathbb{T}^n\times[0,2\pi];\mathbb{R})$ such that*

$$V_0(\omega t)=\mathscr{V}_0(\theta)|_{\theta=\omega t},\ V(\omega t,x)=\mathscr{V}(\theta,x)|_{\theta=\omega t}\,,\ \ \mathbb{T}^n=\mathbb{R}^n/2\pi\mathbb{Z}^n,$$

where $N>200n$.

Assumption B. *Assume $\omega\in[1,2]^n\subset\mathbb{R}^n$ satisfies Diophantine conditions*

$$|\langle k,\omega\rangle|\geq\frac{\gamma}{|k|^{n+1}},\ \ k\in\mathbb{Z}^n\setminus\{0\}, \tag{1.2}$$

where γ is a constant and $0<\gamma\ll 1$.

We recall the reducibility problem for a time dependent linear system

$$\dot{x}=A(t)x,\ x\in\mathbb{R}^n, \tag{1.3}$$

where $A(t)$ is an $n\times n$ real or complex value matrix. If $A(t)$ is time T-periodic and continuous, it follows from Floquet theory that there exists a continuous time T-periodic coordinate change

$$x=P(t)y \tag{1.4}$$

such that (1.3) is changed into a constant system

$$\dot{y}=By, \tag{1.5}$$

where B is an $n\times n$ complex value matrix independent of time t. However, there usually does not exist the change (1.4) such that (1.3) is reduced to (1.5) when $A(t)$ is time quasi-periodic. See [18]. Let us consider a special case: $A(t)=\Lambda+\varepsilon Q(t)$, where Λ is a constant, $Q(t)$ is time quasi-periodic and ε is small. The well known KAM (Kolmogorov–Arnold–Moser) theory can be applied to this case. See [11,17,24,27,28], for example. In recent decades, there have been many literatures dealing with the reducibility of time quasi-periodic, infinite dimensional linear systems via KAM technique. One model is the time-quasi-periodic Schrödinger operator

$$\mathbf{i}\,\dot{u}=(H_0+\varepsilon W(\omega t,x,-\mathbf{i}\,\nabla))u,\ \ x\in\mathbb{R}^d\ \text{ or }\ x\in\mathbb{T}^d=\mathbb{R}^d/2\pi\mathbb{Z}^d, \tag{1.6}$$

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where $H_0 = -\triangle + V(x)$ or an abstract self-adjoint (unbounded) operator while the perturbation W is quasi-periodic in time t and it may or may not depend on x or/and ∇. See [2–4,14–16,18,19,34], and the references therein.

Another model is the time-quasi-periodic wave operator or linear wave equation

$$u_{tt} = (-\triangle + \varepsilon V(\phi_0 + \omega t, x; \omega))u. \tag{1.7}$$

Up to now, the reducibility of (1.7) has not been explicitly dealt with. Note that a reducibility procedure has been included in classical KAM for the existence of lower-dimensional invariant tori for infinitely dimensional Hamiltonian partial differential equations. It can be implicitly derived from the classical KAM [13,25,31,35] that (1.7) with $d = 1$ and subject to Dirichlet boundary conditions or periodic boundary conditions can be reduced to a constant coefficient equation for "most[1]" frequency ω, provided that V is analytic. For $d = 1$ and (1.7) with a finitely smooth potential V and subject to Dirichlet boundary conditions, it has been recently proved that (1.7) can still be reduced to a constant system for "most" frequency ω. See [26].

In this paper, we will prove the following reducibility theorem:

Theorem 1.1. *With Assumptions A, B, for any given $0 < \gamma \ll 1$, there exists an ϵ^* with $0 < \varepsilon^* = \varepsilon^*(n, \gamma) \ll \gamma$, and exists a subset $\Pi \subset [1, 2]^n$ with*

$$mes\,\Pi \geq 1 - O(\gamma^{1/3})$$

such that for any $0 < \varepsilon < \varepsilon^*$ and for any $\omega \in \Pi$, there is a quasi-periodic symplectic change such that

$$u_{tt} - u_{xx} + Mu + \varepsilon(V_0(\omega t)u_{xx} + V(\omega t, x)u) = 0, \quad x \in \mathbb{R}/2\pi\mathbb{Z} \tag{1.8}$$

is reduced to a linear Hamiltonian system

$$\begin{cases} \dot{\tilde{q}} = (\Lambda + \varepsilon \tilde{Q})\tilde{p}, \\ \dot{\tilde{p}} = -(\Lambda + \varepsilon \tilde{Q})\tilde{q}, \end{cases} \tag{1.9}$$

where $\Lambda = diag\left(\Lambda_j : j = 0, 1, 2, \cdots\right)$, $\Lambda_0 = \rho\sqrt{M}$, $\Lambda_j = \rho\sqrt{j^2 + M}E_{22}$, ρ is a constant close to 1, E_{22} is a 2×2 unit matrix, and $\tilde{Q} = diag(\tilde{Q}_i : i = 0, 1, 2, \cdots)$ is independent of time with $\tilde{Q}_0 \in \mathbb{R}$, $\tilde{Q}_i$ being a real 2×2 matrix, and $|\tilde{Q}_i| \leq C/i$, $i = 1, 2, \cdots$. Here $|\cdot|$ denotes the sup-norm for real matrices, $mes\,\Pi$ denotes Lebesgue measure for set Π.

The more exact statement of Theorem 1.1 can be found in Theorem 2.1 in Section 2. From Theorem 1.1, the following two corollaries can be obtained.

Corollary 1.1. *With Assumptions A, B, for $\omega \in \Pi$ and $0 < \varepsilon < \varepsilon^*$, the wave operator*

$$\mathscr{L}u(t, x) = (\partial_t^2 - \partial_x^2 + M + \varepsilon(V_0(\omega t)\partial_x^2 + V(\omega t, x))u(t, x), \quad x \in \mathbb{R}/2\pi\mathbb{Z}$$

[1] Here the word "most" means that for a given set $\Pi \subset \mathbb{R}^n$ with Lebesgue measure which is equal to 1, there exists a subset $\Pi_\varepsilon \subset \Pi$ with measure $\Pi \setminus \Pi_\varepsilon \to 0$ as $\varepsilon \to 0$ such that for "any $\omega \in \Pi_\varepsilon$".

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is of pure point spectrum property and of zero Lyapunov exponent.

Corollary 1.2. *With Assumptions A, B, for any $\omega \in \Pi$ and $0 < \varepsilon < \varepsilon^*$, there exists a unique solution $u(t,x)$ with initial values $(u(0,x), u_t(0,x)) = (u_0(x), v_0(x)) \in \mathscr{H}^N \times \mathscr{H}^{N-1}$, which is almost-periodic in time and*

$$\frac{1}{C}(\|u_0\|_{\mathscr{H}^N} + \|v_0\|_{\mathscr{H}^{N-1}}) \le \|u(t)\|_{\mathscr{H}^N} + \|u_t(t)\|_{\mathscr{H}^{N-1}} \le C(\|u_0\|_{\mathscr{H}^N} + \|v_0\|_{\mathscr{H}^{N-1}}),$$

where $C > 0$ is a constant, $\mathscr{H}^N = \mathscr{H}^N(\mathbb{T}^n)$ is the usual Sobolev space.

Remark 1.1. Since $V_0(\omega t)\partial_{xx}$ appears in (1.1), the perturbation is unbounded. This kind of unbounded perturbation, which is of the highest unboundedness, can come from the linearization of some quasi-linear perturbations. For quasi-linear KdV equations and quasi-linear Schrödinger equations, there has been a progress about KAM theory [5–8,10,20–22,29]. It is still an open problem whether or not there exists KAM theory for quasi-linear wave equations. In the present paper, the potential $V_0(\omega t)$ in (1.1) does not depend on the space variable x. The methods of Baldi–Berti–Montalto [5,29,30] and Feola–Procesi [21] are still valid for the $V_0(\omega t)$ in (1.1).

Remark 1.2. Here we would like to compare the results of Theorem 1.1 with some existent results. As mentioned before, without $V_0(\omega t)$, when $d = 1$ and the potential V is analytic, the reducibility of (1.7) can be implicitly derived from the classical KAM theorems. However, there are some differences between the analytic potential V and the finitely smooth one, not to mention the existence of V_0. In this paper, by elegant variable and symplectic changes for several times, the wave equation (1.1) can be written as a linear Hamiltonian system with Hamiltonian

$$H = \langle \widetilde{\Lambda} z, \bar{z}\rangle + \varepsilon\left[\langle \widetilde{R}^{zz}(\theta)z, z\rangle + \langle \widetilde{R}^{z\bar{z}}(\theta)z, \bar{z}\rangle + \langle \widetilde{R}^{\bar{z}\bar{z}}(\theta)\bar{z}, \bar{z}\rangle\right].$$

See (2.17) for more details. The basic task is to search a series of symplectic coordinate changes to eliminate the perturbations $\widetilde{R}^{zz}(\theta)$, $\widetilde{R}^{z\bar{z}}(\theta)$ and $\widetilde{R}^{\bar{z}\bar{z}}(\theta)$ except the averages of the diagonal of $\widetilde{R}^{z\bar{z}}(\theta)$. To this end, the symplectic coordinate changes are the time-1 map of the flow for the Hamiltonian εF where F is of the form

$$F = \langle F^{zz}(\theta)z, z\rangle + \langle F^{z\bar{z}}(\theta)z, \bar{z}\rangle + \langle F^{\bar{z}\bar{z}}(\theta)\bar{z}, \bar{z}\rangle.$$

- When the potential $V(\theta)$ $(\theta = \omega t)$ is analytic in some strip domain $|\mathrm{Im}\theta| \le s_\nu^*$, (where ν is the KAM iteration step), the perturbations $\widetilde{R}^{zz}(\theta)$, $\widetilde{R}^{z\bar{z}}(\theta)$ and $\widetilde{R}^{\bar{z}\bar{z}}(\theta)$ are also analytic in $|\mathrm{Im}\theta| \le s_\nu^*$. An important fact in this analytic case is that s_ν^*'s have a uniform non-zero below bound:

$$s_\nu^* \ge \frac{s_0}{2}, \; s_0 > 0, \;\text{ for all }\; \nu = 1, 2, \cdots.$$

- When the potential $V(\theta)$ is finitely smooth of order N, by using Jackson–Moser–Zehnder approximate lemma, we can still make sure that $\widetilde{R}^{zz}(\theta)$, $\widetilde{R}^{z\bar{z}}(\theta)$ and $\widetilde{R}^{\bar{z}\bar{z}}(\theta)$ are analytic in $|\mathrm{Im}\theta| \le s_\nu$ at the $\nu-$th KAM step. However, the strip width s_ν's have no non-zero below bound. Actually, s_ν goes to zero very rapidly:

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$$s_\nu = \varepsilon_{\nu+1}^{1/N}, \ \ \varepsilon_\nu = \varepsilon^{(4/3)^\nu}, \ \ \nu = 1, 2, \cdots.$$

- For the analytic case, we can prove the Hamiltonian $\varepsilon F = O(\varepsilon_\nu)$ at the $\nu-$th KAM step, because $s_\nu^* \geq \frac{s_0}{2}$. It follows immediately that the new perturbation is $\{\varepsilon F, \varepsilon R\} = O(\varepsilon_\nu^2) = O(\varepsilon_{\nu+1})$.
- For the finitely smooth case, the situation is much more complicated. At this case, we find $\varepsilon F = O(\varepsilon_\nu^{1-\frac{2(3n+4)}{N}})$ at the $\nu-$th KAM step. Thus, for the finitely smooth potential $V \in C^N$, the new perturbation is $\{\varepsilon F, \varepsilon R\} = O(\varepsilon_\nu^{2-\frac{2(3n+4)}{N}})$. In order to guarantee the quadratic convergence of the KAM iterations, $O(\varepsilon_\nu^{2-\frac{2(3n+4)}{N}}) = O(\varepsilon_\nu^{4/3}) = O(\varepsilon_{\nu+1})$, it is necessary to assume the smoothness order $N >> 1$. It is enough to assume $N > 200n$. Clearly, this is not sharp. In this paper, we do not pursuit the lowest smoothness for the potential V.

Remark 1.3. The reducibility of (1.1) with finitely smooth potential V subject to Dirichlet boundary condition has been derived in a recent paper [26]. However, the results on the reducibility between Dirichlet boundary conditions and periodic boundary conditions are different. For Dirichlet boundary conditions, the eigenvalues λ_j $(j = 1, 2, \cdots)$ are simple. Thus, we can reduce the Hamiltonian

$$H = \langle \widetilde{\Lambda} z, \overline{z} \rangle + \varepsilon(\langle \widetilde{R}^{zz}(\theta) z, z \rangle + \langle \widetilde{R}^{z\overline{z}}(\theta) z, \overline{z} \rangle + \langle \widetilde{R}^{\overline{z}\overline{z}}(\theta) \overline{z}, \overline{z} \rangle)$$

to

$$H_\infty = \langle \widetilde{\widetilde{\Lambda}} z, z \rangle,$$

where $\widetilde{\widetilde{\Lambda}} = diag(\widetilde{\lambda}_j : j = 1, 2, \cdots)$ and $\widetilde{\lambda}_j = \sqrt{j^2 + M} + \xi_j$. Moreover, (1.1) can be reduced to

$$u_{tt} - u_{xx} + M_\xi u = 0,$$

where M_ξ is a Fourier multiplier. However, for periodic boundary conditions, the eigenvalues λ_j $(j = 0, 1, \cdots)$ are double:

$$\lambda_0^\sharp = 1, \ \ \lambda_j^\sharp = 2, \ \ j = 0, 1, \cdots.$$

In this case, the Hamiltonian H can be reduced to

$$H_\infty = \langle (\Lambda + \varepsilon \widetilde{Q}) u, \overline{u} \rangle,$$

where Λ and $\widetilde{Q}$ are matrices defined as (1.9), u is a vector defined as (2.21). Although we can still get some dynamical behaviours from this reducibility, (1.1) can not be reduced to a linear wave equation with a Fourier multiplier as in Dirichlet boundary conditions.

Remark 1.4. Since $\lambda_j^\sharp = 2$, the homological equations are no longer scalar. For example, in order to eliminate the term $\langle R^{u\overline{u}}(\theta) u, \overline{u} \rangle$ (see (2.22)–(2.25) for more details), the homological equations have the form:

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$$\omega \cdot \partial_\theta F - \mathbf{i}(\Lambda F - F\Lambda) = R, \tag{1.10}$$

where $F = F(\theta)$ is the unknown matrix of order 2, Λ is a 2×2 constant matrix, $R = R(\theta)$ is known matrix of order 2. It is more complicated to find the solution of this matrix equation (1.10) than that of scalar homological equations. In this case, the delicate small divisor problem becomes one dealing with the inverse of the matrix

$$A := -\langle k, \omega\rangle(1 \otimes 1) + 1 \otimes \Lambda - \Lambda \otimes 1 \tag{1.11}$$

(see (7.5) for more details). A usual method dealing with (1.11) is to investigate $\partial_\omega^4 \det A$. See [11] and [13], for example. In the present paper, we use the variation principle of eigenvalues to deal with the inverse A^{-1}. The advantage of the variation principle of eigenvalues is that the method dealing with scalar small divisor problems [31] can be recovered.

Remark 1.5. In [9], it is proved that there is a quasi-periodic solution for any d-dimensional nonlinear wave equation with a quasi-periodic in time nonlinearity,

$$u_{tt} - \Delta u - V(x)u = \varepsilon f(\omega t, x, u), \;\; x \in \mathbb{T}^d,$$

where the multiplicative potential V is in $C^q(\mathbb{T}^d; \mathbb{R})$, $\omega \in \mathbb{R}^n$ is a non-resonant frequency vector and $f \in C^q(\mathbb{T}^n \times \mathbb{T}^d \times \mathbb{R}; \mathbb{R})$. Because of the application of multi-scale-analysis, it is not clear whether the obtained quasi-periodic solution is linear stable and has zero Lyapunov exponent. As a corollary of Theorem 1.1, we can prove that the quasi-periodic solution by [9] is linear stable and has zero Lyapunov exponent, when $d = 1$.

Remark 1.6. When $d > 1$, it is a well-known open problem that (1.7) subject to Dirichlet or periodic boundary conditions is reduced to a linear Hamiltonian system with a constant coefficient linear operator. See the series of talks by L.H. Eliasson [38–40]. Also see a recent paper [30] where the perturbation is a finite rank operator.

This paper is organized as follows. In Section 2, we redescribe Theorem 1.1 as Theorem 2.1. In Section 3–10, to prove the main results of the paper, some preliminary work and many lemmas will be given. The proof of Theorem 2.1 is in the last section.

