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Ellipsoidal BGK model for polyatomic molecules near Maxwellians: A dichotomy in the dissipation estimate

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Abstract

We consider the global existence and asymptotic behavior of classical solutions to the ellipsoidal BGK model for polyatomic molecules when the initial data starts sufficiently close to a global polyatomic Maxwellian. We observe that the linearized relaxation operator is decomposed into a truly polyatomic part and an essentially monatomic part, leading to a dichotomy in the dissipative property in the sense that the degeneracy of the dissipation shows an abrupt jump as the relaxation parameter θ reaches zero. Accordingly, we employ two different sets of micro–macro system to derive the full coercivity and close the energy estimate.

Keywords: BGK model; Polyatomic gases; Boltzmann equation; Kinetic theory of gases; Dissipation estimate

1. Introduction

The collective dynamics of rarefied gases at the mesoscopic scale is described by the celebrated Boltzmann equation. But the practical application of the Boltzmann equation has been restricted by its highly resource-consuming features such as the complicated structure of the collision operator, high dimensionality and stiffness problem. In this regard, Bhatnagar, Gross, Krook [4] and, independently Welander [53], suggested a model equation by replacing the collision operator with a relaxation operator which still keeps the most important features of the Boltzmann equation such as the conservation laws, H-theorem and the correct hydrodynamic

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limit to the Euler equation. Ever since it was introduced, the BGK model has been widely used in place of the Boltzmann equation because it reproduces the qualitative features of the Boltzmann dynamics very well at much lower computational costs.

Both the Boltzmann equation and the BGK model are derived under the assumption that the gas consists of monatomic molecules. The necessity of kinetic equations that account for the collisional dynamics of polyatomic molecules is apparent, considering that there are very few elements in the nature which stay stable as monatomic molecules at room temperature. Any attempt for the description of kinematics of polyatomic molecules, however, must allow some simplifying assumptions or phenomenological description because the diversity of the inner configuration of polyatomic molecules make it almost impossible to express the pre-post collision process in an explicit form, except for some special cases. One such formulation is so-called the internal energy formulation where a new variable I is introduced to incorporate the information on the non-translational internal energy due to the molecular structure [1,2,5,6,9,10,29,37,41,42,49].

In this paper, we study the existence and asymptotic behavior for the polyatomic ellipsoidal BGK model, which is a polyatomic generalization of the original BGK model using such internal energy formulation: [1,2]:

$$\begin{aligned} \partial_t F + v\cdot\nabla_x F &= A_{\nu,\theta}(\mathcal{M}_{\nu,\theta}(F) - F), \\ F(0,x,v,I) &= F_0(x,v,I). \end{aligned} \tag{1.1}$$

The velocity-energy distribution function $F(t,x,v,I)$ represents the number density on phase point $(x,v)\in\mathbb{T}^3_x\times\mathbb{R}^3_v$ with non-translational internal energy $I^{2/\delta}$ $(I\geq 0)$ at time $t\geq 0$. The parameter $\delta>0$ measures the degree of excitation of non-translational mode of the molecules such as the rotational or vibrational mode. The collision frequency $A_{\nu,\theta}$ is given by $A_{\nu,\theta}=(\rho^\alpha T_\delta^\beta)/(1-\nu+\theta\nu)$ for some $0\leq\alpha,\beta\leq 1$. (ρ and T_δ are defined below.) Throughout this paper, we fix $\alpha=\beta=1$ for simplicity. The relaxation parameters $-1/2\leq\nu<1$ and $0\leq\theta\leq 1$ are introduced to reproduce the correct Prandtl number and the second viscosity coefficient in the Chapman–Enskog expansion [2]. The number $1/\theta$ is interpreted as the relaxation collision number, which is the average number of collisions needed to transfer the rotational and vibrational internal energy into the translational energy [1,10].

We define the macroscopic density, momentum, stress tensor and total energy by

$$\begin{aligned} \rho(t,x) &= \int_{\mathbb{R}^3\times\mathbb{R}^+} F(t,x,v,I)\,dvdI, \\ U(t,x) &= \frac{1}{\rho}\int_{\mathbb{R}^3\times\mathbb{R}^+} vF(t,x,v,I)\,dvdI, \\ \Theta(t,x) &= \frac{1}{\rho}\int_{\mathbb{R}^3\times\mathbb{R}^+} (v-U)\otimes(v-U)F(t,x,v,I)\,dvdI, \\ E(t,x) &= \int_{\mathbb{R}^3\times\mathbb{R}^+} \left(\frac{1}{2}|v|^2 + I^{\frac{2}{\delta}}\right)F(t,x,v,I)\,dvdI. \end{aligned}$$

The total energy is decomposed further into the following three parts:

$$E = E_{kin} + E_{tr} + E_{I,\delta}$$

where the kinetic energy E_{kin}, the internal energy due to the translational motion E_{tr}, and the internal energy attributed to the internal configuration of the molecules $E_{I,\delta}$ are given respectively by

$$\begin{aligned} E_{kin} &= \frac{1}{2}\rho|U|^2, \\ E_{tr} &= \frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^+} |v-U|^2 F(t,x,v,I)\,dv\,dI, \\ E_{I,\delta} &= \int\limits_{\mathbb{R}^3\times\mathbb{R}^+} I^{\frac{2}{\delta}} F(t,x,v,I)\,dv\,dI. \end{aligned}$$

We also define the total internal energy E_δ:

$$\begin{aligned} E_\delta(t,x) &= E_{tr} + E_{I,\delta} \\ &= \int\limits_{\mathbb{R}^3\times\mathbb{R}^+} \left(\frac{1}{2}|v-U|^2 + I^{\frac{2}{\delta}}\right) F(t,x,v,I)\,dv\,dI, \end{aligned}$$

from which we can define the corresponding temperatures T_δ, T_{tr} and $T_{I,\delta}$ using the equipartition principle:

$$E_\delta = \frac{3+\delta}{2}\rho T_\delta, \quad E_{tr} = \frac{3}{2}\rho T_{tr}, \quad E_{I,\delta} = \frac{\delta}{2}\rho T_{I,\delta}.$$

Consequently, T_δ is represented by a convex combination of T_{tr} and $T_{I,\delta}$:

$$T_\delta = \frac{3}{3+\delta}T_{tr} + \frac{\delta}{3+\delta}T_{I,\delta}.$$

For $-1/2 \le \nu < 1$ and $0 \le \theta \le 1$, we define the relaxation temperature $T_{\nu,\theta}$ and the corrected temperature tensor $\mathcal{T}_\theta$ by

$$\begin{aligned} T_\theta &= \theta T_\delta + (1-\theta)T_{I,\delta}, \\ \mathcal{T}_{\nu,\theta} &= \theta T_\delta Id + (1-\theta)\big\{(1-\nu)T_{tr} Id + \nu\Theta\big\}. \end{aligned}$$

Now, the polyatomic ellipsoidal Maxwellian $\mathcal{M}_{\nu,\theta}$ reads

$$\mathcal{M}_{\nu,\theta}(F) = \frac{\rho\Lambda_\delta}{\sqrt{\det(2\pi\mathcal{T}_{\nu,\theta})}\,T_\theta^{\frac{\delta}{2}}}\exp\left(-\frac{1}{2}(v-U)^\top\mathcal{T}_{\nu,\theta}^{-1}(v-U) - \frac{I^{\frac{2}{\delta}}}{T_\theta}\right),$$

where Λ_δ is the normalizing factor: $\Lambda_\delta = 1/\int_{\mathbb{R}_+} e^{-I^{2/\delta}}dI$.

The relaxation operator satisfies the following cancellation property:

$$\int_{\mathbb{R}^3\times\mathbb{R}^+} (\mathcal{M}_{\nu,\theta}(F)-F)\begin{pmatrix}1\\ v\\ \frac{1}{2}|v|^2+I^{\frac{2}{\delta}}\end{pmatrix}dvdI=0,$$

which leads to the conservation of mass, momentum and energy:

$$\begin{aligned}\int F(t)dxdvdI&=\int F_0dxdvdI,\\ \int F(t)vdxdvdI&=\int F_0vdxdvdI,\\ \int F(t)\left(\frac{|v|^2}{2}+I^{2/\delta}\right)dxdvdI&=\int F_0\left(\frac{|v|^2}{2}+I^{2/\delta}\right)dxdvdI.\end{aligned}\tag{1.2}$$

The H-theorem for this model was established in [2] (see also [9,36]):

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^+} F(t)\ln F(t)dvdI\le 0.$$

In this paper, we study the dynamics of the polyatomic BGK model (1.1) near a global polyatomic Maxwellian:

$$m(v,I)=\frac{\Lambda_\delta}{\sqrt{(2\pi)^3}}e^{-\frac{|v|^2}{2}-I^{2/\delta}}.\tag{1.3}$$

For this, we define the perturbation f around the equilibrium by

$$F=m+\sqrt{m}f,\quad F_0=m+\sqrt{m}f_0\tag{1.4}$$

and rewrite (1.1) as

$$\begin{aligned}\partial_t f+v\cdot\nabla_x f&=L_{\nu,\theta}f+\Gamma_{\nu,\theta}(f),\\ f(0,x,v,I)&=f_0(x,v,I),\end{aligned}$$

where $L_{\nu,\theta}$ denotes the linearized relaxation operator and $\Gamma_{\nu,\theta}(f)$ is the nonlinear perturbation. (See Section 2.) We then analyze this linearized polyatomic BGK model in the framework of nonlinear energy methods developed in, for example, [24–26].

The most important step is to verify the dissipative nature of the linearized relaxation operator $L_{\nu,\theta}$. In this regard, we make a key observation that there exists a dichotomy in the coercive estimate of $L_{\nu,\theta}$ (see Section 3):

$$-(1-\nu+\theta\nu)\langle L_{\nu,\theta}f,f\rangle_{L^2_{v,I}}\ge\theta\|(I-P_p)f\|^2_{L^2_{v,I}}\quad(0<\theta\le 1),$$

and

$$-(1-\nu)\langle L_{\nu,0}f,f\rangle_{L^2_{v,I}}\geq(1-|\nu|)\,\|(I-P_m)f\|^2_{L^2_{v,I}}\qquad(\theta=0).$$

Note that the coefficient in the l.h.s. of the above dissipative estimates changes continuously as θ goes to 0, while the coefficient in the right hand side jumps from θ to $1-|\nu|$ at $\theta=0$. More importantly, the macroscopic projection on the right hand side, which determines the degeneracy of the dissipation, changes abruptly from the projection P_p on

$$span\left\{\sqrt{m},\ v\sqrt{m},\ \frac{(|v|^2-3)+(2I^{2/\delta}-\delta)}{\sqrt{2(3+\delta)}}\sqrt{m}\right\}\quad(0<\theta\leq 1),\tag{1.5}$$

to the projection P_m on

$$span\left\{\sqrt{m},\ v\sqrt{m},\ \frac{|v|^2-3}{\sqrt{6}}\sqrt{m},\ \frac{I^{2/\delta}-\delta}{\sqrt{2\delta}}\sqrt{m}\right\}\qquad(\theta=0).\tag{1.6}$$

Therefore, the degeneracy at $\theta=0$ is strictly stronger than the non-zero θ case.

This agrees well with the similar dichotomy in the nonlinear entropy–entropy production estimate observed in [36], of which the above estimates can be considered as a linearized version:

$$D_{\nu,\theta}(f)\geq\theta A_{\nu,\theta}H(f|\mathcal{M}_{0,1})\qquad(0<\theta\leq 1),$$

and

$$D_{\nu,0}(f)\geq\min\{1-\nu,1+2\nu\}A_{\nu,0}H(f|\mathcal{M}_{0,0})\quad(\theta=0).$$

Here, $D_{\nu,\theta}$ and $H(f|g)$ denote the entropy production functional and the relative entropy for (1.1) respectively. See [36] for the exact definition of the target polyatomic Maxwellians $\mathcal{M}_{0,1}$ and $\mathcal{M}_{0,0}$. In [36], however, it is not clear whether such dichotomy is an intrinsic property of the model, or can be resolved into a better estimate that continuously interpolates the two entropy production estimates.

It is explicitly shown in Section 3 that the linearized relaxation operator $L_{\nu,\theta}$ is divided into a truly polyatomic part and an essentially monatomic part. We then prove that the polyatomic dissipation is strictly stronger than that of the monatomic-like part in the range $0<\theta\leq 1$ so that the whole coercivity is governed by the polyatomic part, whereas the coercivity for $\theta=0$ case is governed solely by the monatomic-like part. This shows that such dichotomy is intrinsic, and cannot be avoided by developing a refined argument.

Recalling that θ^{-1} is interpreted as the average number of collisions needed for the non-translational energy due to the molecular configuration to be transferred, we see that such dichotomy has a nice physical interpretation: when $\theta=0$, the relaxation collision number is infinite, and therefore, no matter how many collisions occur, the exchange between the translational energy and the non-translational energy does not happen, making the kinematics essentially – though not exactly – that of the monatomic gases. (Note that in the kernel of P_m, the translational energy and the non-translational energy are completely split, whereas they are given in an entangled form in the kernel of P_p.) We, however, mention that such physical interpretation alone does not give any hint that there has to be a discontinuity at $\theta=0$.

As a result of such dichotomy, we need to employ two different types of micro–macro decomposition, namely, the polyatomic decomposition:

$$f = P_p f + (I - P_p) f \quad (0 < \theta \le 1),$$

and the monatomic-like decomposition:

$$f = P_m f + (I - P_m) f \quad (\theta = 0).$$

Therefore, we need to study two different sets of micro–macro equations accordingly, in order to fill up the degeneracy and to derive the full coercivity.

1.1. Main result

We define the high-order energy functional $\mathcal{E}(f(t))$:

$$\mathcal{E}\big(f(t)\big) = \frac{1}{2} \sum_{|\alpha|+|\beta| \le N} \|\partial^\alpha_\beta f(t)\|^2_{L^2_{x,v,I}} + \sum_{|\alpha|+|\beta| \le N} \int_0^t \|\partial^\alpha_\beta f(s)\|^2_{L^2_{x,v,I}} ds.$$

Theorem 1.1. *Let $-1/2 \le \nu < 1$, $0 \le \theta \le 1$ and $N \ge 4$. Suppose that $F_0 = m + \sqrt{m} f_0 \ge 0$ has the same mass, momentum and energy with m:*

$$\begin{aligned} \int_{\mathbb{T}^3_x \times \mathbb{R}^3_v \times \mathbb{R}^+_I} f_0 \sqrt{m}\, dx dv dI &= 0, \\ \int_{\mathbb{T}^3_x \times \mathbb{R}^3_v \times \mathbb{R}^+_I} f_0 v \sqrt{m}\, dx dv dI &= 0, \\ \int_{\mathbb{T}^3_x \times \mathbb{R}^3_v \times \mathbb{R}^+_I} f_0 \left\{ \frac{1}{2}|v|^2 + I^{2/\delta} \right\} \sqrt{m}\, dx dv dI &= 0. \end{aligned} \tag{1.7}$$

Then there exist $\varepsilon > 0$ and $C = C(f_0, N, \nu, \theta, \delta) > 0$, such that if $\mathcal{E}(0) < \varepsilon$, then there exists a unique global in time solution f for (2.12) satisfying:

(1) *The distribution function is non-negative for all $t \ge 0$:*

$$F = m + \sqrt{m} f \ge 0,$$

and satisfies the conservation laws:

$$\int_{\mathbb{T}^3_x \times \mathbb{R}^3_v \times \mathbb{R}^+_I} f(x, v, t) \sqrt{m}\, dx dv dI = 0,$$

$$\int_{\mathbb{T}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^+_I} f(x,v,t)v\sqrt{m}\,dxdvdI = 0, \tag{1.8}$$

$$\int_{\mathbb{T}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^+_I} f(x,v,t)\left\{\frac{1}{2}|v|^2 + I^{2/\delta}\right\}\sqrt{m}\,dxdvdI = 0.$$

(2) *The high-order energy functional is uniformly bounded:*

$$\mathcal{E}(t) \le C\mathcal{E}(0).$$

(3) *The initial perturbation decays exponentially fast:*

$$\sum_{|\alpha|+|\beta|\le N} \|\partial^\alpha_\beta f(t)\|^2_{L^2_{x,v,I}} \le Ce^{-Ct}.$$

A brief review on the related literature is in order. We start with the original monatomic BGK model. The first mathematical study of the BGK model was made in [38] where Perthame established the existence of weak solutions under the assumption of finite mass, energy and entropy. Perthame and Pulvirenti then studied the existence of unique mild solutions in a weighted L^∞ space in [39]. These results were extended, for example, to Cauchy problem for L^p data [54], plasma [59] or gases under the influence of external forces [60]. Ukai studied the stationary problem in a bounded interval with a fixed boundary condition in [50]. For the application of BGK type models to various macroscopic limits, see [7,16,31–33,44,45]. The existence of classical solutions and their asymptotic behavior were studied in [3,13,55]. Some error analysis of numerical schemes for BGK model can be found in [28,43].

Recently, the interest on the ES-BGK model [27], which is a generalized version of the monatomic BGK model designed to reproduce the physical Prandtl number, revived after the H-theorem was verified for this model in [2]. (See also [8,58].) For the existence results of this model in various situations, see [17,35,56,57].

The study of the ellipsoidal BGK model for polyatomic molecules is in its initial stage. The H-theorem was shown to hold in [2,9]. Entropy–entropy production estimate for this model was established in [36], where the dichotomy in the entropy dissipation mechanism mentioned above, was first observed. The extension of [39] arguments to the polyatomic case was made in [34]. In the near-equilibrium regime, no existence result is available so far.

We mention that there has been an alternative approach besides the internal energy formulation to construct BGK type model for polyatomic molecules, where the polyatomic gas is treated as a mixture of monatomic gases endowed with discrete levels of internal energy [22,23].

We omit the reference review on the numerical results on BGK type models (monatomic or polyatomic), since they are huge. Interested readers may refer to [1,2,10,15,18,19,21,28,29,40, 41,43] and references therein. For general review on the mathematical and physical theory of kinetic equations, see [11,12,14,20,30,46–48,51,52].

The following are the notations and conventions kept throughout this paper:

- All the constants, usually denoted by C will be defined generically.
- For $\kappa \in \mathbb{R}^3$, $\kappa^\top$ denotes its transpose.

