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Exponential decay and symmetry of solitary waves to Degasperis-Procesi equation

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Abstract

We improve the decay argument by Bona and Li (1997) [5] for solitary waves of general dispersive equations and illustrate it in the proof for the exponential decay of solitary waves to steady Degasperis-Procesi equation in the nonlocal formulation. In addition, we give a method which confirms the symmetry of solitary waves, including those of the maximum height. Finally, we discover how the symmetric structure is connected to the steady structure of solutions to the Degasperis-Procesi equation, and give a more intuitive proof for symmetric solutions to be traveling waves. The improved argument and new method above can be used for the decay rate of solitary waves to many other dispersive equations and will give new perspectives on symmetric solutions for general evolution equations.

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1. Introduction

The Degasperis-Procesi (DP) equation

$$u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0 \tag{1.1}$$

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is a unidirectional model for shallow water waves (see [15]) and can be reformulated as a nonlocal equation

$$\partial_t u + u\partial_x u + \partial_x L(\frac{3}{2}u^2) = 0, \tag{1.2}$$

where the dispersive operator $L = (1-\partial_x^2)^{-1}$ corresponds to the Fourier symbol $m(\xi) = (1+\xi^2)^{-1}$ and a convolution kernel function $K(x) = \frac{1}{2}e^{-|x|}$. Being completely integrable and having bi-Hamiltonian structure [15], this equation together with KdV and Camassa-Holm are three well-known representatives in both integrable system theory and water wave problems. Although firstly put forward from the perspective of integrability, this model was later rigorously derived as a model for shallow water waves and proved to have the same accuracy as the Camassa-Holm equation [13]. The Degasperis-Procesi equation is locally well-posed in the classical Sobolev space H^s, $s > \frac{3}{2}$, in both periodic and non-periodic settings [28], and it allows global weak and classical solutions [29,27] as well as solutions which blow up in the form of wave-breaking [19]. Soliton solutions of Degasperis-Procesi equation can be found by inverse scattering technique [16,12]. Later, traveling wave solutions (both periodic and solitary) to (1.1) were found in [26], and Lenells classified in [22] all possible traveling wave solutions, which include smooth waves, peaked waves, cusped waves, stumped waves and their reasonable composition. Very recently, Arnesen [4] worked on the non-local formulation (1.2) and proved that differentiable, symmetric traveling solutions with uniform bound have the wave speed as the upper bound and are smooth when wave height is strictly smaller than wave speed c. In addition, crests of periodic waves will turn to peaks when the wave height reaches the wave speed.

This paper focuses on solitary waves (steady solutions with decay at infinity) of the Degasperis-Procesi equation and the motivation comes from several aspects. Firstly, Bona and Li studied in [5] the decay and analyticity of solitary waves to a class of evolution equations in the steady form

$$f = k * G(f) \tag{1.3}$$

where k denotes the convolution kernel function, and $G(\cdot)$ is locally bounded and has superlinear growth. The procedure of proving exponential decay of solitary waves mainly involves two steps: step 1 for algebraic decay in some $L^p(\mathbb{R})$ spaces and step 2 for a delicate control of L^1 norm of $|x|^n\phi(x)$, $n \in \mathbb{N}$, to guarantee exponential decay. We hope to simplify this two-step procedure. In fact, the following polynomial type convolution estimate (see also other similar estimates in [5, Lemma 3.1.1])

$$\int_0^\infty \frac{|x|^l}{(1+\epsilon|x|)^m(1+|y-x|)^m}\mathrm{d}x \le B\frac{|y|^l}{(1+\epsilon|y|)^m}, \quad |y| \ge 1, \tag{1.4}$$

is the key for the algebraic decay in [5]. We improve this polynomial type estimate to exponential type estimate. In this way, the algebraic decay estimate of solitary waves can be skipped in the argument by Bona and Li and we can prove the exponential decay directly. In view that the improved exponential type estimate (see Lemma 2.6 below) are independent of the form of dispersive equations, it is expected to simplify the proof for exponential decay of solitary waves for more general dispersive equations as (1.4) does for algebraic decay.