2. Passing to Fourier coefficients

Consider the differential equation:

$$\mathscr{L}u = u_{tt} - u_{xx} + Mu + \varepsilon(V_0(\theta)u_{xx} + V(\omega t, x)u) = 0 \tag{2.1}$$

subject to periodic boundary condition

$$u(t, x) = u(t, x + 2k\pi), \;\; k \in \mathbb{Z}. \tag{2.2}$$

It is well-known that the Sturm–Liouville problem

$$-y'' + My = \lambda y, \;\; ' := \frac{d}{dx}, \;\; x \in \mathbb{R}/2\pi\mathbb{Z}$$

has the eigenvalues and eigenfunctions, respectively,

$$\lambda_k = k^2 + M, \quad k \in \mathbb{Z},$$
$$\phi_k(x) = e^{\mathbf{i}kx}, \quad k \in \mathbb{Z}.$$

Set $-\partial_{xx} + M$ as D, the wave equation can be seen as

$$u_{tt} = -Du + \varepsilon V_0(\omega t)Du - \varepsilon V_1(\omega t, x)u, \tag{2.3}$$

where $V_1(\omega t, x) = V(\omega t, x) + MV_0(\omega t)$. Let $u_t = v$, we have

$$v_t = -(1 - \varepsilon V_0(\omega t))Du - \varepsilon V_1(\omega t, x)u. \tag{2.4}$$

Step 1: Rescale

$$\begin{cases} u = \beta(\theta)|D|^{-\frac{1}{4}}q, \\ v = (\beta(\theta))^{-1}|D|^{\frac{1}{4}}p. \end{cases}$$

Then

$$\begin{cases} q_t = \frac{1}{\beta^2(\theta)}|D|^{\frac{1}{2}}p - \frac{\omega\cdot\partial_\theta\beta(\theta)}{\beta(\theta)}q, \\ p_t = -(1 - \varepsilon V_0(\omega t))\beta^2(\theta)|D|^{\frac{1}{2}}q + \frac{\omega\cdot\partial_\theta\beta(\theta)}{\beta(\theta)}p - \varepsilon|D|^{-\frac{1}{4}}\beta^2(\theta)V_1(\omega t, x)|D|^{-\frac{1}{4}}q. \end{cases}$$

Choose a suitable $\beta(\theta)$, such that $\beta(\theta) = (1 - \varepsilon V_0(\theta))^{-\frac{1}{4}}$. Then

$$\frac{1}{\beta^2(\theta)} = (1 - \varepsilon V_0(\omega t))\beta^2(\theta) \triangleq a_0(\theta).$$

Also, set $\frac{\omega\cdot\partial_\theta\beta(\theta)}{\beta(\theta)} = \varepsilon a_1(\theta)$, $\beta^2(\theta)V_1(\theta, x) = \widetilde{V}_1(\theta, x)$, we have

$$\begin{cases} q_t = a_0(\theta)|D|^{\frac{1}{2}}p - \varepsilon a_1(\theta)q, \\ p_t = -a_0(\theta)|D|^{\frac{1}{2}}q + \varepsilon a_1(\theta)p - \varepsilon|D|^{-\frac{1}{4}}\widetilde{V}_1(\theta, x)|D|^{\frac{1}{4}}q. \end{cases}$$

Clearly, we can see a_0, $\tilde{V}_1 \in C^N(\mathbb{T}^n \times [0, 2\pi], \mathbb{R})$ and $a_1 \in C^{N-1}(\mathbb{T}^n \times [0, 2\pi], \mathbb{R})$.

Step 2: Now we consider the complex variables

$$z = \frac{q - \mathbf{i}p}{\sqrt{2}}, \quad \bar{z} = \frac{q + \mathbf{i}p}{\sqrt{2}}.$$

Then, we have

$$\begin{cases} \omega\cdot\partial_\theta z = \mathbf{i}a_0(\theta)|D|^{\frac{1}{2}}z - \varepsilon a_1(\theta)\bar{z} + \varepsilon\mathbf{i}|D|^{-\frac{1}{4}}\frac{\widetilde{V}_1(\theta,x)}{2}|D|^{-\frac{1}{4}}(z+\bar{z}), \\ \omega\cdot\partial_\theta \bar{z} = -\mathbf{i}a_0(\theta)|D|^{\frac{1}{2}}\bar{z} - \varepsilon a_1(\theta) z - \varepsilon\mathbf{i}|D|^{-\frac{1}{4}}\frac{\widetilde{V}_1(\theta,x)}{2}|D|^{-\frac{1}{4}}(z+\bar{z}). \end{cases} \tag{2.5}$$

Step 3: Now we introduce a time variable change, a diffeomorphism of the torus $\mathbb{T}^n$ of the form

$$\vartheta = \theta + \omega a(\theta), \quad \theta = \vartheta + \omega\tilde{a}(\vartheta). \tag{2.6}$$

For any function $h(\theta,x)$ and $\tilde{h}(\vartheta,x)$, we introduce operators A and A^{-1}, where

$$\begin{aligned} h(\theta,x) &= (A^{-1}h)(\vartheta,x) = [h](\vartheta,x) = h(\vartheta+\omega\tilde{a}(\vartheta),x), \\ \tilde{h}(\vartheta,x) &= (A\tilde{h})(\theta,x) = \tilde{h}(\theta+\omega a(\theta),x). \end{aligned} \tag{2.7}$$

Our aim is to rewrite the equation (2.5) in the new time variable ϑ. Thus, we can set

$$\begin{gathered} z(\theta,t) = z(\vartheta+\omega\tilde{a}(\vartheta),x) = [z](\vartheta,x), \\ a_i(\theta) = a_i(\vartheta+\omega\tilde{a}(\vartheta)) = [a_i](\vartheta),\ i=0,1, \\ \widetilde{V}_1(\theta,x) = \widetilde{V}_1(\vartheta+\omega\tilde{a}(\vartheta),x) = [\widetilde{V}_1](\vartheta,x), \\ 1+\omega\partial_\theta a(\theta) = 1+\omega\partial_\theta a(\vartheta+\omega\tilde{a}(\vartheta)) = [1+\omega\partial_\theta a](\vartheta), \end{gathered} \tag{2.8}$$

$$\mathscr{T}: \begin{cases} \omega\cdot\partial_\vartheta[z] = \mathbf{i}\frac{[a_0]}{[1+\omega\partial_\theta a]}|D|^{\frac{1}{2}}[z] - \varepsilon\frac{[a_1]}{[1+\omega\partial_\theta a]}[\bar{z}] + \varepsilon\mathbf{i}|D|^{-\frac{1}{4}}\frac{[\widetilde{V}_1(\theta,x)]}{2[1+\omega\partial_\theta a]}|D|^{-\frac{1}{4}}([z]+[\bar{z}]), \\ \omega\cdot\partial_\vartheta[\bar{z}] = -\mathbf{i}\frac{[a_0]}{[1+\omega\partial_\theta a]}|D|^{\frac{1}{2}}[\bar{z}] - \varepsilon\frac{[a_1]}{[1+\omega\partial_\theta a]}[z] - \varepsilon\mathbf{i}|D|^{-\frac{1}{4}}\frac{[\widetilde{V}_1(\theta,x)]}{2[1+\omega\partial_\theta a]}|D|^{-\frac{1}{4}}([z]+[\bar{z}]). \end{cases}$$

We want to choose a function a so that $[a_0]$ is proportional to $[1+\omega\partial_\theta a]$. Thus, it is enough to solve the equation

$$\rho(1+\omega\partial_\theta a(\theta)) = a_0(\theta), \quad \rho\in\mathbb{R}. \tag{2.9}$$

Integrating on $\mathbb{T}^n$ we fix the value of ρ as

$$\rho = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} a_0(\theta)d\theta. \tag{2.10}$$

By (2.9), we get

$$a(\theta) = (\omega\cdot\partial_\theta)^{-1}[\frac{a_0}{\rho} - 1](\theta). \tag{2.11}$$

For notational simplicity, rename ϑ, $[z]$, $[\bar{z}]$, $\frac{[a_1]}{[1+\omega\partial_\theta a]}$, $\frac{[\widetilde{V}_0(\theta,x)]}{[1+\omega\partial_\theta a]}$ as θ z, $\bar{z}$, b_0, V. Then, we have

$$\mathscr{T}: \begin{cases} z_t = \mathbf{i}\rho|D|^{\frac{1}{2}}z - \varepsilon b_0\bar{z} + \varepsilon\mathbf{i}|D|^{-\frac{1}{4}}\frac{V}{2}|D|^{-\frac{1}{4}}(z+\bar{z}), \\ \bar{z}_t = -\mathbf{i}\rho|D|^{\frac{1}{2}}\bar{z} - \varepsilon b_0 z - \varepsilon\mathbf{i}|D|^{-\frac{1}{4}}\frac{V}{2}|D|^{-\frac{1}{4}}(z+\bar{z}). \end{cases}$$

By Sobolev embedding theorem and inverse function theorem, we see $a \in C^{N-2n-2}(\mathbb{T}^n \times [0, 2\pi])$ and $\tilde{a} \in C^{N-2n-2}(\mathbb{T}^n \times [0, 2\pi])$. Thus, we can get b_0, $V \in C^{N-2n-3}(\mathbb{T}^n \times [0, 2\pi])$. In the following section, we will rename $N - 2n - 3$ as N for notational simplicity.

Make the ansatz

$$z(t,x) = \mathscr{S}(z_k) = \sum_{k \in \mathbb{Z}} z_k(t)\phi_k(x), \quad \bar{z}(t,x) = \mathscr{S}(\bar{z}_k) = \sum_{k \in \mathbb{Z}} \bar{z}_k(t)\phi_k(x) \tag{2.12}$$

and

$$V(\omega t, x) = \sum_{k \in \mathbb{Z}} v_k(\omega t)\phi_k(x).$$

Then (2.1) can be transformed as

$$\begin{aligned} \frac{dz_k}{dt} &= \mathbf{i}\rho\sqrt{\lambda_k} z_k - \varepsilon b_0 \bar{z}_k + \mathbf{i}\varepsilon \sum_{l \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \frac{c_{jlk} v_j}{2\sqrt[4]{\lambda_k \lambda_l}} (z_l + \bar{z}_l), \\ \frac{d\bar{z}_k}{dt} &= -\mathbf{i}\rho\sqrt{\lambda_k} \bar{z}_k - \varepsilon b_0 z_k - \mathbf{i}\varepsilon \sum_{l \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \frac{c_{jlk} v_j}{2\sqrt[4]{\lambda_k \lambda_l}} (z_l + \bar{z}_l), \end{aligned} \tag{2.13}$$

where

$$c_{jlk} = \int_0^{2\pi} e^{\mathbf{i}(j+l-k)x}\, dx = \begin{cases} 0, & j+l-k \neq 0, \\ 2\pi, & j+l-k = 0. \end{cases} \tag{2.14}$$

Endow a symplectic transformation with $-\mathbf{i} dz \wedge d\bar{z}$. Thus (2.13) is changed into

$$\begin{cases} \dot{z}_k = \mathbf{i}\,\frac{\partial H}{\partial \bar{z}_k}, & k \in \mathbb{Z}, \\ \dot{\bar{z}}_k = -\mathbf{i}\,\frac{\partial H}{\partial z_k}, & k \in \mathbb{Z}, \end{cases} \tag{2.15}$$

where

$$H(z, \bar{z}) = \sum_{k \in \mathbb{Z}} \rho\sqrt{\lambda_k} z_k \bar{z}_k + \varepsilon \mathbf{i} \sum_{k \in \mathbb{Z}} b_0 \Big(\frac{\bar{z}_k^2 - z_k^2}{2}\Big) + \varepsilon \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} c_{jlk} \frac{v_j(\theta)}{2\sqrt[4]{\lambda_k \lambda_l}} (z_l + \bar{z}_l)(z_k + \bar{z}_k). \tag{2.16}$$

For two sequences $x = (x_j \in \mathbb{C},\ j \in \mathbb{Z})$, $y = (y_j \in \mathbb{C},\ j \in \mathbb{Z})$, define

$$\langle x, y \rangle = \sum_{j \in \mathbb{Z}} x_j y_j.$$

Then we can rewrite (2.16) as follows:

$$H(z,\overline{z})=\langle\rho\widetilde{\Lambda}z,\overline{z}\rangle+\varepsilon\mathbf{i}\frac{b_0}{2}\Big(\langle\overline{z},\overline{z}\rangle-\langle z,z\rangle\Big)+\varepsilon\Big[\langle\widetilde{R}^{zz}(\theta)z,z\rangle+\langle\widetilde{R}^{z\overline{z}}(\theta)z,\overline{z}\rangle+\langle\widetilde{R}^{\overline{z}\overline{z}}(\theta)\overline{z},\overline{z}\rangle\Big],\tag{2.17}$$

where

$$\widetilde{\Lambda}=diag\left(\sqrt{\lambda_j}:j\in\mathbb{Z}\right),\ \ \theta=\omega t,$$

$$\widetilde{R}^{zz}(\theta)=\left(\widetilde{R}^{zz}_{kl}(\theta):k,l\in\mathbb{Z}\right),\ \ \widetilde{R}^{zz}_{kl}(\theta)=\frac{1}{2}\sum_{j\in\mathbb{Z}}\frac{c_{jlk}v_j(\theta)}{\sqrt[4]{\lambda_k}\sqrt[4]{\lambda_l}},\tag{2.18}$$

$$\widetilde{R}^{z\overline{z}}(\theta)=\left(\widetilde{R}^{z\overline{z}}_{kl}(\theta):k,l\in\mathbb{Z}\right),\ \ \widetilde{R}^{z\overline{z}}_{kl}(\theta)=\sum_{j\in\mathbb{Z}}\frac{c_{jlk}v_j(\theta)}{\sqrt[4]{\lambda_k}\sqrt[4]{\lambda_l}},\tag{2.19}$$

$$\widetilde{R}^{\overline{z}\,\overline{z}}(\theta)=\left(\widetilde{R}^{\overline{z}\,\overline{z}}_{kl}(\theta):k,l\in\mathbb{Z}\right),\ \ \widetilde{R}^{\overline{z}\,\overline{z}}_{kl}(\theta)=\frac{1}{2}\sum_{j\in\mathbb{Z}}\frac{c_{jlk}v_j(\theta)}{\sqrt[4]{\lambda_k}\sqrt[4]{\lambda_l}}.\tag{2.20}$$

For the sequence $z=(z_j\in\mathbb{C},\ j\in\mathbb{Z})$, we can rewrite z as

$$z=(z_0,\ z_j,\ z_{-j}:\ j=1,2,\cdots)\triangleq u=(u_j:\ j=0,1,2,\cdots),\tag{2.21}$$

where $u_0=z_0,\ u_j=(z_j,\ z_{-j})^T,\ j=1,2,\cdots$. Here $(z_j,\ z_{-j})^T$ denotes the transpose of the vector $(z_j,\ z_{-j})$. Let $\Lambda_0=\sqrt{\lambda_0}$, $\Lambda_j=\begin{pmatrix}\sqrt{\lambda_j} & 0\\ 0 & \sqrt{\lambda_{-j}}\end{pmatrix}$, $j=1,2,\cdots$. Note that $\lambda_j=\lambda_{-j}=j^2+M,\ j=1,2,\cdots$. Then $\Lambda_j=\sqrt{\lambda_j}E_{22},\ j=1,2,\cdots$, where E_{22} is a 2×2 unit matrix. For $u_j=(z_j,z_{-j})^T$ and $\widetilde{u}_j=(\widetilde{z}_j,\widetilde{z}_{-j})^T$, define $u_j\cdot\widetilde{u_j}=z_j\widetilde{z}_j+z_{-j}\widetilde{z}_{-j}$, $j=1,2,\cdots$.
Then we can also rewrite (2.16) as

$$\widetilde{H}=\langle\rho\Lambda u,\overline{u}\rangle+\varepsilon\mathbf{i}\frac{b_0}{2}\Big(\langle\overline{u},\overline{u}\rangle-\langle u,u\rangle\Big)+\varepsilon\Big[\langle R^{uu}(\theta)u,u\rangle+\langle R^{u\overline{u}}(\theta)u,\overline{u}\rangle+\langle R^{\overline{uu}}(\theta)\overline{u},\overline{u}\rangle\Big],\tag{2.22}$$

where

$$\Lambda=diag\left(\Lambda_j:j=0,1,2,\cdots\right),\ \ \theta=\omega t,$$

$$R^{uu}(\theta)=\left(R^{uu}_{kl}(\theta):k,l=0,1,2,\cdots\right),\ R^{u\overline{u}}(\theta)=\left(R^{u\overline{u}}_{kl}(\theta):k,l=0,1,2,\cdots\right),\tag{2.23}$$

$$R^{\overline{u}\,\overline{u}}(\theta)=\left(R^{\overline{u}\,\overline{u}}_{kl}(\theta):k,l=0,1,2,\cdots\right),\ \ R^{uu}_{kl}(\theta)=R^{\overline{u}\,\overline{u}}_{kl}(\theta)=\frac{1}{2}R^{u\overline{u}}_{kl}(\theta),\tag{2.24}$$

where

$$R^{uu}_{kl}(\theta)=\begin{cases} R_{0,0}(\theta), & k=l=0; \\ (R_{0,l}(\theta),\ R_{0,-l}(\theta)), & k=0,\ l=1,2,\cdots; \\ (R_{k,0}(\theta),\ R_{-k,0}(\theta))^T, & l=0,\ k=1,2,\cdots; \\ \begin{pmatrix} R_{k,l}(\theta) & R_{k,-l}(\theta) \\ R_{-k,l}(\theta) & R_{-k,-l}(\theta) \end{pmatrix}, & k,l=1,2,\cdots, \end{cases} \tag{2.25}$$

and

$$R_{k,l}(\theta)=\frac{1}{2}\sum_{j\in\mathbb{Z}}\frac{c_{jlk}v_j(\theta)}{\sqrt[4]{\lambda_k}\sqrt[4]{\lambda_l}},\quad k,l\in\mathbb{Z}.$$

Define a Hilbert space $h_{\widetilde{N}}$ as follows:

$$h_{\widetilde{N}}=\{x=(x_k\in\mathbb{C}:k\in\mathbb{Z}):\|x\|^2_{\widetilde{N}}=\sum_{k\in\mathbb{Z}}|k|^{2N}|x_k|^2\}. \tag{2.26}$$

Similarly define a Hilbert space h_N as follows:

$$h_N=\{y=(y_k:k=0,1,\cdots):\|y\|^2_N=\sum_{k=0}^{\infty}|k|^{2N}|y_k|^2\}, \tag{2.27}$$

where $y_0\in\mathbb{C}$, $y_k=(z_k,\ z_{-k})^T,\ z_k,\ z_{-k}\in\mathbb{C},\ k=1,2,\cdots$, and $|y_k|^2=|z_k|^2+|z_{-k}|^2$. In (2.26) and (2.27), we define $|k|^{2N}=1$, if $k=0$. For $z=(z_0,\ z_j,\ z_{-j}:\ j=1,2,\cdots)\in h_{\widetilde{N}},\ \ u=(u_j: j=0,1,2,\cdots)\in h_N$, where $u_0=z_0$, $u_j=(z_j,\ z_{-j})^T,\ j=1,2,\cdots$. It can be obtained that

$$\|u\|_N=\|z\|_{\widetilde{N}}.$$

Recall that

$$\mathscr{V}(\theta,x)\in C^N(\mathbb{T}^n\times[0,2\pi],\mathbb{R}).$$

Note that the Fourier transformation (2.12) is isometric from $u\in\mathscr{H}^N[0,2\pi]$ to $(u_k:k=0,1,\cdots)\in h_N$, where $\mathscr{H}^N[0,2\pi]$ is the usual Sobolev space.