- For symmetric $n\times n$ matrices A and B, $A\le B$ means that $B-A$ satisfies $k^{\top}\{B-A\}k\ge 0$ for all $k\in\mathbb{R}^n$.
- When there is no risk of confusion, we use $\mathcal{E}(t)$ instead of $\mathcal{E}\big(f(t)\big)$ for simplicity. The latter notation will be employed when the dependency needs to be clarified.
- We slightly abuse the notation to define the summation on the index set $i<j$ by

$$\sum_{i<j} a_{ij} = a_{12}+a_{23}+a_{31}.$$

- $\langle\cdot,\cdot\rangle_{L^2_{v,I}}$ and $\langle\cdot,\cdot\rangle_{L^2_{x,v,I}}$ denote the standard L^2 inner product on $\mathbb{R}^3_v\times\mathbb{R}^+_I$ and $\mathbb{T}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^+_I$ respectively:

$$\begin{aligned}
\langle f,g\rangle_{L^2_{v,I}} &= \int\limits_{\mathbb{R}^3\times\mathbb{R}^+} f(v,I)g(v,I)dvdI,\\
\langle f,g\rangle_{L^2_{x,v,I}} &= \int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^+} f(x,v,I)g(x,v,I)dxdvdI.
\end{aligned}$$

- $\|\cdot\|_{L^2_{v,I}}$ and $\|\cdot\|_{L^2_{x,v,I}}$ denote the standard L^2 norms on $\mathbb{R}^3_v\times\mathbb{R}^+_I$ and $\mathbb{T}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^+_I$ respectively:

$$\begin{aligned}
\|f\|_{L^2_{v,I}} &= \Big(\int\limits_{\mathbb{R}^3\times\mathbb{R}^+} |f(v,I)|^2 dvdI\Big)^{\frac12},\\
\|f\|_{L^2_{x,v,I}} &= \Big(\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^+} |f(x,v,I)|^2 dxdvdI\Big)^{\frac12}.
\end{aligned}$$

- We use the following notations for the multi-indices and differential operators:

$$\alpha=[\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4],\qquad \beta=[\beta_1,\beta_2,\beta_3],$$

  and

$$\partial^{\alpha}_{\beta} = \partial^{\alpha_0}_t\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}\partial^{\beta_4}_{I}.$$

The paper is organized as follows: In Section 2, we consider the linearization of the relaxation operator. Then section 3 is devoted to the coercivity estimate for the linearized relaxation operator. We treat the case $0<\theta\le 1$ and $\theta=0$ separately, yielding different dissipation estimate in each case. In Section 4, we derive various estimates for macroscopic fields. In Section 5, we consider the existence of the local in time classical solution. Section 6 is devoted to the study of the micro–macro systems, where, due to the dichotomy observed in Section 4, the case $0<\theta\le 1$ and $\theta=0$ are considered separately. Finally, we prove the main result in Section 7.

2. Linearization of the polyatomic BGK model

In this section, we carry out the linearization of (1.1) around the normalized global polyatomic Maxwellian (1.3).

2.1. Transitional fields

Let F_η denote the transition from the solution F of (1.1) to the global polyatomic Maxwellian m:

$$F_\eta = \eta F + (1-\eta)m = m + \eta f\sqrt{m} \qquad (0 \leq \eta \leq 1),$$

where f is defined in (1.4). In view of the following identities:

$$\begin{aligned}
&\rho = \int_{\mathbb{R}^3\times\mathbb{R}_+} F\,dvdI, \quad \rho U = \int_{\mathbb{R}^3\times\mathbb{R}_+} Fv\,dvdI,\\
&\rho\mathcal{T}_{\nu,\theta} + \frac{\theta}{3+\delta}\rho|U|^2 Id + (1-\theta)\left\{\frac{1-\nu}{3}\rho|U|^2 Id + \nu\rho U\otimes U\right\}\\
&\qquad = \theta\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+} F\left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{2/\delta}\right)dvdI\right\} Id\\
&\qquad + (1-\theta)\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+} F\left(\frac{1-\nu}{3}|v|^2 Id + \nu v\otimes v\right)dvdI\right\},\\
&\rho T_\theta + \frac{\theta}{3+\delta}\rho|U|^2 = \theta\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+} F\left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{2/\delta}\right)dvdI\right\}\\
&\qquad + (1-\theta)\left\{\frac{2}{\delta}\int_{\mathbb{R}^3\times\mathbb{R}_+} FI^{2/\delta}dvdI\right\},
\end{aligned}$$

we define transitional macroscopic fields: ρ_η, U_η, $\mathcal{T}_{\nu,\theta\eta}$ and $T_{\theta\eta}$ by

$$\begin{aligned}
&\rho_\eta = \int_{\mathbb{R}^3\times\mathbb{R}_+} F_\eta\,dvdI, \quad \rho_\eta U_\eta = \int_{\mathbb{R}^3\times\mathbb{R}_+} F_\eta v\,dvdI,\\
&\rho_\eta\mathcal{T}_{\nu,\theta\eta} + \frac{\theta}{3+\delta}\rho_\eta|U_\eta|^2 Id + (1-\theta)\left\{\frac{1-\nu}{3}\rho_\eta|U_\eta|^2 Id + \nu\rho_\eta U_\eta\otimes U_\eta\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \theta \left\{ \int_{\mathbb{R}^3\times\mathbb{R}_+} F_\eta \left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta} I^{2/\delta} \right) dv dI \right\} Id \\
&+ (1-\theta) \left\{ \int_{\mathbb{R}^3\times\mathbb{R}_+} F_\eta \left(\frac{1-\nu}{3}|v|^2 Id + \nu v\otimes v \right) dv dI \right\}, \qquad (2.1)
\end{aligned}
$$

$$
\begin{aligned}
&\rho_\eta T_{\theta\eta} + \frac{\theta}{3+\delta}\rho_\eta |U_\eta|^2 \\
&= \theta \left\{ \int_{\mathbb{R}^3\times\mathbb{R}_+} F_\eta \left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta} I^{2/\delta} \right) dv dI \right\} \\
&+ (1-\theta) \left\{ \frac{2}{\delta} \int_{\mathbb{R}^3\times\mathbb{R}_+} F_\eta I^{2/\delta} dv dI \right\},
\end{aligned}
$$

and the transitional polyatomic Maxwellian:

$$
\mathcal{M}_{\nu,\theta}(\eta) = \frac{\rho_\eta \Lambda_\delta}{\sqrt{\det(2\pi \mathcal{T}_{\theta\eta})}\, T_{\theta\eta}^{\frac{\delta}{2}}} \exp\left(-\frac{1}{2}(v-U_\eta)^\top \mathcal{T}^{-1}_{\nu,\theta\eta}(v-U_\eta) - \frac{I^{\frac{2}{\delta}}}{T_{\theta\eta}} \right). \qquad (2.2)
$$

For simplicity, we set

$$
\begin{aligned}
A(\eta) &= \rho_\eta, \\
B(\eta) &= \rho_\eta U_\eta, \\
C(\eta) &= \rho_\eta \mathcal{T}_{\nu,\theta\eta} + \frac{\theta}{3+\delta}\rho_\eta |U_\eta|^2 Id + (1-\theta)\left\{ \frac{1-\nu}{3}\rho_\eta |U_\eta|^2 Id + \nu \rho_\eta U_\eta \otimes U_\eta \right\}, \\
D(\eta) &= \rho_\eta T_{\theta\eta} + \frac{\theta}{3+\delta}\rho_\eta |U_\eta|^2.
\end{aligned}
$$

Note that $A(0)=1$, $B(0)=0$, $C(0)=Id$, $D(0)=1$ since $F^0=m$, and the macroscopic fields can be recovered from the following relations:

$$
\begin{aligned}
\rho_\eta &= A(\eta), \\
U_\eta &= \frac{B(\eta)}{A(\eta)}, \\
\mathcal{T}_{\nu,\theta\eta} &= \frac{A(\eta)C(\eta) - \left\{ \frac{\theta}{3+\delta}|B(\eta)|^2 Id + (1-\theta)\left(\frac{1-\nu}{3}|B(\eta)|^2 + \nu B(\eta)\otimes B(\eta) \right) \right\}}{|A(\eta)|^2}, \qquad (2.3) \\
T_{\theta\eta} &= \frac{A(\eta)D(\eta) - \frac{\theta}{3+\delta}|B(\eta)|^2}{|A(\eta)|^2}.
\end{aligned}
$$

The following identity plays an important role throughout the linearization procedure.

Lemma 2.1. *The Jacobian matrix* $J(\eta) = \frac{\partial(\rho_\eta, U_\eta, \mathcal{T}_{\nu,\theta\eta}, T_{\theta\eta})}{\partial(A(\eta), B(\eta), C(\eta), D(\eta))}$ *is given by*

$$J(\eta) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{U_{\eta 1}}{\rho_\eta} & \frac{1}{\rho_\eta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{U_{\eta 2}}{\rho} & 0 & \frac{1}{\rho_\eta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{U_{\eta 3}}{\rho_\eta} & 0 & 0 & \frac{1}{\rho_\eta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda^{11}_\eta & J_+\frac{U_{\eta 1}}{\rho_\eta} & J_-\frac{U_{\eta 2}}{\rho_\eta} & J_-\frac{U_{\eta 3}}{\rho_\eta} & \frac{1}{\rho_\eta} & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda^{22}_\eta & J_-\frac{U_{\eta 1}}{\rho_\eta} & J_+\frac{U_{\eta 2}}{\rho} & J_-\frac{U_{\eta 3}}{\rho_\eta} & 0 & \frac{1}{\rho_\eta} & 0 & 0 & 0 & 0 & 0 \\
\Lambda^{33}_\eta & J_-\frac{U_{\eta 1}}{\rho_\eta} & J_-\frac{U_{\eta 2}}{\rho} & J_+\frac{U_{\eta 3}}{\rho_\eta} & 0 & 0 & \frac{1}{\rho_\eta} & 0 & 0 & 0 & 0 \\
\Lambda^{12}_\eta & -\nu\frac{U_{\eta 2}}{\rho_\eta} & -\nu\frac{U_{\eta 1}}{\rho_\eta} & 0 & 0 & 0 & 0 & \frac{1}{\rho_\eta} & 0 & 0 & 0 \\
\Lambda^{23}_\eta & 0 & -\nu\frac{U_{\eta 3}}{\rho_\eta} & -\nu\frac{U_{\eta 2}}{\rho_\eta} & 0 & 0 & 0 & 0 & \frac{1}{\rho_\eta} & 0 & 0 \\
\Lambda^{31}_\eta & -\nu\frac{U_{\eta 3}}{\rho_\eta} & 0 & -\nu\frac{U_{\eta 1}}{\rho_\eta} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho_\eta} & 0 \\
\Omega_\eta & -\frac{2\theta}{3+\delta}\frac{U_{\eta 1}}{\rho_\eta} & -\frac{2\theta}{3+\delta}\frac{U_{\eta 2}}{\rho_\eta} & -\frac{2\theta}{3+\delta}\frac{U_{3\eta}}{\rho_\eta} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho_\eta}
\end{pmatrix},$$

where Λ^{ij}_η, Ω_η *and* $J_\pm$ *are*

$$\begin{aligned}
\Lambda^{ii}_\eta &= \frac{1}{\rho_\eta}\left\{-\mathcal{T}^{ii}_{\nu,\theta\eta} + \left(\frac{\theta}{3+\delta} + (1-\theta)\frac{1-\nu}{3}\right)|U_\eta|^2 + \nu(1-\theta)U^2_{\eta i}\right\}, \\
\Lambda^{ij}_\eta &= \frac{1}{\rho_\eta}\left\{-\mathcal{T}^{ij}_{\nu,\theta\eta} + \nu(1-\theta)U_{\eta i}U_{\eta j}\right\}, \\
\Omega_\eta &= \frac{1}{\rho}\left(-T_{\theta\eta} + \frac{\theta}{3+\delta}|U_\eta|^2\right), \\
J_+ &= -\left\{\frac{\theta}{3+\delta} + (1-\theta)\frac{1+2\nu}{3}\right\}, \\
J_- &= -\left\{\frac{\theta}{3+\delta} + (1-\theta)\frac{1-\nu}{3}\right\}.
\end{aligned}$$

Proof. It follows from a straightforward computation using the relations (2.3). We omit it. □

The following corollary comes immediately.

Corollary 2.1. *When $F^0 = m$, the Jacobian is given by*

$$J(0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *We have*

$$(1)\ \frac{\partial \mathcal{M}_{\nu,\theta}(\eta)}{\partial \rho_\eta} = \frac{1}{\rho_\eta}\mathcal{M}_{\nu,\theta}(\eta),$$

$$(2)\ \nabla_{U_\eta}\mathcal{M}_{\nu,\theta}(\eta) = \frac{1}{2}\left\{\mathcal{T}^{-1}_{\nu,\theta\eta}(v-U_\eta) + (v-U_\eta)^\top \mathcal{T}^{-1}_{\nu,\theta\eta}\right\}\mathcal{M}_{\nu,\theta}(\eta),$$

$$(3)\ \frac{\partial \mathcal{M}_{\nu,\theta}(\eta)}{\partial \mathcal{T}^{ii}_{\nu,\theta\eta}} = \frac{1}{2}\left\{-\frac{1}{\det \mathcal{T}_{\nu,\theta\eta}}\frac{\partial\big(\det \mathcal{T}_{\nu,\theta\eta}\big)}{\partial \mathcal{T}^{ij}_{\nu,\theta\eta}} + \big\{(v-U_\eta)^\top \mathcal{T}^{-1}_{\nu,\theta\eta}\}_i^2\right\}\mathcal{M}_{\nu,\theta}(\eta),$$

$$(4)\ \frac{\partial \mathcal{M}_{\nu,\theta}(\eta)}{\partial \mathcal{T}^{ij}_{\nu,\theta\eta}} = \frac{1}{2}\left\{-\frac{1}{\det \mathcal{T}_{\nu,\theta\eta}}\frac{\partial\big(\det \mathcal{T}_{\nu,\theta\eta}\big)}{\partial \mathcal{T}^{ij}_{\nu,\theta\eta}} + 2\big\{(v-U_\eta)^\top \mathcal{T}^{-1}_{\nu,\theta\eta}\}_i\big\{\mathcal{T}^{-1}_{\nu,\theta\eta}(v-U_\eta)\big\}_j\right\}$$
$$\times \mathcal{M}_{\nu,\theta}(\eta),$$

$$(5)\ \frac{\partial \mathcal{M}_{\nu,\theta}(\eta)}{\partial T_{\theta\eta}} = \left\{\frac{2I^{2/\delta} - \delta T_{\theta\eta}}{2T^2_{\theta\eta}}\right\}\mathcal{M}_{\nu,\theta}(\eta).$$

Proof. (1), (2) and (5) follow from direct computations. The proofs for (3) and (4) are similar. We only prove (4). We first compute

$$\frac{\partial \mathcal{M}_{\nu,\theta}(\eta)}{\partial \mathcal{T}^{ij}_{\nu,\theta\eta}} = \frac{1}{2}\left\{-\frac{1}{\det(\mathcal{T}_{\nu,\theta\eta})}\frac{\partial\big(\det \mathcal{T}_{\nu,\theta\eta}\big)}{\partial \mathcal{T}^{ij}_{\nu,\theta\eta}} - (v-U_\eta)^\top\left(\frac{\partial \mathcal{T}^{-1}_{\nu,\theta\eta}}{\partial \mathcal{T}^{ij}_{\nu,\theta\eta}}\right)(v-U_\eta)\right\}\mathcal{M}_{\nu,\theta}(\eta).$$

We then observe that for any invertible matrix A

$$\partial\big\{A^{-1}\big\} = -A^{-1}\{\partial A\}A^{-1}, \tag{2.4}$$

which is obtained by applying ∂ on both sides of $AA^{-1} = I$:

$$\big\{\partial A\big\}A^{-1} + A\partial\big\{A^{-1}\big\} = \partial I = 0.$$

Therefore,

$$(v-U_\eta)^\top\left(\frac{\partial\mathcal{T}^{-1}_{\nu,\theta\eta}}{\partial\mathcal{T}^{ij}_{\nu,\theta\eta}}\right)(v-U_\eta)=(v-U_\eta)^\top\mathcal{T}^{-1}_{\nu,\theta\eta}\left(\frac{\partial\mathcal{T}_{\nu,\theta\eta}}{\partial\mathcal{T}^{ij}_{\nu,\theta\eta}}\right)\mathcal{T}^{-1}_{\nu,\theta\eta}(v-U_\eta).$$

Finally, since $\left(\frac{\partial\mathcal{T}_{\nu,\theta\eta}}{\partial\mathcal{T}^{ij}_{\nu,\theta\eta}}\right)$ is a matrix whose only non-zero element is ijth and jith elements, this simplifies further:

$$(v-U_\eta)^\top\mathcal{T}^{-1}_{\nu,\theta\eta}\left(\frac{\partial\mathcal{T}_{\nu,\theta\eta}}{\partial\mathcal{T}^{ij}_{\nu,\theta\eta}}\right)\mathcal{T}^{-1}_{\nu,\theta\eta}(v-U_\eta)=2\big\{(v-U_\eta)^\top\mathcal{T}^{-1}_{\nu,\theta\eta}\}_i\big\{\mathcal{T}^{-1}_{\nu,\theta\eta}(v-U_\eta)\big\}_j.$$

This completes the proof. □

Corollary 2.2. *When $\eta=0$, we have*

$$\begin{aligned}
&(1)\ \frac{\partial\mathcal{M}_{\nu,\theta}(0)}{\partial\rho_\eta}=m,\\
&(2)\ \frac{\partial\mathcal{M}_{\nu,\theta}(0)}{\partial U_{i\eta}}=v_i m \qquad\qquad (i=1,2,3),\\
&(3)\ \frac{\partial\mathcal{M}_{\nu,\theta}(0)}{\partial\mathcal{T}^{ii}_{\nu,\theta\eta}}=\frac{v_i^2-1}{2}m \qquad (1\le i=j\le 3),\\
&(4)\ \frac{\partial\mathcal{M}_{\nu,\theta}(0)}{\partial\mathcal{T}^{ij}_{\nu,\theta\eta}}=v_iv_jm \qquad\qquad (1\le i\neq j\le 3),\\
&(5)\ \frac{\partial\mathcal{M}_{\nu,\theta}(0)}{\partial T_{\theta\eta}}=\left\{\frac{2I^{2/\delta}-\delta}{2}\right\}m.
\end{aligned}$$

Proof. Note that when $\eta=0$, F_η reduces to m. Therefore, the result follows by inserting $\rho_0=1$, $U_0=0$, $\mathcal{T}_0=Id$, $T_0=1$ to Lemma 2.2. □

2.2. Linearized relaxation operator

We consider the transitional polyatomic Maxwellian as a function of η and set

$$g(\eta)=\mathcal{M}_{\nu,\theta}\left(A(\eta),B(\eta),C(\eta),D(\eta)\right).$$

Here, we view C as a 6 dimensional vector $\big(C_{11},C_{22},C_{33},C_{12},C_{23},C_{31}\big)$ by symmetry. Note that $g(\eta)$ depicts the transition from the polyatomic local Maxwellian $\mathcal{M}_{\nu,\theta}(F)$ to the polyatomic global Maxwellian $m(v,I)$. We expand it using the Taylor's theorem:

$$g(1)=g(0)+g'(0)+\int_0^1 g''(\eta)(1-\eta)d\eta. \tag{2.5}$$

Clearly,

$$g(0) = m, \text{ and } g(1) = \mathcal{M}_{\nu,\theta}(F).$$

The calculation of the second and third terms in the right hand side of (2.5) is carried out in the following theorem and Proposition 2.1 respectively.

