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Boundedness and stabilization in a two-species chemotaxis-competition system with signal-dependent diffusion and sensitivity

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Abstract

We consider the following two-species chemotaxis-competition system with signal-dependent diffusion and sensitivity

$$\begin{cases} u_{1t} = \nabla\cdot(d_1(v)\nabla u_1) - \nabla\cdot(\chi_1(v)u_1\nabla v) + \mu_1 u_1(1-u_1-a_1u_2), & x\in\Omega,\ t>0,\\ u_{2t} = \nabla\cdot(d_2(v)\nabla u_2) - \nabla\cdot(\chi_2(v)u_2\nabla v) + \mu_2 u_2(1-u_2-a_2u_1), & x\in\Omega,\ t>0,\\ v_t = \Delta v + b_1u_1 + b_2u_2 - v, & x\in\Omega,\ t>0,\\ u_1(x,0)=u_{10}(x), u_2(x,0)=u_{20}(x), v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{$*$}$$

in a bounded smooth domain $\Omega\subset\mathbb{R}^2$ with homogeneous Neumann boundary conditions, where μ_i, a_i, b_i are positive constants for $i=1,2$, and the functions $d_i(v)$, $\chi_i(v)$ satisfy the following assumptions:

- $(d_i(v),\chi_i(v))\in[C^2[0,\infty)]^2$ with $d_i(v),\chi_i(v)>0$ for all $v\ge 0$, $d_i'(v)<0$ and $\lim\limits_{v\to\infty} d_i(v)=0$;
- $\lim\limits_{v\to\infty}\frac{\chi_i(v)}{d_i(v)}$ and $\lim\limits_{v\to\infty}\frac{d_i'(v)}{d_i(v)}$ exist.

Since $\lim\limits_{v\to\infty} d_i(v)=0$ for $i=1,2$, the diffusion may degenerate, which makes the analysis of system ($*$) much more difficult. To overcome this problem, we shall use the functions $d_i(v)$ as weight functions and

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then employ the weighted energy estimates to establish the boundedness of solutions. Furthermore, by constructing some appropriate Lyapunov functionals, we show that

- If $a_1, a_2 \in (0,1)$ and μ_1, μ_2 are large enough, then the solution (u_1, u_2, v) exponentially converges to $(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}, \frac{b_1+b_2-a_1b_1-a_2b_2}{1-a_1a_2})$ as $t \to \infty$.
- If $a_1 \geq 1$, $a_2 \in (0,1)$ and μ_2 is large enough, the solution (u_1, u_2, v) converges to $(0, 1, b_2)$ as $t \to \infty$ with algebraic decay when $a_1 = 1$, and with exponential decay when $a_1 > 1$.

MSC: 35A01; 35B40; 35B44; 35K57; 35Q92; 92C17

Keywords: Chemotaxis; Boundedness; Stabilization; Signal-dependent diffusion; Signal-dependent sensitivity

1. Introduction

In this paper, we shall consider the following two species chemotaxis-competition system with signal-dependent diffusion and sensitivity

$$\begin{cases} u_{1t} = \nabla \cdot (d_1(v)\nabla u_1) - \nabla \cdot (\chi_1(v)u_1\nabla v) + \mu_1 u_1(1 - u_1 - a_1u_2), & x \in \Omega, \quad t > 0, \\ u_{2t} = \nabla \cdot (d_2(v)\nabla u_2) - \nabla \cdot (\chi_2(v)u_2\nabla v) + \mu_2 u_2(1 - u_2 - a_2u_1), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v + b_1u_1 + b_2u_2 - v, & x \in \Omega, \quad t > 0, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u_1(x,0) = u_{10}(x), u_2(x,0) = u_{20}(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $u_i(x,t)$ with $i = 1,2$ denote the densities of two populations, respectively, and $v(x,t)$ accounts for the concentration of chemical substance. The parameters μ_i, a_i, b_i are positive constants for $i = 1,2$. The terms $\nabla \cdot (d_i(v)\nabla u_i)$ describe the diffusion of two species with coefficient $d_i(v)$ respectively. $-\nabla \cdot (\chi_i(v)u_i\nabla v)$ stand for the chemotaxis with coefficient $\chi_i(v)$ for $i = 1,2$, where the coefficient of diffusion and chemotaxis may depend on the chemical concentration v.

When $d_i(v) = d_i > 0$ and $\chi_i(v) = \chi_i > 0$ are constants, system (1.1) becomes the following two-species chemotaxis system

$$\begin{cases} u_{1t} = d_1\Delta u_1 - \chi_1\nabla \cdot (u_1\nabla v) + \mu_1u_1(1 - u_1 - a_1u_2), \\ u_{2t} = d_2\Delta u_2 - \chi_2\nabla \cdot (u_2\nabla v) + \mu_2u_2(1 - u_2 - a_2u_1), \\ v_t = \Delta v + b_1u_1 + b_2u_2 - v, \end{cases} \tag{1.2}$$

which can be viewed as the generalized classical one species chemotaxis model [32,34] (see the review paper [3] for details). System (1.2) is much more difficult to study than the single species chemotaxis model due to the influence of chemotaxis, diffusion, and the Lotka–Volterra kinetics. We recall some results on the system (1.2). First, the global existence and blow-up of solution of system (1.2) with $\mu_1 = \mu_2 = 0$ has been studied [4,5,8–10,19,36,37]. For the two-species chemotaxis model (1.2) with Lotka–Volterra-type competition (i.e., $\mu_1, \mu_2 > 0$), the situation

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becomes more complicated. If the third equation of system (1.2) is replaced by the elliptic equation $0 = \Delta v + b_1u_1 + b_2u_2 - v$, based on some elaborate comparison techniques, under some appropriate conditions on the parameters μ_i, χ_i for $i = 1, 2$, the global existence and stability of steady states including either competitive exclusion or coexistence have been studied analytically [6,28,33]. More precisely, the solution (u_1, u_2, v) will converge to the coexistence steady state $(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}, \frac{b_1+b_2-a_1b_1-a_2b_2}{1-a_1a_2})$ when $a_1 < 1, a_2 < 1$ (see [6,33]), while when $a_1 > 1$, $a_2 < 1$, the competitive exclusion occurs, which means that $(u_1, u_2, v) \to (0, 1, b_2)$ as $t \to \infty$ in [28]. For the full parabolic system (1.2), the comparison techniques can not be used anymore, Bai & Winkler [2] first established the global existence of classical solution in two dimensions by deriving suitable *a priori* estimate, and then obtained the global stabilization and decay rate of the solutions by means of the construction of suitable energy functionals. Moreover, the boundedness and asymptotic behavior of the two-species chemotaxis model with signal consumption cases also were studied [7,12,16] recently.

When $d_i(v) = d_i > 0$, $\chi_i(v)$ is signal-dependent sensitivity, the global existence and large time behavior of solution for the two species chemotaxis model with competitive kinetics have been studied [21–24]. However, to the best our knowledge, the existed literatures do not provided any qualitative information for the two species chemotaxis-competition model with signal-dependent diffusion and sensitivity as described in system (1.1). If $a_1 = b_2 = 0$, the second species u_2 is decoupled from the system (1.1) and the first and third equation of system (1.1) comprises a one species chemotaxis model with signal-dependent diffusion and sensitivity

$$\begin{cases} u_{1t} = \nabla \cdot (d_1(v)\nabla u_1) - \nabla \cdot (\chi_1(v)u_1\nabla v) + \mu_1u_1(1-u_1), \\ v_t = \Delta v + b_1u_1 - v, \end{cases} \tag{1.3}$$

which has been proposed in [11] to describe the stripe pattern driven by the density-suppressed motility in the case of $\chi_1(v) = -d_1'(v) > 0$. For system (1.3) with $\chi_1(v) = -d_1'(v) > 0$, if $\mu_1 = 0$ and $d_1(v) = \frac{c_0}{v^k}$, $c_0 > 0, k > 0$, Yoon and Kim [35] obtained the existence of the boundedness solution in any dimensions provided $c_0 > 0$ is small. On the other hand, if $d_1(v)$ has a positive lower and upper bound, Tao and Winkler [31] proved the existence of global classical solution in two dimensions and global weak solutions in higher dimensions ($n \geq 3$). If $\mu_1 > 0$, the boundedness, stabilization and pattern formation have been studied in [14] without small assumption in [35] or the lower-upper bound assumption in [31]. Recently, Jin [13] removed the structure assumption $\chi_1(v) = -d_1'(v) > 0$ and obtained the boundedness and large time behavior of solutions for system (1.3).

As recalled above, not many mathematical results of the two species chemotaxis-competition model with signal-dependent diffusion and sensitivity are available up to date. Our goals are to investigate the following two major questions:

(Q1) Whether the two species interaction itself is sufficient to preclude the population overcrowding in spite of the aggregation effect of the chemotaxis?
(Q2) Whether the two species can coexist, exclude or extinct and how does the signal-dependent diffusion and sensitivity affect the dynamics?

To answer the above two questions, the global boundedness and the asymptotic behavior of solutions should be explored. We assume the functions $d_i(z)$, $\chi_i(z) > 0$ with $i = 1, 2$ satisfy the following hypotheses

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(H1) $(d_i(z), \chi_i(z)) \in [C^2([0,\infty)]^2$ with $d_i(z), \chi_i(z) > 0$ for all $z \geq 0$, $d_i'(z) < 0$ and $\lim\limits_{z\to\infty} d_i(z) = 0$.

(H2) $\lim\limits_{z\to\infty} \frac{\chi_i(z)}{d_i(z)}$ and $\lim\limits_{z\to\infty} \frac{d_i'(z)}{d_i(z)}$ exist.

Different from the two-species chemotaxis-competition model with linear diffusion [6,21–24,28,33], the signal-dependent diffusion may degenerate due to $d_i'(v) < 0$ for $i = 1, 2$, which may cause many difficulties to obtain the uniform-in-time bound of solution. To overcome this problem, we use $d_i(v) (i = 1, 2)$ as the weight functions in the energy estimates motivated by the ideas in [13,14]. More precisely, using the weighted diffusive dissipation along with the competition effects, we derive the L^2-boundedness of u_1 and u_2, and then with the help of parabolic regularity theory, obtain the boundedness of v from the third equation of system (1.1), which removes the possibility of degenerate. At last, we apply the Moser iteration method to show the boundedness of u_i $(i = 1, 2)$ and the existence of global classical solutions. Moreover, by constructing Lyapunov functionals, we can show the large time behavior of solutions. The main results are stated as follows.

Theorem 1.1 *(Boundedness). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and the parameters a_i, b_i, μ_i be positive constants for $i = 1, 2$. Suppose that the hypotheses (H1)–(H2) hold. Assume $(u_{10}, u_{20}, v_0) \in [W^{1,p}(\Omega)]^3$ with some $p > 2$ and $u_{10}, u_{20}, v_0 \geq 0 (\not\equiv 0)$. Then the system (1.1) has a unique global classical solution $(u_1, u_2, v) \in [C^0([0,\infty) \times \bar{\Omega}) \cap C^{2,1}((0,\infty) \cap \times \bar{\Omega}) \cap L^\infty_{loc}([0,\infty); W^{1,p}(\Omega))]^3$ satisfying $u_1, u_2, v > 0$ for all $t > 0$. Moreover, the solution satisfies*

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \ \text{ for all } \ t > 0,$$

where $C > 0$ is a constant independent of t.