Now we state a lemma, which will be used in the next section.

Lemma 2.1.

$$\begin{aligned} &\sup_{\theta\in\mathbb{T}^n}\|\sum_{|\alpha|\le N}\partial^\alpha_\theta JR^{uu}(\theta)J\|_{h_N\to h_N}\le C, \\ &\sup_{\theta\in\mathbb{T}^n}\|\sum_{|\alpha|\le N}\partial^\alpha_\theta JR^{u\overline{u}}(\theta)J\|_{h_N\to h_N}\le C, \\ &\sup_{\theta\in\mathbb{T}^n}\|\sum_{|\alpha|\le N}\partial^\alpha_\theta JR^{\overline{u}\,\overline{u}}(\theta)J\|_{h_N\to h_N}\le C, \end{aligned} \tag{2.28}$$

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where $||\cdot||_{h_N\to h_N}$ *is the operator norm from* h_N *to* h_N*, and* $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$*,* $|\alpha|=|\alpha_1|+|\alpha_2|+\cdots+|\alpha_n|$*,* α_j*'s are positive integers, and* $J=diag(J_j:j=0,1,\cdots)$*,* $J_0=\sqrt[4]{\lambda_0}$*,* $J_j=\sqrt[4]{\lambda_j}E_{22}$*,* $j=1,2,\cdots$*.*

Proof. By (2.23), (2.24) and (2.25), we have that

$$\partial_\theta^\alpha JR^{uu}(\theta)J\triangleq(A_{kl}^{uu}(\theta):k,l=0,1,\cdots),$$

where

$$A_{kl}^{uu}(\theta)=\begin{cases}\frac12\sum_{j\in\mathbb{Z}}c_{j\,0\,0}\partial_\theta^\alpha v_j(\theta), & k=l=0;\\ (\frac12\sum_{j\in\mathbb{Z}}c_{j\,l\,0}\partial_\theta^\alpha v_j(\theta),\ \frac12\sum_{j\in\mathbb{Z}}c_{j\,-l\,0}\partial_\theta^\alpha v_j(\theta)), & k=0,\ l=1,2,\cdots;\\ (\frac12\sum_{j\in\mathbb{Z}}c_{j\,0\,k}\partial_\theta^\alpha v_j(\theta),\ \frac12\sum_{j\in\mathbb{Z}}c_{j\,0\,-k}\partial_\theta^\alpha v_j(\theta))^T, & l=0,\ k=1,2,\cdots;\\ \begin{pmatrix}\frac12\sum_{j\in\mathbb{Z}}c_{j\,l\,k}\partial_\theta^\alpha v_j(\theta) & \frac12\sum_{j\in\mathbb{Z}}c_{j\,-l\,k}\partial_\theta^\alpha v_j(\theta)\\ \frac12\sum_{j\in\mathbb{Z}}c_{j\,l\,-k}\partial_\theta^\alpha v_j(\theta) & \frac12\sum_{j\in\mathbb{Z}}c_{j\,-l\,-k}\partial_\theta^\alpha v_j(\theta)\end{pmatrix}, & k,l=1,2,\cdots.\end{cases}$$

For any $u=(u_k:k=0,1,\cdots)\in h_N$,

$$\left(\sum_{|\alpha|\le N}\partial_\theta^\alpha JR^{uu}(\theta)J\right)u=\left(\sum_{k=0}^{\infty}(\sum_{|\alpha|\le N}A_{lk}^{uu})u_k:l=0,1,\cdots\right).\tag{2.29}$$

Suppose $\widetilde{J}=diag(\sqrt[4]{\lambda_j}:j\in\mathbb{Z})$. Then for any $z=(z_k\in\mathbb{C}:k\in Z)\in h_{\widetilde{N}}$,

$$\left(\sum_{|\alpha|\le N}\partial_\theta^\alpha\widetilde{J}\widetilde{R}^{zz}(\theta)\widetilde{J}\right)z=\left(\frac12\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}C_{jlk}(\sum_{|\alpha|\le N}\partial_\theta^\alpha v_j(\theta))z_k:l\in\mathbb{Z}\right).\tag{2.30}$$

A combination of (2.26), (2.27), (2.29) and (2.30) gives

$$\begin{aligned}&\left\|\left(\sum_{|\alpha|\le N}\partial_\theta^\alpha JR^{uu}(\theta)J\right)u\right\|_N^2\\ &=\sum_{l=0}^{\infty}l^{2N}\left|\sum_{k=0}^{\infty}(\sum_{|\alpha|\le N}A_{lk}^{uu})u_k\right|^2\\ &=\sum_{l\in\mathbb{Z}}|l|^{2N}\left|\frac12\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}C_{jlk}(\sum_{|\alpha|\le N}\partial_\theta^\alpha v_j(\theta))z_k\right|^2.\end{aligned}\tag{2.31}$$

Let

$$\gamma_{lj}=\frac{(l+j)j}{l},\quad\text{where }\ l,j=1,2,\cdots.$$

Note that

$$c_{jlk} = \begin{cases} 0, & j+l-k \neq 0, \\ 2\pi, & j+l-k = 0. \end{cases}$$

By (2.31), one has

$$\begin{aligned}
&\left\| \left(\sum_{|\alpha|\le N} \partial_\theta^\alpha J R^{uu}(\theta) J \right) u \right\|_N^2 \\
&= \sum_{l\in\mathbb{Z}} |l|^{2N} \left| \frac{1}{2} \sum_{j\in\mathbb{Z}} C_{jl(l+j)} \Big(\sum_{|\alpha|\le N} \partial_\theta^\alpha v_j(\theta) \Big) z_{l+j} \right|^2 = \frac{1}{4} \left| \sum_{j\in\mathbb{Z}} c_{j0j} \Big(\sum_{|\alpha|\le N} \partial_\theta^\alpha v_j(\theta) \Big) z_j \right|^2 \\
&+ \frac{1}{4} \sum_{l\in\mathbb{Z}\setminus\{0\}} |l|^{2N} \left| c_{0ll} \Big(\sum_{|\alpha|\le N} \partial_\theta^\alpha v_0(\theta) \Big) z_l + \sum_{j\in\mathbb{Z}\setminus\{0\}} C_{jl(l+j)} \Big(\sum_{|\alpha|\le N} \partial_\theta^\alpha v_j(\theta) \Big) z_{l+j} \right|^2 \\
&\le C \left(\sum_{j\in\mathbb{Z}} |j|^{2N} \Big| \sum_{|\alpha|\le N} \partial_\theta^\alpha v_j(\theta) \Big|^2 \right) \Big(\sum_{j\in\mathbb{Z}} |j|^{2N} |z_j|^2 \Big) + C \sum_{l\in\mathbb{Z}\setminus\{0\}} |l|^{2N} \Big| \sum_{|\alpha|\le N} \partial_\theta^\alpha v_0(\theta) \Big|^2 |z_l|^2 \\
&+ C \sum_{l\in\mathbb{Z}\setminus\{0\}} \Big| \sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{1}{\gamma_{lj}^N} \cdot \gamma_{lj}^N |l|^N \Big(\sum_{|\alpha|\le N} \partial_\theta^\alpha v_j(\theta) \Big) z_{l+j} \Big|^2 \\
&\le C ||z||_{\widetilde{N}}^2 + C \sum_{l\in\mathbb{Z}\setminus\{0\}} \left(\sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{1}{\gamma_{lj}^{2N}} \right) \Big(\sum_{j\in\mathbb{Z}\setminus\{0\}} |j|^{2N} \Big| \sum_{|\alpha|\le N} \partial_\theta^\alpha v_j(\theta) \Big|^2 |l+j|^{2N} |z_{l+j}|^2 \Big) \\
&\le C ||z||_{\widetilde{N}}^2 + C \sum_{j\in\mathbb{Z}\setminus\{0\}} |j|^{2N} \Big| \sum_{|\alpha|\le N} \partial_\theta^\alpha v_j(\theta) \Big|^2 \|z\|_N^2 \\
&\le C ||z||_{\widetilde{N}}^2 + C \sup_{(\theta,x)\in\mathbb{T}^n\times[0,2\pi]} \Big| \sum_{|\alpha|\le N} \partial_\theta^\alpha \partial_x^N \mathscr{V}(\theta,x) \Big| \|z\|_N^2 \\
&\le C \|z\|_{\widetilde{N}}^2 = C \|u\|_N^2,
\end{aligned}$$

where C is a universal constant which might be different in different places. It follows that

$$\sup_{\theta\in\mathbb{T}^n} \Big\| \sum_{|\alpha|\le N} \partial_\theta^\alpha J R^{uu}(\theta) J \Big\|_{h_N\to h_N} \le C. \tag{2.32}$$

The proofs of the last two inequalities in (2.28) are similar to that of (2.32). □

Now our goal is to find a symplectic transformation Ψ, such that the term $\varepsilon \mathbf{i} \frac{b_0}{2} \Big(\langle \overline{u}, \overline{u} \rangle - \langle u, u \rangle \Big)$ disappears. To this end, let G be a linear Hamiltonian of the form

$$G = -b_1(\theta) \Big(\langle \Lambda^{-1} u, u \rangle + \langle \Lambda^{-1} \bar{u}, \bar{u} \rangle \Big), \tag{2.33}$$

where $\theta = \omega t$ and $b_1(\theta)$ need to be specified. Moreover, let

$$\Psi = X^t_{\varepsilon G}|_{t=1}, \tag{2.34}$$

where $X^t_{\varepsilon G}$ is the flow of Hamiltonian, $X_{\varepsilon G}$ is the vector field of the Hamiltonian εG with the symplectic $\mathbf{i}du \wedge d\bar{u}$. Let

$$H_0 = \widetilde{H} \circ \Psi. \tag{2.35}$$

Recall that

$$\widetilde{H} = \langle \rho\Lambda u, \overline{u}\rangle + \varepsilon \mathbf{i}\frac{b_0}{2}\Big(\langle \overline{u}, \overline{u}\rangle - \langle u, u\rangle\Big) + \varepsilon\Big[\langle R^{uu}(\theta)u, u\rangle + \langle R^{u\overline{u}}(\theta)u, \overline{u}\rangle + \langle R^{\overline{uu}}(\theta)\overline{u}, \overline{u}\rangle\Big].$$

Then we have $\widetilde{H} = N + \varepsilon Q + \varepsilon R_0$, where

$$N = \langle \rho\Lambda u, \overline{u}\rangle, \quad Q = \mathbf{i}\frac{b_0}{2}\Big(\langle \overline{u}, \overline{u}\rangle - \langle u, u\rangle\Big), \tag{2.36}$$

$$R_0 = \Big[\langle R^{uu}(\theta)u, u\rangle + \langle R^{u\overline{u}}(\theta)u, \overline{u}\rangle + \langle R^{\overline{uu}}(\theta)\overline{u}, \overline{u}\rangle\Big]. \tag{2.37}$$

Since the Hamiltonian $\widetilde{H} = \widetilde{H}(\omega t, u, \overline{u})$ depends on time t, we introduce a fictitious action $I =$ constant, and let $\theta = \omega t$ be angle variable. Then the non-autonomous $\widetilde{H}(\omega t, u, \overline{u})$ can be written as

$$\omega I + \widetilde{H}(\theta, u, \overline{u})$$

with symplectic structure $dI \wedge d\theta + \mathbf{i}\, du \wedge d\overline{u}$. See Section 45 (B) in [1]. By Taylor formula, we have

$$\begin{aligned} H_0 =& \widetilde{H} \circ X^1_{\varepsilon G} \\ =& N + \varepsilon Q + \varepsilon\{N, G\} + \varepsilon^2 \int_0^1 \{Q, G\} \circ X^\tau_{\varepsilon G} d\tau \\ &+ \varepsilon^2 \int_0^1 (1-\tau)\{\{N, G\}, G\} \circ X^\tau_{\varepsilon G} d\tau + \varepsilon R_0 \circ X^1_{\varepsilon G}, \end{aligned} \tag{2.38}$$

where $\{N, G\} = \omega \cdot \partial_\theta b_1\Big(\langle \Lambda^{-1}u, u\rangle + \langle \Lambda^{-1}\bar{u}, \bar{u}\rangle\Big) - i2\rho b_1\Big(\langle \bar{u}, \bar{u}\rangle - \langle u, u\rangle\Big)$. Let $b_1 = \frac{b_0}{4\rho}$, then we have $H_0 = N + R$, where

$$R = \varepsilon\omega \cdot \partial_\theta b_1\Big(\langle \Lambda^{-1}u, u\rangle + \langle \Lambda^{-1}\bar{u}, \bar{u}\rangle\Big) \tag{2.39}$$

$$+ \varepsilon^2 \int_0^1 \{Q, G\} \circ X^\tau_{\varepsilon G} d\tau \tag{2.40}$$

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$$+\varepsilon^2\int_0^1(1-\tau)\{\{N,G\},G\}\circ X_{\varepsilon G}^{\tau}d\tau \tag{2.41}$$

$$+\varepsilon R_0\circ X_{\varepsilon G}^1. \tag{2.42}$$

The aim of the following section is to estimate R.

• Estimate of (2.39).

Let

$$\overline{G}^*=\begin{pmatrix}\frac{\omega\cdot\partial_\theta b_0}{4\rho}\Lambda^{-1} & 0\\ 0 & \frac{\omega\cdot\partial_\theta b_0}{4\rho}\Lambda^{-1}\end{pmatrix},\ \widetilde{u}=\begin{pmatrix}u\\ \overline{u}\end{pmatrix}.$$

Then, we have (2.39)$=\langle\varepsilon\overline{G}^*\widetilde{u},\widetilde{u}\rangle$. Obviously,

$$\sup_{\theta\in\mathbb{T}^n}\|\sum_{|\alpha|\le N-1}\partial_\theta^\alpha J\overline{G}^*(\theta)J\|_{h_N\to h_N}\le C.$$

• Estimate of (2.42).