Theorem 2.3. *$g'(0)$ is given by*

$$g'(0) = (P_{\nu,\theta} f)\sqrt{m},$$

where $P_{\nu,\theta}$ is defined by

$$P_{\nu,\theta} f \equiv \theta P_p f + (1-\theta)\big\{P_m f + \nu(P_1 f + P_2 f)\big\}.$$

(1) P_p: polyatomic projection:

$$\begin{aligned}
P_p f &= \left(\int_{\mathbb{R}^3\times\mathbb{R}_+} f\sqrt{m}\,dv\,dI\right)\sqrt{m} \\
&+ \left(\int_{\mathbb{R}^3\times\mathbb{R}_+} f v\sqrt{m}\,dv\,dI\right)\cdot v\sqrt{m} \\
&+ \left\{\int_{\mathbb{R}^3\times\mathbb{R}_+} f\left(\frac{(|v|^2-3)+(2I^{\frac{2}{\delta}}-\delta)}{\sqrt{2(3+\delta)}}\right)\sqrt{m}\,dv\,dI\right\}\left(\frac{(|v|^2-3)+(2I^{2/\delta}-\delta)}{\sqrt{2(3+\delta)}}\right)\sqrt{m},
\end{aligned}$$

(2) P_m: monatomic-like projection:

$$\begin{aligned}
P_m f &= \left(\int_{\mathbb{R}^3\times\mathbb{R}_+} f\sqrt{m}\,dv\,dI\right)\sqrt{m} \\
&+ \left(\int_{\mathbb{R}^3\times\mathbb{R}_+} f v\sqrt{m}\,dv\,dI\right)\cdot v\sqrt{m} \\
&+ \left\{\int_{\mathbb{R}^3\times\mathbb{R}_+} f\left(\frac{|v|^2-3}{\sqrt{6}}\right)\sqrt{m}\,dv\,dI\right\}\left(\frac{|v|^2-3}{\sqrt{6}}\right)\sqrt{m} \\
&+ \left\{\int_{\mathbb{R}^3\times\mathbb{R}_+} f\left(\frac{2I^{\frac{2}{\delta}}-\delta}{\sqrt{2\delta}}\right)\sqrt{m}\,dv\,dI\right\}\left(\frac{2I^{\frac{2}{\delta}}-\delta}{\sqrt{2\delta}}\right)\sqrt{m},
\end{aligned}$$

(3) P_1 & P_2: non-diagonal projections:

$$P_1 f = \sum_{i<j} \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}_+} f\Big(\frac{3v_i^2 - |v|^2}{3\sqrt{2}}\Big)\sqrt{m}\, dv dI \right\} \left(\frac{3v_1^2 - |v|^2}{3\sqrt{2}}\right)\sqrt{m}\,,$$

$$P_2 f = \sum_{i<j} \left(\int_{\mathbb{R}^3 \times \mathbb{R}_+} f v_i v_j \sqrt{m}\, dv dI \right) v_i v_j \sqrt{m}\,.$$

Proof. By chain rule, we have

$$\begin{aligned} g'(0) &= A'(0)\frac{\partial \mathcal{M}_{\nu,\theta}(0)}{\partial A} + B'(0)\frac{\partial \mathcal{M}_{\nu,\theta}(0)}{\partial B} + C'(0)\frac{\partial \mathcal{M}_{\nu,\theta}(0)}{\partial C} + D'(0)\frac{\partial \mathcal{M}_{\nu,\theta}(0)}{\partial D} \\ &= \nabla_{(A,B,C,D)}\mathcal{M}_{\nu,\theta}(0) \cdot \big(A'(0), B'(0), C'(0), D'(0)\big) \\ &= \left\{\nabla_{(\rho_\eta, U_\eta, \mathcal{T}_{\nu,\theta\eta}, T_{\theta\eta})}\mathcal{M}_{\nu,\theta}(0) J(0)\right\} \big(A'(0), B'(0), C'(0), D'(0)\big)^\top, \end{aligned} \tag{2.6}$$

where $J(\eta)$ denotes the Jacobian matrix between the translational macroscopic fields given in Lemma 2.1. Recalling Corollary 2.1 and Corollary 2.2, we see that

$$\begin{aligned} &\nabla_{(\rho_\eta, U_\eta, \mathcal{T}_{\nu,\theta\eta}, T_{\theta\eta})}\mathcal{M}_{\nu,\theta}(0) J(0) \\ &= \left(1, v_1, v_2, v_3, \frac{v_1^2-1}{2}, \frac{v_2^2-1}{2}, \frac{v_3^2-1}{2}, v_1v_2, v_2v_3, v_3v_1, \frac{2I^{\frac{2}{\delta}} - \delta}{2}\right) m\, J(0) \\ &= \left(1 - \frac{|v|^2-3}{2} - \frac{2I^{\frac{2}{\delta}} - \delta}{2}, v_1, v_2, v_3, \frac{v_1^2-1}{2}, \frac{v_2^2-1}{2}, \frac{v_3^2-1}{2}, v_1v_2, v_2v_3, v_3v_1, \frac{2I^{\frac{2}{\delta}} - \delta}{2}\right) m. \end{aligned}$$

On the other hand, we note that

$$\begin{pmatrix} A'(0) \\ B'(0) \\ C'(0) \\ D'(0) \end{pmatrix} = \begin{pmatrix} \int f\sqrt{m}\, dv dI \\ \int f v_1 \sqrt{m}\, dv dI \\ \int f v_2 \sqrt{m}\, dv dI \\ \int f v_3 \sqrt{m}\, dv dI \\ \theta \int f \left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta} I^{\frac{2}{\delta}}\right)\sqrt{m}\, dv dI + (1-\theta)\left\{\int f\left(\frac{1-\nu}{3}|v|^2 + \nu v_1^2\right)\sqrt{m}\, dv dI\right\} \\ \theta \int f \left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta} I^{\frac{2}{\delta}}\right)\sqrt{m}\, dv dI + (1-\theta)\left\{\int f\left(\frac{1-\nu}{3}|v|^2 + \nu v_1^2\right)\sqrt{m}\, dv dI\right\} \\ \theta \int f \left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta} I^{\frac{2}{\delta}}\right)\sqrt{m}\, dv dI + (1-\theta)\left\{\int f\left(\frac{1-\nu}{3}|v|^2 + \nu v_1^2\right)\sqrt{m}\, dv dI\right\} \\ (1-\theta)\nu \int f v_1 v_2 \sqrt{m}\, dv dI \\ (1-\theta)\nu \int f v_2 v_3 \sqrt{m}\, dv dI \\ (1-\theta)\nu \int f v_3 v_1 \sqrt{m}\, dv dI \\ \theta \int f \left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta} I^{\frac{2}{\delta}}\right)\sqrt{m}\, dv dI + (1-\theta)\left(\frac{2}{\delta}\int f I^{\frac{2}{\delta}}\sqrt{m}\, dv dI\right) \end{pmatrix}$$

Inserting these identities into (2.6), we get

$$
\begin{aligned}
g'(0)m^{-1} = &\left(\int f\sqrt{m}dvdI\right)\Big(1-\frac{|v|^2-3}{2}-\frac{2I^{\frac{2}{\delta}}-\delta}{2}\Big)\\
&+\left(\int fv\sqrt{m}dvdI\right)\cdot v\\
&+\sum_{i=1}^{3}\bigg[\theta\int f\left(\frac{1}{3+\delta}|v|^2+\frac{2}{3+\delta}I^{\frac{2}{\delta}}\right)\sqrt{m}dvdI\\
&\qquad+(1-\theta)\left\{\int f\left(\frac{1-\nu}{3}|v|^2+\nu v_i^2\right)\sqrt{m}dvdI\right\}\bigg]\left(\frac{v_i^2-1}{2}\right)\\
&+(1-\theta)\nu\sum_{i<j}\left(\int fv_iv_j\sqrt{m}dvdI\right)v_iv_j\\
&+\bigg[\theta\int f\left(\frac{1}{3+\delta}|v|^2+\frac{2}{3+\delta}I^{\frac{2}{\delta}}\right)\sqrt{m}dvdI\\
&\qquad+(1-\theta)\left(\frac{2}{\delta}\int fI^{\frac{2}{\delta}}\sqrt{m}dvdI\right)\bigg]\left(\frac{2I^{\frac{2}{\delta}}-\delta}{2}\right)\\
&\equiv I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

For later computation, we further decompose I_1 and I_5 as follows:

$$
\begin{aligned}
I_1^1=\int f\sqrt{m}dvdI,\ I_1^2=-\left(\int f\sqrt{m}dvdI\right)\Big(\frac{|v|^2-3}{2}\Big),\\
I_1^3=-\left(\int f\sqrt{m}dvdI\right)\Big(\frac{2I^{\frac{2}{\delta}}-\delta}{2}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
I_5^1&=\theta\left\{\int f\left(\frac{1}{3+\delta}|v|^2+\frac{2}{3+\delta}I^{\frac{2}{\delta}}\right)\sqrt{m}dvdI\right\}\Big(\frac{2I^{\frac{2}{\delta}}-\delta}{2}\Big)\\
I_5^2&=(1-\theta)\left(\frac{2}{\delta}\int fI^{\frac{2}{\delta}}\sqrt{m}dvdI\right)\Big(\frac{2I^{\frac{2}{\delta}}-\delta}{2}\Big).
\end{aligned}
$$

We now rearrange these terms so that (1) the polyatomic part and monatomic-like part are separated, and (2) the orthogonality between the components are clearly revealed, as is given in the statement of the Lemma 3.2 later. For this, we need some preliminary calculations:

• **Step I:** $I_3 = A_1 + A_2 + A_3$.
First we compute the summation in I_3 to obtain

$$\begin{aligned} I_3 = \theta &\left\{\int f\left(\frac{|v|^2 + 2I^{\frac{2}{\delta}}}{3+\delta}\right)\sqrt{m}dvdI\right\}\frac{|v|^2-3}{2} \\ &+ (1-\theta)\frac{1-\nu}{3}\left(\int f|v|^2\sqrt{m}dvdI\right)\frac{|v|^2-3}{2} \\ &+ (1-\theta)\nu\sum_{1\le i\le 3}\left(\int f\, v_i^2\sqrt{m}dvdI\right)\frac{v_i^2-1}{2}. \end{aligned} \tag{2.7}$$

The last term can be decomposed as

$$\begin{aligned} \sum_{1\le i\le 3}\left(\int f\, v_i^2 dvdI\right)\frac{v_i^2-1}{2} = &\sum_{1\le i\le 3}\left(\int f\left(\frac{3v_i^2-|v|^2}{3}\right)\sqrt{m}dvdI\right)\frac{3v_i^2-|v|^2}{6} \\ &+ \sum_{1\le i\le 3}\left\{\int f\left(\frac{3v_i^2-|v|^2}{3}\right)\sqrt{m}dvdI\right\}\frac{|v|^2-3}{6} \\ &+ \sum_{1\le i\le 3}\left\{\int f\left(\frac{|v|^2}{3}\right)\sqrt{m}dvdI\right\}\frac{3v_i^2-|v|^2}{6} \\ &+ \sum_{1\le i\le 3}\left\{\int f\left(\frac{|v|^2}{3}\right)\sqrt{m}dvdI\right\}\frac{|v|^2-3}{6}. \end{aligned} \tag{2.8}$$

The second and third terms vanish due to

$$\sum_{1\le i\le 3}\left\{3v_i^2 - |v|^2\right\} = 0,$$

and the last term is

$$\left\{\int f|v|^2\sqrt{m}dvdI\right\}\frac{|v|^2-3}{6},$$

so that (2.8) is reduced to

$$\begin{aligned} \sum_{1\le i\le 3}\left(\int f v_i^2\sqrt{m}dvdI\right)\frac{v_i^2-1}{2} = &\sum_{1\le i\le 3}\left\{\int f\left(\frac{3v_i^2-|v|^2}{3}\right)\sqrt{m}dvdI\right\}\frac{3v_i^2-|v|^2}{6} \\ &+ \left(\int f|v|^2\sqrt{m}dvdI\right)\frac{|v|^2-3}{6}. \end{aligned}$$

We plug this into (2.7),

$$\begin{aligned}I_3 = \theta &\left\{\int f\left(\frac{|v|^2+2I^{\frac{2}{\delta}}}{3+\delta}\right)\sqrt{m}dvdI\right\}\frac{|v|^2-3}{2}\\ &+(1-\theta)\frac{1-\nu}{3}\left(\int f\,|v|^2dvdI\right)\frac{|v|^2-3}{2}\\ &+(1-\theta)\nu\sum_{1\le i\le 3}\left\{\int f\Big(\frac{3v_i^2-|v|^2}{3}\Big)\sqrt{m}dvdI\right\}\frac{3v_1^2-|v|^2}{6}\\ &+(1-\theta)\nu\left(\int f\,|v|^2dvdI\right)\frac{|v|^2-3}{6},\end{aligned}$$

and put together the second and fourth terms to get

$$\begin{aligned}I_3 = \theta &\left\{\int f\left(\frac{|v|^2+2I^{\frac{2}{\delta}}}{3+\delta}\right)\sqrt{m}dvdI\right\}\frac{|v|^2-3}{2}\\ &+(1-\theta)\left(\int f\frac{|v|^2}{3}dvdI\right)\frac{|v|^2-3}{2}\\ &+\nu(1-\theta)\sum_{1\le i\le 3}\left\{\int f\Big(\frac{3v_i^2-|v|^2}{3}\Big)\sqrt{m}dvdI\right\}\frac{3v_i^2-|v|^2}{6}\\ &\equiv A^1+A^2+A^3.\end{aligned}$$

• **Step II:** $\theta(I_1^2+I_1^3)+A^1+I_5^1$.
We combine the first term of I_5 with A^1:

$$A^1+I_5^1=\theta\left\{\int f\left(\frac{|v|^2+2I^{\frac{2}{\delta}}}{3+\delta}\right)\sqrt{m}\,dvdI\right\}\left(\frac{(|v|^2-3)+(2I^{2/\delta}-\delta)}{2}\right).$$

Therefore, adding θ portion of the second, third term of I_1 to I_5+A^1, we obtain

$$\begin{aligned}&\theta\left(I_1^2+I_1^3\right)+A^1+I_5^1\\ &\quad=-\theta\left\{\int f\sqrt{m}\,dvdI\right\}\left(\frac{(|v|^2-3)+(2I^{2/\delta}-\delta)}{2}\right)\\ &\quad+\theta\left\{\int f\left(\frac{|v|^2+2I^{\frac{2}{\delta}}}{3+\delta}\right)\sqrt{m}\,dvdI\right\}\left(\frac{(|v|^2-3)+(2I^{2/\delta}-\delta)}{2}\right)\\ &\quad=\theta\left\{\int f\left(\frac{(|v|^2-3)+(2I^{\frac{2}{\delta}}-\delta)}{3+\delta}\right)\sqrt{m}\,dvdI\right\}\left(\frac{(|v|^2-3)+(2I^{2/\delta}-\delta)}{2}\right).\end{aligned}$$

• **Step III:** Now, we rewrite $I_1 + \cdots + I_5$ as

$$\begin{aligned} & I_1 + I_2 + I_3 + I_4 + I_5 \\ & \quad = \big(I_1^1 + I_1^2 + I_1^3\big) + I_2 + \big(A_1 + A_2 + A_3\big) + I_4 + (I_5^1 + I_5^2) \\ & \quad = I_1^1 + I_2 + \Big\{\theta\big(I_1^2 + I_1^3\big) + A_1 + I_5^1\Big\} + A_2 + A_3 + (1-\theta)\big(I_1^2 + I_1^3\big) + I_4 + I_5^2 \end{aligned}$$

and insert the above computations in step I and step II, to derive

$$\begin{aligned} & I_1 + \cdots + I_5 \\ & \quad = \left(\int f\sqrt{m}dvdI\right) \\ & \quad + \left(\int fv\sqrt{m}dvdI\right)\cdot v \\ & \quad + \theta\left\{\int f\left(\frac{(|v|^2-3)+(2I^{\frac{2}{\delta}}-\delta)}{3+\delta}\right)\sqrt{m}\,dvdI\right\}\left(\frac{(|v|^2-3)+(2I^{2/\delta}-\delta)}{2}\right) \\ & \quad + (1-\theta)\left(\int f\frac{|v|^2}{3}\sqrt{m}dvdI\right)\left(\frac{|v|^2-3}{2}\right) \\ & \quad + \nu(1-\theta)\sum_{1\le i\le 3}\left\{\int f\Big(\frac{3v_i^2-|v|^2}{3}\Big)\sqrt{m}dvdI\right\}\frac{3v_1^2-|v|^2}{6} \\ & \quad - (1-\theta)\left\{\int f\sqrt{m}dvdI\right\}\left(\frac{|v|^2-3}{2}\right) \\ & \quad - (1-\theta)\left\{\int f\sqrt{m}dvdI\right\}\left(\frac{2I^{\frac{2}{\delta}}-\delta}{2}\right) \\ & \quad + (1-\theta)\nu\sum_{i<j}\left(\int fv_iv_j\sqrt{m}dvdI\right)v_iv_j \\ & \quad + (1-\theta)\left\{\frac{2}{\delta}\left(\int fI^{\frac{2}{\delta}}dvdI\right)\right\}\left(\frac{2I^{\frac{2}{\delta}}-\delta}{2}\right). \end{aligned}$$