Theorem 1.2 *(Stabilization and convergence rate). Let (u_1, u_2, v) be the solution obtained in Theorem 1.1. Then we have the following results:*

(1) If $a_1, a_2 \in (0, 1)$ and μ_1, μ_2 satisfy

$$\max_{0\leq z\leq\infty} \left\{ \frac{u_1^*\chi_1^2(z)}{4a_1\mu_1 d_1(z)} + \frac{u_2^*\chi_2^2(z)}{4a_2\mu_2 d_2(z)}\xi_1 \right\} < \frac{4\xi_1 - a_1a_2(1+\xi_1)^2}{a_1b_1^2\xi_1 + a_2b_2^2 - a_1a_2b_1b_2(1+\xi_1)}, \tag{1.4}$$

where $\xi_1 > 0$ is a constant fulfilling $4\xi_1 - a_1a_2(1+\xi_1)^2 > 0$ and

$$u_1^* = \frac{1-a_1}{1-a_1a_2}, \quad u_2^* = \frac{1-a_2}{1-a_1a_2}, \quad v^* = \frac{b_1+b_2-a_1b_1-a_2b_2}{1-a_1a_2},$$

then for all $t > 0$, the classical solution (u_1, u_2, v) of system (1.1) satisfies

$$\|u_1(\cdot,t) - u_1^*\|_{L^\infty(\Omega)} + \|u_2(\cdot,t) - u_2^*\|_{L^\infty(\Omega)} + \|v(\cdot,t) - v^*\|_{L^\infty(\Omega)} \leq C_1 e^{-\lambda_1 t},$$

where C_1 and λ_1 are positive constants independent of t.

(2) If $a_1 > 1$ and $a_2 \in (0,1)$ and suppose that for some $a_1^* \in (1, a_1]$ and $a_1^* a_2 < 1$, we have

$$\mu_2 > \max_{0 \le z \le \infty} \frac{(a_1^* b_1^2 \xi_2 + a_2 b_2^2 - a_1^* a_2 b_1 b_2 (1+\xi_2))\xi_2 \chi_2^2(z)}{4a_2(4\xi_2 - a_1^* a_2 (1+\xi_2)^2) d_2(z)} \tag{1.5}$$

with some constant $\xi_2 > 0$ satisfying $4\xi_2 - a_1^* a_2 (1+\xi_2)^2 > 0$, then it holds that

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|v(\cdot,t) - b_2\|_{L^\infty(\Omega)} \le C_2 e^{-\lambda_2 t}$$

for all $t > 0$, where C_2 and λ_2 are positive constants independent of t.

(3) If $a_1 = 1$, $a_2 \in (0,1)$ and μ_2 satisfies (1.5) with $a_1^* = 1$, then there exist constant $C_3, \lambda_3 > 0$ independent of t such that for all $t > 0$

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|v(\cdot,t) - b_2\|_{L^\infty(\Omega)} \le C_3(t+1)^{-\lambda_3}.$$

Remark 1.1. When $d_i(z)$ and $\chi_i(z)$ are constants for $i = 1, 2$, Theorem 1.2 covers the results derived by Bai and Winkler in [2].

Notation. Without confusion, we use $\int_\Omega f$ and $\int_0^t \int_\Omega f$ to represent $\int_\Omega f(\cdot,t)dx$ and $\int_0^t \int_\Omega f(\cdot,s)dxds$ for short, respectively. Moreover, we denote $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$, and $c_i (i = 1, 2, 3, \cdots)$ stand for generic constants which may alter from line to line.

2. Boundedness of solutions: proof of Theorem 1.1

In this section, we focus on proving the global existence of classical of solution for system (1.1). We first obtain the local existence of solution based on the Schauder fixed point theorem. After that, we shall obtain *a priori* estimates to extend the local solution to the global one. To this end, we shall use the motility function $d_i(v)$ $(i = 1, 2)$ as weight functions to obtain the boundedness of $\|u_1(\cdot,t)\|_{L^2}$ and $\|u_2(\cdot,t)\|_{L^2}$ inspired by the ideas in [13,14]. With the L^2-norm of u_1 and u_2, we can obtain the uniform boundedness of $\|v(\cdot,t)\|_{L^\infty}$ immediately thanks to the parabolic regularity, hence the possibility of degeneration is excluded. Then we can show $\|u_1(\cdot,t)\|_{L^\infty}$ and $\|u_2(\cdot,t)\|_{L^\infty}$ is uniform bounded via conventional methods for chemotaxis-competition models with linear diffusion.

Based on the Schauder fixed point theorem, the local existence of solutions to system (1.1) can be established by the similar arguments as in [14, Lemma 2.1], we omit the details for convenience.

Lemma 2.1 *(Local existence). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and the constants a_i, b_i, μ_i be positive for $i = 1, 2$. Suppose that $(u_{10}, u_{20}, v_0) \in [W^{1,p}(\Omega)]^3$ with some $p > 2$ and $(u_{10}, u_{20}, v_0) \ge 0 (\not\equiv 0)$ and the hypotheses (H1)–(H2) hold. Then there exists $T_{max} \in (0, \infty]$ such that system (1.1) has a unique classical solution (u_1, u_2, v) fulfilling $u_1, u_2, v > 0$ for all $t > 0$ and*

$$\begin{aligned} &u_1 \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})),\\ &u_2 \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})),\\ &v \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})) \cap L^\infty_{loc}([0, T_{max}); W^{1,p}(\Omega)). \end{aligned}$$

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Moreover, either $T_{max} = \infty$ *or*

$$\|u_1(\cdot,t)\|_{L^\infty} + \|u_2(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} \to \infty \ \ as \ \ t \nearrow T_{max}.$$

2.1. Lower order estimates

Lemma 2.2. *Let the assumptions in Lemma 2.1 hold. There exists a constant* $C > 0$ *independent of* t *such that the solution of system (1.1) fulfills*

$$\int_\Omega u_1 + \int_\Omega u_2 \le C, \qquad \forall t \in (0, T_{\max}) \tag{2.1}$$

and

$$\int_t^{t+\tau}\int_\Omega u_1^2 + \int_t^{t+\tau}\int_\Omega u_2^2 \le C, \qquad \forall t \in (0, T_{\max} - \tau),$$

where $\tau = \min\{1, \frac{T_{\max}}{2}\}$.

Proof. The integration of the first equation of system (1.1) with respect to x gives

$$\frac{d}{dt}\int_\Omega u_1 = \mu_1 \int_\Omega u_1 - \mu_1 \int_\Omega u_1^2 - a_1\mu_1 \int_\Omega u_1 u_2, \quad \forall t \in (0, T_{\max}).$$

Since $u_i \ge 0$ for $i = 1, 2$, by using Young's inequality and picking $c_1 = \frac{(\mu_1+1)^2}{2\mu_1}|\Omega|$, we derive that

$$\frac{d}{dt}\int_\Omega u_1 + \int_\Omega u_1 + \frac{\mu_1}{2}\int_\Omega u_1^2 \le c_1, \quad \forall t \in (0, T_{\max}), \tag{2.2}$$

which, together with Gronwall's inequality, leads to

$$\int_\Omega u_1 \le c_2, \quad \forall t \in (0, T_{\max}), \tag{2.3}$$

where $c_2 = c_1 + \|u_{10}\|_{L^1}$. Moreover, integrating (2.2) over $(t, t+\tau)$ for all $t \in (0, T_{\max} - \tau)$ and noting (2.3), we obtain

$$\frac{\mu_1}{2}\int_t^{t+\tau}\int_\Omega u_1^2 \le c_1\tau - \int_\Omega u_1(\cdot, t+\tau) + \int_\Omega u_1(\cdot, t) - \int_t^{t+\tau}\int_\Omega u_1 \le c_1\tau + c_2,$$

which implies

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$$\int_t^{t+\tau}\int_\Omega u_1^2 \le c_3, \quad \forall t\in(0,T_{\max}-\tau), \tag{2.4}$$

where $c_3 = \frac{2(c_1\tau+c_2)}{\mu_1}$. Similarly, from the second equation of system (1.1), we can derive

$$\frac{d}{dt}\int_\Omega u_2 + \int_\Omega u_2 + \frac{\mu_2}{2}\int_\Omega u_2^2 \le c_4, \quad \forall t\in(0,T_{\max}) \tag{2.5}$$

with $c_4 = \frac{(\mu_2+1)^2}{2\mu_2}|\Omega|$. From (2.5), we can find a constant $c_5 = c_4 + \|u_{20}\|_{L^1} > 0$ such that

$$\int_\Omega u_2 \le c_5, \quad \forall t\in(0,T_{\max})$$

and

$$\int_t^{t+\tau}\int_\Omega u_2^2 \le \frac{2(c_4\tau+c_5)}{\mu_2}, \quad \forall t\in(0,T_{\max}-\tau).$$

This together with (2.3) and (2.4), completes the proof. $\square$

Lemma 2.3. *Suppose the assumptions in Lemma 2.1 hold. Then the solution (u_1,u_2,v) of system (1.1) satisfies*

$$\int_\Omega |\nabla v|^2 \le C, \qquad \forall t\in(0,T_{\max}) \tag{2.6}$$

and

$$\int_t^{t+\tau}\int_\Omega |\Delta v|^2 \le C, \qquad \forall t\in(0,T_{\max}-\tau), \tag{2.7}$$

where $\tau = \min\{1, \frac{T_{\max}}{2}\}$ and $C>0$ is a constant independent of t.

Proof. Multiplying the third equation of system (1.1) by $-\Delta v$, then integrating the result with respect to x and using Young's inequality, we end up with

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 &= -\int_\Omega |\Delta v|^2 - b_1\int_\Omega u_1\Delta v - b_2\int_\Omega u_2\Delta v + \int_\Omega v\Delta v\\ &\le -\frac{1}{2}\int_\Omega |\Delta v|^2 + b_1^2\int_\Omega u_1^2 + b_2^2\int_\Omega u_2^2 - \int_\Omega |\nabla v|^2, \qquad \forall t\in(0,T_{max}),\end{aligned}$$

that is

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 + \int_\Omega |\Delta v|^2 + 2\int_\Omega |\nabla v|^2 \le 2b_1^2 \int_\Omega u_1^2 + 2b_2^2 \int_\Omega u_2^2, \qquad \forall t \in (0, T_{max}). \tag{2.8}$$

Integrating the first and second equations of system (1.1) with respect to x, respectively, and using Young's inequality, we end up with

$$\frac{d}{dt}\int_\Omega u_1 + \int_\Omega u_1 + \frac{\mu_1}{2}\int_\Omega u_1^2 \le c_1, \quad \forall t \in (0, T_{\max}) \tag{2.9}$$

and

$$\frac{d}{dt}\int_\Omega u_2 + \int_\Omega u_2 + \frac{\mu_2}{2}\int_\Omega u_2^2 \le c_2, \quad \forall t \in (0, T_{\max}) \tag{2.10}$$

where $c_1 = \frac{(\mu_1+1)^2}{2\mu_1}|\Omega|$ and $c_2 = \frac{(\mu_2+1)^2}{2\mu_2}|\Omega|$. We multiply (2.9) by $\frac{4b_1^2}{\mu_1}$ and (2.10) by $\frac{4b_2^2}{\mu_2}$, respectively, and add the resulting estimates to (2.8), to obtain

$$y' + y + \int_\Omega |\Delta v|^2 \le \frac{4c_1 b_1^2}{\mu_1} + \frac{4c_2 b_2^2}{\mu_2}, \quad \forall t \in (0, T_{\max}) \tag{2.11}$$

where $y = \int_\Omega |\nabla v|^2 + \frac{4b_1^2}{\mu_1}\int_\Omega u_1 + \frac{4b_2^2}{\mu_2}\int_\Omega u_2$. Then applying the Gronwall's inequality to (2.11) and noting (2.1), one has (2.6). Moreover, integrating (2.11) over $(t, t+\tau)$ for all $t \in (0, T_{\max} - \tau)$ and using (2.6), we obtain (2.7). Then the proof of Lemma 2.3 is completed. □

2.2. *Boundedness of u_1, u_2 in L^2 and v in L^∞*

Lemma 2.4. *Let the assumptions in Lemma 2.1 hold. Then it holds that*

$$\|u_1(\cdot,t)\|_{L^2} + \|u_2(\cdot,t)\|_{L^2} + \|v(\cdot,t)\|_{L^\infty} \le C, \qquad \forall t \in (0, T_{max}), \tag{2.12}$$

where the constant $C > 0$ independent of t.