Let

$$\widehat{R}=\begin{pmatrix}R^{uu}(\theta,\omega) & \frac12 R^{u\,\overline{u}}(\theta,\omega)\\ \frac12 R^{u\overline{u}}(\theta,\omega) & R^{\overline{uu}}(\theta,\omega)\end{pmatrix},\ \mathscr{J}=\begin{pmatrix}0 & -\mathbf{i}\,id\\ \mathbf{i}\,id & 0\end{pmatrix}$$

and

$$\overline{G}=\begin{pmatrix}-\frac{b_0}{4\rho}\Lambda^{-1} & 0\\ 0 & -\frac{b_0}{4\rho}\Lambda^{-1}\end{pmatrix}. \tag{2.43}$$

Then we have

$$R_0=\langle\widehat{R}(\theta)\begin{pmatrix}u\\ \overline{u}\end{pmatrix},\begin{pmatrix}u\\ \overline{u}\end{pmatrix}\rangle.$$

It follows that

$$\varepsilon^2\{R_0,G\}=4\varepsilon^2\langle\widehat{R}(\theta)\mathscr{J}\overline{G}(\theta)\widetilde{u},\widetilde{u}\rangle. \tag{2.44}$$

Let $\widehat{G}=\mathscr{J}\overline{G}(\theta)$ and $[\widehat{R},\widehat{G}]=\widehat{R}\widehat{G}+(\widehat{R}\widehat{G})^T$. By Taylor formula, we have

$$(2.42)=\varepsilon\langle R_1^*\tilde{u},\tilde{u}\rangle,$$

where

$$R_1^*=\widehat{R}+2^2\varepsilon\widehat{R}\widehat{G}+\sum_{j=2}^{\infty}\frac{2^{j+1}\varepsilon^j}{j!}\underbrace{[\cdots[\widehat{R},\widehat{G}],\cdots,\widehat{G}]}_{j-1-\text{fold}}\widehat{G}. \tag{2.45}$$

Thus, we can see

$$\sup_{\theta\in\mathbb{T}^n}\|\sum_{|\alpha|\le N}\partial_\theta^\alpha JR_1^*(\theta)J\|_{h_N\to h_N}\le C.$$

• Estimate of (2.40).

$$\{Q,G\}=-\frac{b_0^2}{\rho}\langle\Lambda^{-1}u,\bar u\rangle=\langle K^*\widetilde u,\widetilde u\rangle, \tag{2.46}$$

where

$$K_1^*=\begin{pmatrix}0 & -\frac{b_0^2}{2\rho}\Lambda^{-1}\\ -\frac{b_0^2}{2\rho}\Lambda^{-1} & 0\end{pmatrix}. \tag{2.47}$$

By Taylor formula, we have

$$(2.40)=\varepsilon^2\langle K^*\tilde u,\tilde u\rangle,$$

where

$$K^*=K_1^*+\sum_{j=2}^{\infty}\frac{2^{j-1}\varepsilon^j}{j!}\underbrace{[\cdots K_1^*,\cdots,\widehat G]}_{j-2-\text{fold}}\widehat G. \tag{2.48}$$

Now we have

$$\sup_{\theta\in\mathbb{T}^n}\|\sum_{|\alpha|\le N-1}\partial_\theta^\alpha JK_1^*(\theta)J\|_{h_N\to h_N}\le C.$$

• Estimate of (2.41).
By directly calculation, we have

$$\{\{N,G\},G\}=\langle H_1^*\bar u,\bar u\rangle, \tag{2.49}$$

where

$$H_1^*=\begin{pmatrix}0 & \frac{b_0^2}{2\rho}\Lambda^{-1}\\ \frac{b_0^2}{2\rho}\Lambda^{-1} & 0\end{pmatrix}. \tag{2.50}$$

By Taylor formula, we have

$$(2.41)=\varepsilon^2\langle H^*\tilde u,\tilde u\rangle,$$

where

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$$H^* = \frac{H_1^*}{2} + \sum_{j=3}^{\infty} \frac{2^{j-2}\varepsilon^{j-1}}{j!} \underbrace{[\cdots H_1^*, \cdots, \widehat{G}]}_{j-3-\text{fold}} \widehat{G}. \tag{2.51}$$

Now we have

$$\sup_{\theta\in\mathbb{T}^n} \| \sum_{|\alpha|\le N} \partial_\theta^\alpha J H_1^*(\theta) J \|_{h_N\to h_N} \le C.$$

To simplify the notation, we rename $N-1$ as N

$$\sup_{\theta\in\mathbb{T}^n} \| \sum_{|\alpha|\le N} \partial_\theta^\alpha J R J \|_{h_N\to h_N} \le C.$$

Now, Theorem 1.1 can be transformed into a more exact expression.

Theorem 2.1. *With Assumptions A, B, for given $1 \gg \gamma > 0$, there exists ϵ^* with $0 < \varepsilon^* = \varepsilon^*(n,\gamma) \ll \gamma$, and exists a subset $\Pi \subset [1,2]^n$ with*

$$mes\,\Pi \ge 1 - O(\gamma^{1/3})$$

such that for any $0 < \varepsilon < \varepsilon^*$ and any $\omega \in \Pi$, there is a time-quasi-periodic symplectic change

$$\begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \Phi(\omega t) \begin{pmatrix} \widetilde{u} \\ \overline{\widetilde{u}} \end{pmatrix}$$

such that the Hamiltonian system (2.22) is changed into

$$\begin{cases} \dot{\widetilde{u}}_k = \mathbf{i}\,\frac{\partial \widetilde{H}}{\partial \overline{\widetilde{u}}_k}, & k\in\mathbb{Z}, \\ \dot{\overline{\widetilde{u}}}_k = -\mathbf{i}\,\frac{\partial \widetilde{H}}{\partial \widetilde{u}_k}, & k\in\mathbb{Z}, \end{cases}$$

where

$$\widetilde{H}(\widetilde{u}, \overline{\widetilde{u}}) = \Lambda_0^\infty \widetilde{u}_0 \overline{\widetilde{u}}_0 + \sum_{j=1}^{\infty} (\Lambda_j^\infty \widetilde{u}_j) \cdot \overline{\widetilde{u}}_j,$$

$$\Lambda_0^\infty = \rho\sqrt{\lambda_0} + \varepsilon Q_0, \;\; \Lambda_j^\infty = \rho\sqrt{\lambda_j} E_{22} + \varepsilon Q_j$$

with

- **(i)** *Q_0 and Q_k $(k = 1, 2, \cdots)$ are independent of time t, and $Q_0 \in \mathbb{R}$, Q_k is a 2×2 real matrix $(k = 1, 2, \cdots)$;*
- **(ii)** *$\widetilde{Q} = diag(Q_j)$ satisfies $\| J\widetilde{Q}J \|_{h_N\to h_N} \le C$, $J = diag(J_j : j = 0, 1, \cdots)$, $J_0 = \sqrt[4]{\lambda_0}$, $J_j = \sqrt[4]{\lambda_j} E_{22}$, $j = 1, 2, \cdots$;*

(iii) $\Phi = \Phi(\omega t)$ *is quasi-periodic in time and close to the identity map:*

$$\|\Phi(\omega t) - id\|_{h_N \to h_N} \leq C\varepsilon,$$

where id *is the identity map from* $h_N \to h_N$.

3. Analytical approximation lemma

We need to find a series of operators which are analytic in some complex strip domains to approximate the operators $R^{uu}(\theta)$, $R^{u\overline{u}}(\theta)$ and $R^{\overline{u}\,\overline{u}}(\theta)$. To this end, we cite an approximation lemma (see [23,32,33] for the details). This method is used in [36], too.

We start by recalling some definitions and setting some new notations. Assume X is a Banach space with the norm $||\cdot||_X$. First recall that $C^\mu(\mathbb{R}^n; X)$ for $0 < \mu < 1$ denotes the space of bounded Hölder continuous functions $f : \mathbb{R}^n \mapsto X$ with the form

$$\|f\|_{C^\mu,X} = \sup_{0<|x-y|<1} \frac{\|f(x) - f(y)\|_X}{|x-y|^\mu} + \sup_{x\in\mathbb{R}^n} \|f(x)\|_X.$$

If $\mu = 0$, then $\|f\|_{C^\mu,X}$ denotes the sup-norm. For $\ell = k + \mu$ with $k \in \mathbb{N}$ and $0 \leq \mu < 1$, we denote by $C^\ell(\mathbb{R}^n; X)$ the space of functions $f : \mathbb{R}^n \mapsto X$ with Hölder continuous partial derivatives, i.e., $\partial^\alpha f \in C^\mu(\mathbb{R}^n; X_\alpha)$ for all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ with the assumption that $|\alpha| := |\alpha_1| + \cdots + |\alpha_n| \leq k$ and X_α is the Banach space of bounded operators $T : \prod^{|\alpha|}(\mathbb{R}^n) \mapsto X$ with the norm

$$\|T\|_{X_\alpha} = \sup\{||T(u_1, u_2, \cdots, u_{|\alpha|})||_X : \|u_i\| = 1,\ 1 \leq i \leq |\alpha|\}.$$

We define the norm

$$||f||_{C^\ell} = \sup_{|\alpha|\leq\ell} ||\partial^\alpha f||_{C^\mu,X_\alpha}.$$

Lemma 3.1. *(Jackson–Moser–Zehnder) Let* $f \in C^\ell(\mathbb{R}^n; X)$ *for some* $\ell > 0$ *with finite* C^ℓ *norm over* $\mathbb{R}^n$*. Let* ϕ *be a radial-symmetric,* C^∞ *function, having as support the closure of the unit ball centered at the origin, where* ϕ *is completely flat and takes value 1. Let* $K = \widehat{\phi}$ *be its Fourier transform. For all* $\sigma > 0$*, we define*

$$f_\sigma(x) := K_\sigma * f = \frac{1}{\sigma^n} \int_{\mathbb{R}^n} K(\frac{x-y}{\sigma}) f(y) dy.$$

Then there exists a constant $C \geq 1$ *depending only on* ℓ *and* n *such that the following holds: for any* $\sigma > 0$*, the function* $f_\sigma(x)$ *is a real-analytic function from* $\mathbb{C}^n/(\pi\mathbb{Z})^n$ *to* X *such that if* Δ_σ^n *denotes the* n*-dimensional complex strip of width* σ*,*

$$\Delta_\sigma^n := \{x \in \mathbb{C}^n \big| |\mathrm{Im} x_j| \leq \sigma,\ 1 \leq j \leq n\},$$

then for any $\alpha \in \mathbb{N}^n$ *such that* $|\alpha| \leq \ell$ *one has*

$$\sup_{x\in\Delta_\sigma^n} ||\partial^\alpha f_\sigma(x) - \sum_{|\beta|\le \ell-|\alpha|} \frac{\partial^{\beta+\alpha} f(\mathrm{Re}x)}{\beta!}(\sqrt{-1}\mathrm{Im}x)^\beta||_{X_\alpha} \le C||f||_{C^\ell}\sigma^{\ell-|\alpha|},$$

and for all $0\le s\le\sigma$*,*

$$\sup_{x\in\Delta_s^n} \|\partial^\alpha f_\sigma(x) - \partial^\alpha f_s(x)\|_{X_\alpha} \le C||f||_{C^\ell}\sigma^{\ell-|\alpha|}.$$

The function f_σ *preserves periodicity (i.e., if* f *is T-periodic in any of its variable* x_j*, so is* f_σ*). Finally, if* f *depends on some parameters* $\xi\in\Pi\subset\mathbb{R}^n$ *and*

$$||f(x,\xi)||^{\mathcal{L}}_{C^\ell(X)} := \sup_{\xi\in\Pi} ||\partial_\xi f(x,\xi)||_{C^\ell(X)}$$

are uniformly bounded by a constant C*, then all the above estimates hold true with* $\|\cdot\|$ *replaced by* $\|\cdot\|^{\mathcal{L}}$*.*

The proof of this lemma consists in a direct check which is based on standard tools from calculus and complex analysis. It is used to deal with KAM theory for finite smooth systems by Zehnder [37]. Also see [12] and [36] and references therein, for example. For simplicity of notation, we shall replace $\|\cdot\|_X$ by $\|\cdot\|$. Now let us apply this lemma to the perturbation $P(\phi)$.

Fix a sequence of fast decreasing numbers $s_\upsilon \downarrow 0$, $\upsilon\ge 0$, and $s_0\le \frac{1}{2}$. For an X-valued function $P(\phi)$, construct a sequence of real analytic functions $P^{(\upsilon)}(\phi)$ such that the following conclusions hold:

(1) $P^{(\upsilon)}(\phi)$ is real analytic on the complex strip $\mathbb{T}^n_{s_\upsilon}$ of the width s_υ around $\mathbb{T}^n$.

(2) The sequence of functions $P^{(\upsilon)}(\phi)$ satisfies the bounds:

$$\sup_{\phi\in\mathbb{T}^n} \|P^{(\upsilon)}(\phi) - P(\phi)\| \le C\|P\|_{C^\ell} s_\upsilon^\ell, \tag{3.1}$$

$$\sup_{\phi\in\mathbb{T}^n_{s_{\upsilon+1}}} \|P^{(\upsilon+1)}(\phi) - P^{(\upsilon)}(\phi)\| \le C\|P\|_{C^\ell} s_\upsilon^\ell, \tag{3.2}$$

where C denotes (different) constants depending only on n and ℓ.

(3) The first approximate $P^{(0)}$ is "small" with the perturbation P. Precisely speaking, for arbitrary $\phi\in\mathbb{T}^n_{s_0}$, we have

$$\|P^{(0)}(\phi)\| \le C\|P\|_{C^\ell}, \tag{3.3}$$

where the constant C is independent of s_0, and the last inequality holds true due to the hypothesis that $s_0\le\frac{1}{2}$.

(4) From the first inequality (3.1), we have the equality below. For any arbitrary $\phi\in\mathbb{T}^n$,

$$P(\phi) = P^{(0)}(\phi) + \sum_{\upsilon=0}^{+\infty}(P^{(\upsilon+1)}(\phi) - P^{(\upsilon)}(\phi)). \tag{3.4}$$

Now take a sequence of real numbers $\{s_\upsilon \geq 0\}_{\upsilon=0}^{\infty}$ with $s_\upsilon > s_{\upsilon+1}$ going fast to zero. Let $R^{p,q}(\theta) = P(\theta)$ for $p, q \in \{u, \overline{u}\}$. Then by (3.4) and (2.28), for $p, q \in \{u, \overline{u}\}$, we have,

$$R^{p,q}(\theta) = R_0^{p,q}(\theta) + \sum_{l=1}^{\infty} R_l^{p,q}(\theta), \tag{3.5}$$

where $R_0^{p,q}(\theta)$ is analytic in $\mathbb{T}_{s_0}^n$ with

$$\sup_{\theta\in\mathbb{T}_{s_0}^n} \|R_0^{p,q}(\theta)\|_{h_N\to h_N} \leq C, \tag{3.6}$$

and $R_l^{p,q}(\theta)$ $(l \geq 1)$ is analytic in $\mathbb{T}_{s_l}^n$ with

$$\sup_{\theta\in\mathbb{T}_{s_l}^n} \|JR_l^{p,q}(\theta)J\|_{h_N\to h_N} \leq Cs_l^N. \tag{3.7}$$

4. Iterative parameters of domains

Let

- $\varepsilon_0 = \varepsilon$, $\varepsilon_\nu = \varepsilon^{(\frac{4}{3})^\nu}$, $\nu = 0, 1, 2, \cdots$, which measures the size of perturbations at $\nu - th$ step.
- $s_\nu = \varepsilon_{\nu+1}^{1/N}$, $\nu = 0, 1, 2, \cdots$, which measures the strip-width of the analytic domain $\mathbb{T}_{s_\nu}^n$, $\mathbb{T}_{s_\nu}^n = \{\theta \in \mathbb{C}^n/2\pi\mathbb{Z}^n : |Im\theta| \leq s_\nu\}$.
- $C(\nu)$ is a constant which may be different in different places, and it is of the form

  $$C(\nu) = C_1 2^{C_2\nu},$$

  where C_1, C_2 are constants.
- $K_\nu = 100 s_\nu^{-1} 2^\nu |\log\varepsilon|$.
- $\gamma_\nu = \frac{\gamma}{2^\nu}$, $0 < \gamma \ll 1$.
- A family of subsets $\Pi_\nu \subset [1,2]^n$ with $[1,2]^n \supset \Pi_0 \supset \cdots \supset \Pi_\nu \supset \cdots$, and

  $$mes\,\Pi_\nu \geq mes\,\Pi_{\nu-1} - C\gamma_{\nu-1}^{1/3}.$$

- For an operator-value (or a vector-value) function $B(\theta, \omega)$, whose domain is $(\theta, \omega) \in \mathbb{T}_{s_\nu}^n \times \Pi_\nu$. Set

  $$\|B\|_{\mathbb{T}_{s_\nu}^n\times\Pi_\nu} = \sup_{(\theta,\omega)\in\mathbb{T}_{s_\nu}^n\times\Pi_\nu} \|B(\theta,\omega)\|_{h_N\to h_N},$$

  where $\|\cdot\|_{h_N\to h_N}$ is the operator norm, and set

  $$\|B\|_{\mathbb{T}_{s_\nu}^n\times\Pi_\nu}^{\mathcal{L}} = \sup_{(\theta,\omega)\in\mathbb{T}_{s_\nu}^n\times\Pi_\nu} \|\partial_\omega B(\theta,\tau)\|_{h_N\to h_N}.$$

5. Iterative lemma

In the following, for a function $f(\omega)$, denote by ∂_ω the derivative of $f(\omega)$ with respect to ω in Whitney's sense.