Note that the 4th and the 6th terms on the r.h.s. put together give

$$(1-\theta)\left(\int f\frac{|v|^2-3}{\sqrt{6}}dvdI\right)\frac{|v|^2-3}{\sqrt{6}}.$$

Likewise, the 7th and the 9th term on the r.h.s. can be combined to yield

$$(1-\theta)\left\{\int f\left(\frac{2I^{\frac{2}{\delta}}-\delta}{\sqrt{2\delta}}\right)\sqrt{m}dvdI\right\}\left(\frac{2I^{\frac{2}{\delta}}-\delta}{\sqrt{2\delta}}\right).$$

In conclusion,

$$
\begin{aligned}
I_1+\cdots+I_5 &= \left(\int f\sqrt{m}dvdI\right)\\
&+\left(\int fv\sqrt{m}dvdI\right)\cdot v\\
&+\theta\left\{\int f\left(\frac{(|v|^2-3)+(2I^{\frac{2}{\delta}}-\delta)}{\sqrt{2(3+\delta)}}\right)\sqrt{m}\,dvdI\right\}\left(\frac{(|v|^2-3)+(2I^{2/\delta}-\delta)}{\sqrt{2(3+\delta)}}\right)\\
&+(1-\theta)\left\{\int f\left(\frac{|v|^2-3}{\sqrt{6}}\right)\sqrt{m}dvdI\right\}\left(\frac{|v|^2-3}{\sqrt{6}}\right)\\
&+(1-\theta)\left\{\int f\left(\frac{2I^{\frac{2}{\delta}}-\delta}{\sqrt{2\delta}}\right)\sqrt{m}dvdI\right\}\left(\frac{2I^{\frac{2}{\delta}}-\delta}{\sqrt{2\delta}}\right)\\
&+\nu(1-\theta)\sum_i\left\{\int f\Big(\frac{3v_i^2-|v|^2}{3\sqrt{2}}\Big)\sqrt{m}dvdI\right\}\left(\frac{3v_i^2-|v|^2}{3\sqrt{2}}\right)\\
&+\nu(1-\theta)\sum_{i<j}\left\{\int fv_iv_j\sqrt{m}dvdI\right\}v_iv_j.
\end{aligned}
$$

Finally, we split the first two terms as

$$
\begin{aligned}
&\left(\int f\sqrt{m}dvdI\right)+\left(\int fv\sqrt{m}dvdI\right)\cdot v\\
&\qquad=\theta\left\{\left(\int f\sqrt{m}dvdI\right)+\left(\int fv\sqrt{m}dvdI\right)\cdot v\right\}\\
&\qquad+(1-\theta)\left\{\left(\int f\sqrt{m}dvdI\right)+\left(\int fv\sqrt{m}dvdI\right)\cdot v\right\},
\end{aligned}
$$

and gather terms with θ and $(1-\theta)$ separately, which are P_pf and $P_mf+\nu(P_1+P_2)f$ respectively. $\square$

We now move on to the nonlinear term. In the following, the polynomials $P_{ij}^{\mathcal{M}}$, $R_{ij}^{\mathcal{M}}$ are generically defined in the sense that their exact form may vary line after line, but can be explicitly computed in principle. Note that explicit form is not relevant as long as they satisfy the structural assumptions $\mathcal{H}_{\mathcal{M}}$ below.

Proposition 2.1. *$g''(\eta)$ is given by*

$$
g''(\eta)=\sum_{i,j}\left\{\int_0^1\frac{P_{i,j}^{\mathcal{M}}(\rho_\eta,v-U_\eta,\mathcal{T}_{\nu,\theta\eta}^{-1},I^{2/\delta},T_{\theta\eta})}{R_{i,j}^{\mathcal{M}}(\rho_\eta,\det\mathcal{T}_{\nu,\theta\eta},T_{\theta\eta})}\mathcal{M}_{\nu,\theta}(\eta)(1-\eta)d\eta\right\}\langle f,e_i\rangle_{L^2_{v,I}}\langle f,e_j\rangle_{L^2_{v,I}},
$$

where $P^{\mathcal{M}}_{i,j}(x_1,\dots,x_n)$ and $R^{\mathcal{M}}_{i,j}(x_1,\dots,x_n)$ are generically defined polynomials satisfying the following structural assumptions $(\mathcal{H}_{\mathcal{M}})$:

- $(\mathcal{H}_{\mathcal{M}}1)$ *$P^{\mathcal{M}}_{i,j}$ is a polynomial such that $P_{i,j}(0,0,\dots,0)=0$,*
- $(\mathcal{H}_{\mathcal{M}}2)$ *$R^{\mathcal{M}}_{i,j}$ is a monomial,*

and

$$\begin{aligned}
e_1 &= \sqrt{m},\\
e_{i+1} &= v_i\sqrt{m} \quad (i=1,2,3),\\
e_j &= \left\{\theta\left(\frac{1}{3+\delta}|v|^2+\frac{2}{3+\delta}I^{\frac{2}{\delta}}\right)+(1-\theta)\left(\frac{1-\nu}{3}|v|^2+\nu v_j^2\right)\right\}\sqrt{m} \quad (j=5,6,7),\\
e_8 &= \nu(1-\theta)v_1v_2\sqrt{m},\\
e_9 &= \nu(1-\theta)v_2v_3\sqrt{m},\\
e_{10} &= \nu(1-\theta)v_3v_1\sqrt{m},\\
e_{11} &= \left\{\theta\left(\frac{1}{3+\delta}|v|^2+\frac{2}{3+\delta}I^{\frac{2}{\delta}}\right)+(1-\theta)\left(\frac{2}{\delta}I^{\frac{2}{\delta}}\right)\right\}\sqrt{m}.
\end{aligned}$$

Proof. For a matrix A, let A_k and $A_{k\ell}$ denote the k-th column of A and the $k\ell$ element respectively. For simplicity, we set

$$\begin{aligned}
X_\eta &= \big(\rho_\eta, U_\eta, \mathcal{T}_{\nu,\theta\eta}, T_{\theta\eta}\big),\\
Y(\eta) &= \big(A(\eta), B(\eta), C(\eta), D(\eta)\big).
\end{aligned}$$

Observe that each component of $Y(\eta)$ takes the form

$$\int F_\eta P(v,I)\,dv\,dI = \int (m+\eta\sqrt{m}f)P(v,I)\,dv\,dI$$

for some polynomial P. Therefore, $Y'(\eta)$ does not depend on η, and we can write

$$Y'(\eta) = Y'(0).$$

Hence, applying the chain rule, we compute

$$\begin{aligned}
g'(\eta) &= A'(0)\frac{\partial\mathcal{M}_{\nu,\theta}(\eta)}{\partial A}+B'(0)\frac{\partial\mathcal{M}_{\nu,\theta}(\eta)}{\partial B}+C'(0)\frac{\partial\mathcal{M}_{\nu,\theta}(\eta)}{\partial C}+D'(0)\frac{\partial\mathcal{M}_{\nu,\theta}(\eta)}{\partial D}\\
&= \nabla_{(A,B,C,D)}\mathcal{M}_{\nu,\theta}(\eta)\cdot\big(A'(0),B'(0),C'(0),D'(0)\big)\\
&= \nabla_{(\rho_\eta,U_\eta,\mathcal{T}_{\nu,\theta\eta},T_{\theta\eta})}\mathcal{M}_{\nu,\theta}(\eta)J(\eta)Y'(0)^\top\\
&= \sum_i\left\{\nabla_{(\rho_\eta,U_\eta,\mathcal{T}_{\nu,\theta\eta},T_{\theta\eta})}\mathcal{M}_{\nu,\theta}(\eta)\cdot J_i(\eta)\right\}Y_i'(0).
\end{aligned}$$

Taking the derivative again,

$$
\begin{aligned}
g''(\eta) &= \sum_i \left\{ \left(\nabla_{(\rho_\eta, U_\eta, \mathcal{T}_{\nu,\theta\eta}, T_{\theta\eta})} \mathcal{M}_{\nu,\theta}(\eta) \right)' \cdot J_i(\eta) \right\} Y_i'(0) \\
&\quad + \sum_i \left\{ \nabla_{(\rho_\eta, U_\eta, \mathcal{T}_{\nu,\theta\eta}, T_{\theta\eta})} \mathcal{M}_{\nu,\theta}(\eta) \cdot \big(J_i(\eta)\big)' \right\} Y_i'(0) \\
&= I + II.
\end{aligned}
$$

Now, since we have

$$
\begin{aligned}
\left(\nabla_{(\rho_\eta, U_\eta, \mathcal{T}_{\nu,\theta\eta}, T_{\theta\eta})} \mathcal{M}_{\nu,\theta}(\eta) \right)' &= \nabla_{X_\eta} \left\{ \nabla_{X_\eta} \mathcal{M}_{\nu,\theta}(\eta) \right\} J(\eta) \big\{ Y'(\eta) \big\}^\top \\
&= \nabla_{X_\eta} \left\{ \nabla_{X_\eta} \mathcal{M}_{\nu,\theta}(\eta) \right\} J(\eta) \big\{ Y'(0) \big\}^\top \\
&= (a_1, \cdots, a_{11}),
\end{aligned}
\tag{2.9}
$$

where

$$
\begin{aligned}
a_k &= \left\{ \nabla_{X_\eta} \left\{ \nabla_{X_\eta} \mathcal{M}_{\nu,\theta}(\eta) \right\} J(\eta) \right\}_k \cdot Y'(0) \\
&= \sum_\ell \left\{ \nabla_{X_\eta} \left\{ \nabla_{X_\eta} \mathcal{M}_{\nu,\theta}(\eta) \right\} J(\eta) \right\}_{k\ell} Y_\ell'(0),
\end{aligned}
$$

and

$$
\begin{aligned}
J_i'(\eta) &= \nabla_{X_\eta} J_i(\eta) J(\eta) \left\{ Y'(\eta) \right\}^\top \\
&= \nabla_{X_\eta} J_i(\eta) J(\eta) Y'(0)^\top \\
&= \sum_j \nabla_{X_\eta} J_i(\eta) \cdot J_j(\eta) Y_j(0),
\end{aligned}
\tag{2.10}
$$

we can derive the following expression for I:

$$
\begin{aligned}
I &= \sum_i \left\{ \Big(\sum_k \sum_\ell \left\{ \nabla_{X_\eta} \left\{ \nabla_{X_\eta} \mathcal{M}_{\nu,\theta}(\eta) \right\} J(\eta) \right\}_{k\ell} Y_\ell'(0) \Big) J_{ki}(\eta) \right\} Y_i'(0) \\
&= \sum_{i,k,\ell} A_{k\ell} J_{ki}(\eta) Y_\ell'(0) Y_i'(0) \\
&= \sum_{i,k,\ell} A_{k\ell} J_{ki}(\eta) \langle f, e_\ell \rangle_{L^2_{v,I}} \langle f, e_i \rangle_{L^2_{v,I}}
\end{aligned}
$$

with

$$
A_{k\ell} = \left\{ \nabla_{X_\eta} \left\{ \nabla_{X_\eta} \mathcal{M}_{\nu,\theta}(\eta) \right\} J(\eta) \right\}_{k\ell}.
$$

In view of Lemma 2.1 and Lemma 2.2, it can be easily verified that $A_{k\ell}J_{ki}(\eta)$ takes the following form:

$$\frac{P^{\mathcal{M}}_{i,j}(\rho_\eta, U_\eta, v-U_\eta, \mathcal{T}^{-1}_{\nu,\theta\eta}, I^{2/\delta}, T_{\theta\eta})}{R^{\mathcal{M}}_{i,j}(\rho_\eta, \det\mathcal{T}_{\nu,\theta\eta}, T_{\theta\eta})}\mathcal{M}_{\nu,\theta}(\eta)$$

for some polynomials $P^{\mathcal{M}}_{i,j}$, $R^{\mathcal{M}}_{i,j}$ satisfying the structural assumptions. *II* can be treated in a similar manner. □

Finally, we consider the linearization of the collision frequency.

Lemma 2.4. *The collision frequency $A_{\nu,\theta}$ can be linearized around the normalized global Maxwellian as follows:*

$$A_{\nu,\theta} = \frac{1}{1-\nu+\theta\nu}\left\{1 + \sum_{2\le i\le 7,11} a_i\langle f, e_i\rangle_{L^2_{v,I}}\right\},$$

where

$$a_1 = -\frac{2}{3+\delta}\int_0^1 |U_\eta|^2 d\eta, \quad a_i = \frac{4}{3+\delta}\int_0^1 U_{\eta,i}d\eta \quad (i=2,3,4),$$
$$a_i = \frac{1}{3+\delta} \quad (i=5,6,7), \quad a_{11} = \frac{\delta}{3+\delta}.$$

Proof. We compute

$$\begin{aligned}\rho T_\delta &= \int_{\mathbb{R}^3\times\mathbb{R}_+} F\left(\frac{1}{3+\delta}|v-U|^2 + \frac{2}{3+\delta}I^{2/\delta}\right)dvdI \\ &= \int_{\mathbb{R}^3\times\mathbb{R}_+} (m+\sqrt{m}f)\left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{2/\delta}\right)dvdI + \frac{2}{3+\delta}\rho|U|^2 \\ &= 1 + \int_{\mathbb{R}^3\times\mathbb{R}_+} \sqrt{m}f\left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{2/\delta}\right)dvdI + \frac{2}{3+\delta}\rho|U|^2.\end{aligned} \tag{2.11}$$

Then, observe

$$|v|^2 + 2I^{2/\delta} = e_5 + e_6 + e_7 + \delta e_{11}$$

to write the second term in the last line of (2.11) as

$$\int_{\mathbb{R}^3\times\mathbb{R}_+} \sqrt{m}f\left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{2/\delta}\right)dvdI = \frac{1}{3+\delta}\sum_{i=5,6,7}\langle f, e_i\rangle_{L^2_{v,I}} + \frac{\delta}{3+\delta}\langle f, e_{11}\rangle_{L^2_{v,I}}.$$

For the third term, we define

$$R(\eta)=\rho_\eta|U_\eta|^2=\frac{|B(\eta)|^2}{A(\eta)}.$$

Note that $R(1)=\rho|U|^2$ and $R(0)=0$. Therefore, applying Taylor's theorem and the chain rule with

$$A'(\eta)=A'(0),\quad B'(\eta)=B'(0),$$

yields

$$\begin{aligned}
R(1)&=R(0)+\int_0^1 R'(\eta)d\eta\\
&=\int_0^1\left\{\frac{\partial R}{\partial A}A'(0)+\frac{\partial R}{\partial B}B'(0)\right\}d\eta\\
&=-\left(\int_0^1\frac{|B(\eta)|^2}{|A(\eta)|^2}d\eta\right)A'(0)+\left(\int_0^1\frac{2B(\eta)}{A(\eta)}d\eta\right)\cdot B'(0)\\
&=-\left(\int_0^1|U_\eta|^2d\eta\right)\langle f,e_1\rangle_{L^2_{v,I}}+\sum_{i=2,3,4}\left(\int_0^1 2U_{\eta,i}d\eta\right)\langle f,e_i\rangle_{L^2_{v,I}}.
\end{aligned}$$

This completes the proof. □

2.3. Linearized polyatomic BGK model

Now we finish our linearization process. To further simplify the presentation of the linearized relaxation operator, we denote

$$\mathcal{Q}^{\mathcal{M}}_{ij}(f)=\frac{1}{\sqrt{m}}\int_0^1\frac{P^{\mathcal{M}}_{i,j}(\rho_\eta,v-U_\eta,\mathcal{T}^{-1}_{v,\theta\eta},I^{2/\delta},T_{\theta\eta})}{R^{\mathcal{M}}_{ij}(\rho_\eta,\det\mathcal{T}_{v,\theta\eta},T_{\theta\eta})}\mathcal{M}_\eta(\eta)(1-\eta)d\eta,$$

$$\mathcal{Q}^A_i(f)=\begin{cases}-\frac{2}{3+\delta}\int_0^1|U_\eta|^2d\eta & (i=1)\\ \frac{4}{3+\delta}\int_0^1 U_{\eta,i}d\eta & (i=2,3,4)\\ \frac{1}{3+\delta} & (i=5,6,7)\\ \frac{\delta}{3+\delta} & (i=11)\\ 0 & \text{otherwise}\end{cases}$$

so that we can represent the linearized operators more succinctly as

$$\mathcal{M}_{\nu,\theta}(F) - F = \big(P_{\nu,\theta} f - f\big)\sqrt{m} + \sum_{ij} \mathcal{Q}^{\mathcal{M}}_{ij} \langle f, e_i\rangle_{L^2_{v,I}} \langle f, e_j\rangle_{L^2_{v,I}} \sqrt{m},$$

and

$$A_{\nu,\theta} = \frac{1}{1-\nu+\theta\nu}\left\{1 + \sum_i \mathcal{Q}^A_i \langle f, e_i\rangle_{L^2_{v,I}}\right\}.$$

We summarize all the computations of this section in the following proposition.