The second aspect for motivation is related to symmetry issues of the highest solitary wave to nonlinear dispersive equations. Traveling waves solutions are often studied by *a priori* assuming that they are even or symmetric, and it raises the question whether there exist asymmetric traveling waves. For dispersive equations where complete integrity is unknown, the inverse scattering technique for obtaining exact solutions will fail. In this case, the symmetry of solutions is often obtained by the classical method of moving planes put forward by Aleksandrov [1] and Serrin [25] (see also [14] about this method for water waves). However, two obstacles will appear when applying the method of moving planes: one is to remove the *a priori* monotonicity condition on solitary waves (essentially, this condition assumes that the wave has only one crest); the other is to prove the symmetry for the solitary wave of the maximum height. These difficulties can be well-illustrated by the symmetry problem of supercritical solitary wave solutions to the steady Whitham equation (see [7])

$$\phi(c-\phi) = K_w * \phi^2, \tag{1.5}$$

where K_w denotes the kernel function for the Whitham equation. The monotonicity condition on solitary waves was removed by using the exponential decay of solitary waves and inspired by the idea in [8] for integral equations induced from fractional Laplacian. However, when the solitary wave ϕ reaches the maximum height $\frac{c}{2}$ at the crest, the left side of (1.5) will generate a factor $c - \phi(x) - \phi(2\lambda - x)$ for some $\lambda \in \mathbb{R}$ close to that crest. This factor approaches 0 as x approaches λ and causes singularity when it is moved to the right side of the equation. In this case the argument for symmetry in [6] fails to give a contradiction. This obstacle also appears for DP equation in the nonlocal form when ϕ reaches the maximum height c. In this paper, we get around this obstacle by studying the local structure of the solitary wave near the crest $\phi = c$, and then manage to prove the symmetry also for the highest wave. This new idea is expected to work after modification for the symmetry of the highest solitary wave to the Whitham and other dispersive equations. It is worth to point out that the appearance of a peak at the crest for the highest wave is in line with the fact that Stokes waves of the extreme form present peaks at crests for the governing equation of water waves ([2,3], see also [9,10] for the motion of particles in Stokes waves).

The third aspect of motivation comes from the classification of symmetric solutions to general evolution equations. In [17], the authors put forward a principle for a class of equations for which solutions with *a priori* spatial symmetry must be traveling waves. This principle was later extended to cover nonlocal equations and differential systems in [6], where two new principles were also found. The Degasperis-Procesi equation satisfies the principle in [17] so that symmetric solutions must be traveling waves. The beautiful proof in [17], however, is quite constructive and does not give further information about how symmetric structure is related to the steady structure of those waves. In this paper, we study the two restriction conditions that symmetric solutions satisfy and find that each of them determines one aspect of the steady structure of these solutions: the fixed shape of wave profile and the constant propagation speed. In this way, we give a more intuitive, straightforward proof for symmetric solutions to be traveling waves. This idea can be used for a family of equations whose structure satisfies Principle P1 in [17,6], including KdV and Benjamin-Ono equation. The corresponding results for symmetric waves to be traveling waves for the full water wave problem can be found in [17,6,21].

The final aspect of motivation comes from the classification of solitary waves to the Degasperis-Procesi equation. Inserting the ansatz $u(t,x) = \phi(x-ct)$ for traveling wave solutions into (1.1), one obtains the Degasperis-Procesi equation in steady form with some integration parameter a. According to the value of a, all possible traveling wave solutions, periodic or solitary, were completely classified by Lenells in [22], including smooth waves, peaked waves, cusped waves, stumped waves and their proper composition. In this paper, we work on DP equation in the nonlocal form (1.2) and get the following steady equation

$$\frac{\phi}{3}(2c-\phi) = L\phi^2 + a, \tag{1.6}$$

where a denotes the integration constant. Unlike the Whitham equation and many others, it is not possible to use Galiean transformation to remove the constant a in (1.6). However, we prove that the constant a must be trivially 0 for solitary waves with decay (meaning that $\phi(x) \to 0$ in as $|x| \to \infty$) so that these waves actually solve the steady equation

$$\frac{\phi}{3}(2c-\phi) = L\phi^2. \tag{1.7}$$

In addition, we prove that these waves are symmetric with respect to the only symmetric axis at crest at some point and are strictly monotone on each side of the crest. Therefore, a solitary solution ϕ with decay only has one crest at a single point, excluding stumped solutions in [22] and the possibility to compose solitary waves with different propagation speeds into new solitary waves.[1] Moreover, the peaked wave defined and found in [22] is only locally symmetric at the peak of a solitary wave, so our result improves this local symmetry near the peak to global symmetry for the whole solitary wave. It is worth to point out that this finding does not contradict with the fact that the Degasperis-Procesi allows for multipeakon solutions [11,23], which are not steady solutions.