Lemma 5.1. *For $p, q \in \{u, \overline{u}\}$, let $R^{p,q}_{0,0} = R^{p,q}_0$, $R^{p,q}_{l,0} = R^{p,q}_l$, where $R^{p,q}_0$, $R^{p,q}_l$ are defined by (3.5), (3.6) and (3.7). Assume that we have a family of Hamiltonian functions H_ν:*

$$H_\nu = \Lambda_0^{(\nu)} u_0 \overline{u}_0 + \sum_{j=1}^{\infty} (\Lambda_j^{(\nu)} u_j) \cdot \overline{u}_j + \sum_{l \geq \nu}^{\infty} \varepsilon_l (\langle R^{uu}_{l,\nu} u, u\rangle + \langle R^{u\overline{u}}_{l,\nu} u, \overline{u}\rangle + \langle R^{\overline{u}\,\overline{u}}_{l,\nu} \overline{u}, \overline{u}\rangle),$$
$$\nu = 0, 1, \cdots, m, \tag{5.1}$$

where $R^{uu}_{l,\nu}$, $R^{u\overline{u}}_{l,\nu}$, $R^{\overline{uu}}_{l,\nu}$ are operator-valued functions defined on the domain $\mathbb{T}^n_{s_\nu} \times \Pi_\nu$, and

$$\theta = \omega t, \quad \omega = (\omega_1, \omega_2, \cdots \omega_n).$$

$(A1)_\nu$

$$\Lambda_0^{(0)} = \rho\sqrt{\lambda_0}, \quad \Lambda_0^{(\nu)} = \rho\sqrt{\lambda_0} + \sum_{i=0}^{\nu-1} \varepsilon_i \mu_0^{(i)}, \quad \nu \geq 1; \tag{5.2}$$

$$\Lambda_j^{(0)} = \rho\sqrt{\lambda_j} E_{22}, \quad \Lambda_j^{(\nu)} = \rho\sqrt{\lambda_j} E_{22} + \sum_{i=0}^{\nu-1} \varepsilon_i \mu_j^{(i)}, \quad j = 1, 2, \cdots, \quad \nu \geq 1, \tag{5.3}$$

where

(i) $\mu_0^{(i)} = \mu_0^{(i)}(\omega) : \Pi_i \to \mathbb{R}$ *with*

$$|\mu_0^{(i)}|_{\Pi_i} := \sup_{\omega \in \Pi_i} |\mu_0^{(i)}(\omega)| \leq C(i), \; 0 \leq i \leq \nu - 1, \tag{5.4}$$

$$|\mu_0^{(i)}|^{\mathcal{L}}_{\Pi_i} := \sup_{\omega \in \Pi_i} \max_{1 \leq l \leq n} |\partial_{\omega_l} \mu_0^{(i)}(\omega)| \leq C(i), \; 0 \leq i \leq \nu - 1. \tag{5.5}$$

Here $|\cdot|$ denotes the absolute value of a function.

(ii) $\mu_j^{(i)} = \mu_j^{(i)}(\omega)$ *($j = 1, 2, \cdots, \; 0 \leq i \leq \nu - 1, \; \nu \geq 1$) are 2×2 real symmetry matrices with*

$$|\mu_j^{(i)}|_{\Pi_i} := \sup_{\omega \in \Pi_i} |\mu_j^{(i)}(\omega)| \leq C(i)/j, \tag{5.6}$$

$$|\mu_j^{(i)}|^{\mathcal{L}}_{\Pi_i} := \sup_{\omega \in \Pi_i} \max_{1 \leq l \leq n} |\partial_{\omega_l} \mu_j^{(i)}(\omega)| \leq C(i)/j. \tag{5.7}$$

Here $|\cdot|$ denotes the sup-norm for real matrices.

$(A2)_\nu$ *For $p, q \in \{u, \overline{u}\}$, $R^{p,q}_{l,\nu} = R^{p,q}_{l,\nu}(\theta, \omega)$ is defined in $\mathbb{T}^n_{s_l} \times \Pi_\nu$ with $l \geq \nu$, and is analytic in θ for fixed $\omega \in \Pi_\nu$, and*

$$\|JR^{p,q}_{l,\nu}J\|_{\mathbb{T}^n_{s_l}\times\Pi_\nu}\le C(\nu), \tag{5.8}$$

$$\|JR^{p,q}_{l,\nu}J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_l}\times\Pi_\nu}\le C(\nu). \tag{5.9}$$

Then there exists a compact set $\Pi_{m+1}\subset\Pi_m$ with

$$mes\,\Pi_{m+1}\ge mes\,\Pi_m - C\gamma_m^{1/3}, \tag{5.10}$$

and exists a symplectic coordinate change

$$\Psi_m:\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}\to\mathbb{T}^n_{s_m}\times\Pi_m, \tag{5.11}$$

$$||\Psi_m - id||_{h_N\to h_N}\le\varepsilon^{1/2},\ (\theta,\omega)\in\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1} \tag{5.12}$$

such that the Hamiltonian function H_m is changed into

$$\begin{aligned} H_{m+1} &\triangleq H_m\circ\Psi_m\\ &=\Lambda_0^{(m+1)}u_0\overline{u}_0+\sum_{j=1}^{\infty}(\Lambda_j^{(m+1)}u_j)\cdot\overline{u}_j+\sum_{l\ge m+1}^{\infty}\varepsilon_l\Big[\langle R^{uu}_{l,m+1}u,u\rangle\\ &\quad+\langle R^{u\overline{u}}_{l,m+1}u,\overline{u}\rangle+\langle R^{\overline{u}\,\overline{u}}_{l,m+1}\overline{u},\overline{u}\rangle\Big], \end{aligned} \tag{5.13}$$

which is defined on the domain $\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}$, and $\Lambda_j^{(m+1)}$'s satisfy the assumptions $(A1)_{m+1}$ and $R^{p,q}_{l,m+1}(p,q\in\{u,\overline{u}\})$ satisfy the assumptions $(A2)_{m+1}$.

6. Derivation of homological equations

Our end is to find a symplectic transformation Ψ_ν such that the terms $R^{uu}_{l,\nu}$, $R^{u\overline{u}}_{l,\nu}$, $R^{\overline{u}\,\overline{u}}_{l,\nu}$ (with $l=\nu$) disappear. To this end, let F be a linear Hamiltonian of the form

$$F=\langle F^{uu}(\theta,\omega)u,u\rangle+\langle F^{u\overline{u}}(\theta,\omega)u,\overline{u}\rangle+\langle F^{\overline{u}\,\overline{u}}(\theta,\omega)\overline{u},\overline{u}\rangle, \tag{6.1}$$

where $\theta=\omega t$, $(F^{uu}(\theta,\omega))^T=F^{uu}(\theta,\omega)$, $(F^{u\overline{u}}(\theta,\omega))^T=F^{u\overline{u}}(\theta,\omega)$, $(F^{\overline{uu}}(\theta,\omega))^T=F^{\overline{uu}}(\theta,\omega)$. Moreover, let

$$\Psi=\Psi_m=X^t_{\varepsilon_m F}\big|_{t=1}, \tag{6.2}$$

where $X^t_{\varepsilon_m F}$ is the flow of the Hamiltonian, $X_{\varepsilon_m F}$ is the vector field of the Hamiltonian $\varepsilon_m F$ with the symplectic structure $\mathbf{i}\,du\wedge d\overline{u}$. Let

$$H_{m+1}=H_m\circ\Psi_m. \tag{6.3}$$

By (5.1), we have

$$H_m=N_m+R_m \tag{6.4}$$

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with

$$N_m = \omega I + \Lambda_0^{(m)} u_0 \overline{u}_0 + \sum_{j=1}^{\infty} (\Lambda_j^{(m)} u_j) \cdot \overline{u}_j, \tag{6.5}$$

$$R_m = \sum_{l=m}^{\infty} \varepsilon_l R_{lm}, \tag{6.6}$$

$$R_{lm} = \langle R_{l,m}^{uu}(\theta) u, u \rangle + \langle R_{l,m}^{u\overline{u}}(\theta) u, \overline{u} \rangle + \langle R_{l,m}^{\overline{u}\,\overline{u}}(\theta) \overline{u}, \overline{u} \rangle, \tag{6.7}$$

where $(R_{l,m}^{uu}(\theta))^T = R_{l,m}^{uu}(\theta)$, $(R_{l,m}^{u\overline{u}}(\theta))^T = R_{l,m}^{u\overline{u}}(\theta)$, $(R_{l,m}^{\overline{u}\overline{u}}(\theta))^T = R_{l,m}^{\overline{u}\overline{u}}(\theta)$.

Recall that the sequence $z = (z_j \in \mathbb{C},\ j \in \mathbb{Z})$ can be rewritten as

$$z = (z_0, z_j, z_{-j} : j = 1, 2, \cdots) = u = (u_j : j = 0, 1, 2, \cdots),$$

where $u_0 = z_0$, $u_j = (z_j, z_{-j})^T,\ j = 1, 2, \cdots$. Suppose $\{\cdot, \cdot\}$ is the Poisson bracket with respect to $\mathbf{i}\, dz \wedge d\overline{z}$, i.e.

$$\{H(z, \overline{z}), F(z, \overline{z})\} = \mathbf{i} \left(\frac{\partial H}{\partial z} \cdot \frac{\partial F}{\partial \overline{z}} - \frac{\partial H}{\partial \overline{z}} \cdot \frac{\partial F}{\partial z} \right).$$

Define

$$\frac{\partial H}{\partial u_0} = \frac{\partial H}{\partial z_0}, \quad \frac{\partial H}{\partial u_j} = (\frac{\partial H}{\partial z_j}, \frac{\partial H}{\partial z_{-j}})^T,\ \ j = 1, 2, \cdots, \ \sum_{j=0}^{\infty} \frac{\partial H}{\partial u_j} \cdot \frac{\partial F}{\partial \overline{u}_j} \triangleq \frac{\partial H}{\partial u} \cdot \frac{\partial F}{\partial \overline{u}}.$$

We can verify that

$$\{H(z, \overline{z}), F(z, \overline{z})\} = \{H(u, \overline{u}), F(u, \overline{u})\} = \mathbf{i} \left(\frac{\partial H}{\partial u} \cdot \frac{\partial F}{\partial \overline{u}} - \frac{\partial H}{\partial \overline{u}} \cdot \frac{\partial F}{\partial u} \right).$$

So $\{\cdot, \cdot\}$ is also the Poisson bracket with respect to $\mathbf{i}\, du \wedge d\overline{u}$. By combination of (6.1)–(6.7) and Taylor formula, we have

$$\begin{aligned} H_{m+1} &= H_m \circ X_{\varepsilon_m F}^1 \\ &= N_m + \varepsilon_m \{N_m, F\} + \varepsilon_m^2 \int_0^1 (1-\tau) \{\{N_m, F\}, F\} \circ X_{\varepsilon_m F}^{\tau} d\tau + \varepsilon_m \omega \cdot \partial_\theta F \\ &\quad + \varepsilon_m R_{mm} + (\sum_{l=m+1}^{\infty} \varepsilon_l R_{lm}) \circ X_{\varepsilon_m F}^1 + \varepsilon_m^2 \int_0^1 \{R_{mm}, F\} \circ X_{\varepsilon_m F}^{\tau} d\tau. \end{aligned} \tag{6.8}$$

Let Γ_{K_m} be a truncation operator. For any

$$f(\theta) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(k) e^{\mathbf{i} \langle k, \theta \rangle},\ \theta \in \mathbb{T}^n.$$

Define, for any given $K_m > 0$,

$$\Gamma_{K_m} f(\theta) = (\Gamma_{K_m} f)(\theta) \triangleq \sum_{|k| \le K_m} \widehat{f}(k) e^{\mathbf{i}\langle k,\theta\rangle},$$

$$(1-\Gamma_{K_m}) f(\theta) = ((1-\Gamma_{K_m}) f)(\theta) \triangleq \sum_{|k| > K_m} \widehat{f}(k) e^{\mathbf{i}\langle k,\theta\rangle}.$$

Then

$$f(\theta) = \Gamma_{K_m} f(\theta) + (1-\Gamma_{K_m}) f(\theta).$$

Let

$$\{N_m, F\} + \Gamma_{K_m} R_{mm} = \langle [R^{u\overline{u}}_{mm}] u, \overline{u} \rangle, \tag{6.9}$$

where

$$[R^{u\overline{u}}_{mm}] := diag\left(\widehat{R}^{u\overline{u}}_{mmjj}(0) : j = 0, 1, 2, \cdots \right), \tag{6.10}$$

and $R^{u\overline{u}}_{mmij}(\theta)$ is the matrix element of $R^{u\overline{u}}_{m,m}(\theta)$ and $\widehat{R}^{u\overline{u}}_{mmij}(k)$ is the k-Fourier coefficient of $R^{u\overline{u}}_{mmij}(\theta)$. Then

$$H_{m+1} = N_{m+1} + C_{m+1} R_{m+1}, \tag{6.11}$$

where

$$N_{m+1} = N_m + \varepsilon_m \langle [R^{u\overline{u}}_{mm}] u, \overline{u} \rangle = \Lambda_0^{(m+1)} u_0 \overline{u}_0 + \sum_{j=1}^{\infty} (\Lambda_j^{(m+1)} u_j) \cdot \overline{u}_j, \tag{6.12}$$

$$\Lambda_j^{(m+1)} = \Lambda_j^{(m)} + \varepsilon_m \widehat{R}^{u\overline{u}}_{mmjj}(0) = \Lambda_j^{(m)} + \sum_{l=0}^{m} \varepsilon_l \mu_j^{(l)}, \ \mu_j^{(m)} := \widehat{R}^{u\overline{u}}_{mmjj}(0), \tag{6.13}$$

$$C_{m+1} R_{m+1} = \varepsilon_m (1-\Gamma_{K_m}) R_{mm} \tag{6.14}$$

$$+ \varepsilon_m^2 \int_0^1 (1-\tau) \{\{N_m, F\}, F\} \circ X^{\tau}_{\varepsilon_m F} d\tau \tag{6.15}$$

$$+ \varepsilon_m^2 \int_0^1 \{R_{mm}, F\} \circ X^{\tau}_{\varepsilon_m F} d\tau \tag{6.16}$$

$$+ \left(\sum_{l=m+1}^{\infty} \varepsilon_l R_{lm} \right) \circ X^1_{\varepsilon_m F}. \tag{6.17}$$

The equation (6.9) is called the homological equation. Developing the Poisson bracket $\{N_m, F\}$ and comparing the coefficients of $u_i u_j, u_i \overline{u}_j, \overline{u}_i \overline{u}_j (i, j = 0, 1, 2, \cdots)$, we get

$$\omega \cdot \partial_\theta F^{uu}(\theta,\omega) + \mathbf{i}(\Lambda^{(m)} F^{uu}(\theta,\omega) + F^{uu}(\theta,\omega)\Lambda^{(m)}) = \Gamma_{K_m} R^{uu}_{mm}(\theta), \tag{6.18}$$

$$\omega \cdot \partial_\theta F^{\bar{u}\bar{u}}(\theta,\omega) - \mathbf{i}(\Lambda^{(m)} F^{\bar{u}\bar{u}}(\theta,\omega) + F^{\bar{u}\bar{u}}(\theta,\omega)\Lambda^{(m)}) = \Gamma_{K_m} R^{\bar{u}\bar{u}}_{mm}(\theta), \tag{6.19}$$

$$\omega \cdot \partial_\theta F^{u\bar{u}}(\theta,\omega) + \mathbf{i}(F^{u\bar{u}}(\theta,\omega)\Lambda^{(m)} - \Lambda^{(m)} F^{u\bar{u}}(\theta,\omega)) = \Gamma_{K_m} R^{u\bar{u}}_{mm}(\theta) - [R_{mm}], \tag{6.20}$$

where

$$\Lambda^{(m)} = diag(\Lambda^{(m)}_j : j = 0, 1, 2, \cdots), \tag{6.21}$$

and we assume

$$\Gamma_{K_m} F^{uu}(\theta,\omega) = F^{uu}(\theta,\omega),\ \Gamma_{K_m} F^{u\bar{u}}(\theta,\omega) = F^{u\bar{u}}(\theta,\omega),\ \Gamma_{K_m} F^{\bar{u}\bar{u}}(\theta,\omega) = F^{\bar{u}\bar{u}}(\theta,\omega).$$

$F^{uu}_{ij}(\theta)$, $F^{u\bar{u}}_{ij}(\theta)$, $F^{\bar{u}\bar{u}}_{ij}(\theta)$ are written as the matrix elements of $F^{uu}(\theta,\omega)$, $F^{u\bar{u}}(\theta,\omega)$, $F^{\bar{u}\bar{u}}(\theta,\omega)$, respectively. More exactly, for $p, q \in \{u, \bar{u}\}$,

$$F^{pq}_{ij}(\theta) = \begin{cases} a_{0,0}(\theta), & i = j = 0; \\ (a_{0,j}(\theta),\ a_{0,-j}(\theta)), & i = 0,\ j = 1, 2, \cdots; \\ (a_{i,0}(\theta),\ a_{-i,0}(\theta))^T, & j = 0,\ i = 1, 2, \cdots; \\ \begin{pmatrix} a_{i,j}(\theta) & a_{i,-j}(\theta) \\ a_{-i,j}(\theta) & a_{-i,-j}(\theta) \end{pmatrix}, & i, j = 1, 2, \cdots, \end{cases}$$

where $a_{i,j}(\theta) : \mathbb{T}^n_{s_m} \to \mathbb{R},\ i, j = 0, 1, 2, \cdots$. Then (6.18)–(6.20) can be rewritten as:

$$\omega \cdot \partial_\theta F^{uu}_{ij}(\theta) + \mathbf{i}(\Lambda^{(m)}_i F^{uu}_{ij}(\theta) + F^{uu}_{ij}(\theta))\Lambda^{(m)}_j = \Gamma_{K_m} R^{uu}_{mmij}(\theta), \tag{6.22}$$

$$\omega \cdot \partial_\theta F^{\bar{u}\bar{u}}_{ij}(\theta) - \mathbf{i}(\Lambda^{(m)}_i F^{\bar{u}\bar{u}}_{ij}(\theta) + F^{\bar{u}\bar{u}}_{ij}(\theta)\Lambda^{(m)}_j) = \Gamma_{K_m} R^{\bar{u}\bar{u}}_{mmij}(\theta), \tag{6.23}$$

$$\omega \cdot \partial_\theta F^{u\bar{u}}_{ij}(\theta) - \mathbf{i}(\Lambda^{(m)}_i F^{u\bar{u}}_{ij}(\theta) - F^{u\bar{u}}_{ij}(\theta)\Lambda^{(m)}_j) = \Gamma_{K_m} R^{u\bar{u}}_{mmij}(\theta),\ i \neq j, \tag{6.24}$$

$$\omega \cdot \partial_\theta F^{u\bar{u}}_{ii}(\theta) - \mathbf{i}(\Lambda^{(m)}_i F^{u\bar{u}}_{ii}(\theta) - F^{u\bar{u}}_{ii}(\theta)\Lambda^{(m)}_i) = \Gamma_{K_m} R^{u\bar{u}}_{mmii}(\theta) - \widehat{R}_{mmii}(0), \tag{6.25}$$

where $i, j = 0, 1, 2, \cdots$.