Proposition 2.2. *The polyatomic relaxation operator can be linearized around the global polyatomic Maxwellian m as follows:*

$$\begin{aligned} A_{\nu,\theta}\big(\mathcal{M}_{\nu,\theta}(F) - F\big) &= \frac{1}{1-\nu+\theta\nu}\Big(1 + \sum_i \mathcal{Q}^A_i \langle f, e_i\rangle_{L^2_{v,I}}\Big) \\ &\quad \times \Big\{\big(P_{\nu,\theta} f - f\big) + \sum_{i,j} \mathcal{Q}^{\mathcal{M}}_{ij} \langle f, e_i\rangle_{L^2_{v,I}} \langle f, e_j\rangle_{L^2_{v,I}}\Big\}\sqrt{m}, \end{aligned}$$

where

$$P_{\nu,\theta} f = \theta P_p f + (1-\theta)\{P_m f + \nu(P_1 f + P_2 f)\}.$$

We now substitute $F = m + \sqrt{m} f$ into (1.1) and apply Proposition 2.2 to obtain the perturbed polyatomic BGK model:

$$\begin{aligned} &\partial_t f + v\cdot\nabla_x f = L_{\nu,\theta} f + \Gamma_{\nu,\theta}(f), \\ &f(0,x,v,I) = f_0(x,v,I), \end{aligned} \tag{2.12}$$

where

$$f_0(x,v,I) = \frac{F_0(x,v,I) - m(v,I)}{\sqrt{m(v,I)}}.$$

The linearized relaxation operator $L_{\nu,\theta}$ is defined as follows:

$$L_{\nu,\theta} f = \frac{1}{1-\nu+\theta\nu}\left\{P_{\nu,\theta} f - f\right\},$$

where the precise form of the polyatomic projection $P_{\nu,\theta}$ is stated in Theorem 2.3. The nonlinear perturbation $\Gamma_{\nu,\theta}(f)$ is given by

$$\begin{aligned} \Gamma_{\nu,\theta}(f) &= \sum_i \mathcal{Q}^A_i \langle f, e_i\rangle_{L^2_{v,I}} \left\{P_{\nu,\theta} f - f\right\} \\ &\quad + \sum_{i,j} \mathcal{Q}^{\mathcal{M}}_{i,j} \langle f, e_i\rangle_{L^2_{v,I}} \langle f, e_j\rangle_{L^2_{v,I}} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,j,k}\mathcal{Q}^A_i\mathcal{Q}^{\mathcal{M}}_{j,k}\langle f,e_i\rangle_{L^2_{v,I}}\langle f,e_j\rangle_{L^2_{v,I}}\langle f,e_k\rangle_{L^2_{v,I}}\\
&\equiv \Gamma_1(f,f)+\Gamma_2(f,f)+\Gamma_3(f,f,f).
\end{aligned}
$$

The conservation laws in (1.2) now take the following form:

$$
\begin{aligned}
\int f(t)\sqrt{m}\,dxdvdI &= \int f_0\sqrt{m}\,dxdvdI,\\
\int f(t)v\sqrt{m}\,dxdvdI &= \int f_0 v\sqrt{m}\,dxdvdI,\\
\int f(t)\left\{\frac{1}{2}|v|^2+I^{2/\delta}\right\}\sqrt{m}\,dxdvdI &= \int f_0\left\{\frac{1}{2}|v|^2+I^{2/\delta}\right\}\sqrt{m}\,dxdvdI.
\end{aligned}
\tag{2.13}
$$

Therefore, if initial data shares the same mass, momentum and energy with m, the conservation laws reduce to (1.8).

3. Coercivity of the linearized relaxation operator

The main goal of this section is to establish the following dissipative property of the linearized polyatomic relaxation operator. Note that the coefficient and the degeneracy in the right hand side see an abrupt jump at $\theta=0$.

Theorem 3.1. *Let $-1/2\le\nu<1$ and $0\le\theta\le1$. Then we have the following dichotomy.*

(1) *For $0<\theta\le1$, $L_{\nu,\theta}$ satisfies*

$$
(1-\nu+\theta\nu)\langle L_{\nu,\theta}f,f\rangle_{L^2_{x,v,I}}\le-\theta\|(I-P_p)f\|^2_{L^2_{x,v,I}}.
$$

(2) *If $\theta=0$, $L_{\nu,0}$ satisfies*

$$
(1-\nu)\langle L_{\nu,0}f,f\rangle_{L^2_{x,v,I}}\le-(1-|\nu|)\,\|(I-P_m)f\|^2_{L^2_{x,v,I}}.
$$

Before proving this theorem, we first need to establish several technical lemmas.

Lemma 3.2. *The projection operators P_p, P_m, P_1, P_2 satisfy*

(1) *P_p, P_m, P_1 and P_2 are orthogonal projections:*

$$
P_p^2=P_p,\ \ P_m^2=P_m,\ \ P_1^2=P_1,\ \ P_2^2=P_2.
$$

(2) *P_m, P_1 and P_2 are mutually orthogonal in the following sense:*

$$
P_mP_1=P_1P_m=P_mP_2=P_2P_m=P_1P_2=P_2P_1=0.
$$

Proof. (1) The first, second and the last identities follow from the fact that the sets (1.5), (1.6) and

$$\left\{v_1v_2\sqrt{m},\ v_2v_3\sqrt{m},\ v_3v_1\sqrt{m}\right\}$$

are orthonormal, which can be checked by a direct calculation. The identity for P_1 needs more consideration. We first compute

$$\left\langle(3v_i^2-|v|^2)\sqrt{m},(3v_i^2-|v|^2)\sqrt{m}\right\rangle_{L^2_{v,I}}=12 \qquad (1=1,2,3),$$
$$\left\langle(3v_i^2-|v|^2)\sqrt{m},(3v_j^2-|v|^2)\sqrt{m}\right\rangle_{L^2_{v,I}}=-6 \qquad (i\neq j).$$

Let us denote $c_i(v)=(3v_i^2-|v|^2)/3\sqrt{2}$, and use the above computations to see that

$$\begin{aligned}
P_1^2f&=P_1\left\{\langle f,c_1\rangle_{L^2_{v,I}}c_1+\langle f,c_2\rangle_{L^2_{v,I}}c_2+\langle f,c_3\rangle_{L^2_{v,I}}c_3\right\}\\
&=\langle f,c_1\rangle_{L^2_{v,I}}\left\{P_1c_1\right\}+\langle f,c_2\rangle_{L^2_{v,I}}\left\{P_2c_2\right\}+\langle f,c_3\rangle_{L^2_{v,I}}\left\{P_3c_3\right\}\\
&=\frac{1}{3}\langle f,c_1\rangle_{L^2_{v,I}}\left\{2c_1-c_2-c_3\right\}\\
&+\frac{1}{3}\langle f,c_2\rangle_{L^2_{v,I}}\left\{-c_1+2c_2-c_3\right\}\\
&+\frac{1}{3}\langle f,c_3\rangle_{L^2_{v,I}}\left\{-c_1-c_2+2c_3\right\}.
\end{aligned}$$

The last term can be rewritten as

$$\left\langle f,\frac{2c_1-c_2-c_3}{3}\right\rangle_{L^2_{v,I}}c_1+\left\langle f,\frac{-c_1+2c_2-c_3}{3}\right\rangle_{L^2_{v,I}}c_2+\left\langle f,\frac{-c_1-c_2+2c_3}{3}\right\rangle_{L^2_{v,I}}c_3,$$

which, in view of $c_1+c_2+c_3=0$, is

$$\langle f,c_1\rangle_{L^2_{v,I}}c_1+\langle f,c_2\rangle_{L^2_{v,I}}c_2+\langle f,c_3\rangle_{L^2_{v,I}}c_3.$$

Therefore, we have $P_1^2f=P_1f$.

(2) We observe from direct computation that the following quantities all vanish:

$$\begin{aligned}
&\langle\big(2I^{2/\delta}-\delta)\big)\sqrt{m},\sqrt{m}\,\rangle_{L^2_{v,I}},\ \langle\big(2I^{2/\delta}-\delta)\big)\sqrt{m},v_\ell\sqrt{m}\,\rangle_{L^2_{v,I}},\\
&\langle\big(2I^{2/\delta}-\delta)\big)\sqrt{m},\big(|v|^2-3\big)\sqrt{m}\,\rangle_{L^2_{v,I}},\ \langle\sqrt{m},(3v_i^2-|v|^2)\sqrt{m}\,\rangle_{L^2_{v,I}},\\
&\langle v_\ell\sqrt{m},(3v_i^2-|v|^2)\sqrt{m}\,\rangle_{L^2_{v,I}},\ \langle(|v|^2-3)\sqrt{m},(3v_i^2-|v|^2)\sqrt{m}\,\rangle_{L^2_{v,I}},\\
&\langle v_iv_j\sqrt{m},(3v_k^2-|v|^2)\sqrt{m}\rangle_{L^2_{v,I}},\ \langle v_iv_j\sqrt{m},\big(2I^{2/\delta}-\delta)\big)\sqrt{m}\rangle_{L^2_{v,I}},
\end{aligned}$$

which implies (2). □

Lemma 3.3. *For* $0 \le \theta \le 1$, *we have*

$$
\begin{aligned}
&-(1-\nu+\theta\nu)\langle L_{\nu,\theta} f, f\rangle_{L^2_{x,v,I}} \\
&\quad = \theta\|(I-P_p)f\|^2_{L^2_{x,v,I}} + (1-\theta)\left\{\|(I-P_m)f\|^2_{L^2_{x,v,I}} - \nu\|(P_1+P_2)f\|^2_{L^2_{x,v,I}}\right\}.
\end{aligned}
$$

Proof. From the definition of $L_{\nu,\theta}$, we have

$$
\begin{aligned}
&-(1-\nu+\theta\nu)\langle L_{\nu,\theta} f, f\rangle_{L^2_v} \\
&\quad = -\theta\langle P_p f - f, f\rangle_{L^2_{v,I}} - (1-\theta)\langle P_m f - f + \nu(P_1+P_2)f, f\rangle_{L^2_{v,I}} \\
&\quad \equiv \theta I + (1-\theta) II.
\end{aligned} \tag{3.1}
$$

Then, the desired result follows from (1) and (2) below.

(1) **The estimate of I:** Lemma 3.2 (1) immediately gives

$$
\langle P_p f - f, f\rangle_{L^2_{v,I}} = -\|(I-P_p)f\|^2_{L^2_{v,I}}.
$$

(2) **The estimate of II:** As in the previous case, we have from Lemma 3.2 (1)

$$
\langle P_m f - f, f\rangle_{L^2_{v,I}} = -\|(I-P_m f)\|^2_{L^2_{v,I}}.
$$

On the other hand, we observe from Lemma 3.2 that

$$
\begin{aligned}
(P_1+P_2)^2 &= P_1^2 + P_1P_2 + P_2P_1 + P_2^2 \\
&= P_1 + P_2,
\end{aligned}
$$

to derive

$$
\begin{aligned}
\langle (P_1+P_2)f, f\rangle_{L^2_{v,I}} &= \langle (P_1+P_2)^2 f, f\rangle_{L^2_{v,I}} \\
&= \langle (P_1+P_2)f, (P_1+P_2)f\rangle_{L^2_{v,I}} \\
&= \|(P_1+P_2)f\|^2_{L^2_{v,I}},
\end{aligned}
$$

where we used the symmetry of $P_1 + P_2$. We combine these estimates to derive

$$
\begin{aligned}
II &= -\langle P_m f - f + \nu(P_1+P_2)f, f\rangle_{L^2_{v,I}} \\
&= -\langle P_m f - f, f\rangle_{L^2_{v,I}} - \nu\langle (P_1+P_2)f, f\rangle_{L^2_{v,I}} \\
&= \|(I-P_m f)\|^2_{L^2_{v,I}} - \nu\|(P_1+P_2)f\|^2_{L^2_{v,I}}. \qquad \square
\end{aligned}
$$

An immediate but important ramification of the above dissipation estimate is that the null space of the linearized relaxation operator has the following dichotomy:

Proposition 3.1. *For $0 \le \theta \le 1$ and $-1/2 \le \nu < 1$, the kernel of the linearized relaxation operator is given by*

$$Ker\{L_{\nu,\theta}\} = span\left\{\sqrt{m},\ v\sqrt{m},\ \frac{(|v|^2-3)+(2I^{2/\delta}-\delta)}{\sqrt{2(3+\delta)}}\sqrt{m}\right\} \quad (\theta \neq 0), \tag{3.2}$$

and

$$Ker\{L_{\nu,0}\} = span\left\{\sqrt{m},\ v\sqrt{m},\ \frac{|v|^2-3}{\sqrt{6}}\sqrt{m},\ \frac{I^{2/\delta}-\delta}{\sqrt{2\delta}}\sqrt{m}\right\} \qquad (\theta = 0). \tag{3.3}$$

Proof. For simplicity, set

$$\begin{aligned} A(f) &= \|(I-P_p)f\|^2_{L^2_{x,v,I}} \\ B(f) &= \|(I-P_m)f\|^2_{L^2_{x,v,I}} - \nu\|(P_1+P_2)f\|^2_{L^2_{x,v,I}}, \end{aligned}$$

so that, in view of Lemma 3.3, we write

$$-(1-\nu+\theta\nu)\langle L_{\nu,\theta}f, f\rangle_{L^2_{x,v,I}} = \theta A(f) + (1-\theta)B(f). \tag{3.4}$$

The non-negativity of $A(f)$ is clear. We claim that it is the case for $B(f)$ too:

Claim. $B(f) \ge 0$ *for* $-1/2 \le \nu < 1$.

Proof of the claim. Lemma 3.2 says $(P_1+P_2) \perp P_m$, so that

$$\|(P_1+P_2)f\|_{L^2_{v,I}} = \|(P_1+P_2)(I-P_m)f\|^2_{L^2_{v,I}}. \tag{3.5}$$

Then, since $(P_1+P_2)^2 = P_1+P_2$, we see that

$$\begin{aligned} \|(P_1+P_2)(I-P_m)f\|^2_{L^2_{v,I}} &= \langle (P_1+P_2)(I-P_m)f, (P_1+P_2)(I-P_m)f\rangle_{L^2_{v,I}} \\ &= \langle (P_1+P_2)^2(I-P_m)f, (I-P_m)f\rangle_{L^2_{v,I}} \\ &= \langle (P_1+P_2)(I-P_m)f, (I-P_m)f\rangle_{L^2_{v,I}} \\ &\le \|(P_1+P_2)(I-P_m)f\|_{L^2_{v,I}}\|(I-P_m)f\|_{L^2_{v,I}}. \end{aligned} \tag{3.6}$$

Therefore, (3.5) and (3.6) give

$$\|(P_1+P_2)f\|_{L^2_{v,I}} \le \|(I-P_m)f\|_{L^2_{v,I}}. \tag{3.7}$$

Hence, we have

$$\begin{aligned} B(f) &\geq \|(I-P_m)f\|^2_{L^2_{v,I}} - |\nu|\|(I-P_m)f\|^2_{L^2_{v,I}} \\ &= (1-|\nu|)\|(I-P_m f)\|^2_{L^2_{v,I}} \\ &\geq 0. \end{aligned} \tag{3.8}$$

This proves the claim. □

Now we return to the proof of the proposition. Consider

$$L_{\nu,\theta} f = \theta A(f) + (1-\theta)B(f) = 0. \tag{3.9}$$

We divide it into the following two cases:

(1) (The case $\theta = 0$): In this case, (3.9) reduces to

$$B(f) = 0.$$

That is,

$$\|(I-P_m)f\|^2_{L^2_{x,v,I}} + \nu\|(P_1+P_2)f\|^2_{L^2_{x,v,I}} = 0,$$

which, in view of (3.7), implies

$$\|(I-P_m)f\|^2_{L^2_v} = -\nu\|(P_1+P_2)f\|_{L^2_v} \leq |\nu|\|(I-P_m)f\|_{L^2_v}.$$

Therefore,

$$(1-|\nu|)\|(I-P_m)f\|^2_{L^2_{x,v,I}} \leq 0,$$

so that

$$f = P_m f. \tag{3.10}$$

From this, we can conclude that, when $\theta = 0$, the kernel of $L_{\nu,0}$ is given by (3.2).

(2) (The case $\theta \neq 0$): Since both A and B are non-negative, we have from (3.9)

$$A(f) = B(f) = 0.$$

First,

$$A(f) = \|(I-P_p)f\|^2_{L^2_{x,v,I}} = 0,$$

clearly gives

$$f = P_p f.$$

On the other hand, it was shown in the previous case that $B(f) = 0$ implies

$$f = P_m f.$$

Therefore, when $0 < \theta \le 1$, we have

$$f = P_p f = P_m f.$$

Hence, the kernel is given by the intersection of (3.2) and (3.3). This gives the desired result since (3.2) is a subspace of (3.3). □

We are now ready to prove the main theorem of this section.

3.1. *Proof of Theorem 3.1*

(1) (The case of $0 < \theta \le 1$): In the proof of Proposition 3.1, we have shown that the degeneracy of $B(f)$ is strictly bigger than that of $A(f)$. Therefore, we can ignore B in (3.4) to obtain

$$-(1-\nu+\theta\nu)\langle L_{\nu,\theta} f, f\rangle_{L^2_{v,I}} \ge \theta \|(I-P_p)f\|^2_{L^2_{v,I}}.$$

(2) (The case of $\theta = 0$): In this case, we are left with

$$-(1-\nu)\langle L_{\nu,0} f, f\rangle_{L^2_{v,I}} = B(f).$$

Recall that we have shown in (3.8) that

$$B(f) \ge (1-|\nu|)\|(I-P_m)f\|^2_{L^2_{v,I}},$$

to see

$$-(1-\nu)\langle L_{\nu,0} f, f\rangle_{L^2_{v,I}} \ge (1-|\nu|)\|(I-P_m)f\|^2_{L^2_{v,I}}.$$

This completes the proof.

4. Estimates on the macroscopic fields

4.1. *Estimates on the macroscopic fields*

In this section, we establish estimates on the macroscopic fields which will be crucially used to control the nonlinear term $\Gamma_{\nu,\theta}(f)$.