Proof. Based on the hypothesis (H2), we can find two positive constants K_1 and K_2 such that

$$\frac{|d_1'(z)|}{d_1(z)} \le K_1 \ \text{ and } \ \frac{|\chi_1(z)|}{d_1(z)} \le K_2 \ \text{ for all } \ z \ge 0 \ \text{ and } \ t \in (0, T_{max}). \tag{2.13}$$

Noting the facts that $u_1, u_2 > 0$, we apply u_1 as a test function to the first equation of (1.1) and integrate the result over Ω to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u_1^2+\int_\Omega d_1(v)|\nabla u_1|^2+\mu_1\int_\Omega u_1^3+a_1\mu_1\int_\Omega u_1^2u_2\\
&=\int_\Omega \chi_1(v)u_1\nabla u_1\cdot\nabla v+\mu_1\int u_1^2\\
&\le\frac{1}{2}\int_\Omega d_1(v)|\nabla u_1|^2+\frac{1}{2}\int_\Omega \frac{\chi_1^2(v)}{d_1(v)}u_1^2|\nabla v|^2+\frac{\mu_1}{2}\int_\Omega u_1^3+\frac{16\mu_1}{27}|\Omega|
\end{aligned}
$$

for all $t\in(0,T_{max})$, that is

$$
\frac{d}{dt}\int_\Omega u_1^2+\int_\Omega d_1(v)|\nabla u_1|^2+\mu_1\int_\Omega u_1^3\le\int_\Omega \frac{\chi_1^2(v)}{d_1(v)}u_1^2|\nabla v|^2+c_1,\quad \forall t\in(0,T_{max}), \tag{2.14}
$$

where $c_1=\frac{32\mu_1}{27}|\Omega|$. Motivated by the ideas in [13,14], we shall use the term $\int_\Omega d_1(v)|\nabla u_1|^2$ to control $\int_\Omega \frac{\chi_1^2(v)}{d_1(v)}u_1^2|\nabla v|^2$. To this end, we first note that

$$
\nabla(d_1^{\frac{1}{2}}(v)u_1)=d_1^{\frac{1}{2}}(v)\nabla u_1+\frac{1}{2}\frac{d_1'(v)}{d_1^{\frac{1}{2}}(v)}u_1\nabla v,\qquad \forall t\in(0,T_{max}),
$$

which gives

$$
\frac{1}{2}|\nabla(d_1^{\frac{1}{2}}(v)u_1)|^2-\frac{1}{4}\frac{|d_1'(v)|^2}{d_1(v)}u_1^2|\nabla v|^2\le d_1(v)|\nabla u_1|^2,\qquad \forall t\in(0,T_{max}) \tag{2.15}
$$

by noting the inequality $\frac{1}{2}A^2-B^2\le|A-B|^2$. Substituting (2.15) into (2.14), and using the fact (2.13), one derives

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u_1^2+\frac{1}{2}\int_\Omega |\nabla(d_1^{\frac{1}{2}}(v)u_1)|^2+\mu_1\int_\Omega u_1^3\\
&\le\frac{1}{4}\int_\Omega \frac{|d_1'(v)|^2}{d_1(v)}u_1^2|\nabla v|^2+\int_\Omega \frac{\chi_1^2(v)}{d_1(v)}u_1^2|\nabla v|^2+c_1\\
&\le\left(\frac{K_1^2}{4}+K_2^2\right)\int_\Omega d_1(v)u_1^2|\nabla v|^2+c_1\\
&\le\left(\frac{K_1^2}{4}+K_2^2\right)\left(\int_\Omega |d_1^{\frac{1}{2}}(v)u_1|^4\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla v|^4\right)^{\frac{1}{2}}+c_1,\qquad \forall t\in(0,T_{max}).
\end{aligned} \tag{2.16}
$$

Since $d_1'(v)<0$ in the hypothesis (H1), we have $d_1(v)\le d_1(0)=:c_2$. Then, applying the Gagliardo–Nirenberg inequality, one has

$$\begin{aligned}\left(\int_\Omega |d_1^{\frac{1}{2}}(v)u_1|^4\right)^{\frac{1}{2}} &= \|d_1^{\frac{1}{2}}(v)u_1\|_{L^4}^2 \\ &\le c_3\left(\|\nabla(d_1^{\frac{1}{2}}(v)u_1)\|_{L^2}\|d_1^{\frac{1}{2}}(v)u_1\|_{L^2}+\|d_1^{\frac{1}{2}}(v)u_1\|_{L^2}^2\right) \\ &\le c_4\left(\|\nabla(d_1^{\frac{1}{2}}(v)u_1)\|_{L^2}\|u_1\|_{L^2}+\|u_1\|_{L^2}^2\right), \quad \forall t\in(0,T_{max}),\end{aligned} \tag{2.17}$$

where $c_4 = c_3(c_2^{\frac{1}{2}}+c_2)$. To proceed, we recall the inequality [14, Lemma 2.5]

$$\|\nabla v\|_{L^4} \le c_5(\|\Delta v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}^{\frac{1}{2}}+\|\nabla v\|_{L^2}), \qquad \forall t\in(0,T_{max}),$$

which together with the estimate (2.6), gives

$$\begin{aligned}\left(\int_\Omega |\nabla v|^4\right)^{\frac{1}{2}} = \|\nabla v\|_{L^4}^2 &\le c_6\left(\|\Delta v\|_{L^2}\|\nabla v\|_{L^2}+\|\nabla v\|_{L^2}^2\right) \\ &\le c_7\left(\|\Delta v\|_{L^2}+1\right), \qquad \forall t\in(0,T_{max})\,.\end{aligned} \tag{2.18}$$

Combining (2.17) with (2.18), and using the Young's inequality, one derives that

$$\begin{aligned}&\left(\frac{K_1^2}{4}+K_2^2\right)\left(\int_\Omega |d_1^{\frac{1}{2}}(v)u_1|^4\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla v|^4\right)^{\frac{1}{2}} \\ &\le \left(\frac{K_1^2}{4}+K_2^2\right)c_4c_7(\|\nabla(d_1^{\frac{1}{2}}(v)u_1)\|_{L^2}\|u_1\|_{L^2}+\|u_1\|_{L^2}^2)(\|\Delta v\|_{L^2}+1) \\ &= \left(\frac{K_1^2}{4}+K_2^2\right)c_4c_7(\|\nabla(d_1^{\frac{1}{2}}(v)u_1)\|_{L^2}\|u_1\|_{L^2}\|\Delta v\|_{L^2} \\ &\quad +\|\nabla(d_1^{\frac{1}{2}}(v)u_1)\|_{L^2}\|u_1\|_{L^2}+\|u_1\|_{L^2}^2\|\Delta v\|_{L^2}+\|u_1\|_{L^2}^2) \\ &\le \frac{1}{2}\|\nabla(d_1^{\frac{1}{2}}(v)u_1)\|_{L^2}^2+c_8\|u_1\|_{L^2}^2\|\Delta v\|_{L^2}^2+c_9^2\|u_1\|_{L^2}^2, \qquad \forall t\in(0,T_{max}),\end{aligned} \tag{2.19}$$

where $c_8 = 2c_4^2c_7^2\left(\frac{K_1^2}{4}+K_2^2\right)^2$ and $c_9 = c_4c_7\left(\frac{K_1^2}{4}+K_2^2\right)+\frac{1}{2}$. Substituting (2.19) into (2.16) and choosing $c_{10} = c_1+\frac{4c_9^6}{27\mu_1^2}|\Omega|$, one has

$$\frac{d}{dt}\|u_1\|_{L^2}^2 \le c_8\|\Delta v\|_{L^2}^2\|u_1\|_{L^2}^2+c_{10}, \qquad \forall t\in(0,T_{max}), \tag{2.20}$$

which, together with the results in Lemmas 2.2 and 2.3, allows us to obtain (2.12). To be exact, for all $t\in(0,T_{max})$, in the cases $t\in(0,\tau)$ and $t>\tau$ with $\tau=\min\{1,\frac{1}{2}T_{max}\}$, we can find $t_0\in((t-\tau)_+,t)$ such that $t_0\ge 0$ and

$$\int_\Omega u_1^2(x,t_0) \le c_{11}. \tag{2.21}$$

Moreover, Lemma 2.3 shows that

$$\int_{t_0}^{t_0+\tau} \int_\Omega |\Delta v|^2 \le c_{12}, \qquad \forall t_0 \in (0, T_{max} - \tau). \tag{2.22}$$

Next, integrating (2.20) over (t_0, t), and applying (2.21) and (2.22), we derive

$$\begin{aligned} \|u_1(\cdot,t)\|_{L^2}^2 &\le \|u_1(\cdot,t_0)\|_{L^2}^2 \cdot e^{c_8 \int_{t_0}^t \|\Delta v(\cdot,\rho)\|_{L^2} d\rho} + c_{10} \int_{t_0}^t e^{c_8 \int_\rho^t \|\Delta v(\cdot,s)\|_{L^2} ds} d\rho \\ &\le c_{11} e^{c_8 c_{12}} + c_{10} e^{c_8 c_{12}} \end{aligned}$$

for all $t \in (0, T_{max} - \tau)$, which yields

$$\|u_1(\cdot,t)\|_{L^2} \le c_{13}, \qquad \forall t \in (0, T_{max}). \tag{2.23}$$

Similarly, we can obtain

$$\|u_2(\cdot,t)\|_{L^2} \le c_{14}, \qquad \forall t \in (0, T_{max}). \tag{2.24}$$

Then, the combination of (2.23) and (2.24) gives

$$\|b_1 u_1(\cdot,t) + b_2 u_2(\cdot,t)\|_{L^2} \le b_1 c_{13} + b_2 c_{14}. \tag{2.25}$$

By applying the parabolic regularity to the third equation of (1.1), and using (2.25) we obtain

$$\|v(\cdot,t)\|_{W^{1,4}} \le c_{15}, \qquad \forall t \in (0, T_{max}), \tag{2.26}$$

which gives $\|v(\cdot,t)\|_{L^\infty} \le c_{16}$ for all $t \in (0, T_{max})$ directly. Then we complete the proof of Lemma 2.4. □

2.3. Boundedness of u_1, u_2 in L^∞, and v in $W^{1,\infty}$

Lemma 2.5. *Assume that the conditions in Lemma 2.1 hold. Then there exists a constant $C > 0$ independent of t, such that*

$$\|u_1(\cdot,t)\|_{L^\infty} + \|u_2(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} \le C, \qquad \forall t \in (0, T_{max}). \tag{2.27}$$

Proof. Multiplying the first equation of (1.1) by u_1^3 and integrating the result equation over Ω, we have