7. Solutions of homological equations

Lemma 7.1. *There exists a compact subset $\Pi^{+-}_{m+1} \subset \Pi_m$ with*

$$mes(\Pi^{+-}_{m+1}) \geq mes\,\Pi_m - C\gamma_m^{1/3} \tag{7.1}$$

such that for any $\omega \in \Pi^{+-}_{m+1}$, the equation (6.20) has a unique solution $F^{u\bar{u}}(\theta,\omega)$, which is defined on the domain $\mathbb{T}^n_{s_{m+1}} \times \Pi^{+-}_{m+1}$, with

$$\|JF^{u\bar{u}}(\theta,\omega)J\|_{\mathbb{T}^n_{s_{m+1}} \times \Pi^{+-}_{m+1}} \leq C(m+1)\varepsilon_m^{-\frac{2(3n+4)}{N}}, \tag{7.2}$$

$$\|JF^{u\bar{u}}(\theta,\omega)J\|^{\mathcal{L}}_{\mathbb{T}^n_{s_{m+1}} \times \Pi^{+-}_{m+1}} \leq C(m+1)\varepsilon_m^{-\frac{6(3n+4)}{N}}. \tag{7.3}$$

Proof. By passing to Fourier coefficients, we can rewrite (6.24) as

$$-\langle k,\omega\rangle \widehat{F}^{u\bar{u}}_{ij}(k)+(\Lambda^{(m)}_i \widehat{F}^{u\bar{u}}_{ij}(k)-\widehat{F}^{u\bar{u}}_{ij}(k)\Lambda^{(m)}_j)=\mathbf{i}\,\widehat{R}^{u\bar{u}}_{mmij}(k), \tag{7.4}$$

where $i, j=0,1,2,\cdots, i\neq j, k\in\mathbb{Z}^n$ with $|k|\leq K_m$. In the following, we always by "1" denote the identity from some finite dimensional space to itself. By applying "vec" to both sides of (7.4), we have

$$(-\langle k,\omega\rangle(1\otimes 1)+1\otimes\Lambda^{(m)}_i-(\Lambda^{(m)}_j)^T\otimes 1)\mathrm{vec}\,\widehat{F}^{u\bar{u}}_{ij}(k)=\mathrm{vec}\,(\mathbf{i}\,\widehat{R}^{u\bar{u}}_{mmij}(k)), \tag{7.5}$$

where $A\otimes B$ is the tensor product of A and B. Let μ^{ml}_{kij} be the l-th eigenvalue of $1\otimes\Lambda^{(m)}_i-(\Lambda^{(m)}_j)^T\otimes 1,\ l=1,2,3,4$. Let

$$A_k=|k|^{2n+4}+8,$$

and

$$Q^{(m)}_{kijl}\triangleq\left\{\omega\in\Pi_m\middle|\,|-\langle k,\omega\rangle+\mu^{ml}_{kij}|<\frac{(|i-j|+1)\gamma_m}{A_k}\right\}, \tag{7.6}$$

where $i, j=0,1,2,\cdots, l=1,2,3,4, k\in\mathbb{Z}^n$ with $|k|\leq K_m$, and $k\neq 0$ when $i=j$. Let

$$\Pi^{+-}_{m+1}=\Pi_m\diagdown\bigcup_{|k|\leq K_m}\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcup_{l=1}^{4}Q^{(m)}_{kijl}.$$

Then for any $\omega\in\Pi^{+-}_{m+1}$, we have

$$\left|-\langle k,\omega\rangle+\mu^{ml}_{kij}\right|\geq\frac{(|i-j|+1)\gamma_m}{A_k}. \tag{7.7}$$

Then

$$||(-\langle k,\omega\rangle(1\otimes 1)+1\otimes\Lambda^{(m)}_i-(\Lambda^{(m)}_j)^T\otimes 1)^{-1}||_2\leq\frac{A_k}{(|i-j|+1)\gamma_m}. \tag{7.8}$$

Here $||\cdot||_2$ denotes the spectral norm of matrices. Recall that $R^{u\bar{u}}_{mm}(\theta)$ is analytic in the domain $\mathbb{T}^n_{s_m}$ for any $\omega\in\Pi_m$,

$$||\widehat{R}^{u\bar{u}}_{mmij}(k)||_2\leq\frac{C(m)}{\sqrt{ij}}e^{-s_m|k|}, \tag{7.9}$$

which implies that

$$||\mathrm{vec}\,(\mathbf{i}\,\widehat{R}^{u\bar{u}}_{mmij}(k))||_2\leq\frac{C(m)}{\sqrt{ij}}e^{-s_m|k|}.$$

By (7.5), we have

$$||\text{vec}\, \widehat{F}_{ij}^{u\overline{u}}(k)||_2 \leq \frac{A_k}{(|i-j|+1)\gamma_m}||\text{vec}(\mathbf{i}\, \widehat{R}_{mmij}^{u\overline{u}}(k))||_2 \leq \frac{A_k}{\gamma_m(|i-j|+1)}\frac{C(m)e^{-s_m|k|}}{\sqrt{ij}}.$$

Then

$$||\widehat{F}_{ij}^{u\overline{u}}(k)||_2 \leq \frac{(|k|^{2n+4}+8)}{\gamma_m(|i-j|+1)}\frac{C(m)e^{-s_m|k|}}{\sqrt{ij}},\ i\neq j. \tag{7.10}$$

Now we need the following lemmas:

Lemma 7.2. *[11] For $0<\delta<1$, $\nu>1$, one has*

$$\sum_{k\in\mathbb{Z}^n} e^{-2|k|\delta}|k|^{\nu} < \left(\frac{\nu}{e}\right)^{\nu}\frac{(1+e)^n}{\delta^{\nu+n}}.$$

Lemma 7.3. *If $A=(A_{ij})$ is a bounded linear operator on h_N, then also $B=(B_{ij}: i,j=0,1,2,\cdots)$ with*

$$||B_{ij}||_2 \leq \frac{|A_{ij}|}{|i-j|},\ i,j=0,1,2,\cdots,\ i\neq j,$$

and $\|B\|\leq C\|A\|$, where $\|\cdot\|$ is $h_N\to h_N$ operator norm,

$$B_{ij}=\begin{cases} b_{0,0}, & i=j=0,\\ (b_{0,j},\ b_{0,-j}), & i=0,\ j=1,2,\cdots,\\ (b_{i,0},\ b_{-i,0})^T, & j=0,\ i=1,2,\cdots,\\ \begin{pmatrix} b_{i,j} & b_{i,-j}\\ b_{-i,j} & b_{-i,-j}\end{pmatrix}, & i,j=1,2,\cdots,\end{cases}$$

with $b_{i,j}\in\mathbb{R}$, $i,j=0,1,2,\cdots$.

The proof of this result is similar to that of Theorem A.1 of [31] and so is omitted. See [31] for the details.

Therefore, by (7.10), we have

$$\begin{aligned}
&\sup_{\theta\in\mathbb{T}^n_{s'_m}\times\Pi_{m+1}}(||J_iF_{ij}^{u\overline{u}}(\theta,\omega)J_j||_2)\\
&\leq\left(\sum_{|k|\leq K_m}(|k|^{2n+4}+8)e^{-(s_m-s'_m)|k|}\right)\frac{C(m)}{\gamma_m(|i-j|+1)}\\
&\leq C\left(\frac{2n+4}{e}\right)^{2n+4}(1+e)^n\left(\frac{2}{s_m-s'_m}\right)^{3n+4}\frac{C(m)}{\gamma_m(|i-j|+1)}\quad\text{(by Lemma 7.2)}\\
&\leq C\frac{C(m)}{(s_m-s'_m)^{3n+4}}\frac{1}{\gamma_m(|i-j|+1)}
\end{aligned}$$

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$$\leq C\varepsilon_m^{-\frac{2(3n+4)}{N}} \frac{C(m)}{\gamma_m(|i-j|+1)},$$

where C is a constant depending on n, $s'_m = s_m - \frac{s_m - s_{m+1}}{4}$. By Lemma 7.3, we have

$$\|JF^{u\bar{u}}(\theta,\omega)J\|_{\mathbb{T}^n_{s'_m}\times\Pi^{+-}_{m+1}} \leq C\,C(m)\gamma_m^{-1}\varepsilon_m^{-\frac{2(3n+4)}{N}} \leq C(m+1)\varepsilon_m^{-\frac{2(3n+4)}{N}}. \tag{7.11}$$

It follows $s'_m > s_{m+1}$ that

$$\|JF^{u\bar{u}}(\theta,\omega)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi^{+-}_{m+1}} \leq \|JF^{u\bar{u}}(\theta,\omega)J\|_{\mathbb{T}^n_{s'_m}\times\Pi^{+-}_{m+1}} \leq C(m+1)\varepsilon_m^{-\frac{2(3n+4)}{N}}.$$

Applying ∂_{ω_l} $(l=1,2,\cdots,n)$ to both sides of (7.4), we have

$$-\langle k,\omega\rangle\partial_{\omega_l}\widehat{F}^{u\bar{u}}_{ij}(k) + (\Lambda_i^{(m)}\partial_{\omega_l}\widehat{F}^{u\bar{u}}_{ij}(k) - \partial_{\omega_l}\widehat{F}^{u\bar{u}}_{ij}(k)\Lambda_j^{(m)}) = \mathbf{i}\,\partial_{\omega_l}\widehat{R}^{u\bar{u}}_{mmij}(k) + (*), \tag{7.12}$$

where

$$(*) = k_l\widehat{F}^{u\bar{u}}_{ij}(k) - \partial_{\omega_l}\Lambda_i^{(m)}\widehat{F}^{u\bar{u}}_{ij}(k) + \widehat{F}^{u\bar{u}}_{ij}(k)\partial_{\omega_l}\Lambda_j^{(m)}. \tag{7.13}$$

By applying "vec" to both sides of (7.12), we have

$$(-\langle k,\omega\rangle(1\otimes 1) + 1\otimes\Lambda_i^{(m)} - (\Lambda_j^{(m)})^T\otimes 1)\mathrm{vec}\,\partial_{\omega_l}\widehat{F}^{u\bar{u}}_{ij}(k) = \mathrm{vec}\,(\mathbf{i}\,\partial_{\omega_l}\widehat{R}^{u\bar{u}}_{mmij}(k) + (*)). \tag{7.14}$$

Recalling $|k| \leq K_m = 100 s_m^{-1} 2^m |\log\varepsilon|$, and using (5.2)–(5.7) with $\nu = m$, using (7.13), we have, on $\omega \in \Pi_{m+1}$,

$$||(*)||_2 \leq CK_m||\widehat{F}^{u\bar{u}}_{ij}(k)||_2. \tag{7.15}$$

According to (5.9),

$$||\partial_{\omega_l}\widehat{R}^{u\bar{u}}_{mmij}(k)||_2 \leq \frac{C(m)e^{-s'_m|k|}}{\sqrt{ij}}. \tag{7.16}$$

By (7.10), (7.14), (7.15) and (7.16), we have

$$||J_i\partial_\omega\widehat{F}^{u\bar{u}}_{ij}(k)J_j||_2 \leq \frac{A_k^2CK_mC(m)e^{-s'_m|k|}}{\gamma_m^2(|i-j|+1)} \quad \text{for } i\neq j. \tag{7.17}$$

Note that $s_m > s'_m > s_{m+1}$. Again using Lemma 7.2 and Lemma 7.3, we have

$$\|JF^{u\bar{u}}(\theta,\omega)J\|^{\mathscr{L}}_{\mathbb{T}_{s_{m+1}}\times\Pi^{+-}_{m+1}} = \|J\partial_\omega F^{u\bar{u}}(\theta,\omega)J\|_{\mathbb{T}_{s_{m+1}}\times\Pi^{+-}_{m+1}} \leq C(m+1)\varepsilon_m^{-\frac{6(3n+4)}{N}}. \tag{7.18}$$

The proof of the measure estimate (7.1) will be postponed to Section 10. This completes the proof of Lemma 7.1. □

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Lemma 7.4. *There exists a compact subset* $\Pi_{m+1}^{++} \subset \Pi_m$ *with*

$$mes(\Pi_{m+1}^{++}) \geq mes\,\Pi_m - C\gamma_m^{1/3} \tag{7.19}$$

such that for any $\omega \in \Pi_{m+1}^{++}$*, the equation* (6.18) *has a unique solution* $F^{uu}(\theta)$*, which is defined on the domain* $\mathbb{T}_{s_{m+1}}^n \times \Pi_{m+1}^{++}$*, with*

$$\|JF^{uu}(\theta,\omega)J\|_{\mathbb{T}_{s_{m+1}}^n \times \Pi_{m+1}^{++}} \leq C(m+1)\varepsilon_m^{-\frac{2(3n+4)}{N}},$$

$$\|JF^{uu}(\theta,\omega)J\|_{\mathbb{T}_{s_{m+1}}^n \times \Pi_{m+1}^{++}}^{\mathscr{L}} \leq C(m+1)\varepsilon_m^{-\frac{6(3n+4)}{N}}.$$

Lemma 7.5. *There exists a compact subset* $\Pi_{m+1}^{--} \subset \Pi_m$ *with*

$$mes(\Pi_{m+1}^{--}) \geq mes\,\Pi_m - C\gamma_m^{1/3} \tag{7.20}$$

such that for any $\omega \in \Pi_{m+1}^{--}$*, the equation* (6.19) *has a unique solution* $F^{\overline{uu}}(\theta)$*, which is defined on the domain* $\mathbb{T}_{s_{m+1}}^n \times \Pi_{m+1}^{--}$ *with*

$$\|JF^{\overline{uu}}(\theta,\omega)J\|_{\mathbb{T}_{s_{m+1}}^n \times \Pi_{m+1}^{--}} \leq C(m+1)\varepsilon_m^{-\frac{2(3n+4)}{N}},$$

$$\|JF^{\overline{uu}}(\theta,\omega)J\|_{\mathbb{T}_{s_{m+1}}^n \times \Pi_{m+1}^{--}}^{\mathscr{L}} \leq C(m+1)\varepsilon_m^{-\frac{6(3n+4)}{N}}.$$

The proofs of Lemma 7.4 and Lemma 7.5 are simpler than that of Lemma 7.1, so we omit them.