Lemma 4.1. *Assume $\mathcal{E}(t)$ is sufficiently small, then there exists a positive constant $C > 0$ such that*

$$\begin{aligned}
&(1)\ |\rho_\eta(x,t) - 1| \le C\sqrt{\mathcal{E}(t)},\\
&(2)\ |U^i_\eta(x,t)| \le C\sqrt{\mathcal{E}(t)} \qquad (1 \le i \le 3),\\
&(3)\ |\mathcal{T}^{ii}_{\nu,\theta\eta}(x,t) - 1| \le C\sqrt{\mathcal{E}(t)} \qquad (1 \le i \le 3),\\
&(4)\ |\mathcal{T}^{ij}_{\nu,\theta\eta}(x,t)| \le C\sqrt{\mathcal{E}(t)} \qquad (1 \le i < j \le 3),\\
&(5)\ |T_{\theta\eta}(x,t) - 1| \le C\sqrt{\mathcal{E}(t)}.
\end{aligned}$$

Proof. (1) Since

$$|F_\eta - m| = |\eta f\sqrt{m}| \le |f|\sqrt{m},$$

Hölder inequality and Sobolev embedding yield

$$|\rho_\eta(x,t) - 1| = \int |f|\sqrt{m}\,dv\,dI \le C\|f\|_{L^2_{v,I}} \le C\sqrt{\mathcal{E}(t)}.$$

(2) Note that

$$\int F_\eta v\,dv\,dI = \int (m + \eta f\sqrt{m})v\,dv\,dI = \eta\int f v\sqrt{m}\,dv\,dI.$$

Therefore, recalling the lower bound estimate of ρ_η in (1), and employing Hölder inequality and Sobolev embedding, we have

$$|U_\eta| \le \frac{1}{\rho_\eta}\left|\int_{\mathbb{R}^3} f v\sqrt{m}\,dv\,dI\right| \le \frac{C\|f\|_{L^2_{v,I}}}{1 - \sqrt{\mathcal{E}(t)}} \le C\sqrt{\mathcal{E}(t)}$$

for sufficiently small $\mathcal{E}(t)$.

(3) We recall (2.1) to write the diagonal elements of $\rho\mathcal{T}_{\nu,\theta,\eta}$ as

$$\begin{aligned}
&\rho_\eta\mathcal{T}^{ii}_{\nu,\theta\eta}\\
&= \theta\left[\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+} F_\eta\left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{2/\delta}\right)dv\,dI\right\} - \frac{1}{3+\delta}\rho_\eta|U_\eta|^2\right]\\
&+ (1-\theta)\left[\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+} F_\eta\left(\frac{1-\nu}{3}|v|^2 + \nu v_i^2\right)dv\,dI\right\} - \left\{\frac{1-\nu}{3}\rho_\eta|U_\eta|^2 + \nu\rho_\eta U^2_{\eta,i}\right\}\right].
\end{aligned}$$

Therefore, using

$$\int m\left(\frac{1}{3+\delta}|v|^2+\frac{2}{3+\delta}I^{2/\delta}\right)dvdI=1,$$

we have

$$\begin{aligned}
&\rho_\eta\mathcal{T}^{ii}_{\nu,\theta\eta}\\
&=1+\theta\left[\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+}\eta f\sqrt{m}\left(\frac{1}{3+\delta}|v|^2+\frac{2}{3+\delta}I^{2/\delta}\right)dvdI\right\}-\frac{\theta}{3+\delta}\rho_\eta|U_\eta|^2\right]\\
&+(1-\theta)\left[\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+}\eta f\sqrt{m}\left(\frac{1-\nu}{3}|v|^2+\nu v_i^2\right)dvdI\right\}-\left\{\frac{1-\nu}{3}\rho_\eta|U_\eta|^2+\nu\rho_\eta U^2_{\eta,i}\right\}\right]\\
&\le 1+\theta\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+}\eta f\sqrt{m}\left(\frac{1}{3+\delta}|v|^2+\frac{2}{3+\delta}I^{2/\delta}\right)dvdI\right\}\\
&+(1-\theta)\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+}\eta f\sqrt{m}\left(\frac{1-\nu}{3}|v|^2+\nu v_i^2\right)dvdI\right\}\\
&\le 1+C_{\theta,\delta}\|f\|_{L^2_{v,I}}.
\end{aligned}$$

In the last line, we used Hölder inequality. Now, the estimate (1) above on ρ_η and Sobolev embedding gives

$$\begin{aligned}
\mathcal{T}^{ii}_{\nu,\theta\eta}-1&\le\frac{1-\rho_\theta+C_{\theta,\delta}\|f\|_{L^2_{v,I}}}{\rho_\theta}\\
&\le\frac{C\sqrt{\mathcal{E}(t)}+C_{\theta,\delta}\|f\|_{L^2_{v,I}}}{1-\sqrt{\mathcal{E}(t)}}\\
&\le C_{\theta,\delta}\sqrt{\mathcal{E}(t)}.
\end{aligned}\tag{4.1}$$

Similarly, we compute

$$\begin{aligned}
&\rho_\eta\mathcal{T}^{ii}_{\nu,\theta\eta}\\
&=1+\theta\left[\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+}\eta f\sqrt{m}\left(\frac{1}{3+\delta}|v|^2+\frac{2}{3+\delta}I^{2/\delta}\right)dvdI\right\}-\frac{1}{3+\delta}\rho_\eta|U_\eta|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&+(1-\theta)\left[\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+}\eta f\sqrt{m}\left(\frac{1-\nu}{3}|v|^2+\nu v_i^2\right)dvdI\right\}-\left\{\frac{1-\nu}{3}\rho_\eta|U_\eta|^2+\nu\rho_\eta U_{\eta,i}^2\right\}\right]\\
&\geq 1-\theta\left\{C_\delta\|f\|_{L^2_{v,I}}+\frac{\theta}{3+\delta}\frac{\|f\|^2_{L^2_{v,I}}}{1-\sqrt{\mathcal{E}(t)}}\right\}-(1-\theta)\left\{\eta C_\nu\|f\|_{L^2_{v,I}}+C_\nu\frac{\|f\|^2_{L^2_{v,I}}}{1-\sqrt{\mathcal{E}(t)}}\right\}\\
&\geq 1-C_{\theta,\delta,\nu}\|f\|^2_{L^2_{v,I}}\\
&\geq 1-C_{\theta,\delta,\nu}\sqrt{\mathcal{E}(t)},
\end{aligned}
$$

yielding

$$
\begin{aligned}
\mathcal{T}^{ii}_{\nu,\theta\eta}-1&\geq\frac{1-\rho_\theta-C_{\theta,\delta,\nu}\sqrt{\mathcal{E}(t)}}{\rho_\theta}\\
&\geq\frac{-C_{\theta,\delta,\nu}\sqrt{\mathcal{E}(t)}}{1+\sqrt{\mathcal{E}(t)}}\\
&\geq -C_{\theta,\delta,\nu}\sqrt{\mathcal{E}(t)}.
\end{aligned}
\tag{4.2}
$$

(4.1) and (4.2) give the desired result for $\mathcal{T}^{ii}_{\nu,\theta\eta}$ $(i=1,2,3)$.

(4) Non-diagonal entries of $\mathcal{T}_{\nu,\theta\eta}$ are given by

$$
\begin{aligned}
\rho_\eta\mathcal{T}^{ij}_{\nu,\theta\eta}&=(1-\theta)\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+}F_\eta\left(\nu v_iv_j\right)dvdI-\nu\rho_\eta U^i_\eta U^j_\eta\right\}\\
&=(1-\theta)\left\{\int_{\mathbb{R}^3\times\mathbb{R}_+}\left(m+\eta f\sqrt{m}\right)\left(\nu v_iv_j\right)dvdI-\nu\rho_\eta U^i_\eta U^j_\eta\right\}\\
&=(1-\theta)\left\{\eta\nu\int_{\mathbb{R}^3\times\mathbb{R}_+}f\sqrt{m}v_iv_jdvdI-\nu\rho_\eta U^i_\eta U^j_\eta\right\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left|\rho_\eta\mathcal{T}^{ij}_{\nu,\theta\eta}\right|&\leq|\nu|\left|\int_{\mathbb{R}^3\times\mathbb{R}^+}fv_iv_j\sqrt{m}dvdI\right|+|\nu|\rho_\eta|U^i_\eta||U^j_\eta|\\
&\leq|\nu|C\|f\|_{L^2_{v,I}}+|\nu|C\mathcal{E}(t)\\
&\leq|\nu|C\sqrt{\mathcal{E}(t)}.
\end{aligned}
$$

(5) The estimate follows by similar argument using the following identity

$$T_{\theta\eta} = \theta\left\{\frac{1}{\rho_\eta}\int\limits_{\mathbb{R}^3\times\mathbb{R}_+} F_\eta\left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{2/\delta}\right)dvdI - \frac{1}{3+\delta}|U_\eta|^2\right\}$$
$$+ (1-\theta)\left\{\frac{2}{\delta}\frac{1}{\rho_\eta}\int\limits_{\mathbb{R}^3\times\mathbb{R}_+} F_\eta I^{2/\delta}dvdI\right\}.$$

We omit it. $\square$

In the following, we estimate the derivatives of the macroscopic fields.

Lemma 4.2. *For sufficiently small $\mathcal{E}(t)$, we have*

$$(1)\ |\partial^\alpha\rho_\eta(x,t)| \le C_\alpha\sqrt{\mathcal{E}(t)},$$
$$(2)\ |\partial^\alpha U_\eta(x,t)| \le C_\alpha\sqrt{\mathcal{E}(t)},$$
$$(3)\ |\partial^\alpha \mathcal{T}^{ij}_{v,\theta\eta}(x,t)| \le C_\alpha\sqrt{\mathcal{E}(t)},$$
$$(4)\ |\partial^\alpha T_{\theta\eta}(x,t)| \le C_\alpha\sqrt{\mathcal{E}(t)}.$$

Here ∂ denotes derivatives in x,t.

Proof. (1) Since $\partial^\alpha\int m dvdI = 0$, we have

$$|\partial^\alpha\rho_\eta| = \left|\partial^\alpha\left(\int\limits_{\mathbb{R}^3\times\mathbb{R}_+}\big(m+\eta f\sqrt{m}\big)dvdI\right)\right| = \eta\int|\partial^\alpha f|\sqrt{m}dvdI \le \|\partial^\alpha f\|_{L^2_{v,I}}.$$

(2) We apply ∂^α to $U = \frac{1}{\rho}\int f v\sqrt{m}dv$ and use Leibniz rule to derive

$$|\partial^\alpha U_\eta| \le \frac{C_{|\alpha|}}{\rho_\eta^{2|\alpha|}}\left(\sum_{|\alpha_1|\le N}\int\limits_{\mathbb{R}^3\times\mathbb{R}_+}|\partial^{\alpha_1}f||v|\sqrt{m}dvdI\right)\left(1+\sum_{|\alpha_2|\le N}|\partial^{\alpha_2}\rho_\eta|\right)^{|\alpha|}.$$

Then, we have from Hölder inequality and Sobolev embedding that

$$|\partial^\alpha U_\eta| \le \frac{C_\alpha}{(1-\mathcal{E}(t))^{2|\alpha|}}\left(\sum_{|\alpha_1|\le N}\|\partial^{\alpha_1}f\|_{L^2_{v,I}}\right)\left(1+\sum_{|\alpha_2|\le N}\|\partial^{\alpha_2}f\|_{L^2_{v,I}}\right)^{|\alpha|}$$
$$\le C_\alpha\left\{\sum_{|\alpha_1|\le N}\|\partial^{\alpha_1}f\|_{L^2_{v,I}} + \left(\sum_{|\alpha_1|\le N}\|\partial^{\alpha_1}f\|_{L^2_{v,I}}\right)^{|\alpha|}\right\}$$

$$\leq C_\alpha\sqrt{\mathcal{E}(t)}$$

for sufficiently small $\mathcal{E}(t)$.

(3) Similar argument as in (2) above, applied to $\rho\mathcal{T}_{\nu,\theta\eta}$ gives

$$\begin{aligned}
&|\partial^\alpha\mathcal{T}^{ij}_{\nu,\theta\eta}|\\
&\leq\frac{C_{|\alpha|,\delta,\theta}}{\rho_\eta^{2|\alpha|}}\left(\sum_{|\alpha_1|\leq N}\int_{\mathbb{R}^3\times\mathbb{R}_+}|\partial^{\alpha_1}f|\left\{|v|^2+2I^{\delta/2}\right\}\sqrt{m}\,dv\,dI\right)\left(1+\sum_{|\alpha_2|\leq N}|\partial^{\alpha_2}\rho_\eta|\right)^{|\alpha|}\\
&+\frac{C_{|\alpha|}}{\rho_\eta^{2|\alpha|}}\left(\sum_{|\alpha_1|\leq N}\int_{\mathbb{R}^3\times\mathbb{R}_+}|\partial^{\alpha_1}f||v|\sqrt{m}\,dv\,dI\right)\left(1+\sum_{|\alpha_2|\leq N}|\partial^{\alpha_2}\rho_\eta|\right)^{|\alpha|}\\
&\leq C_{|\alpha|}\left\{\sum_{|\alpha_1|\leq N}\|\partial^{\alpha_1}f\|_{L^2_{v,I}}+\left(\sum_{|\alpha_1|\leq N}\|\partial^{\alpha_1}f\|_{L^2_{v,I}}\right)^{|\alpha|}\right\}.
\end{aligned}$$

(4) The estimate for $\partial^\alpha T_{\theta\eta}$ is similar. We omit it. $\square$

Lemma 4.3. *Let $\mathcal{E}(t)$ be sufficiently small. Then the determinant of $\mathcal{T}_{\nu,\theta\eta}$ satisfies*

$$\begin{aligned}
&(1)\ \left|\partial^\alpha\det\left(\mathcal{T}_{\nu,\theta\eta}\right)\right|\leq C_\alpha\sqrt{\mathcal{E}(t)},\\
&(2)\ \left|\det\left(\mathcal{T}_{\nu,\theta\eta}\right)\right|\geq 1-C_\alpha\sqrt{\mathcal{E}(t)},
\end{aligned}$$

for some $C_\alpha>0$.

Proof. (1) A straightforward computation gives

$$\begin{aligned}
\det(\mathcal{T}_{\nu,\theta\eta})&=\mathcal{T}^{11}_{\nu,\theta\eta}\mathcal{T}^{22}_{\nu,\theta\eta}\mathcal{T}^{33}_{\nu,\theta\eta}+2\mathcal{T}^{12}_{\nu,\theta\eta}\mathcal{T}^{23}_{\nu,\theta\eta}\mathcal{T}^{31}_{\nu,\theta\eta}\\
&\quad-\left\{\mathcal{T}^{23}_{\nu,\theta\eta}\right\}^2\mathcal{T}^{11}_{\nu,\theta\eta}-\left\{\mathcal{T}^{31}_{\nu,\theta\eta}\right\}^2\mathcal{T}^{22}_{\nu,\theta\eta}-\left\{\mathcal{T}^{12}_{\nu,\theta\eta}\right\}\mathcal{T}^{33}_{\nu,\theta\eta}.
\end{aligned}\tag{4.3}$$

Therefore, $\partial^\alpha\det\mathcal{T}_{\nu,\theta\eta}$ takes the following form:

$$\partial^\alpha\det(\mathcal{T}_{\nu,\theta\eta})=\sum_{\alpha=\alpha_1+\alpha_2+\alpha_3}C_{ij\ell kmn}\partial^{\alpha_1}\mathcal{T}^{ij}_{\nu,\theta\eta}\partial^{\alpha_2}\mathcal{T}^{\ell k}_{\nu,\theta\eta}\partial^{\alpha_3}\mathcal{T}^{mn}_{\nu,\theta\eta}.$$

Now, we recall from Lemma 4.1 and Lemma 4.2 that

$$\mathcal{T}^{ii}_{\nu,\theta\eta}=1+o\big(\sqrt{\mathcal{E}(t)}\big)\ (i=1,2,3),\quad \mathcal{T}^{ij}_{\nu,\theta\eta}=o\left(\sqrt{\mathcal{E}(t)}\right)\ (i\neq j),\tag{4.4}$$

and

$$\left|\partial^\alpha\mathcal{T}^{ij}_{\nu,\theta\eta}\right|\leq C_\alpha\sqrt{\mathcal{E}(t)},\tag{4.5}$$

to deduce

$$|\partial^\alpha \det \mathcal{T}_{\nu,\theta\eta}| \le C_\alpha \sqrt{\mathcal{E}(t)}.$$

(2) Inserting (4.4) and (4.5) into (4.3), we get

$$\begin{aligned}\det \mathcal{T}_{\nu,\theta\eta} &= \{1+o(\mathcal{E}(t))\}^3 - 2\left\{o\big(\sqrt{\mathcal{E}(t)}\big)\right\}^3 - 3\left\{o\big(\sqrt{\mathcal{E}(t)}\big)\right\}^2\left\{1+o\big(\sqrt{\mathcal{E}(t)}\big)\right\}\\ &\ge 1 - C_\alpha\sqrt{\mathcal{E}(t)},\end{aligned}$$

for sufficiently small $\mathcal{E}(t)$. □

Lemma 4.4. *Let $0 \le \theta \le 1$ and $-1/2 < \nu < 1$. Suppose $\mathcal{E}(t)$ is sufficiently small. Then, there exist positive constants C_1, C_2 such that*

$$\begin{aligned}&(1)\ X^\top\{\mathcal{T}^{-1}_{\nu,\theta\eta}\}Y \le \{1 - C_1\mathcal{E}(t)\}^{-1}\|X\|\|Y\|,\\ &(2)\ X^\top\{\mathcal{T}^{-1}_{\nu,\theta\eta}\}X \ge \{1 + C_2\mathcal{E}(t)\}^{-1}\|X\|^2,\end{aligned}$$

for any X, Y in $\mathbb{R}^3$.