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\int_\Omega u_1^4+3\int_\Omega d_1(v)u_1^2|\nabla u_1|^2+\mu_1\int_\Omega u_1^5+a_1\mu_1\int_\Omega u_1^4u_2\\&=3\int_\Omega \chi_1(v)u_1^3\nabla u_1\cdot\nabla v+\mu_1\int_\Omega u_1^4\end{aligned}\tag{2.28}$$

for all $t\in(0,T_{max})$. Noting the fact that $\|v(\cdot,t)\|_{L^\infty}\le c_1$ in Lemma 2.4 and the hypotheses (H1)–(H2), one obtains

$$d_1(v)>d_1(c_1):=c_2>0 \quad\text{and}\quad |\chi_1(v)|\le c_3, \qquad \forall\, t\in(0,T_{max}).$$

This allows us to rewrite (2.28) as follows, for all $t\in(0,T_{max})$

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\int_\Omega u_1^4+3c_2\int_\Omega u_1^2|\nabla u_1|^2+\frac{1}{4}\int_\Omega u_1^4\\&\le 3c_3\int_\Omega u_1^3|\nabla u_1||\nabla v|+\left(\frac{1}{4}+\mu_1\right)\int_\Omega u_1^4-\mu_1\int_\Omega u_1^5\\&\le\frac{3c_2}{2}\int_\Omega u_1^2|\nabla u_1|^2+\frac{3c_3^2}{2c_2}\int_\Omega u_1^4|\nabla v|^2+c_4,\end{aligned}\tag{2.29}$$

where $c_4=\left(\frac{4\mu_1+1}{5}\right)^5\frac{|\Omega|}{4\mu_1^4}$. For the second term on the right hand side of (2.29), we invoke the result $\|\nabla v(\cdot,t)\|_{L^4}\le c_5$ from (2.26). Then, using Cauchy–Schwarz inequality and the Gagliardo–Nirenberg inequality, and applying the fact $\|u_1^2\|_{L^1}=\|u_1\|_{L^2}^2\le c_6$ (see Lemma 2.4), one has

$$\begin{aligned}\frac{3c_3^2}{2c_2}\int_\Omega u_1^4|\nabla v|^2&\le\frac{3c_3^2}{2c_2}\left(\int_\Omega u_1^8\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla v|^4\right)^{\frac{1}{2}}\\&\le\frac{3c_3^2c_5^2}{2c_2}\|u_1^2\|_{L^4}^2\\&\le c_7\left(\|\nabla u_1^2\|_{L^2}^{\frac{3}{2}}\|u_1^2\|_{L^1}^{\frac{1}{2}}+\|u_1^2\|_{L^1}^2\right)\\&\le\frac{3c_2}{8}\|\nabla u_1^2\|_{L^2}^2+c_8,\qquad\forall t\in(0,T_{max}),\end{aligned}\tag{2.30}$$

where $c_8=c_6^2c_7+\frac{2c_7^4c_6^2}{c_2^3}$. Substituting (2.30) into (2.29), we obtain

$$\frac{d}{dt}\int_\Omega u_1^4+\int_\Omega u_1^4\le 4(c_4+c_8),\qquad\forall t\in(0,T_{max}),$$

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which, together with Gronwall's inequality, implies

$$\|u_1(\cdot,t)\|_{L^4}^4 \le \|u_1(\cdot,0)\|_{L^4}^4 + 4(c_4+c_8), \qquad \forall t \in (0, T_{max}). \tag{2.31}$$

In addition, applying the similar process to the second equation of system (1.1), we obtain

$$\|u_2(\cdot,t)\|_{L^4}^4 \le \|u_2(\cdot,0)\|_{L^4}^4 + 4c_9, \qquad \forall t \in (0, T_{max}). \tag{2.32}$$

With (2.31) and (2.32) in hand, we can apply the parabolic regularity to the third equation again to find a constant $c_{10} > 0$ such that $\|\nabla v(\cdot,t)\|_{W^{1,\infty}} \le c_{10}$ in two dimensional spaces. With these results in hand and using the well-known Moser iteration procedure (cf. [29,32]), we have (2.27). □

Proof of Theorem 1.1. Theorem 1.1 is a consequence of the combination of Lemma 2.1 and Lemma 2.5. □

3. Stabilization and convergence rate

In this section, we will show that the classical solution of (1.1) converges to the constant steady state by constructing some Lyapunov functionals under certain conditions motivated by some ideas in [2,15,30].

3.1. Competitive coexistence: $a_1 \in (0,1)$ and $a_2 \in (0,1)$

In this subsection, we shall study the asymptotic behavior of the solution of system (1.1) with $a_i \in (0,1)$ for $i = 1,2$ based on the following energy functional

$$\begin{aligned}\mathcal{E}_1(t) := &\int_\Omega \left(u_1 - u_1^* - u_1^* \ln\frac{u_1}{u_1^*}\right) + \zeta_1 \int_\Omega \left(u_2 - u_2^* - u_2^* \ln\frac{u_2}{u_2^*}\right)\\ &+ \frac{\eta_1}{2}\int_\Omega (v-v^*)^2,\end{aligned} \tag{3.1}$$

where ζ_1 and η_1 are two positive constants which will be chosen later. Here (u_1^*, u_2^*, v^*) is the constant steady state defined by

$$u_1^* = \frac{1-a_1}{1-a_1a_2}, \quad u_2^* = \frac{1-a_2}{1-a_1a_2}, \quad v^* = \frac{b_1+b_2-a_1b_1-a_2b_2}{1-a_1a_2}.$$

Lemma 3.1. *Assume that $a_i \in (0,1)$ $(i=1,2)$ and $\mathcal{E}_1(t)$ is defined by (3.1). Then $\mathcal{E}_1(t) \ge 0$ for all $t > 0$. Moreover, if μ_1 and μ_2 satisfy (1.4), there exists a constant $\varepsilon_1 > 0$ such that*

$$\frac{d}{dt}\mathcal{E}_1(t) \le -\varepsilon_1 \mathcal{F}_1(t), \tag{3.2}$$

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where

$$\mathcal{F}_1(t) := \int_\Omega (u_1 - u_1^*)^2 + \int_\Omega (u_2 - u_2^*)^2 + \int_\Omega (v - v^*)^2. \tag{3.3}$$

Proof. Since $0 < a_1 < 1$ and $0 < a_2 < 1$, we can find a constant $\xi_1 > 0$ such that

$$4\xi_1 - a_1 a_2 (1 + \xi_1)^2 > 0, \tag{3.4}$$

and hence

$$a_1 b_1^2 \xi_1 + a_2 b_2^2 - a_1 a_2 b_1 b_2 (1 + \xi_1) > 0 \tag{3.5}$$

by noting the discriminant of (3.5) is negative ($\Delta := a_1 a_2 b_2^2 [a_1 a_2 (1 + \xi_1)^2 - 4\xi_1] < 0$).

Since μ_1, μ_2 satisfy (1.4), there exists a constant $\eta_1 > 0$ such that

$$\eta_1 \in \left(\frac{u_2^* a_1 \mu_1 \xi_1 \chi_2^2(v)}{4 a_2 \mu_2 d_2(v)} + \frac{u_1^* \chi_1^2(v)}{4 d_1(v)}, \; \frac{a_1 \mu_1 (4\xi_1 - a_1 a_2 (1 + \xi_1)^2)}{a_1 b_1^2 \xi_1 + a_2 b_2^2 - a_1 a_2 b_1 b_2 (1 + \xi_1)} \right). \tag{3.6}$$

Choosing $\zeta_1 = \frac{a_1 \mu_1 \xi_1}{a_2 \mu_2}$ and η_1 defined by (3.6), we can rewrite $\mathcal{E}_1(t)$ as follows

$$\mathcal{E}_1(t) = I_1(t) + \frac{a_1 \mu_1 \xi_1}{a_2 \mu_2} I_2(t) + \eta_1 I_3(t), \tag{3.7}$$

where

$$\begin{cases} I_1(t) &= \int_\Omega (u_1 - u_1^* - u_1^* \ln \frac{u_1}{u_1^*}), \\ I_2(t) &= \int_\Omega (u_2 - u_2^* - u_2^* \ln \frac{u_2}{u_2^*}), \\ I_3(t) &= \frac{1}{2} \int_\Omega (v - v^*)^2. \end{cases}$$

Then, applying the similar arguments as in [2, pp. 568–569], we can derive that I_1 and I_2 are nonnegative and hence $\mathcal{E}_1(t) \geq 0$.

Next, we shall prove (3.2). To this end, we first show that

$$\begin{aligned}\frac{d}{dt}I_1(t)=&\int\limits_\Omega\left(u_{1t}-\frac{u_1^*}{u_1}u_{1t}\right)\\ =&\mu_1\int\limits_\Omega(u_1-u_1^*)(1-u_1-a_1u_2)+u_1^*\int\limits_\Omega d_1(v)\nabla u_1\cdot\nabla\left(\frac{1}{u_1}\right)\\ &-u_1^*\int\limits_\Omega\chi_1(v)u_1\nabla v\cdot\nabla\left(\frac{1}{u_1}\right)\\ =&-\mu_1\int\limits_\Omega(u_1-u_1^*)^2-a_1\mu_1\int\limits_\Omega(u_1-u_1^*)(u_2-u_2^*)\\ &-u_1^*\int\limits_\Omega d_1(v)\left|\frac{\nabla u_1}{u_1}\right|^2+u_1^*\int\limits_\Omega\chi_1(v)\frac{\nabla u_1}{u_1}\cdot\nabla v,\end{aligned}\tag{3.8}$$

where we have used the fact that $u_1^*+a_1u_2^*=1$. Similarly, noting $a_2u_1^*+u_2^*=1$, we deduce

$$\begin{aligned}\frac{d}{dt}I_2(t)=&\int\limits_\Omega\left(u_{2t}-\frac{u_2^*}{u_2}u_{2t}\right)\\ =&-\mu_2\int\limits_\Omega(u_2-u_2^*)^2-a_2\mu_2\int\limits_\Omega(u_1-u_1^*)(u_2-u_2^*)\\ &-u_2^*\int\limits_\Omega d_2(v)\left|\frac{\nabla u_2}{u_2}\right|^2+u_2^*\int\limits_\Omega\chi_2(v)\frac{\nabla u_2}{u_2}\cdot\nabla v.\end{aligned}\tag{3.9}$$

Moreover, with the identity $b_1u_1^*+b_2u_2^*=v^*$, it holds that

$$\begin{aligned}\frac{d}{dt}I_3(t)=&\int\limits_\Omega(v-v^*)v_t\\ =&-\int\limits_\Omega|\nabla v|^2+b_1\int\limits_\Omega(u_1-u_1^*)(v-v^*)\\ &+b_2\int\limits_\Omega(u_2-u_2^*)(v-v^*)-\int\limits_\Omega(v-v^*)^2.\end{aligned}\tag{3.10}$$

Substituting (3.8)–(3.10) into (3.7), one has

$$\frac{d}{dt}\mathcal{E}_1(t)=-\int\limits_\Omega X_1A_1X_1^T-\int\limits_\Omega Y_1B_1Y_1^T,\tag{3.11}$$