We now state the structure of this paper. Section 2 starts with an estimate where the kernel $K(\cdot)$ is convoluted with exponential type functions. Based on this lemma, we prove that solitary solutions decay exponentially at infinity and the decay rate is at least as good as the decay rate of the kernel $K(\cdot)$. Section 3 focuses on the symmetry of solitary waves. In particular, we prove symmetry for solitary waves with height smaller than the wave speed in section 3.1, while the wave with the maximum height are treated in section 3.2. Finally, we give a new proof in section 4 for the classification principle that classical symmetric solutions to the Degasperis-Procesi equation must be traveling wave solutions.[2]

2. Exponential decay of solitary waves at infinity

For a traveling wave solution $u(t,x) = \phi(x-ct)$ with speed c, the sign of c distinguishes only the direction of the propagation of the wave. So, we will only work with $c > 0$ in the following. As mentioned above, direct calculation by Fourier analysis gives that

$$\mathscr{F}[Lf](\xi) = \frac{1}{1+\xi^2}\mathscr{F}[f](\xi) = \mathscr{F}[K * f](\xi), \tag{2.1}$$

[1] This is because solitary waves with different propagation speeds will separate from each other during later propagation so that their composition will not be a solitary solution to the steady equation (1.7).

[2] Such classification principle could also be formulated similarly in the weak setting with distribution theory, see [17,20], but it is not our focus here.

where $\mathscr{F}$ denotes the usual Fourier transform and $K(x) = \frac{1}{2}e^{-|x|}$ denotes the convolution kernel of L. By definition, the operator L lifts a L^∞-bounded function to a continuous function (see [18,4] for details), so we will work with continuous solutions in the following. We also need some elementary concepts from topology (see [24] for details). A pointed space is a topological space with a distinguished point called *basepoint*. A map g from a pointed space (X, x_0) to another pointed space (Y, y_0) is a *homomorphism* if g is a continuous map from X to Y and preserves the basepoints, namely $g(x_0) = y_0$. In particular, we call g a homomorphism on (X, x_0) if it is a homomorphism from (X, x_0) to itself. We can choose the origin as basepoint so that $(\mathbb{R}, 0)$ forms a pointed space with the usual Euclidean metric topology. We start with the proof for integration constant a to vanish for solitary waves to steady Degasperis-Procesi equation (1.6), which follows directly from the lemma below for the structure of general convolution equations.

Lemma 2.1. *Let G be a homomorphism on the pointed space $(\mathbb{R}, 0)$. Let $k \in L^1(\mathbb{R})$ decay at infinity and H be a continuous function on $\mathbb{R}$. If the following convolution equation*

$$f = k * G(f) + H(f) \tag{2.2}$$

has a solution $f(x)$ which is continuous and decays at infinity. Then, H is a homomorphism on $(\mathbb{R}, 0)$.

Proof. It suffices to prove that H preserves the origin as the basepoint, i.e., $H(0) = 0$. Since $f(x)$ decays at infinity, we only need to prove that $k * G(f)$ vanishes as $|x| \to \infty$ in (2.2). Note that

$$k * G(f)(x) = \int\limits_{|x-y|<N} k(y)[G(f)](x-y)\mathrm{d}y + \int\limits_{|x-y|>N} k(y)[G(f)](x-y)\mathrm{d}y \tag{2.3}$$

for some $N > 0$. For any small $\eta > 0$, we can choose N large enough such that $|f(x)| < \frac{\eta}{2}$ for all $|x| > N$. Then, we have

$$\left| \int\limits_{|x-y|>N} k(y)[G(f)](x-y)\mathrm{d}y \right| < \left| \sup_{|f|<\frac{\eta}{2}} G(f) \right| \int\limits_{|x-y|>N} |k(y)|\mathrm{d}y \le \left| \sup_{|f|<\frac{\eta}{2}} G(f) \right| \|k\|_{L^1(\mathbb{R})}. \tag{2.4}$$

Note that k decays at infinity, so we can fix the above N and η, and choose $M_1 > 0$ large enough such that $k(y) < \frac{\eta}{8NG(\|f\|_{L^\infty(\mathbb{R})})}$ for all $|y| > M_1$. Then, for any y such that $|x-y| < N$ and $|x| > M_1 + N$, we have

$$M_1 < |x| - N < |y| < |x| + N.$$

Therefore, for $|x| > M_1 + N$, we have

$$\left| \int\limits_{|x-y|<N} k(y)[G(f)](x-y)\mathrm{d}y \right| < 2NG(\|f\|_{L^\infty(\mathbb{R})}) \sup_{|y|>M_1} k(y) < \frac{\eta}{4}. \tag{2.5}$$

Now, for any small $\epsilon > 0$, we can choose $\eta < \epsilon$ sufficiently small so that $\left|\sup_{|f|<\frac{\eta}{2}} G(f)\right| < \frac{\epsilon}{4\|k\|_{L^1(\mathbb{R})}}$ due to the fact that G is a homomorphism on $(\mathbb{R}, 0)$. Then, we insert (2.3), (2.4), (2.5) into (2.2), and get

$$|H(f)(x)| \leq |f(x)| + |k * G(f)(x)| < \epsilon \tag{2.6}$$

for all $|x| > M_1 + N$. The lemma then follows directly from the decay of f at infinity and the continuity of H. $\square$

Remark 2.2. The Galilean transform as a usual trick to remove integration constants fails here. The idea in the Lemma 2.1 is expected to work for more general settings where the kernel function is integrable and has decay at infinity, such as the Whitham equation [6].