Proof. We start with proving the following claim:

Claim. *For sufficiently small $\mathcal{E}(t)$, we have*

$$\left\{1 - C_1\sqrt{\mathcal{E}(t)}\right\} Id \le \mathcal{T}_{\nu,\theta\eta} \le \left\{1 + C_2\sqrt{\mathcal{E}(t)}\right\} Id. \tag{4.6}$$

Proof of the claim. For $\kappa \in \mathbb{R}^3$, we have

$$\kappa^\top \mathcal{T}_{\nu,\theta\eta}\kappa = \sum_{i=1,2,3} \mathcal{T}^{ii}_{\nu,\theta\eta}\kappa_i^2 + \sum_{1\le i,j\le 3} \mathcal{T}^{ij}_{\nu,\theta\eta}\kappa_i\kappa_j.$$

In view of Lemma 4.1 (3), (4), this gives

$$\begin{aligned}\kappa^\top \mathcal{T}_{\nu,\theta\eta}\kappa &= \sum_{i=1,2,3}\left\{1 + C\sqrt{\mathcal{E}(t)}\right\}\kappa_i^2 + C\sum_{1\le i,j\le 3}\sqrt{\mathcal{E}(t)}\kappa_i\kappa_j\\ &= \sum_{i=1,2,3}\kappa_i^2 + C\sqrt{\mathcal{E}(t)}\left\{\sum_{1\le i\le 3}\kappa_i^2 + \sum_{1\le i,j\le 3}\kappa_i\kappa_j\right\}\\ &\le \left\{1 + C_1\sqrt{\mathcal{E}(t)}\right\}|\kappa|^2.\end{aligned}$$

Likewise,

$$\kappa^\top \mathcal{T}_{\nu,\theta\eta}\kappa \ge \left\{1 - C_2\sqrt{\mathcal{E}(t)}\right\}|\kappa|^2.$$

This completes the proof of the claim. □

(1) Let $\{\lambda_i\}$ denote the eigenvalues of $\mathcal{T}_{\nu,\theta\eta}$ so that we can write

$$\mathcal{T}_{\nu,\theta\eta} = P^\top diag\{\lambda_1, \cdots, \lambda_{11}\}P,$$

for some orthogonal matrix P. Here $diag\{a, b, \cdots\}$ denotes the diagonal matrix whose diagonal entries are $a, b, \cdots$. Therefore, since the above claim implies

$$1 - C_1\mathcal{E}(t) \le \lambda_i \le 1 + C_2\mathcal{E}(t) \qquad (1 \le i \le 11)$$

for sufficiently small $\mathcal{E}(t)$, we have

$$\begin{aligned}
X^\top \mathcal{T}^{-1}_{\nu,\theta\eta} Y &= X^\top \Big[P^\top diag\{\lambda_1^{-1}, \cdots, \lambda_{11}^{-1}\}P\Big] Y \\
&= \{PX\}^\top diag\{\lambda_1^{-1}, \cdots, \lambda_{11}^{-1}\}\{PY\} \\
&= \sum_i \lambda_i^{-1}\{PX\}_i\{PY\}_i \\
&\le \max\{\lambda_i^{-1}\}\|PX\|\|PY\| \\
&= \max\{\lambda_i^{-1}\}\|X\|\|Y\|. \\
&\le \{1 - C_1\mathcal{E}(t)\}^{-1}\|X\|\|Y\|.
\end{aligned}$$

(2) Lower bound can be computed in a similar way as follows:

$$\begin{aligned}
X^\top \mathcal{T}^{-1}_{\nu,\theta\eta} X &= X^\top P^\top diag\{\lambda_1^{-1}, \cdots, \lambda_{11}^{-1}\}PX \\
&= \sum_i \lambda_i^{-1} |\{PX\}_i|^2 \\
&\ge \min\{\lambda_i^{-1}\}\|PX\|^2 \\
&= \min\{\lambda_i^{-1}\}\|X\|^2 \\
&\ge \{1 + C_2\mathcal{E}(t)\}^{-1}\|X\|^2. \qquad \square
\end{aligned}$$

In the next lemma, we prove an estimate for derivatives of $\mathcal{T}_{\nu,\theta\eta}$.

Lemma 4.5. *Let* $0 \le \theta \le 1$ *and* $-1/2 < \nu < 1$. *Assume* $\mathcal{E}(t)$ *is sufficiently small. Then we have*

$$\|\partial^\alpha\big(\mathcal{T}^{-1}_{\nu,\theta\eta}\big)\| \le C_\alpha\sqrt{\mathcal{E}(t)}.$$

Proof. Applying $\partial\left\{\mathcal{T}^{-1}_{\nu,\theta,\eta}\right\} = -\mathcal{T}^{-1}_{\nu,\theta,\eta}\left\{\partial\mathcal{T}_{\nu,\theta,\eta}\right\}\mathcal{T}^{-1}_{\nu,\theta,\eta}$ recursively, we can derive $\partial^\alpha\left\{\mathcal{T}^{-1}_{\nu,\theta,\eta}\right\} = P\big(\mathcal{T}^{-1}_{\nu,\theta,\eta}, \partial\mathcal{T}_{\nu,\theta,\eta}, \cdots, \partial^\alpha\mathcal{T}_{\nu,\theta,\eta}\big)$ for some polynomial P. Therefore, by Lemma 4.1 (4), (5) and Lemma 4.2 (3), we get the desired result. $\square$

5. Estimates on nonlinear perturbation and local existence

In this section, we establish the local in time existence of smooth solutions. For this, we first need to estimate the nonlinear part.

Lemma 5.1. *The nonlinear perturbation $\Gamma_{\nu,\theta}(f)$ satisfies:*

$$
\begin{aligned}
(1)\ \left|\int \partial^{\alpha}_{\beta}\Gamma_{\nu,\theta}(f)g\,dv\,dI\right| &\le C\sum_{|\alpha_1|+|\alpha_2|\le|\alpha|}\|\partial^{\alpha_1}f\|_{L^2_{v,I}}\|\partial^{\alpha_2}f\|_{L^2_{v,I}}\|g\|_{L^2_{v,I}}\\
&\quad + C\sum_{\substack{|\alpha_1|+|\alpha_2|\le|\alpha|\\|\beta_1|\le|\beta|}}\|\partial^{\alpha_1}f\|_{L^2_{v,I}}\|\partial^{\alpha_2}_{\beta_1}f\|_{L^2_{v,I}}\|g\|_{L^2_{v,I}}\\
&\quad + C\sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|\le|\alpha|}\|\partial^{\alpha_1}f\|_{L^2_{v,I}}\|\partial^{\alpha_2}f\|_{L^2_{v,I}}\|\partial^{\alpha_3}f\|_{L^2_{v,I}}\|h\|_{L^2_{v,I}},
\end{aligned}
$$

$$
(2)\ \left\|\int \Gamma_{1,2}(f,g)h\,dv\,dI\right\|_{L^2_x}+\left\|\int \Gamma_{1,2}(g,h)h\,dv\,dI\right\|_{L^2_x}\le C\sup_{x,v,I}|\nu_{v,I}h|\sup_x\|f\|_{L^2_{v,I}}\|g\|_{L^2_{x,v,I}},
$$

$$
\begin{aligned}
&\left\|\int \Gamma_3(f,g,h)r\,dv\,dI\right\|_{L^2_x}+\left\|\int \Gamma_3(g,f,h)r\,dv\,dI\right\|_{L^2_x}+\left\|\int \Gamma_3(g,h,f)r\,dv\,dI\right\|_{L^2_x}\\
&\qquad\le C\sup_{x,v,I}|\nu_{v,I}r|\sup_x\|f\|_{L^2_{v,I}}\sup_x\|g\|_{L^2_{v,I}}\|h\|_{L^2_{x,v,I}},
\end{aligned}
$$

where $\nu_{v,I}=(1+|v|^2)(1+I)$.

Proof. We prove this lemma only for Γ_2. Other terms can be treated similarly. We first need to estimate $\partial^{\alpha}_{\beta}Q^{\mathcal{M}}_{ij}$.

Claim. $\left|\partial^{\alpha}_{\beta}Q^{\mathcal{M}}_{ij}\right|\le C_{\alpha,\beta}\exp\left(-\frac{|v|^2}{12}-\frac{I^{2/\delta}}{6}\right)$.

Proof of the claim. Note that there exist a homogeneous polynomial $P_{\alpha,\beta}$ and a monomial $M_{\alpha,\beta}$ such that

$$
\begin{aligned}
\left|\partial^{\alpha}_{\beta}\mathcal{M}_{\nu,\theta}\left(\rho_\eta,U_\eta,\mathcal{T}_{\nu,\theta\eta},T_{\theta\eta},\right)\right| &= \frac{\left|P_{\alpha,\beta}\left(\partial\rho_\eta,\partial U_\eta,\partial(v-U_\eta),\partial\mathcal{T}_{\nu,\theta\eta},\partial T_{\theta\eta},\partial I^{2/\delta}\right)\right|}{M_{\alpha,\beta}\left(\det(\mathcal{T}_{\nu,\theta\eta}),T_{\theta\eta}\right)}\\
&\quad\times\exp\left(-\frac{1}{2}(v-U_\eta)^{\top}\mathcal{T}^{-1}_{\nu,\theta\eta}(v-U_\eta)-\frac{I^{2/\delta}}{T_{\theta\eta}}\right).
\end{aligned}
$$

Here we slightly abused the notation to let ∂ denote any of $\partial^{\bar\alpha}_{\bar\beta}$ such that $\bar\alpha\le|\alpha|$ and $\bar\beta\le|\beta|$. Recalling the upper and lower bound estimates on the macroscopic fields in Lemma 4.1, Lemma 4.2, the determinant estimates in Lemma 4.3 and the estimates on the temperature tensor made in Lemma 4.4, Lemma 4.5, we have

$$\frac{\left|P_{\alpha,\beta}\left(\partial\rho_\eta, \partial U_\eta, \partial(v-U_\eta), \partial\mathcal{T}_{v,\theta\eta}, \partial T_{\theta\eta}, I^{2\delta}\right)\right|}{M_{\alpha,\beta}\left(\det(\mathcal{T}_{v,\theta\eta}), T_{\theta\eta}\right)} \le C_{\alpha,\beta}(1+|v|^2+I^{2/\delta})^{m(\alpha)},$$

for some $C_{\alpha,\beta}, m(\alpha) > 0$. On the other hand, Lemma 4.1 (2), (5) and Lemma 4.4 (2) give the following lower bound:

$$\begin{aligned}\frac{1}{2}(v-U_\eta)^\top\mathcal{T}^{-1}_{v,\theta\eta}(v-U_\eta) + \frac{I^{2/\delta}}{T_{\theta\eta}} &\ge \left(\frac{2}{3}+2\varepsilon\right)\frac{|v-U_\eta|^2}{2} + \left(\frac{2}{3}+\varepsilon\right)I^{2/\delta}\\ &\ge \left(\frac{1}{3}+\varepsilon\right)|v|^2 + \left(\frac{2}{3}+\varepsilon\right)I^{2/\delta} + o\big(\mathcal{E}(t)\big),\end{aligned}$$

for some small $\varepsilon > 0$. This gives

$$\begin{aligned}\left|\partial^\alpha_\beta Q^{\mathcal{M}}_{ij}\right| &\le C_{\alpha,\beta}\exp\left(\frac{1}{4}|v|^2+\frac{I^{2/\delta}}{2}\right)(1+|v|^2+I^{2/\delta})^{m(\alpha)}\\ &\quad\times\exp\left(-\left(\frac{1}{3}+\varepsilon\right)|v|^2-\left(\frac{2}{3}+\varepsilon\right)I^{2/\delta}+o\big(\mathcal{E}(t)\big)\right)\\ &\le C_{\alpha,\beta}\exp\left(-\frac{|v|^2}{12}-\frac{I^{2/\delta}}{6}\right).\end{aligned} \tag{5.1}$$

This completes the proof of the claim. □

We now return to the proof of the lemma. In view of (5.1), we denote throughout this proof

$$\widetilde{m}(v,I) = \exp\left(-\frac{|v|^2}{12}-\frac{I^{2/\delta}}{6}\right)$$

for simplicity.

(1) From (5.1) and Hölder inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}_+}|\partial^\alpha_\beta\Gamma_2(f)g|\,dvdI\\ &\le \sum_{\substack{|\alpha_1|+|\alpha_2|+|\alpha_3|\\=|\alpha|}}\int_{\mathbb{R}^3\times\mathbb{R}_+}\left|\partial^{\alpha_1}_{\beta_1}Q^{\mathcal{M}}_{ij}\langle\partial^{\alpha_2}f,e_i\rangle_{L^2_{v,I}}\langle\partial^{\alpha_2}f,e_j\rangle_{L^2_{v,I}}\,g\right|dvdI\\ &\le C\sum_{\substack{|\alpha_1|+|\alpha_2|+|\alpha_3|\\=|\alpha|}}\left(\int_{\mathbb{R}^3\times\mathbb{R}_+}\left|\partial^{\alpha_1}_{\beta_1}Q^{\mathcal{M}}_{ij}\;g\right|dvdI\right)\|\partial^{\alpha_2}f\|_{L^2_{v,I}}\|\partial^{\alpha_3}f\|_{L^2_{v,I}}\\ &\le C_{\alpha,\beta}\sum_{\substack{|\alpha_1|+|\alpha_2|+|\alpha_3|\\=|\alpha|}}\left(\int_{\mathbb{R}^3\times\mathbb{R}_+}\widetilde{m}\,|g|\,dvdI\right)\|\partial^{\alpha_2}f\|_{L^2_{v,I}}\|\partial^{\alpha_3}f\|_{L^2_{v,I}}\end{aligned}$$

$$
\begin{aligned}
&\le C_{\alpha,\beta} \sum_{\substack{|\alpha_1|+|\alpha_2|+|\alpha_3|\\=|\alpha|}} \|\partial^{\alpha_2} f\|_{L^2_{v,I}} \|\partial^{\alpha_3} f\|_{L^2_{v,I}} \|g\|_{L^2_{v,I}} \\
&\le C_{\alpha,\beta} \sum_{|\alpha_1|+|\alpha_2|\le|\alpha|} \|\partial^{\alpha_1} f\|_{L^2_{v,I}} \|\partial^{\alpha_2} f\|_{L^2_{v,I}} \|g\|_{L^2_{v,I}}.
\end{aligned}
$$

Here we omitted $\sum_{ij}$ for simplicity of presentation.

(2) Note that when $\alpha = \beta = 0$, we have much simpler estimate: $|Q^{\mathcal{M}}_{ij}| \le C\widetilde{m}$ directly from Lemma 4.1, Lemma 4.3 (2) and Lemma 4.4 (2). Therefore, applying Hölder inequality, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^3\times\mathbb{R}_+} \Gamma_2(f,g)h\,dv\,dI &\le C\int_{\mathbb{R}^3} \|f\|_{L^2_{v,I}} \|g\|_{L^2_{v,I}} \left(\int_{\mathbb{R}^3\times\mathbb{R}_+} \widetilde{m}|h|\,dv\,dI\right) \\
&\le C\|f\|_{L^2_{v,I}} \|g\|_{L^2_{v,I}} \|h\|_{L^2_{v,I}} \\
&\le C \sup_{v,I} |\nu_{v,I} h| \sup_{x} \|f\|_{L^2_{v,I}} \|g\|_{L^2_{v,I}}.
\end{aligned}
$$

We then take L^2_x norm to get the result. The proofs for other terms are similar. □

5.1. *Local existence*

Now, the local existence theorem can be proved by standard arguments (see, e.g., [25]).

Theorem 5.2. *Let $0 \le \theta \le 1$ and $-1/2 < \nu < 1$. Let $F_0 = m + \sqrt{m} f_0 \ge 0$ and f_0 satisfy (1.8). Then there exist $M_0 > 0$, $T_* > 0$, such that if $\mathcal{E}(0) \le \frac{M_0}{2}$, then there is a unique solution $f(t,x,v,I)$ to (2.12) defined on $[0,T_*)$, such that:*

(1) *The high order energy $\mathcal{E}\big(f(t)\big)$ is continuous in $[0,T*)$ and uniformly bounded:*

$$
\sup_{0\le t\le T_*} \mathcal{E}\big(f(t)\big) \le M_0.
$$

(2) *The distribution function is non-negative on $[0,T_*)$:*

$$
F(t,x,v,I) = m + \sqrt{m} f(t,x,v,I) \ge 0.
$$

(3) *The conservation laws (1.8) hold for all $[0,T_*)$.*

Proof. We consider the following scheme:

$$
\partial_t F^{n+1} + v\cdot\nabla_x F^{n+1} = \frac{\rho^n T^n_\delta}{1-\nu+\theta\nu}\left\{\mathcal{M}_{\nu,\theta}(F^n) - F^{n+1}\right\}, \tag{5.2}
$$

with

$$\mathcal{M}_{\nu,\theta}(F^n) = \frac{\rho^n \Lambda_\delta}{\sqrt{\det(2\pi \mathcal{T}^n_{\nu,\theta})\{T^n_\theta\}^{\delta/2}}} \exp\left(\frac{1}{2}(v-U^n)\left\{\mathcal{T}^n_{\nu,\theta}\right\}^{-1}(v-U^n) - \frac{I^{2/\delta}}{T^n_\theta}\right),$$

where ρ^n, U^n, $\mathcal{T}^n_{\nu,\theta}$ and T^n_θ denote the local density, bulk velocity and the temperature tensor associated with $F^n = m + \sqrt{m} f^n$. Making use of Lemma 5.1, the local existence follows from a standard argument. (See [25].) The only difference from the usual proof is that the strict positiveness of $\mathcal{T}^n_{\nu,\theta}$ and T^n_δ should be secured in each step, so that $\mathcal{M}_{\nu,\theta}(F^n)$ is well-defined. This is guaranteed by Lemma 4.1 (5) and Lemma 4.4. □

6. Micro–macro system

In this section, we study the micro–macro system of (2.12) to fill up the degeneracy in the dissipation estimates in Theorem 3.1. The dichotomy in the dissipation estimate observed in Theorem 3.1 indicates that we should employ two different sets of micro–macro decomposition.