Let

$$\Pi_{m+1} = \Pi_{m+1}^{+-} \bigcap \Pi_{m+1}^{++} \bigcap \Pi_{m+1}^{--}.$$

By (7.1), (7.19) and (7.20), we have

$$mes\,\Pi_{m+1} \geq mes\,\Pi_m - C\gamma_m^{1/3}.$$

8. Coordinate change Ψ by $\varepsilon_m F$

Recall $\Psi = \Psi_m = X_{\varepsilon_m F}^t\big|_{t=1}$, where $X_{\varepsilon_m F}^t$ is the flow of the Hamiltonian $\varepsilon_m F$ and $X_{\varepsilon_m F}$ is the vector field with symplectic $\mathbf{i}\,du \wedge d\overline{u}$. So

$$\mathbf{i}\,\dot{u} = \varepsilon_m \frac{\partial F}{\partial \overline{u}}, \ \ -\mathbf{i}\,\dot{\overline{u}} = \varepsilon_m \frac{\partial F}{\partial u}, \ \ \dot{\theta} = \omega.$$

More exactly,

$$\begin{cases} \mathbf{i}\dot{u} = \varepsilon_m(F^{u\overline{u}}(\theta,\omega)u + 2F^{\overline{u}\,\overline{u}}(\theta,\omega)\overline{u}),\ \theta=\omega t, \\ -\mathbf{i}\dot{\overline{u}} = \varepsilon_m(2F^{uu}(\theta,\omega)u + F^{u\overline{u}}(\theta,\omega)\overline{u}),\ \theta=\omega t, \\ \dot{\theta}=\omega. \end{cases}$$

Let $\widetilde{u} = \begin{pmatrix} u \\ \overline{u} \end{pmatrix}$,

$$B_m = \begin{pmatrix} -\mathbf{i}\,F^{u\overline{u}}(\theta,\omega) & -2\mathbf{i}\,F^{\overline{u}\,\overline{u}}(\theta,\omega) \\ 2\mathbf{i}\,F^{uu}(\theta,\omega) & \mathbf{i}\,F^{u\overline{u}}(\theta,\omega) \end{pmatrix}. \text{ Recall that } \theta=\omega t. \tag{8.1}$$

Then

$$\frac{d\widetilde{u}(t)}{dt} = \varepsilon_m B_m(\theta)\widetilde{u},\ \ \dot{\theta}=\omega. \tag{8.2}$$

Let $\widetilde{u}(0)=\widetilde{u}_0 \in h_N \times h_N$, $\theta(0)=\theta_0 \in \mathbb{T}^n_{s_{m+1}}$ be initial value. Then

$$\begin{cases} \widetilde{u}(t) = \widetilde{u}_0 + \int_0^t \varepsilon_m B_m(\theta_0+\omega s)\widetilde{u}(s)ds, \\ \theta(t) = \theta_0 + \omega t. \end{cases} \tag{8.3}$$

By Lemmas 7.1, 7.4 and 7.5,

$$\|JB_m(\theta)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m+1)\varepsilon_m^{-\frac{2(3n+4)}{N}}, \tag{8.4}$$

$$\|JB_m(\theta)J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m+1)\varepsilon_m^{-\frac{6(3n+4)}{N}}. \tag{8.5}$$

It follows from (8.3) that

$$\widetilde{u}(t)-\widetilde{u}_0 = \int_0^t \varepsilon_m B_m(\theta_0+\omega s)\widetilde{u}_0 ds + \int_0^t \varepsilon_m B_m(\theta_0+\omega s)(\widetilde{u}(s)-\widetilde{u}_0)ds.$$

Moreover, for $t \in [0,1]$, $\|\widetilde{u}_0\|_N \le 1$,

$$\|\widetilde{u}(t)-\widetilde{u}_0\|_N \le \varepsilon_m C(m+1)\varepsilon_m^{-\frac{2(3n+4)}{N}} + \int_0^t \varepsilon_m\|B_m(\theta_0+\omega s)\|\,\|\widetilde{u}(s)-\widetilde{u}_0\|_N ds, \tag{8.6}$$

where $\|\cdot\|$ is the operator norm from $h_N\times h_N \to h_N\times h_N$. By Gronwall's inequality,

$$\|\widetilde{u}(t)-\widetilde{u}_0\|_N \le C(m+1)\varepsilon_m^{1-\frac{2(3n+4)}{N}} \exp\left(\int_0^t \varepsilon_m\|B_m(\theta_0+\omega s)\|ds\right) \le \varepsilon_m^{1/2}. \tag{8.7}$$

Thus,

$$\Psi_m : \mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1} \to \mathbb{T}^n_{s_m}\times\Pi_m, \tag{8.8}$$

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and

$$\|\Psi_m - id\|_{h_N \to h_N} \le \varepsilon_m^{1/2}. \tag{8.9}$$

Since (8.2) is linear, Ψ_m is a linear coordinate change. According to (8.3), construct Picard sequence:

$$\begin{cases} \widetilde{u}_0(t) = \widetilde{u}_0, \\ \widetilde{u}_{j+1}(t) = \widetilde{u}_0 + \int_0^t \varepsilon_m B(\theta_0 + \omega s)\widetilde{u}_j(s)ds, \ j = 0, 1, 2, \cdots. \end{cases}$$

By (8.9), this sequence with $t = 1$ goes to

$$\Psi_m(u_0) = \widetilde{u}(1) = (id + P_m(\theta_0))u_0, \tag{8.10}$$

where id is the identity from $h_N \times h_N \to h_N \times h_N$, and $P_m(\theta_0)$ is an operator form $h_N \times h_N \to h_N \times h_N$ for any fixed $\theta_0 \in \mathbb{T}^n_{s_{m+1}}$, $\omega \in \Pi_{m+1}$, and is analytic in $\theta_0 \in \mathbb{T}^n_{s_{m+1}}$ with

$$\|P_m(\theta_0)\|_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \le \varepsilon_m^{1/2}. \tag{8.11}$$

Note that (8.2) is a Hamiltonian system, so $P_m(\theta_0)$ is a symplectic linear operator from $h_N \times h_N$ to $h_N \times h_N$.

9. Estimates of the remainders

The aim of this section is devoted to the estimates of the remainders:

$$C_{m+1}R_{m+1} = (6.14) + \cdots + (6.17).$$

- Estimate of (6.14).
  By (6.7), let

$$\widetilde{R}_{mm} = \widetilde{R}_{mm}(\theta) = \begin{pmatrix} R^{uu}_{m,m}(\theta) & \frac{1}{2}R^{u\bar{u}}_{m,m}(\theta) \\ \frac{1}{2}R^{u\bar{u}}_{m,m}(\theta) & R^{\bar{u}\bar{u}}_{m,m}(\theta) \end{pmatrix},$$

  then

$$R_{mm} = \langle \widetilde{R}_{mm}\begin{pmatrix} u \\ \bar{u} \end{pmatrix}, \begin{pmatrix} u \\ \bar{u} \end{pmatrix}\rangle.$$

  So

$$(1 - \Gamma_{K_m})R_{mm} \triangleq \langle (1 - \Gamma_{K_m})\widetilde{R}_{mm}\begin{pmatrix} u \\ \bar{u} \end{pmatrix}, \begin{pmatrix} u \\ \bar{u} \end{pmatrix}\rangle.$$

  By the definition of truncation operator Γ_{K_m},

$$(1 - \Gamma_{K_m})\widetilde{R}_{mm} = \sum_{|k| > K_m} \widehat{\widetilde{R}}_{mm}(k)e^{\mathbf{i}\langle k, \theta\rangle}, \ \theta \in \mathbb{T}^n_{s_m}, \ \omega \in \Pi_m.$$

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Since $\widetilde{R}_{mm} = \widetilde{R}_{mm}(\theta)$ is analytic in $\theta \in \mathbb{T}^n_{s_m}$,

$$\begin{aligned}
&\sup_{(\theta,\omega)\in\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \|J(1-\Gamma_{K_m})\widetilde{R}_{mm}J\|^2_{h_N\to h_N} \le \sum_{|k|>K_m} \|J\widehat{\widetilde{R}}_{mm}(k)J\|^2_N e^{2|k|s_{m+1}} \\
&\le \|J\widetilde{R}_{mm}J\|^2_{\mathbb{T}^n_{s_m}\times\Pi_m} \sum_{|k|>K_m} e^{-2(s_m-s_{m+1})|k|} \\
&\le C^2(m)\varepsilon_m^{-1} e^{-2K_m(s_m-s_{m+1})} \text{ (by (5.8))} \\
&\le C^2(m)\varepsilon_m^2,
\end{aligned}$$

which leads to

$$\|J(1-\Gamma_{K_m})\widetilde{R}_{mm}J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le \varepsilon_m C(m+1).$$

Thus,

$$\|\varepsilon_m J(1-\Gamma_{K_m})\widetilde{R}_{mm}J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le \varepsilon_m^2 C(m+1) \le \varepsilon_{m+1}C(m+1).$$

Similarly,

$$\|\varepsilon_m J(1-\Gamma_{K_m})\widetilde{R}_{mm}J\|^{\mathcal{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le \varepsilon_m^2 C(m+1) \le \varepsilon_{m+1}C(m+1).$$

- Estimate of (6.16).
  Let

$$S_m = \begin{pmatrix} F^{uu}(\theta,\omega) & \frac{1}{2}F^{u\bar{u}}(\theta,\omega) \\ \frac{1}{2}F^{u\bar{u}}(\theta,\omega) & F^{\bar{u}\bar{u}}(\theta,\omega) \end{pmatrix}.$$

Then we have

$$F = \langle S_m(\theta)\begin{pmatrix} u \\ \bar{u}\end{pmatrix}, \begin{pmatrix} u \\ \bar{u}\end{pmatrix}\rangle = \langle S_m\widetilde{u}, \widetilde{u}\rangle,\ \widetilde{u} = \begin{pmatrix} u \\ \bar{u}\end{pmatrix}.$$

Then

$$\varepsilon_m^2\{R_{mm}, F\} = 4\varepsilon_m^2\langle \widetilde{R}_{mm}(\theta)\mathscr{J} S_m(\theta)\widetilde{u}, \widetilde{u}\rangle. \tag{9.1}$$

Note $\mathbb{T}^n_{s_m} \times \Pi_m \supset \mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}$. By (5.8) and (5.9) with $l = m, v = m$,

$$\|J\widetilde{R}_{mm}(\theta)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le \|J\widetilde{R}_{mm}(\theta)J\|_{\mathbb{T}^n_{s_m}\times\Pi_m} \le C(m), \tag{9.2}$$

$$\|J\widetilde{R}_{mm}(\theta)J\|^{\mathcal{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m). \tag{9.3}$$

Let $\widetilde{S}_m(\theta) = \mathscr{J} S_m(\theta)$. Then by Lemmas 7.1, 7.4 and 7.5, we have

$$\|J\widetilde{S}_m(\theta)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \leq C(m+1)\varepsilon_m^{-\frac{2(3n+4)}{N}}, \tag{9.4}$$

$$\|J\widetilde{S}_m(\theta)J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \leq C(m+1)\varepsilon_m^{-\frac{6(3n+4)}{N}}, \tag{9.5}$$

and

$$\|J\widetilde{R}_{mm}\mathscr{J} S_m J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} = \|J\widetilde{R}_{mm}\widetilde{S}_m J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \leq C(m)C(m+1)\varepsilon_m^{-\frac{2(3n+4)}{N}}. \tag{9.6}$$

Set

$$[\widetilde{R}_{mm}, \widetilde{S}_m] = \widetilde{R}_{mm}\widetilde{S}_m + (\widetilde{R}_{mm}\widetilde{S}_m)^T.$$

Note that the vector field is linear. So, by Taylor formula, one has

$$(6.16) = \varepsilon_m^2\langle \widetilde{R}^*_m(\theta)\widetilde{u}, \widetilde{u}\rangle,$$

where

$$\widetilde{R}^*_m(\theta) = 2^2\widetilde{R}_{mm}\widetilde{S}_m + \sum_{j=2}^{\infty}\frac{2^{j+1}\varepsilon_m^{j-1}}{j!}\underbrace{[\cdots[\widetilde{R}_{mm}, \widetilde{S}_m], \cdots, \widetilde{S}_m]}_{j-1-\text{fold}}\widetilde{S}_m.$$

By (9.2) and (9.4),

$$\begin{aligned}\|J\widetilde{R}^*_m(\theta)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} &\leq \sum_{j=1}^{\infty}\frac{C(m)C(m+1)\varepsilon_m^{j-1}(\varepsilon_m^{-\frac{2(3n+4)}{N}})^j}{j!}\\ &\leq C(m)C(m+1)\varepsilon_m^{-\frac{2(3n+4)}{N}}.\end{aligned}$$

By (9.3) and (9.5),

$$\|J\widetilde{R}^*_m(\theta)J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \leq C(m)C(m+1)\varepsilon_m^{-\frac{6(3n+4)}{N}}.$$

Thus,

$$||\varepsilon_m^2 J\widetilde{R}^*_m J||_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \leq C(m)C(m+1)\varepsilon_m^{2-\frac{2(3n+4)}{N}} \leq C(m+1)\varepsilon_{m+1}, \tag{9.7}$$

and

$$||\varepsilon_m^2 J\widetilde{R}^*_m J||^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \leq C(m)C(m+1)\varepsilon_m^{2-\frac{6(3n+4)}{N}} \leq C(m+1)\varepsilon_{m+1}. \tag{9.8}$$

- Estimate of (6.15)
  By (6.9),

$$\{N_m, F\} = \langle [R_{mm}^{u\overline{u}}]u, \overline{u}\rangle - \Gamma_{K_m} R_{mm} \triangleq R_{mm}^*.$$

Thus,

$$(6.15) = \varepsilon_m^2 \int_0^1 (1-\tau)\{R_{mm}^*, F\} \circ X_{\varepsilon_m F}^\tau d\omega. \tag{9.9}$$

Note R_{mm}^* is a quadratic polynomial in u and $\overline{u}$. So we write

$$R_{mm}^* = \langle \mathscr{R}_m(\theta,\omega)\widetilde{u}, \widetilde{u}\rangle, \ \widetilde{u} = \begin{pmatrix} u \\ \overline{u} \end{pmatrix}. \tag{9.10}$$

By (5.6) and (5.7) with $l = \nu = m$, and with (9.4) and (9.5),

$$\|J\mathscr{R}_m J\|_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \le C(m)\varepsilon_m^{-\frac{2(3n+4)}{N}}, \quad \|J\mathscr{R}_m J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \le C(m)\varepsilon_m^{-\frac{6(3n+4)}{N}}, \tag{9.11}$$

where $\|\cdot\|$ is the operator norm in $h_N \times h_N \to h_N \times h_N$. Recall $F = \langle S_m(\theta,\omega)\widetilde{u}, \widetilde{u}\rangle$. Set

$$[\mathscr{R}_m, \widetilde{S}_m] = \mathscr{R}_m \widetilde{S}_m + (\mathscr{R}_m \widetilde{S}_m)^T. \tag{9.12}$$

Using Taylor formula to (9.9), we get

$$\begin{aligned}
(6.15) &= \frac{\varepsilon_m^2}{2!}\{R_{mm}^*, F\} + \cdots + \frac{\varepsilon_m^j}{j!}\underbrace{\{\cdots\{R_{mm}^*, F\}, \cdots, F\}}_{j-\text{fold}} + \cdots \\
&= \left\langle \left(\sum_{j=2}^{\infty} \frac{2^j \varepsilon_m^j}{j!} \underbrace{[\cdots[\mathscr{R}_m, \widetilde{S}_m], \cdots, \widetilde{S}_m]}_{j-1-\text{fold}} \widetilde{S}_m \right) \widetilde{u}, \widetilde{u} \right\rangle \\
&\triangleq \langle \mathscr{R}^{**}(\theta,\omega)\widetilde{u}, \widetilde{u}\rangle.
\end{aligned}$$

By (9.4), (9.11) and (9.12), we have

$$\begin{aligned}
&\|J\mathscr{R}^{**}(\theta,\omega)J\|_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \\
&\le \sum_{j=2}^{\infty} \frac{2^{j+1}}{j!} \|J\mathscr{R}_m(\theta,\omega)J\|_{\mathbb{T}^n_{s_m} \times \Pi_m} (\|J\widetilde{S}_m J\|_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \varepsilon_m)^j \\
&\le \sum_{j=2}^{\infty} \frac{C(m)}{j!} \left(\varepsilon_m C(m+1) \varepsilon_m^{-\frac{2(3n+4)}{N}} \right)^j \\
&\le C(m+1)\varepsilon_m^{4/3} = C(m+1)\varepsilon_{m+1}.
\end{aligned} \tag{9.13}$$

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Similarly,

$$\|J\mathscr{R}^{**}(\theta,\omega)J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m+1)\varepsilon_{m+1}. \tag{9.14}$$

- Estimate of (6.17)

$$(6.17) = \sum_{l=m+1}^{\infty} \varepsilon_l (R_{lm}\circ X^1_{\varepsilon_m F}). \tag{9.15}$$

We can write

$$R_{lm} = \langle \widetilde{R}_{lm}(\theta)\widetilde{u}, \widetilde{u}\rangle.$$