As a direct consequence of Lemma 2.1, we have the following corollary for the integration constant a to be trivially 0.

Corollary 2.3. *The integration constant a in* (1.6) *vanishes for continuous solitary waves with decay.*

Proof. By using Lemma 2.1 with $G(\phi) = \phi^2$ and $H(f)(x) = a$, we see that $\lim_{x\to\infty} H(f)(x) = 0$ which implies $a = 0$. $\square$

To proceed, we first give the lower and upper bounds of solitary waves.

Lemma 2.4. *Nontrivial continuous solitary waves with decay to* (1.7) *satisfy*

$$0 < \phi \leq \sup_{x\in\mathbb{R}} \phi < 2c. \tag{2.7}$$

Proof. The strict positiveness of $K(x)$ implies that L is a strictly monotone operator on continuous bounded functions, i.e., $Lf > Lg$ if $f \geq g$ but $f \neq g$. In addition, straightforward calculation shows that $LC = C$ for any constant C. Therefore, we derive from (1.2) that

$$\phi^2 - 2c\phi = -3L\phi^2 < 0, \tag{2.8}$$

which implies $\phi \in (0, 2c)$. The decay of ϕ indicates that $\sup_{x\in\mathbb{R}} \phi$ must be reached at some finite $x_0 \in \mathbb{R}$ so that (2.7) follows. $\square$

Remark 2.5. A recent work [4] by Arnesen shows that all L^∞-bounded traveling waves has wave speed c as upper bound. However, we do not need this better upper bound for the estimate of decay rate of solitary waves.

With the above bounds for solitary waves ready, the decay argument by Bona-Li in [5] could be used to prove the exponential decay for solitary waves. To proceed, we recall the two-step procedure by Bona and Li: firstly derive algebraic decay of solitary waves; then improve the algebraic decay to exponential decay by making delicate control of some $L^p(\mathbb{R})$ norm of solitary waves with monomial weight $|x|^n$ for each $n \in \mathbb{N}$. The key in the algebraic decay is a convolution estimate for functions of polynomial type. In fact, let $F_1(x)$ and $F_2(x)$ be given by $F_1(x) := \frac{|x|^l}{(1+\sigma|x|)^m}$ and $F_2(x) := (1+|x|)^{-m}$. Then, it is proved essentially by Bona and Li that

$$F_1 * F_2(x) \lesssim F_1(x) \tag{2.9}$$

where $\lesssim$ means $\leq$ up to some constant relying on the indices l and m. Intuitively, this statement claims that the convolution of two polynomial functions of negative order could be controlled by the one with higher order. We find that this philosophy also holds if polynomials are replaced by exponential functions in proper formulation. In particular, let $G_1(x) := \frac{e^{l|x|}}{(1+\sigma e^{|x|})^m}$ and $G_2(x) := e^{-m|x|}$. Then it is true that

$$G_1 * G_2(x) \lesssim G_1(x). \tag{2.10}$$

With this new estimate (2.10) for exponential functions, neither the algebraic decay of solitary waves nor the delicate control of the $L^1(\mathbb{R})$ norm of $|x|^n\phi$ for each $n \in \mathbb{N}$ is needed, while the exponential decay of solitary waves could be directly obtained. In this way, the proof for exponential decay can be considerably simplified. We formulate the new estimate for exponential functions in the following Lemma.

Lemma 2.6 *(Convolution estimate of exponential type). For $0 < l < m$ and any $\sigma > 0$, the following inequatlity holds*

$$\int_{\mathbb{R}} \frac{e^{l|x|}}{(1+\sigma e^{|x|})^m e^{m|x-y|}} \mathrm{d}x \leq B \frac{e^{l|y|}}{(1+\sigma e^{|y|})^m}, \quad y \in \mathbb{R}, \tag{2.11}$$

where $B = (\min\{l, m-l\})^{-1}$.