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where X_1 and Y_1 are vector functions defined as

$$X_1(x,t) := (u_1(x,t) - u_1^*, u_2(x,t) - u_2^*, v(x,t) - v^*)$$

and

$$Y_1(x,t) := \left(\frac{|\nabla u_1(x,t)|}{u_1(x,t)}, \frac{|\nabla u_2(x,t)|}{u_2(x,t)}, |\nabla v(x,t)| \right)$$

in $\Omega \times (0,\infty)$, and the matrices A_1 and B_1 are defined by

$$A_1 := \begin{pmatrix} \mu_1 & \frac{a_1\mu_1(1+\xi_1)}{2} & -\frac{\eta_1 b_1}{2} \\ \frac{a_1\mu_1(1+\xi_1)}{2} & \frac{a_1\mu_1\xi_1}{a_2} & -\frac{\eta_1 b_2}{2} \\ -\frac{\eta_1 b_1}{2} & -\frac{\eta_1 b_2}{2} & \eta_1 \end{pmatrix}$$

and

$$B_1 := \begin{pmatrix} u_1^* d_1(v) & 0 & -\frac{u_1^*\chi_1(v)}{2} \\ 0 & \frac{u_2^* a_1\mu_1\xi_1 d_2(v)}{a_2\mu_2} & -\frac{u_2^* a_1\mu_1\xi_1\chi_2(v)}{2a_2\mu_2} \\ -\frac{u_1^*\chi_1(v)}{2} & -\frac{u_2^* a_1\mu_1\xi_1\chi_2(v)}{2a_2\mu_2} & \eta_1 \end{pmatrix}.$$

Next, we prove that the matrices A_1 and B_1 are positive definite. Owing to (3.4) and (3.6), one derives

$$|\mu_1| > 0$$

and

$$\begin{vmatrix} \mu_1 & \frac{a_1\mu_1(1+\xi_1)}{2} \\ \frac{a_1\mu_1(1+\xi_1)}{2} & \frac{a_1\mu_1\xi_1}{a_2} \end{vmatrix} = \frac{a_1\mu_1^2}{4a_2}\left(4\xi_1 - a_1a_2(1+\xi_1)^2\right) > 0$$

as well as

$$|A_1| = \frac{\mu_1\eta_1}{4a_2}\left(a_1\mu_1(4\xi_1 - a_1a_2(1+\xi_1)^2) - \eta_1(a_1b_1^2\xi_1 + a_2b_2^2 - a_1a_2b_1b_2(1+\xi_1))\right) > 0$$

due to $a_i > 0$ $(i = 1,2)$. Hence the matrix A_1 is positive definite by using the Sylvester's criterion. On the other hand, noting $\zeta_1 = \frac{a_1\mu_1\xi_1}{a_2\mu_2} > 0$, we obtain

$$|u_1^* d_1(v)| > 0 \quad \text{and} \quad \begin{vmatrix} u_1^* d_1(v) & 0 \\ 0 & u_2^*\zeta_1 d_2(v) \end{vmatrix} = u_1^* u_2^* \zeta_1 d_1(v) d_2(v) > 0,$$

which together with (3.6) gives

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$$
\begin{aligned}
|B_1| &= u_1^* d_1(v) \begin{vmatrix} \frac{u_2^* a_1\mu_1\xi_1 d_2(v)}{a_2\mu_2} & -\frac{u_2^* a_1\mu_1\xi_1\chi_2(v)}{2a_2\mu_2} \\ -\frac{u_2^* a_1\mu_1\xi_1\chi_2(v)}{2a_2\mu_2} & \eta_1 \end{vmatrix} \\
&\quad - \frac{u_1^*\chi_1(v)}{2} \begin{vmatrix} 0 & \frac{u_2^* a_1\mu_1\xi_1 d_2(v)}{a_2\mu_2} \\ -\frac{u_1^*\chi_1(v)}{2} & -\frac{u_2^* a_1\mu_1\xi_1\chi_2(v)}{2a_2\mu_2} \end{vmatrix} \\
&= \frac{u_1^* u_2^* a_1\mu_1\xi_1}{a_2\mu_2}\left(d_1(v)d_2(v)\eta_1 - \frac{u_2^* a_1\mu_1\xi_1 d_1(v)\chi_2^2(v)}{4a_2\mu_2} - \frac{u_1^* d_2(v)\chi_1^2(v)}{4}\right) \\
&> 0.
\end{aligned}
$$

Hence applying Sylvester's criterion again, we show the positive definiteness of B_1.

Since the matrices A_1, B_1 are positive definite, we can find a constant ε_1 independent of t, such that

$$X_1(x,t)A_1X_1^T(x,t) \geq \varepsilon_1 |X_1(x,t)|^2$$

and

$$Y_1(x,t)B_1Y_1^T(x,t) \geq 0,$$

which together with (3.11) yields (3.2). Hence, the proof is completed. □

With Lemma 3.1 in hand, we in fact show that $\mathcal{E}_1(t)$ is a Lyapuonv functional under the assumption (1.4). Next, we shall apply the LaSalle's invariant principle (e.g. see [18, Theorem 3] or [27, pp. 198–199, Theorem 5.24]) to show the solution (u_1, u_2, v) of (1.1) converges to the steady state (u_1^*, u_2^*, v^*) in the sense of L^∞-norm motivated by the ideas in [15].

Lemma 3.2. *Suppose the assumptions in Lemma 3.1 hold. Then we obtain*

$$\|u_1(\cdot,t) - u_1^*\|_{L^\infty} + \|u_2(\cdot,t) - u_2^*\|_{L^\infty} + \|v(\cdot,t) - v^*\|_{L^\infty} \to 0 \ \text{ as } \ t \to \infty. \tag{3.12}$$

Proof. Denote $Z(t) := (u_1, u_2, v)(t)$ be the unique global classical solution to system (1.1) with initial data $Z_0 = (u_{10}, u_{20}, v_0)$, which defines a semi-flow (or trajectory) on $X = [W^{1,p}(\bar{\Omega})]^3$ with $p > 2$ (see [1]) due to Theorem 1.1. Suppose $\mathcal{E}_1(Z) = \mathcal{E}_1(t)$, that is

$$
\begin{aligned}
\mathcal{E}_1(u_1, u_2, v) &= \int_\Omega \left(u_1 - u_1^* - u_1^* \ln \frac{u_1}{u_1^*}\right) + \frac{a_1\mu_1\xi_1}{a_2\mu_2} \int_\Omega \left(u_2 - u_2^* - u_2^* \ln \frac{u_2}{u_2^*}\right) \\
&\quad + \frac{\eta_1}{2}\int_\Omega (v - v^*)^2.
\end{aligned}
$$

For a given $c^* > 0$, we have $c - c^* - c^* \ln \frac{c}{c^*} > 0$ for all positive constants $c \neq c^*$. Hence, it yields $\mathcal{E}_1(Z) > 0$ for all $Z \neq (u_1^*, u_2^*, v^*)$ and $\mathcal{E}_1(Z) = 0$ if and only if $Z = (u_1^*, u_2^*, v^*)$. Moreover, we derive from (3.2)–(3.3) that $\frac{d}{dt}\mathcal{E}_1(Z) \leq 0$, where $\frac{d}{dt}\mathcal{E}_1(Z) = 0$ if and only if $Z = (u_1^*, u_2^*, v^*)$. Then applying the LaSalle's invariance principle (e.g. see [18, Theorem 3] or [27, pp. 198–199, Theorem 5.24]), we obtain (3.12) directly. □

3.2. Competitive exclusion: $a_1 \geq 1$ and $a_2 < 1$

When $a_1 \geq 1$ and $a_2 < 1$, we shall show that the competitive exclusion will occur based on the following energy functional

$$\mathcal{E}_2(t) := \int_\Omega u_1 + \zeta_2 \int_\Omega (u_2 - 1 - \ln u_2) + \frac{\eta_2}{2} \int_\Omega (v - b_2)^2, \tag{3.13}$$

where ζ_2 and η_2 are some positive constants, which will be chosen later.

Lemma 3.3. *Assume that $a_1 \geq 1$ and $a_2 \in (0,1)$. Let $\mathcal{E}_2(t)$ be defined as (3.13). Then it holds that $\mathcal{E}_2(t) \geq 0$ for all $t > 0$. Moreover, if μ_2 satisfies (1.5) with some constant $a_1^* \in (1, a_1]$ and $a_1^* a_2 < 1$, then we have*

$$\frac{d}{dt}\mathcal{E}_2(t) \leq -\varepsilon_2 \mathcal{F}_2(t) - \mu_1(a_1^* - 1)\int_\Omega u_1, \tag{3.14}$$

where ε_2 is a positive constant and

$$\mathcal{F}_2(t) := \int_\Omega u_1^2 + \int_\Omega (u_2 - 1)^2 + \int_\Omega (v - b_2)^2.$$

Proof. First, by using the similar way in Lemma 3.1, we apply the Taylor formula to derive that $\int_\Omega (u_2 - 1 - \ln u_2) \geq 0$, which implies that $\mathcal{E}_2(t) \geq 0$.

Next, we shall show (3.14) holds under the condition (1.5). Since $a_1^* \in (1, a_1]$ and $a_1^* a_2 < 1$, we can find a constant $\xi_2 > 0$ such that $4\xi_2 - a_1^* a_2 (1 + \xi_2)^2 > 0$, and hence

$$a_1^* b_1^2 \xi_2 + a_2 b_2^2 - a_1^* a_2 b_1 b_2 (1 + \xi_2) > 0 \tag{3.15}$$

for the discriminant of (3.15) is negative. Then we choose $\zeta_2 = \frac{a_1^* \mu_1 \xi_2}{a_2 \mu_2}$ and

$$\eta_2 \in \left(\frac{a_1^* \mu_1 \xi_2 \chi_2^2(v)}{4 a_2 \mu_2 d_2(v)}, \ \frac{a_1^* \mu_1 (4\xi_2 - a_1^* a_2 (1 + \xi_2)^2)}{a_1^* b_1^2 \xi_2 + a_2 b_2^2 - a_1^* a_2 b_1 b_2 (1 + \xi_2)} \right) \tag{3.16}$$

such that $\mathcal{E}_2(t)$ can be rewritten as

$$\begin{aligned} \mathcal{E}_2(t) &= \int_\Omega u_1 + \zeta_2 \int_\Omega (u_2 - 1 - \ln u_2) + \frac{\eta_2}{2} \int_\Omega (v - b_2)^2 \\ &= J_1(t) + \zeta_2 J_2(t) + \eta_2 J_3(t), \end{aligned} \tag{3.17}$$

where

$$\begin{cases} J_1(t) &= \int_\Omega u_1, \\ J_2(t) &= \int_\Omega (u_2 - 1 - \ln u_2), \\ J_3(t) &= \frac{1}{2}\int_\Omega (v - b_2)^2. \end{cases}$$

Accordingly, we have

$$\begin{aligned} \frac{d}{dt}J_1(t) = \frac{d}{dt}\int_\Omega u_1 =& \mu_1 \int_\Omega u_1(1 - u_1 - a_1 u_2) \\ =& -\mu_1 \int_\Omega u_1^2 - a_1^*\mu_1 \int_\Omega u_1(u_2 - 1) - \mu_1(a_1 - a_1^*)\int_\Omega u_1 u_2 \\ & - \mu_1(a_1^* - 1)\int_\Omega u_1, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \frac{d}{dt}J_2(t) =& -\mu_2 \int_\Omega (u_2 - 1)^2 - a_2\mu_2 \int_\Omega u_1(u_2 - 1) - \int_\Omega d_2(v)\left|\frac{\nabla u_2}{u_2}\right|^2 \\ & + \int_\Omega \chi_2(v)\frac{\nabla u_2}{u_2}\cdot \nabla v, \end{aligned} \tag{3.19}$$

as well as

$$\begin{aligned} \frac{d}{dt}J_3(t) = \int_\Omega (v - b_2)v_t =& -\int_\Omega |\nabla v|^2 - \int_\Omega (v - b_2)^2 \\ & + b_1\int_\Omega u_1(v - b_2) + b_2 \int_\Omega (u_2 - 1)(v - b_2). \end{aligned} \tag{3.20}$$