6.1. *Micro–macro system I* $(0 < \theta \le 1)$

Define

$$a(x,t) = \int_{\mathbb{R}^3_v \times \mathbb{R}^+_I} f\sqrt{m}\,dv\,dI,$$

$$b_i(x,t) = \int_{\mathbb{R}^3_v \times \mathbb{R}^+_I} f v_i \sqrt{m}\,dv\,dI \quad (i = 1, 2, 3),$$

$$c(x,t) = \int_{\mathbb{R}^3_v \times \mathbb{R}^+_I} f\left(\frac{(|v|^2-3)+(2I^{2/\delta}-\delta)}{\sqrt{2(3+\delta)}}\right)\sqrt{m}\,dv\,dI,$$

so that the polyatomic projection operator P_p is written

$$P_p f = a(x,t)\sqrt{m} + \sum_i b_i(x,t) v_i \sqrt{m} + c(x,t)\left(\frac{(|v|^2-3)+(2I^{2/\delta}-\delta)}{\sqrt{2(3+\delta)}}\right)\sqrt{m}.$$

Since $L_{\nu,\theta}\{P_p f\} = 0$ for $0 < \theta \le 1$ by Proposition 3.1, the linearized polyatomic BGK model (2.12) is decomposed into the macroscopic part and the microscopic parts as follows:

$$\{\partial_t + v\cdot\nabla_x\}\{P_p f\} = -\{\partial_t + v\cdot\nabla_x\}\{(I-P_p)f\} + L_{\nu,\theta}\{(I-P_p)f\} + \Gamma_{\nu,\theta}(f).$$

We then expand the l.h.s. and r.h.s. with respect to the following basis $(1 \le i, j \le 3)$:

$$\left\{\sqrt{m},\ v_i\sqrt{m},\ v_iv_j\sqrt{m},\ v_i^2\sqrt{m},\ v_i\left\{|v|^2 + (2I^{2/\delta}-\delta)\right\}\sqrt{m},\ (2I^{2/\delta}-\delta)\sqrt{m}\right\}. \qquad (6.1)$$

Comparing the coefficients on both sides, we derive the following micro–macro system:

$$\begin{aligned}
\partial_t a - 3A_\delta \partial_t c &= \ell_a + h_a, \\
\partial_t b_i + \partial_{x_i} a - 3A_\delta \partial_{x_i} c &= \ell_{abi} + h_{abi}, \\
\partial_{x_i} b_j + \partial_{x_j} b_i &= \ell_{ij} + h_{ij} \quad (i \neq j), \\
A_\delta \partial_t c + \partial_{x_i} b_i &= \ell_{bci} + h_{bci}, \\
A_\delta \partial_{x_i} c &= \ell_{ci} + h_{ci}, \\
A_\delta \partial_t c &= \ell_{ct} + h_{ct},
\end{aligned} \tag{6.2}$$

for $i, j = 1, 2, 3$. Here, $A_\delta = 1/\sqrt{2(3+\delta)}$, and ℓ_a, ℓ_{abci}, ℓ_{ij}, ℓ_{bci}, ℓ_{ci} and ℓ_{ct} denote the coefficients of projection of $-\{\partial_t + v \cdot \nabla_x\}\{(I - P_m)f\} + L_{\nu,\theta}\{(I - P_m)f\}$ onto the basis (6.1), and $h_a, h_{abci}, h_{ij}, h_{bci}, h_{ci}$ and h_{ct} are the projection of $\Gamma_{\nu,\theta}(f)$ onto (6.1). Adding the last two equations to the first and second line, we get

$$\begin{aligned}
\partial_t a &= (\ell_a + h_{ct}) + 3\{\ell_{ct} + h_{ct}\}, \\
\partial_t b_i + \partial_{x_i} a &= \left\{\ell_{abi} + h_{abi}\right\} + 3\{\ell_{ci} + h_{ci}\}, \\
\partial_{x_i} b_j + \partial_{x_j} b_i &= \ell_{ij} + h_{ij} \quad (i \neq j), \\
A_\delta \partial_t c + \partial_{x_i} b_i &= \ell_{bci} + h_{bci}, \\
A_\delta \partial_{x_i} c &= \ell_{ci} + h_{ci}, \\
A_\delta \partial_t c &= \ell_{ct} + h_{ct}.
\end{aligned} \tag{6.3}$$

The first 5 lines are, up to constant multiplication on the l.h.s. and additional slight complication in r.h.s., identical to the micro–macro system derived in [24,25] and the last line is easy to estimate. Hence, following the same line of argument, we arrive at

$$\sum_{|\alpha| \le N} \left\{ \|\partial^\alpha a\|^2_{L^2_x} + \|\partial^\alpha b\|^2_{L^2_x} + C_\delta \|\partial^\alpha c\|^2_{L^2_x} \right\} \le C \sum_{|\alpha| \le N-1} \|\partial^\alpha (\ell_{\nu,\theta} + h_{\nu,\theta})\|^2_{L^2_x}. \tag{6.4}$$

Note that we have slightly abused the notation to denote $\ell_{\nu,\theta} = (\ell_a, \ell_{abi}, \ell_{ij}, \ell_{bci}, \ell_{ci}, \ell_{ct})$ and $h_{\nu,\theta} = (h_a, h_{abci}, h_{ij}, h_{bci}, h_{ci}, h_{ct})$. Now, $\ell_{\nu,\theta}$ and $h_{\nu,\theta}$ can be controlled in a standard way using Lemma 5.1 as follows:

$$\sum_{|\alpha| \le N-1} \|\partial^\alpha (\ell_{\nu,\theta} + h_{\nu,\theta})\|^2_{L^2_x} \le C \sum_{|\alpha| \le N} \|(I - P_p)\partial^\alpha f\|^2_{L^2_{x,v,I}} + C\sqrt{M_0} \sum_{|\alpha| \le N} \|\partial^\alpha f\|^2_{L^2_{x,v,I}}.$$

Combining this with (6.4), we derive

$$\begin{aligned}
\sum_{|\alpha| \le N} \|\partial^\alpha P_p f\|^2_{L^2_{x,v,I}} &\le \sum_{|\alpha| \le N} \left\{ \|\partial^\alpha a\|^2_{L^2_{x,v,I}} + \|\partial^\alpha b\|^2_{L^2_{x,v,I}} + \|\partial^\alpha c\|^2_{L^2_{x,v,I}} \right\} \\
&\le C \sum_{|\alpha| \le N} \|\partial^\alpha (I - P_p) f\|^2_{L^2_{x,v,I}} + C\sqrt{M_0} \sum_{|\alpha| \le N} \|\partial^\alpha f\|^2_{L^2_{x,v,I}},
\end{aligned}$$

which gives

$$\sum_{|\alpha|\leq N}\|P_p\partial^\alpha f\|^2_{L^2_{x,v,I}}\leq C\sum_{|\alpha|\leq N}\|(I-P_p)\partial^\alpha f\|^2_{L^2_{x,v,I}}. \tag{6.5}$$

Then, we conclude from Proposition 3.1 and (6.5) that there exists $C_{\nu,\theta}>0$ such that

$$\sum_{|\alpha|\leq N}\langle L_{\nu,\theta}\partial^\alpha f,\partial^\alpha f\rangle_{L^2_{x,v,I}}\leq -C_{\nu,\theta}\sum_{|\alpha|\leq N}\|\partial^\alpha f(t)\|^2_{L^2_{x,v,I}} \tag{6.6}$$

for sufficiently small $\mathcal{E}(t)$.

6.2. *Micro–macro system II* ($\theta=0$)

Recalling Proposition 3.1, we see that in this case, (2.12) should be decomposed with respect to the monatomic-like projection $P_m f$. In view of this observation, we define

$$\begin{aligned}
a(x,t)&=\int_{\mathbb{R}^3_v\times\mathbb{R}^+_I} f\sqrt{m}\,dv\,dI,\\
b_i(x,t)&=\int_{\mathbb{R}^3_v\times\mathbb{R}^+_I} f v_i\sqrt{m}\,dv\,dI \quad (i=1,2,3),\\
c(x,t)&=\int_{\mathbb{R}^3_v\times\mathbb{R}^+_I} f\left(\frac{|v|^2-3}{\sqrt{6}}\right)\sqrt{m}\,dv\,dI,\\
d(x,t)&=\int_{\mathbb{R}^3_v\times\mathbb{R}^+_I} f\left(\frac{2I^{2/\delta}-\delta}{\sqrt{2\delta}}\right)\sqrt{m}\,dv\,dI,
\end{aligned}$$

to write

$$P_m f=a\sqrt{m}+\sum_i b_iv_i\sqrt{m}+c\left(\frac{|v|^2-3}{\sqrt{6}}\right)\sqrt{m}+d\left(\frac{2I^{2/\delta}-\delta}{\sqrt{2\delta}}\right)\sqrt{m}.$$

We recall from Proposition 3.1 that $L_{\nu,0}(P_m f)=0$, and divide (2.12) into the macroscopic part and the microscopic part as follows:

$$\{\partial_t+v\cdot\nabla_x\}\{P_m f\}=-\{\partial_t+v\cdot\nabla_x\}\{(I-P_m)f\}+L_{\nu,0}\{(I-P_m)f\}+\Gamma_{\nu,0}(f).$$

Comparing coefficients corresponding to the following basis:

$$\left\{\sqrt{m},\,v_i\sqrt{m},\,v_iv_j\sqrt{m},\,v_i^2\sqrt{m},\,v_i|v|^2\sqrt{m},\,\left(2I^{2/\delta}-\delta\right)\sqrt{m},\,\left(2I^{2/\delta}-\delta\right)v_i\sqrt{m}\right\}, \tag{6.7}$$

we obtain $(i, j = 1, 2, 3)$:

$$\begin{aligned}\partial_t a - 3/\sqrt{6}\partial_t c &= \ell_a + h_a,\\ \partial_t b_i + \partial_{x_i} a - 3/\sqrt{6}\partial_{x_i} c &= \ell_{abci} + h_{abci},\\ \partial_{x_i} b_j + \partial_{x_j} b_i &= \ell_{ij} + h_{ij},\\ \partial_{x_i} b_i + 1/\sqrt{6}\partial_t c &= \ell_{bci} + h_{bci},\\ 1/\sqrt{6}\partial_{x_i} c &= \ell_{ci} + h_{ci},\\ \partial_t d &= \ell_{dt} + h_{dt},\\ \partial_{x_i} d &= \ell_{dxi} + h_{dxi},\end{aligned} \tag{6.8}$$

where ℓ_a, ℓ_{abci}, ℓ_{ij}, ℓ_{bci}, ℓ_{ci}, ℓ_{dt} and ℓ_{dxi} are obtained by taking the inner product of $-\{\partial_t + v\cdot\nabla_x\}\{(I-P_m)f\} + L_{\nu,0}\{(I-P_m)f\}$ with the basis in (6.7), and h_a, h_{abci}, h_{ij}, h_{bci}, h_{ci}, h_{dt} and h_{dxi} are the inner product of $\Gamma_{\nu,0}(f)$ with (6.7). Setting $\tilde{a} = a - 3/\sqrt{6}c$ and $\tilde{c} = 1/\sqrt{6}c$, we derive from (6.8)

$$\begin{aligned}\partial_t \tilde{a} &= \ell_a + h_a,\\ \partial_t b_i + \partial_{x_i}\tilde{a} &= \ell_{abi} + h_{abi},\\ \partial_{x_i} b_j + \partial_{x_j} b_i &= \ell_{ij} + h_{ij},\\ \partial_{x_i} b_i + \partial_t \tilde{c} &= \ell_{bc1} + h_{bc1},\\ \partial_{x_i}\tilde{c} &= \ell_{ci} + h_{ci},\\ \partial_t d &= \ell_{dt} + h_{dt},\\ \partial_{x_i} d &= \ell_{dxi} + h_{dxi}.\end{aligned} \tag{6.9}$$

Except for the last two lines, which are decoupled from the other equations, and therefore, estimated easily, this is identical to the micro–macro system for the usual Boltzmann equation or BGK model. Therefore, we can derive

$$\sum_{|\alpha|\le N}\left\{\|\partial^\alpha\widetilde{a}\|^2_{L^2_x} + \|\partial^\alpha b\|^2_{L^2_x} + \|\partial^\alpha\widetilde{c}\|^2_{L^2_x} + \|\partial^\alpha d\|^2_{L^2_x}\right\} \le C\sum_{|\alpha|\le N-1}\|\partial^\alpha(\ell_{\nu,0}+h_{\nu,0})\|^2_{L^2_x}.$$

Here we used the simplified notation again: $\ell_{\nu,0} = (\ell_a, \ell_{abci}, \ell_{ij}, \ell_{bci}, \ell_{ci}, \ell_{dt}, \ell_{dxi})$ and $h_{\nu,0} = (h_a, h_{abci}, h_{ij}, h_{bci}, h_{ci}, h_{dt}, h_{dxi})$. We now take $\frac{3\sqrt{6}}{10} < \varepsilon^2 < \frac{2}{\sqrt{6}}$ and set

$$C_\varepsilon = \min\left\{(1-\sqrt{6}\varepsilon^2),\ 3/5 - \sqrt{6}/(2\varepsilon^2)\right\} > 0$$

to obtain

$$C_\varepsilon\sum_{|\alpha|\le N}\left\{\|\partial^\alpha a\|^2_{L^2_x} + \|\partial^\alpha b\|^2_{L^2_x} + \|\partial^\alpha c\|^2_{L^2_x} + \|\partial^\alpha d\|^2_{L^2_x}\right\} \le C\sum_{|\alpha|\le N-1}\|\partial^\alpha(\ell_{\nu,0}+h_{\nu,0})\|^2_{L^2_x}.$$

Then, by a similar argument as in the previous case, we can control $P_m f$ by $(I - P_m)f$:

$$\sum_{|\alpha|\leq N} \|P_m\partial^\alpha f\|^2_{L^2_{x,v,I}} \leq C\sum_{|\alpha|\leq N} \|(I-P_m)\partial^\alpha f\|^2_{L^2_{x,v,I}}, \tag{6.10}$$

which, combined with the dissipation estimate in Theorem 3.1 (2), implies

$$\sum_{|\alpha|\leq N} \langle L_{\nu,0}\partial^\alpha f, \partial^\alpha f\rangle_{L^2_{x,v,I}} \leq -C_\nu \sum_{|\alpha|\leq N} \|\partial^\alpha f(t)\|^2_{L^2_{x,v,I}}. \tag{6.11}$$

7. Proof of Theorem 1.1

We have derived all the necessary estimates to close the energy estimate. Let f be the smooth local in time solution obtained in Theorem 5.2. Take derivatives on x, t and I of (2.12):

$$\partial_t\partial^\alpha f + v\cdot\nabla_x\partial^\alpha f = L_{\nu,\theta}\partial^\alpha f + \partial^\alpha\Gamma_{\nu,\theta}(f),$$

and take inner product with $\partial^\alpha f$ to get

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f\|^2_{L^2_{x,v,I}} \leq \langle L_{\nu,\theta}\partial^\alpha f, \partial^\alpha f\rangle_{L^2_{x,v,I}} + \langle \partial^\alpha\Gamma_{\nu,\theta}(f), \partial^\alpha f\rangle_{L^2_{x,v,I}}.$$

Making use of the coercivity estimate in the previous section yields

$$E^\alpha_0: \quad \frac{1}{2}\frac{d}{dt}\|\partial^\alpha f\|^2_{L^2_{x,v,I}} + C\sum_{|\alpha|\leq N}\|\partial^\alpha f\|^2_{L^2_{x,v,I}} \leq C\sqrt{\mathcal{E}(t)}\mathcal{D}(t),$$

where $\mathcal{D}(t) = \sum \|\partial^\alpha f\|^2_{L^2_{x,v,I}}$. For the energy estimate involving velocity derivatives, we apply ∂^α_β to (2.12)

$$\left\{\partial_t + v\cdot\nabla_x + a_{\nu,\theta}\right\}\partial^\alpha_\beta f = \sum_{|\beta_1|\neq 0}\partial_{\beta_1}v\cdot\nabla_x\partial^\alpha_{\beta-\beta_1}f + \partial_\beta P_{\nu,\theta}\partial^\alpha f + \partial^\alpha_\beta\Gamma_{\nu,\theta}(f,f),$$

where $a_{\nu,\theta} = 1/(1-\nu+\nu\theta)$. Then, take inner product with $\partial^\alpha_\beta f$ and use Hölder inequality with Lemma 5.1 to derive

$$\begin{aligned} E^\alpha_\beta: \quad \frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta f\|^2_{L^2_{x,v,I}} + a_{\nu,\theta}\|\partial^\alpha_\beta f\|^2_{L^2_{x,v,I}} &\leq C\sum_i \|\partial^{\alpha+e_i}_{\beta-e_i} f\|_{L^2_{x,v,I}}\|\partial^\alpha_\beta f\|_{L^2_{x,v,I}} \\ &\quad + C\|\partial^\alpha f\|_{L^2_{x,v,I}}\|\partial^\alpha_\beta f\|_{L^2_{x,v,I}} + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t), \end{aligned}$$

where e_i $(i = 1,2,3)$ is the standard basis of $\mathbb{R}^3_x$. By Young's inequality, we can split the first two terms in the r.h.s. as

$$\begin{aligned} \|\partial^{\alpha+e_i}_{\beta-e_i} f\|_{L^2_{x,v,I}}\|\partial^\alpha_\beta f\|_{L^2_{x,v,I}} &\leq C_\varepsilon\|\partial^{\alpha+e_i}_{\beta-e_i} f\|^2_{L^2_{x,v,I}} + \varepsilon\|\partial^\alpha_\beta f\|^2_{L^2_{x,v,I}} \\ \|\partial^\alpha f\|_{L^2_{x,v,I}}\|\partial^\alpha_\beta f\|_{L^2_{x,v,I}} &\leq C_\varepsilon\|\partial^\alpha f\|^2_{L^2_{x,v,I}} + \varepsilon\|\partial^\alpha_\beta f\|^2_{L^2_{x,v,I}} \end{aligned}$$

whose ε terms can be absorbed in the production term in the l.h.s. to get

$$E^\alpha_\beta: \quad \frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta f\|^2_{L^2_{x,v,I}} + \frac{1}{2}a_{\nu,\theta}\|\partial^\alpha_\beta f\|^2_{L^2_{x,v,I}} \le C_\varepsilon \sum_i \|\partial^{\alpha+e_i}_{\beta-e_i} f\|^2_{L^2_{x,v,I}} + C_\varepsilon \|\partial^\alpha f\|^2_{L^2_{x,v,I}} + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

We then observe that the r.h.s. of $\sum_{|\beta|=m+1} E^\alpha_\beta$ can be absorbed into the production terms in the lower order estimate: $C_m \sum_{|\beta|\le m} E^\alpha_\beta$ for sufficiently large C_m. This observation enables one to find constants C^1_m, C^2_m inductively such that

$$\sum_{\substack{|\alpha|+|\beta|\le N,\\ |\beta|\le m}} \left\{ C^1_m \frac{d}{dt}\|\partial^\alpha_\beta f\|^2_{L^2_{x,v,I}} + C^2_m \|\partial^\alpha_\beta f\|^2_{L^2_{x,v,I}} \right\} \le C_N \sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

Now, the standard continuity argument gives the global existence for (2.12) [25]. This completes the proof.

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