Then, by Taylor formula, one has

$$R_{lm}\circ X^1_{\varepsilon_m F} = R_{lm} + \sum_{j=1}^{\infty}\frac{1}{j!}\langle \widetilde{R}_{lmj}\widetilde{u}, \widetilde{u}\rangle,$$

where

$$\widetilde{R}_{lmj} = 2^{j+1}\underbrace{[\cdots[\widetilde{R}_{lm}, \widetilde{S}_m], \cdots]}_{j-1-\text{fold}}\widetilde{S}_m\varepsilon_m^j.$$

By (5.8), (5.9),

$$\|J\widetilde{R}_{lm}J\|_{\mathbb{T}^n_{s_l}\times\Pi_m} \le C(l), \quad \|J\widetilde{R}_{lm}J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_l}\times\Pi_m} \le C(l).$$

Combining the last inequalities with (9.4) and (9.5), one has

$$\begin{aligned}
&\|J\widetilde{R}_{lmj}J\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}\\
&\le \|J\widetilde{R}_{lm}J\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}(||J\widetilde{S}_mJ||_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}4\varepsilon_m)^j\\
&\le C^2(m)(\varepsilon_m\varepsilon_m^{-\frac{2(3n+4)}{N}})^j,
\end{aligned}$$

where $||J^{-1}||_{\mathbb{T}^n_{sl}\times\Pi_{m+1}} \le C$ is used, and

$$\begin{aligned}
&||J\widetilde{R}_{lmj}J||^{\mathscr{L}}_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}\\
&\le ||J\widetilde{R}_{lm}J||^{\mathscr{L}}_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}(||J\widetilde{S}_mJ||_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}4\varepsilon_m)^j\\
&\quad +||J\widetilde{R}_{lm}J||_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}(||J\widetilde{S}_mJ||^{\mathscr{L}}_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}\varepsilon_m)^j\\
&\le C^2(m)(\varepsilon_m\varepsilon_m^{-\frac{6(3n+4)}{N}})^j.
\end{aligned}$$

Thus, let

$$\overline{R}_{l,m+1} := \widetilde{R}_{lm} + \sum_{j=1}^{\infty} \frac{1}{j!} \widetilde{R}_{lmj},$$

then

$$(6.17) = \sum_{l=m+1}^{\infty} \varepsilon_l \langle \overline{R}_{l,m+1} \widetilde{u}, \widetilde{u} \rangle \tag{9.16}$$

and

$$||J\overline{R}_{l,m+1}J||_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}} \le C^2(m) \le C(m+1), \;\; ||J\overline{R}_{l,m+1}J||^{\mathcal{L}}_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}} \le C^2(m) \le C(m+1). \tag{9.17}$$

As a whole, the remainder R_{m+1} can be written as

$$C_{m+1}R_{m+1} = \sum_{l=m+1}^{\infty} \varepsilon_l (\langle R^{uu}_{l,\nu}(\theta)u, u\rangle + \langle R^{u\overline{u}}_{l,\nu}(\theta)u, \overline{u}\rangle) + \langle R^{\overline{u}\overline{u}}_{l,\nu}(\theta)\overline{u}, \overline{u}\rangle), \;\; \nu = m+1,$$

where, for $p, q \in \{u, \overline{u}\}$, $R^{p,q}_{l,\nu}$ satisfies (5.8) and (5.9) with $\nu = m+1, l \ge m+1$. This shows that Assumption $(A2)_\nu$ with $\nu = m+1$ holds true.
By (6.13), we know

$$\mu_j^{(m)} = \widehat{R}^{u\overline{u}}_{mmjj}(0).$$

Taking $p = u, q = \overline{u}$ into (5.8) and (5.9), we have

$$|\mu_j^{(m)}|_{\Pi_m} \le |R^{u\overline{u}}_{mmjj}(\theta,\omega)|/j \le C(m)/j,$$
$$|\mu_j^{(m)}|^{\mathcal{L}}_{\Pi_m} \le |\partial_\omega R^{u\overline{u}}_{mmjj}(\theta,\omega)|/j \le C(m)/j.$$

This shows that Assumption $(A1)_\nu$ with $\nu = m+1$ holds true.

10. Estimates of measure

In this section, C denotes a universal constant, which may be different in different places.

Lemma 10.1. *If $|i|, |j| \gg 1$, then*

$$\mu^{ml}_{ijk} = \rho\sqrt{\lambda_i} - \rho\sqrt{\lambda_j} + O(\frac{\varepsilon_0}{|i|}) + O(\frac{\varepsilon_0}{|j|}), \tag{10.1}$$

where $\lambda_k = k^2 + M,\ k \in \mathbb{Z},\ \mu^{ml}_{kij}$ is the l-th eigenvalue of $1 \otimes \Lambda_i^{(m)} - (\Lambda_j^{(m)})^T \otimes 1,\ i, j = 1, 2, \cdots,\ i \ne j,\ l = 1, 2, 3, 4$ (for more details, see Section 7, the proof of Lemma 7.1).

Proof. Recall that

$$\Lambda_i^{(m)} = \rho\sqrt{\lambda_i}E_{22} + O(\frac{\varepsilon_0}{|i|}),\ i \neq 0.$$

By computation, we have

$$\begin{aligned} 1\otimes\Lambda_i^{(m)} - (\Lambda_j^{(m)})^T \otimes 1 &= \rho\sqrt{\lambda_i}(E_{22}\otimes E_{22}) - \rho\sqrt{\lambda_j}(E_{22}\otimes E_{22}) + E_{22}\otimes G_i + G_j\otimes E_{22} \\ &= \rho(\sqrt{\lambda_i}-\sqrt{\lambda_j})E_{44} + E_{22}\otimes G_i + G_j \otimes E_{22}, \end{aligned} \tag{10.2}$$

where G_i is a 2×2 matrix such that $|G_i| \leq \frac{C\varepsilon_0}{|i|}$. Then

$$|1\otimes\Lambda_i^{(m)} - (\Lambda_j^{(m)})^T\otimes 1 - \rho(\sqrt{\lambda_i}-\sqrt{\lambda_j})E_{44}| \leq (\frac{C}{|i|}+\frac{C}{|j|})\varepsilon_0.$$

Note that $1\otimes\Lambda_i^{(m)} - (\Lambda_j^{(m)})^T\otimes 1$ is Hermitian. By the perturbation theory for eigenvalue of matrices, we obtain (10.1). □

Now let us return to (7.6)

$$Q_{kijl}^{(m)} \triangleq \left\{\omega\in\Pi_m \middle| |-\langle k,\omega\rangle + \mu_{kij}^{ml}| < \frac{(|i-j|+1)\gamma_m}{A_k}\right\},\quad A_k = |k|^{2n+4}+8. \tag{10.3}$$

Case 1. $i\neq j$. If $Q_{kijl}^{(m)} = \varnothing$, then $mes\,Q_{kijl}^{(m)} = 0$. So we assume $Q_{kijl}^{(m)} \neq \varnothing$. Then there exists $\omega\in\Pi_m$ such that

$$|-\langle k,\omega\rangle + \mu_{kij}^{ml}| < \frac{|i-j|+1}{A_k}\gamma_m. \tag{10.4}$$

(1.1) $k\neq 0$.
By Lemma 10.1,

$$|\mu_{kij}^{ml}| = |\rho\sqrt{\lambda_i} - \rho\sqrt{\lambda_j} + O(\frac{\varepsilon_0}{|i|}) + O(\frac{\varepsilon_0}{|j|})| \geq \frac{1}{2}|\sqrt{\lambda_i}-\sqrt{\lambda_j}|. \tag{10.5}$$

Furthermore, it is easy to verify that

$$|\sqrt{\lambda_i}-\sqrt{\lambda_j}| \geq \frac{4(|i-j|+1)\gamma_m}{A_k}. \tag{10.6}$$

Then by (10.4), (10.5) and (10.6), one has

$$\begin{aligned} |\langle k,\omega\rangle| &\geq |\mu_{kij}^{ml}| - \frac{(|i-j|+1)\gamma_m}{A_k} \geq \frac{1}{2}|\sqrt{\lambda_i}-\sqrt{\lambda_j}| - \frac{(|i-j|+1)\gamma_m}{A_k} \\ &\geq \frac{1}{4}|\sqrt{\lambda_i}-\sqrt{\lambda_j}| \geq \frac{1}{C}|i-j|. \end{aligned}$$

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So

$$|i-j| \le C|\langle k,\omega\rangle|. \tag{10.7}$$

(1.1.1) $i \ge i_0,\ j \ge j_0$.
By (10.1), we have that, when $\omega \in \Pi_m$ such that (10.4) holds true, the following inequality holds true:

$$\begin{aligned} |-\langle k,\omega\rangle+\rho i-\rho j| &= |(-\langle k,\omega\rangle+\mu_{kij}^{ml})+(\rho i-\rho j-\mu_{kij}^{ml})| \\ &\le \frac{|i-j|+1}{A_k}\gamma_m+\frac{C_1(M)}{i}+\frac{C_2(M)}{j} \\ &\le \frac{|i-j|+1}{A_k}\gamma_m+\frac{C_1(M)}{i_0}+\frac{C_2(M)}{j_0}, \end{aligned} \tag{10.8}$$

where $C_1(M)>0$ and $C_2(M)>0$ are constants.
Thus

$$Q_{kijl}^{(m)} \subset \left\{\omega\in\Pi_m \Big| |-\langle k,\omega\rangle+\rho\widetilde{l}| < \frac{|\widetilde{l}|+1}{A_k}\gamma_m+\frac{C_1(M)}{i_0}+\frac{C_2(M)}{j_0}\right\} \triangleq \widetilde{Q}_{k\widetilde{l}}. \tag{10.9}$$

By (10.7), one has

$$|\widetilde{l}| \le C|\langle k,\omega\rangle| \le C|k|. \tag{10.10}$$

Note that $k \neq 0$. Then

$$\frac{d(-\langle k,\omega\rangle+\rho\widetilde{l})}{d\omega} > \frac{1}{2}|k| \ge \frac{1}{2}.$$

It follows that

$$mes\,\widetilde{Q}_{k\widetilde{l}} \le 4\left(\frac{|\widetilde{l}|+1}{A_k}\gamma_m+\frac{C_1(M)}{i_0}+\frac{C_2(M)}{j_0}\right). \tag{10.11}$$

Take

$$j_0 = i_0 = |k|^{n+2}\gamma_m^{-1/3}. \tag{10.12}$$

Then

$$\begin{aligned} mes \bigcup_{1\le\widetilde{l}\le C|k|} \widetilde{Q}_{k\widetilde{l}} &\le \frac{C|k|\gamma_m}{A_k}+C\sum_{1\le|\widetilde{l}|\le C|k|}\left(\frac{C_1(M)}{i_0}+\frac{C_2(M)}{j_0}\right) \\ &\le \frac{C|k|\gamma_m}{A_k}+\gamma_m^{1/3}\frac{C|k|}{|k|^{n+2}} \\ &\le \frac{C\gamma_m^{1/3}}{|k|^{n+1}}. \end{aligned}$$

It follows from (10.9) that

$$mes \bigcup_{\substack{i \geq i_0 \\ j \geq j_0 \\ |i-j| \leq C|k|}} Q_{kijl}^{(m)} \leq \frac{C\gamma_m^{1/3}}{|k|^{n+1}}. \tag{10.13}$$

(1.1.2) $i \leq i_0$ or $j \leq j_0$.
By (10.7), one has $|i-j| \leq C|k|$. In addition, $1 \otimes \Lambda_i^{(m)} - (\Lambda_j^{(m)})^T \otimes 1$ is obviously Hermitian. Then by the variation of eigenvalues for Hermitian matrix, we have

$$\left|\frac{d}{d\omega}(-\langle k,\omega\rangle + \mu_{ijk}^{ml})\right| \geq |k| - \left|\frac{d\mu_{ijk}^{ml}}{d\omega}\right| \geq \frac{1}{2}.$$

Therefore,

$$mes \bigcup_{\substack{1 \leq i \leq i_0 \\ |i-j| \leq C|k|}} Q_{kijl}^{(m)} \leq \sum_{\substack{1 \leq i \leq i_0 \\ |i-j| \leq C|k|}} \frac{4(|i-j|+1)\gamma_m}{A_k} \leq \frac{C|k|\gamma_m i_0}{A_k}$$

$$\leq C|k|^{n+3}\gamma_m^{2/3}\frac{1}{A_k} \leq \frac{C\gamma_m^{2/3}}{|k|^{n+1}}. \tag{10.14}$$

Similarly, one has

$$mes \bigcup_{\substack{1 \leq j \leq j_0 \\ |i-j| \leq C|k|}} Q_{kijl}^{(m)} \leq \frac{C\gamma_m^{2/3}}{|k|^{n+1}}. \tag{10.15}$$

(1.2) $k = 0$.
By (10.5) and (10.6), one has $Q_{kijl}^{(m)} = \varnothing$, then

$$mes\, Q_{kijl}^{(m)} = 0. \tag{10.16}$$

Case 2. $i = j$, one has $k \neq 0$.
At this time, by Lemma 10.1,

$$-\langle k,\omega\rangle + \mu_{kij}^{ml} = -\langle k,\omega\rangle + O(\frac{\varepsilon_0}{|i|}). \tag{10.17}$$

(2.1) Suppose $\left|\langle k,\omega\rangle\right| \geq \frac{2\gamma_m^{2/3}}{A_k}$.

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(2.1.1) $i > \frac{C\varepsilon_0 A_k}{\gamma_m^{2/3}}$.
By (10.17), one has

$$|-\langle k,\omega\rangle + \mu_{kij}^{ml}| \geq \frac{2\gamma_m^{2/3}}{A_k} - \frac{C\varepsilon_0}{i} > \frac{\gamma_m^{2/3}}{A_k}.$$

It follows from (10.4) that $Q_{kiil}^{(m)} = \emptyset$. Then

$$mes\, Q_{kiil}^{(m)} = 0. \tag{10.18}$$

(2.1.2) $i \leq \frac{C\varepsilon_0 A_k}{\gamma_m^{2/3}} \triangleq \widetilde{k}$.
Note that

$$\frac{d(-\langle k,\omega\rangle + \mu_{kij}^{ml})}{d\omega} = |k| + O(\frac{\varepsilon_0}{|i|}) \geq \frac{1}{2}.$$

Then

$$mes \bigcup_{i\leq\widetilde{k}} Q_{kiil}^{(m)} \leq \frac{4\widetilde{k}\gamma_m}{A_k} \leq C\gamma_m^{1/3}. \tag{10.19}$$

(2.2) Suppose $|\langle k,\omega\rangle| < \frac{2\gamma_m^{2/3}}{A_k}$.
Let

$$\widetilde{Q}_k = \left\{ \omega\in\Pi_m \middle| |\langle k,\omega\rangle| < \frac{2\gamma_m^{2/3}}{A_k} \right\}.$$

Note that $|\frac{d(\langle k,\omega\rangle)}{d\omega}| = |k| \geq 1$. Then

$$mes\, \widetilde{Q}_k \leq \frac{4\gamma_m^{2/3}}{A_k},$$

and

$$mes \bigcup_{k\in\mathbb{Z}^n\setminus\{0\}} \widetilde{Q}_k \leq \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{C\gamma_m^{2/3}}{A_k} \leq C\gamma_m^{1/3}. \tag{10.20}$$

Combining (10.13), (10.14), (10.15), (10.16), (10.18), (10.19) and (10.20), we have

$$mes \bigcup_{|k|\leq K_m} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcup_{l=1}^{4} Q_{kijl}^{(m)} \leq C\gamma_m^{1/3}. \tag{10.21}$$

Let

$$\Pi_{m+1}^{+-} = \Pi_m \backslash \bigcup_{|k|\le K_m} \bigcup_{i,j=1}^{\infty} \bigcup_{l=1}^{4} Q_{kijl}^{(m)}.$$

Then we have proved the following Lemma 10.2.

Lemma 10.2.

$$mes\Pi_{m+1}^{+-} \ge mes\Pi_m - C\gamma_m^{1/3}.$$

11. Proof of theorems

Theorem 2.1 is a more exact statement of Theorem 1.1. Let

$$\Pi_\infty = \bigcap_{m=1}^{\infty} \Pi_m,$$

and

$$\Psi_\infty = \lim_{m\to\infty} \Psi_0 \circ \Psi_1 \circ \cdots \circ \Psi_m.$$

By (5.11) and (5.12), one has

$$\Psi_\infty : \mathbb{T}^n \times \Pi_\infty \to \mathbb{T}^n \times \Pi_\infty,$$
$$||\Psi_\infty - id|| \le \varepsilon^{1/2},$$

and, by (5.13),

$$H_\infty = H \circ \Psi_\infty = \sum_{j=0}^{\infty} \langle \Lambda_j^\infty u_j, \bar{u}_j \rangle,$$

where $\Lambda_j^\infty = \rho\Lambda_j^{(0)} + Q_j^{(0)}$, and $Q_j^{(0)}$ is independent of time, $Q_0 \in \mathbb{R}$, $Q_j \in gl(\mathbb{R}, 2)$ with $j \ne 0$.
This completes the proof of Theorem 2.1.

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