Consequently, substituting (3.18)–(3.20) into (3.17) gives that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_2(t) =& -X_2 A_2 X_2^T - Y_2 B_2 Y_2^T - \mu_1(a_1 - a_1^*)\int_\Omega u_1 u_2 - \mu_1(a_1^* - 1)\int_\Omega u_1 \\ \le& -X_2 A_2 X_2^T - Y_2 B_2 Y_2^T - \mu_1(a_1^* - 1)\int_\Omega u_1, \end{aligned} \tag{3.21}$$

where $X_2 = (u_1, u_2 - 1, v - b_2)$ and $Y_2 = \left(\frac{|\nabla u_2|}{u_2}, |\nabla v|\right)$, and

$$A_2 := \begin{pmatrix} \mu_1 & \frac{a_1^*\mu_1(1+\xi_2)}{2} & -\frac{\eta_2 b_1}{2} \\ \frac{a_1^*\mu_1(1+\xi_2)}{2} & \frac{a_1^*\mu_1\xi_2}{a_2} & -\frac{\eta_2 b_2}{2} \\ -\frac{\eta_2 b_1}{2} & -\frac{\eta_2 b_2}{2} & \eta_2 \end{pmatrix} \quad \text{and} \quad B_2 := \begin{pmatrix} \frac{a_1^*\mu_1\xi_2 d_2(v)}{a_2\mu_2} & -\frac{a_1^*\mu_1\xi_2\chi_2(v)}{2a_2\mu_2} \\ -\frac{a_1^*\mu_1\xi_2\chi_2(v)}{2a_2\mu_2} & \eta_2 \end{pmatrix}.$$

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Next, we consider the leading principle minors of A_2 and B_2. Noting $4\xi_2 - a_1^* a_2(1+\xi_2)^2 > 0$, it follows that

$$|\mu_1| > 0$$

and

$$\begin{vmatrix} \mu_1 & \frac{a_1^*\mu_1(1+\xi_2)}{2} \\ \frac{a_1^*\mu_1(1+\xi_2)}{2} & \frac{a_1^*\mu_1\xi_2}{a_2} \end{vmatrix} = \frac{a_1^*\mu_1^2}{4a_2}\left(4\xi_2 - a_1^* a_2(1+\xi_2)^2\right) > 0.$$

Furthermore, since η_2 satisfies (3.16), we obtain

$$|A_2| = \frac{\mu_1\eta_2}{4a_2}\left(a_1^*\mu_1(4\xi_2 - a_1^* a_2(1+\xi_2)^2) - \eta_2(a_1^* b_1^2\xi_2 + a_2 b_2^2 - a_1^* a_2 b_1 b_2(1+\xi_2))\right) > 0.$$

Continuously, we know that

$$\left|\frac{a_1^*\mu_1\xi_2 d_2(v)}{a_2\mu_2}\right| > 0$$

and

$$|B_2| = \frac{a_1^*\mu_1\xi_2 d_2(v)}{a_2\mu_2}\left(\eta_2 - \frac{a_1^*\mu_1\xi_2\chi_2^2(v)}{4a_2\mu_2 d_2(v)}\right) > 0$$

thanks to $\eta_2 > \frac{a_1^*\mu_1\xi_2\chi_2^2(v)}{4a_2\mu_2 d_2(v)}$. It follows from Sylvester's criterion that A_2 and B_2 are positive definite. Consequently, (3.21) together with the positivity of u_i $(i = 1, 2)$ shows (3.14). $\square$

Lemma 3.4. *Suppose the assumptions in Lemma 3.3 hold. Then it holds that*

$$\|u_1(\cdot,t)\|_{L^\infty} + \|u_2(\cdot,t) - 1\|_{L^\infty} + \|v(\cdot,t) - b_2\|_{L^\infty} \to 0 \ \ as \ \ t \to +\infty. \tag{3.22}$$

Proof. From Lemma 3.3, we derive that $\mathcal{E}_2(t) \ge 0$, $\frac{d}{dt}\mathcal{E}_2(t) \le 0$ and $\frac{d}{dt}\mathcal{E}_2(t) = 0$ if and only if $(u_1, u_2, v) = (0, 1, b_2)$. Then applying LaSalle's invariance principle again as in Lemma 3.2, we obtain (3.22) immediately. $\square$

3.3. Convergence rates

In this subsection, we shall show the convergence rates of solutions in L^∞-norm. To this aim, we first derive the decay rate of solution with L^p-norm for some $p \ge 1$ based on the energy functionals constructed in Lemma 3.1 and Lemma 3.3. Then using the higher energy estimate of solution, we shall show that there exists a positive constant of C independent of t such that $\|\nabla u_1\|_{L^4} + \|\nabla u_2\|_{L^4} + \|v\|_{W^{1,\infty}} \le C$ for some $t > 1$, which combined with the decay rate of solution with L^p-norm and the Gagliardo–Nirenberg inequality gives convergence rates of solutions in L^∞-norm. First, we improve the regularity of solutions as follows.

Lemma 3.5. *Let (u_1, u_2, v) be the nonnegative global classical solution of system (1.1) obtained in Theorem 1.1. Then there exist $\sigma \in (0, 1)$ and $C > 0$ such that*

$$\|v\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le C, \text{ for all } t > 1. \tag{3.23}$$

Proof. The Theorem 1.1 entails us to find three positive constants c_1, c_2, c_3 such that

$$0 < u_1(x,t), u_2(x,t) \le c_1, 0 < v(x,t) \le c_2 \text{ and } |\nabla v(x,t)| \le c_3$$

for all $x \in \Omega$ and $t > 0$. Next, we shall apply the Hölder regularity for quasilinear parabolic equations [25, Theorem 1.3 and Remark 1.4] to show that

$$\|u_1\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le c_4, \text{ for all } t > 1. \tag{3.24}$$

In fact, we can rewrite the first equation of system (1.1) as follows

$$u_{1t} = \nabla \cdot f(x,t,\nabla u_1) + g(x,t) \qquad \text{for all } x \in \Omega \text{ and } t > 0$$

with

$$f(x,t,\nabla u_1) := d_1(v)\nabla u_1 - \chi_1(v)u_1\nabla v$$

and

$$g(x,t) := \mu_1 u_1(1 - u_1 - a_1 u_2).$$

Using the assumptions in (H1) and the Young's inequality, then we obtain

$$\begin{aligned} f(x,t,\nabla u_1)\cdot\nabla u_1 &= d_1(v)|\nabla u_1|^2 - \chi_1(v)u_1\nabla v\cdot\nabla u_1 \\ &\ge d_1(v)|\nabla u_1|^2 - |\chi_1(v)|u_1|\nabla v||\nabla u_1| \\ &\ge \frac{d_1(v)}{2}|\nabla u_1|^2 - \frac{|\chi_1(v)|^2}{2d_1(v)}u_1^2|\nabla v|^2 \\ &\ge \frac{d_1(c_2)}{2}|\nabla u_1|^2 - c_5 \end{aligned} \tag{3.25}$$

and

$$|f(x,t,\nabla u_1)| \le d(0)|\nabla u_1| + c_6, \text{ and } |g(x,t)| \le c_7 \tag{3.26}$$

for all $x \in \Omega$ and $t > 0$. Then the application of Hölder regularity, we obtain (3.24) directly by noting (3.25) and (3.26). Similarly, from the second equation of system (1.1), we have

$$\|u_2\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le c_8$$

for all $t > 1$. Then applying the standard parabolic schauder theory [17] to the third equation of (1.1), one has (3.23). Then the proof of Lemma 3.5 is completed. □

Lemma 3.6. *Assume that $\Omega \subset \mathbb{R}^2$ and (u_1, u_2, v) is a global classical solution to system (1.1). Let μ_i, $a_i \geq 0$ for $i = 1, 2$. Then, for all $p > 1$, there exists a constant $C > 0$ such that for all $t > 1$*

$$\|\nabla u_1(\cdot, t)\|_{L^{2p}} + \|\nabla u_2(\cdot, t)\|_{L^{2p}} \leq C.$$

Proof. We differentiate the first equation of (1.1) once and test the result with $|\nabla u_1|^{p-2}\nabla u_1$ to obtain

$$\begin{aligned}\frac{1}{2p}\frac{d}{dt}\int |\nabla u_1|^{2p} = &\int_\Omega |\nabla u_1|^{2p-2}\nabla u_1 \cdot \nabla(\nabla \cdot (d_1(v)\nabla u_1)) \\ &- \int_\Omega |\nabla u_1|^{2p-2}\nabla u_1 \cdot \nabla(\nabla \cdot (\chi_1(v)u_1\nabla v)) \\ &+ \mu_1 \int_\Omega |\nabla u_1|^{2p-2}\nabla u_1 \cdot \nabla(u_1 - u_1^2 - a_1 u_1 u_2) \\ =&: G_1 + G_2 + G_3.\end{aligned} \tag{3.27}$$

Using the identity $\nabla\Delta u_1 \cdot \nabla u_1 = \frac{1}{2}\Delta|\nabla u_1|^2 - |D^2 u_1|^2$, the term G_1 can be rewritten as follows

$$\begin{aligned}G_1 &= -\int_\Omega |\nabla u_1|^{2p-2}\Delta u_1 \nabla \cdot (d_1(v)\nabla u_1) - \int_\Omega \nabla|\nabla u_1|^{2p-2} \cdot \nabla u_1 \nabla \cdot (d_1(v)\nabla u_1) \\ &= \int_\Omega d_1(v)|\nabla u_1|^{2p-2}\nabla\Delta u_1 \cdot \nabla u_1 - \int_\Omega d_1'(v)\nabla|\nabla u_1|^{2p-2} \cdot \nabla u_1 \nabla u_1 \cdot \nabla v \\ &= \frac{1}{2}\int_\Omega d_1(v)|\nabla u_1|^{2p-2}\Delta|\nabla u_1|^2 - \int_\Omega d_1(v)|\nabla u_1|^{2p-2}|D^2 u_1|^2 \\ &\quad - (p-1)\int_\Omega d_1'(v)|\nabla u_1|^{2p-4}\nabla|\nabla u_1|^2 \cdot \nabla u_1 \nabla u_1 \cdot \nabla v.\end{aligned} \tag{3.28}$$

To proceed, we integrate the first term in the right hand side of (3.28) by parts

$$\begin{aligned}&\frac{1}{2}\int_\Omega d_1(v)|\nabla u_1|^{2p-2}\Delta|\nabla u_1|^2 \\ &= \frac{1}{2}\int_{\partial\Omega} d_1(v)|\nabla u_1|^{2p-2}\frac{\partial|\nabla u_1|^2}{\partial \nu}dS - \frac{1}{2}\int_\Omega d_1(v)\nabla|\nabla u_1|^{2p-2} \cdot \nabla|\nabla u_1|^2 \\ &\quad - \frac{1}{2}\int_\Omega d_1'(v)|\nabla u_1|^{2p-2}\nabla|\nabla u_1|^2 \cdot \nabla v.\end{aligned} \tag{3.29}$$

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Noting the fact that $\|v(\cdot,t)\|_{W^{1,\infty}}$ is bounded and the hypothesis $d_1(v) \in C^2([0,\infty))$, then we can find positive constants c_1, c_2, c_3 and c_4 such that

$$0 < c_1 \le d_1(v) \le c_2, \qquad \|\nabla v(\cdot,t)\|_{L^\infty} \le c_3, \qquad \text{and} \qquad |d_1'(v)| \le c_4.$$

Invoking the inequality $\frac{\partial|\nabla u_1|^2}{\partial \nu} \le 2\kappa|\nabla u_1|^2$ with some constant $\kappa > 0$ on $\partial\Omega$ ([20, Lemma 4.2]) and the trace inequality [26, Remark 52.9]

$$\|\phi\|_{L^2(\partial\Omega)} \le \varepsilon\|\nabla\phi\|_{L^2(\Omega)} + c_\varepsilon\|\phi\|_{L^2(\Omega)}$$

for any $\varepsilon > 0$, we get

$$\begin{aligned}
\frac{1}{2}\int_{\partial\Omega} d_1(v)|\nabla u_1|^{2p-2}\frac{\partial|\nabla u_1|^2}{\partial\nu}dS &\le c_2\kappa\int_{\partial\Omega}|\nabla u_1|^{2p}dS \\
&= c_2\kappa\||\nabla u_1|^p\|^2_{L^2(\partial\Omega)} \\
&\le \frac{2c_1(p-1)}{3p^2}\|\nabla|\nabla u_1|^p\|^2_{L^2(\Omega)} + c_5\||\nabla u_1|^p\|^2_{L^2(\Omega)},
\end{aligned}$$

which together with (3.29) implies that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega d_1(v)|\nabla u_1|^{2p-2}\Delta|\nabla u_1|^2 \\
&\le -\frac{c_1(p-1)}{3}\int_\Omega |\nabla u_1|^{2p-4}|\nabla|\nabla u_1|^2|^2 + \frac{c_3c_4}{2}\int_\Omega |\nabla u_1|^{2p-2}|\nabla|\nabla u_1|^2| \\
&\quad + c_5\int_\Omega |\nabla u_1|^{2p}.
\end{aligned} \tag{3.30}$$

Then substituting (3.30) into (3.28) and using the Young's inequality, one has

$$\begin{aligned}
G_1 &\le -\frac{c_1(p-1)}{3}\int_\Omega |\nabla u_1|^{2p-4}|\nabla|\nabla u_1|^2|^2 - c_1\int_\Omega |\nabla u_1|^{2p-2}|D^2u_1|^2 \\
&\quad + c_6\int_\Omega |\nabla u_1|^{2p-2}|\nabla|\nabla u_1|^2| + c_5\int_\Omega |\nabla u_1|^{2p} \\
&\le -\frac{c_1(p-1)}{6}\int_\Omega |\nabla u_1|^{2p-4}|\nabla|\nabla u_1|^2|^2 - c_1\int_\Omega |\nabla u_1|^{2p-2}|D^2u_1|^2 \\
&\quad + c_7\int_\Omega |\nabla u_1|^{2p}
\end{aligned} \tag{3.31}$$

with $c_7 = \frac{3c_6^2}{2c_1(p-1)} + c_5$. Furthermore, noting the boundedness of u_1, v and ∇v and the fact $\chi_1(\cdot) \in C^2([0,\infty))$, one has

$$\nabla \cdot (\chi_1(v)u_1\nabla v) \le c_8(1 + |\nabla u_1| + |\Delta v|).$$

Noting $|\Delta u_1| \le \sqrt{n}|D^2 u_1|$ and using the Cauchy–Schwarz inequality, we can find a positive constant $c_9 = \frac{9c_8^2(6(p-1)+n)}{2c_1}$ such that

$$\begin{aligned}
G_2 =& \int_\Omega \nabla|\nabla u_1|^{2p-2} \cdot \nabla u_1 \nabla \cdot (\chi_1(v)u_1\nabla v) + \int_\Omega |\nabla u_1|^{2p-2}\Delta u_1 \nabla \cdot (\chi_1(v)u_1\nabla v) \\
\le& c_8(p-1)\int_\Omega |\nabla u_1|^{2p-3}|\nabla|\nabla u_1|^2| + c_8(p-1)\int_\Omega |\nabla u_1|^{2p-2}|\nabla|\nabla u_1|^2| \\
&+ c_8(p-1)\int_\Omega |\nabla u_1|^{2p-3}|\nabla|\nabla u_1|^2||\Delta v| + c_8\int_\Omega |\nabla u_1|^{2p-2}|\Delta u_1| \\
&+ c_8\int_\Omega |\nabla u_1|^{2p-1}|\Delta u_1| + c_8\int_\Omega |\nabla u_1|^{2p-2}|\Delta u_1||\Delta v| \\
\le& \frac{c_1(p-1)}{12}\int_\Omega |\nabla u_1|^{2p-4}|\nabla|\nabla u_1|^2|^2 + c_9\int_\Omega |\Delta v|^{2p} \\
&+ \frac{c_1}{2}\int_\Omega |\nabla u_1|^{2p-2}|D^2u_1|^2 + c_9\int_\Omega |\nabla u_1|^{2p} + c_9|\Omega|.
\end{aligned} \tag{3.32}$$

Using the boundedness of u_i $(i = 1, 2)$ again, we can estimate G_3 as follows

$$\begin{aligned}
G_3 =& \int_\Omega (\mu_1 - 2\mu_1 u_1 - a_1\mu_1 u_2)|\nabla u_1|^{2p} - a_1\mu_1\int_\Omega u_1|\nabla u_1|^{2p-2}\nabla u_1 \cdot \nabla u_2 \\
\le& c_{10}\int_\Omega |\nabla u_1|^{2p} + c_{10}\int_\Omega |\nabla u_1|^{2p-1}|\nabla u_2| \\
\le& c_{11}\int_\Omega |\nabla u_1|^{2p} + \frac{1}{2p}\int_\Omega |\nabla u_2|^{2p}
\end{aligned} \tag{3.33}$$

with $c_{11} = c_{10}\left(1 + \frac{2p-1}{2p}c_{10}^{\frac{1}{2p-1}}\right)$. Substituting (3.31)–(3.33) into (3.27), we derive

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$$\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla u_1|^{2p} + \frac{1}{p}\int_\Omega |\nabla u_1|^{2p} \\
&\le -\frac{c_1(p-1)}{12}\int_\Omega |\nabla u_1|^{2p-4}|\nabla|\nabla u_1|^2|^2 - \frac{c_1}{2}\int_\Omega |\nabla u_1|^{2p-2}|D^2u_1|^2 \\
&\quad + c_9\int_\Omega |\Delta v|^{2p} + c_{12}\int_\Omega |\nabla u_1|^{2p} + \frac{1}{2p}\int_\Omega |\nabla u_2|^{2p} + c_9|\Omega|,
\end{aligned} \tag{3.34}$$

where $c_{12} = \frac{1}{p} + c_7 + c_9 + c_{11}$. On the other hand, we use the Young's inequality and the boundedness of $\|u_1(\cdot,t)\|_{L^\infty}$ derived in Theorem 1.1 to get

$$\begin{aligned}
\int_\Omega |\nabla u_1|^{2p} &= \int_\Omega |\nabla u_1|^{2p-2}\nabla u_1 \cdot \nabla u_1 \\
&= -(p-1)\int_\Omega u_1|\nabla u_1|^{2p-4}\nabla|\nabla u_1|^2 \cdot \nabla u_1 - \int_\Omega u_1|\nabla u_1|^{2p-2}\Delta u_1 \\
&\le \frac{c_1(p-1)}{24c_{12}}\int_\Omega |\nabla u_1|^{2p-4}|\nabla|\nabla u_1|^2|^2 + \frac{c_1}{4c_{12}}\int_\Omega |\nabla u_1|^{2p-2}|D^2u_1|^2 \\
&\quad + \frac{1}{2}\int_\Omega |\nabla u_1|^{2p} + c_{13},
\end{aligned}$$

which implies

$$\begin{aligned}
c_{12}\int_\Omega |\nabla u_1|^{2p} \le &\frac{c_1(p-1)}{12}\int_\Omega |\nabla u_1|^{2p-4}|\nabla|\nabla u_1|^2|^2 \\
&+ \frac{c_1}{2}\int_\Omega |\nabla u_1|^{2p-2}|D^2u_1|^2 + 2c_{12}c_{13}.
\end{aligned} \tag{3.35}$$

Combining (3.34) with (3.35), one has

$$\frac{d}{dt}\int_\Omega |\nabla u_1|^{2p} + 2\int_\Omega |\nabla u_1|^{2p} \le 2pc_9\int_\Omega |\Delta v|^{2p} + \int_\Omega |\nabla u_2|^{2p} + c_{14}, \tag{3.36}$$

where $c_{14} = 2p(2c_{12}c_{13} + c_9|\Omega|)$. Furthermore, applying the similar arguments to u_2, one derives

$$\frac{d}{dt}\int_\Omega |\nabla u_2|^{2p} + 2\int_\Omega |\nabla u_2|^{2p} \le c_{15}\int_\Omega |\Delta v|^{2p} + \int_\Omega |\nabla u_1|^{2p} + c_{15},$$

which, together with (3.36), gives

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega |\nabla u_1|^{2p} + \int_\Omega |\nabla u_2|^{2p}\right) + \left(\int_\Omega |\nabla u_1|^{2p} + \int_\Omega |\nabla u_2|^{2p}\right) \\
&\le (2pc_9 + c_{15}) \int_\Omega |\Delta v|^{2p} + c_{14} + c_{15}.
\end{aligned}
\tag{3.37}
$$

We combine (3.37) with (3.23) to obtain that

$$\|\nabla u_1(\cdot,t)\|_{L^{2p}} + \|\nabla u_2(\cdot,t)\|_{L^{2p}} \le c_{16}$$

for all $t > 1$. Then, the proof of Lemma 3.6 is completed. □

Next, we show the convergence rate of solution solving system (1.1) for the case: $0 < a_1 < 1$ and $0 < a_2 < 1$.

Lemma 3.7 *(Decay rate: $a_1, a_2 \in (0,1)$). Let the assumptions in Lemma 3.1 hold true. Then there exist two positive constants λ_1 and C_1, such that*

$$\|u_1(\cdot,t) - u_1^*\|_{L^\infty} + \|u_2(\cdot,t) - u_2^*\|_{L^\infty} + \|v(\cdot,t) - v^*\|_{L^\infty} \le C_1 e^{-\lambda_1 t} \tag{3.38}$$

holds for all $t > 0$.

Proof. We shall use a nice idea in [2, Lemma 3.7] to prove this lemma. First, we introduce the function $\varphi(w) := w - u_1^* \ln w$ for $w > 0$ and use the L'Hôpital's rule to derive

$$\lim_{w\to u_1^*} \frac{\varphi(w) - \varphi(u_1^*)}{(w - u_1^*)^2} = \lim_{w\to u_1^*} \frac{\varphi'(w)}{2(w - u_1^*)} = \frac{1}{2u_1^*}.$$

Noting the fact $\|u_1(\cdot,t) - u_1^*\|_{L^\infty} \to 0$ as $t \to \infty$ (see Lemma 3.2), we can find a $t_1 > 0$ such that for all $t > t_1$

$$\int_\Omega \left(u_1 - u_1^* - u_1^* \ln \frac{u_1}{u_1^*}\right) = \int_\Omega (\varphi(u_1) - \varphi(u_1^*)) \le \frac{1}{u_1^*} \int_\Omega (u_1 - u_1^*)^2 \tag{3.39}$$

and

$$\int_\Omega \left(u_1 - u_1^* - u_1^* \ln \frac{u_1}{u_1^*}\right) \ge \frac{1}{4u_1^*} \int_\Omega (u_1 - u_1^*)^2. \tag{3.40}$$

Similarly, the fact $\|u_2(\cdot,t) - u_2^*\|_{L^\infty} \to 0$ as $t \to \infty$ implies that there exists $t_2 > 0$ such that for all $t > t_2$

$$\frac{1}{4u_2^*} \int_\Omega (u_2 - u_2^*)^2 \le \int_\Omega \left(u_2 - u_2^* - u_2^* \ln \frac{u_2}{u_2^*}\right) \le \frac{1}{u_2^*} \int_\Omega (u_2 - u_2^*)^2. \tag{3.41}$$

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Let $t_3 = \max\{1, t_1, t_2\}$. Noting that $\mathcal{E}_1(t)$ and $\mathcal{F}_1(t)$ are defined in (3.1) and (3.3), then the combination of (3.39) and the right inequality of (3.41) gives that $\mathcal{E}_1(t) \le \left(\frac{1}{u_1^*} + \frac{\zeta_1}{u_2^*} + \frac{\eta_1}{2}\right)\mathcal{F}_1(t)$ for all $t > t_3$, which together with (3.2) implies that

$$\frac{d}{dt}\mathcal{E}_1(t) \le -\varepsilon_1 \mathcal{F}_1(t) \le -c_1\mathcal{E}_1(t), \quad \forall t > t_3, \tag{3.42}$$

where $c_1 = \frac{2u_1^* u_2^* \varepsilon_1}{2\zeta_1 u_1^* + 2u_2^* + \eta_1 u_1^* u_2^*}$. Therefore, invoking Gronwall's inequality to (3.42) and noting $\mathcal{E}_1(t_3)$ is bounded, we obtain

$$\mathcal{E}_1(t) \le c_2 e^{-c_1 t}, \quad \forall t > t_3,$$

which combined with (3.40) and the left inequality of (3.41) gives

$$\int_\Omega (u_1 - u_1^*)^2 + \int_\Omega (u_2 - u_2^*)^2 + \int_\Omega (v - v^*)^2 \le c_3 e^{-c_1 t}, \quad \forall t > t_3. \tag{3.43}$$

Choosing $p = 2$ in Lemma 3.6 and using the boundedness of $\|v\|_{W^{1,\infty}}$ in Theorem 1.1, we have $\|u_1(\cdot,t)\|_{W^{1,4}} + \|u_2(\cdot,t)\|_{W^{1,4}} + \|v(\cdot,t)\|_{W^{1,4}} \le c_4$ for all $t > t_3$. Accordingly, applying the Gagliardo–Nirenberg inequality to $u_1 - u_1^*$, one has

$$\|u_1 - u_1^*\|_{L^\infty} \le c_5(\|\nabla u_1\|_{L^4}^{\frac{2}{3}}\|u_1 - u_1^*\|_{L^2}^{\frac{1}{3}} + \|u_1 - u_1^*\|_{L^2}) \le c_6\|u_1 - u_1^*\|_{L^2}^{\frac{1}{3}}. \tag{3.44}$$

Using the similar arguments, we can derive

$$\|u_2 - u_2^*\|_{L^\infty} \le c_7\|u_2 - u_2^*\|_{L^2}^{\frac{1}{3}} \text{ and } \|v - v^*\|_{L^\infty} \le c_8\|v - v^*\|_{L^2}^{\frac{1}{3}}. \tag{3.45}$$

The combination of (3.43)–(3.45) shows

$$\|u_1 - u_1^*\|_{L^\infty} + \|u_2 - u_2^*\|_{L^\infty} + \|v - v_*\|_{L^\infty} \le c_9 e^{-\frac{c_1}{6}t},$$

which yields (3.38) with $\lambda_1 = \frac{c_1}{6}$ for $t > t_3$. Furthermore, by choosing the constant C_1 large enough, the result (3.38) becomes valid for $t > 0$. □

Next, we show the decay rate of solutions for the case $a_1 > 1, a_2 \in (0, 1)$, in which case the solution will converge to $(0, 1, b_2)$.

Lemma 3.8 *(Decay rate: $a_1 > 1, a_2 \in (0, 1)$). Let the assumptions in Lemma 3.3 hold true. Suppose $a_1 > 1$. Then there exist two positive constants λ_2 and C_2, such that*

$$\|u_1(\cdot,t)\|_{L^\infty} + \|u_2(\cdot,t) - 1\|_{L^\infty} + \|v(\cdot,t) - b_2\|_{L^\infty} \le C_2 e^{-\lambda_2 t} \tag{3.46}$$

holds for all $t > 0$.

Proof. Using a similar argument as in obtaining (3.41), we can find a $t_0 > 1$ such that

$$\frac{1}{4}\int_\Omega (u_2-1)^2 \le \int_\Omega (u_2 - 1 - \ln u_2) \le \int_\Omega (u_2-1)^2, \quad \forall t > t_0. \tag{3.47}$$

Defined $\mathcal{F}_3(t) := \int_\Omega u_1 + \int_\Omega (u_2-1)^2 + \int_\Omega (v-b_2)^2$. In view of (3.13), it follows from the right inequality of (3.47) that there exists a constant $c_1 > 0$ such that

$$\mathcal{E}_2(t) \le c_1 \mathcal{F}_3(t), \quad \forall t > t_0. \tag{3.48}$$

Noting that $a_1^* > 1$ and $0 < a_2 < 1$, and combining (3.14) with (3.48), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_2(t) &\le -\varepsilon_2 \mathcal{F}_2(t) - \mu_1(a_1^*-1)\int_\Omega u_1 \\ &\le -c_2\mathcal{F}_3(t) \le -c_3\mathcal{E}_2(t), \quad \forall t > t_0,\end{aligned} \tag{3.49}$$

which, together with Gronwall inequality, gives

$$\mathcal{E}_2(t) \le c_4 e^{-c_3 t}, \quad \forall t > t_0$$

due to the fact $\mathcal{E}_2(t_0)$ is bounded. Then we use the definition of $\mathcal{E}_2(t)$ in (3.13) and the left inequality of (3.47) to derive

$$\int_\Omega u_1 + \int_\Omega (u_2-1)^2 + \int_\Omega (v-b_2)^2 \le c_5 e^{-c_3 t}, \quad \forall t > t_0. \tag{3.50}$$

Moreover, Lemma 3.6 shows that $\|\nabla u_1(\cdot,t)\|_{L^4} + \|\nabla u_2(\cdot,t)\|_{L^4} \le c_6$ for all $t > t_0$. Using Theorem 1.1 and the Gagliardo–Nirenberg inequality, we derive that

$$\|u_1\|_{L^\infty} \le c_7\left(\|\nabla u_1\|_{L^4}^{\frac{4}{5}}\|u_1\|_{L^1}^{\frac{1}{5}} + \|u_1\|_{L^1}\right) \le c_8\|u_1\|_{L^1}^{\frac{1}{6}}. \tag{3.51}$$

Similarly, one has

$$\|u_2-1\|_{L^\infty} \le c_9\left(\|\nabla u_2\|_{L^4}^{\frac{2}{3}}\|u_2-1\|_{L^2}^{\frac{1}{3}} + \|u_2-1\|_{L^2}\right) \le c_{10}\|u_2-1\|_{L^2}^{\frac{1}{3}} \tag{3.52}$$

and

$$\|v-b_2\|_{L^\infty} \le c_{11}\left(\|\nabla v\|_{L^4}^{\frac{2}{3}}\|v-b_2\|_{L^2}^{\frac{1}{3}} + \|v-b_2\|_{L^2}\right) \le c_{12}\|v-b_2\|_{L^2}^{\frac{1}{3}}. \tag{3.53}$$

Combining (3.51)–(3.53) with (3.50) gives that

$$\|u_1\|_{L^\infty} + \|u_2-1\|_{L^\infty} + \|v-b_2\|_{L^\infty} \le c_{13}e^{-\frac{c_3 t}{6}}, \quad \forall t > t_0,$$

which implies that (3.46) holds by choosing C_2 large enough. Then the proof of Lemma 3.8 is completed. □

When $a_1 = 1$, then $a_1^* = 1$, then the term $-\mu_1(a_1^* - 1)\int_\Omega u_1$ in (3.14) will disappear. Then we can not use (3.14) to obtain the inequality as in (3.49), which is important to obtain the exponential decay. However, in fact we can still obtain the algebraical decay as follows.

Lemma 3.9. *Suppose the assumptions in Lemma 3.3 hold and $a_1 = 1$. There exist positive constants C_3 and λ_3, such that*

$$\|u_1(\cdot,t)\|_{L^\infty} + \|u_2(\cdot,t) - 1\|_{L^\infty} + \|v(\cdot,t) - b_2\|_{L^\infty} \le C_3(t+1)^{-\lambda_3} \tag{3.54}$$

holds true for all $t > 0$.

Proof. Since $a_1^* = 1$, then (3.14) can be rewritten as

$$\frac{d}{dt}\mathcal{E}_2(t) \le -\varepsilon_2 \mathcal{F}_2(t), \quad \forall t > 0. \tag{3.55}$$

Moreover, noting the definition of $\mathcal{E}_2(t)$ in (3.13) and the fact (3.47), and using Hölder inequality and the boundedness of u_1, u_2 and v, we have

$$\begin{aligned}\mathcal{E}_2(t) \le & c_1\left(\int_\Omega u_1 + \int_\Omega (u_2-1)^2 + \int_\Omega (v-b_2)^2\right)\\ \le & c_1|\Omega|^{\frac12}\left(\int_\Omega u_1^2\right)^{\frac12} + c_2\left(\int_\Omega (u_2-1)^2\right)^{\frac12} + c_3\left(\int_\Omega (v-b_2)^2\right)^{\frac12}\\ \le & c_4\mathcal{F}_2^{\frac12}(t),\end{aligned}$$

which together with (3.55) gives

$$\frac{d}{dt}\mathcal{E}_2(t) \le -\varepsilon_2\mathcal{F}_2(t) \le -\frac{\varepsilon_2}{c_4^2}\mathcal{E}_2^2(t), \quad \forall t > t_0. \tag{3.56}$$

Then solving the ODI (3.56), we end up with

$$\mathcal{E}_2(t) \le \frac{c_3}{t+1}, \quad \forall t > t_0. \tag{3.57}$$

Recalling the inequalities (3.47), (3.57) allows us to find a constant $c_4 > 0$ such that

$$\int_\Omega u_1 + \int_\Omega (u_2-1)^2 + \int_\Omega (v-b_2)^2 \le \frac{c_4}{t+1}, \quad \forall t > t_0. \tag{3.58}$$

By the similar way in Lemma 3.8, we apply the Gagliardo–Nirenberg inequality to derive that

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$$\|u_1\|_{L^\infty} + \|u_2 - 1\|_{L^\infty} + \|v - b_2\|_{L^\infty} \le c_4 \left(\|u_1\|_{L^1}^{\frac{1}{6}} + \|u_2 - 1\|_{L^2}^{\frac{1}{3}} + \|v - b_2\|_{L^2}^{\frac{1}{3}} \right)$$

for all $t > t_0$, which together with (3.58) gives (3.54) by picking a suitably large constant. □

Proof of Theorem 1.2. The combination of Lemmas 3.7–3.9 gives Theorem 1.2. □

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