Proof. By symmetry of the structure in (2.11), it suffices to prove for the case $y > 0$. Note that

$$\int_0^\infty \frac{e^{l|x|}}{(1+\sigma e^{|x|})^m e^{m|x-y|}} \mathrm{d}x = \left(\int_0^y + \int_y^\infty\right) \frac{e^{lx}}{(1+\sigma e^{x})^m e^{m|x-y|}} \mathrm{d}x =: I_1 + I_2.$$

For I_1, we have

$$I_1 = \int_0^y \frac{e^{lx}}{(1+\sigma e^x)^m e^{m(y-x)}} \mathrm{d}x \leq \frac{e^{ly}-1}{e^{my}(\sigma + e^{-y})^m l} \leq \frac{e^{ly}}{l(1+\sigma e^y)^m}.$$

For I_2, we have

$$I_2 = \int_y^\infty \frac{e^{lx}}{(1+\sigma e^x)^m e^{m(x-y)}} \mathrm{d}x \leq \frac{e^{my}}{(1+\sigma e^y)^m} \int_y^\infty e^{(l-m)x} \mathrm{d}x \leq \frac{(m-l)^{-1} e^{ly}}{(1+\sigma e^y)^m}.$$

On the other hand, we have

$$\int_{-\infty}^{0} \frac{e^{l|x|}}{(1+\sigma e^{|x|})^m e^{m|x-y|}} \mathrm{d}x = \left(\int_{0}^{y} + \int_{y}^{\infty} \right) \frac{e^{lx}}{(1+\sigma e^{x})^m e^{m(y+x)}} \mathrm{d}x =: I_3 + I_4.$$

For I_3, we have

$$I_3 \leq \frac{e^{-my}}{(\sigma + e^{-y})^m} \frac{1}{2m-l} (1 - e^{(l-2m)y}) < \frac{e^{ly}}{(\sigma e^y + 1)^m} \frac{1}{2m-l}, \tag{2.12}$$

where in the last inequality we used the fact $0 < 1 - e^{(l-2m)y} < e^{ly}$. For I_4, we have

$$I_4 \leq \frac{e^{-my}}{(\sigma e^y + 1)^m} \frac{1}{m-l} e^{(l-m)y} < \frac{e^{ly}}{(\sigma e^y + 1)^m} \frac{1}{m-l}. \tag{2.13}$$

The inequality (2.11) and hence this lemma follow directly. □

We now illustrate how the estimate of exponential type could be used to prove directly the exponential decay of solitary solutions ϕ to the Degasperis-Procesi equation (1.2). For convenience, we introduce the notation $M := \sup_{x\in\mathbb{R}} \phi$.

Theorem 2.7 *(Exponential decay of solitary waves). The image of the map $x \mapsto e^{|x|}\phi(x)$ is a bounded, simply connected set in $[0, \infty)$.*

Proof. We first prove that

$$e^{\alpha|\cdot|}\phi(\cdot) \in L^q(\mathbb{R}) \tag{2.14}$$

holds for any $\alpha \in (0, 1)$ and $q > 1$. Since $e^{\alpha|x|}K(x) \in L^p(\mathbb{R})$ for any $\alpha \in (0, 1)$ and $p > 0$, we can introduce a constant $C_{\alpha,p}$ given by

$$C_{\alpha,p} := 3(2c - M)^{-1} \| e^{\alpha|\cdot|} K(\cdot) \|_{L_p(\mathbb{R})},$$

where p is chosen to be the conjugate of q, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. By (1.7) and Hölder's inequality, we have

$$\phi = \frac{3}{2c - \phi} \int_{\mathbb{R}} \left[K(x-y) e^{\alpha|x-y|} \right] \frac{\phi^2(y)}{e^{\alpha|x-y|}} \mathrm{d}y \leq C_{\alpha,p} \left(\int_{\mathbb{R}} \frac{|\phi^2(y)|^q}{e^{\alpha q|x-y|}} \mathrm{d}y \right)^{\frac{1}{q}}. \tag{2.15}$$

Let $l \in [0, \alpha)$ and define

$$h_\varepsilon(x) := \frac{e^{l|x|}}{(1 + \epsilon e^{|x|})^\alpha} \phi(x) \tag{2.16}$$

for small $\varepsilon \in (0, 1)$. Then, for each fixed $\varepsilon \in (0, 1)$, the function h_ε is bounded in $L_q(\mathbb{R})$ by the choice of l and boundedness of ϕ. We now prove that $\{h_\varepsilon \mid \varepsilon \in (0, 1)\}$ is uniformly bounded in $L_q(\mathbb{R})$, which then implies that $\lim_{\varepsilon\to 0} h_\varepsilon = e^{l|x|}\phi$ belongs to $L_q(\mathbb{R})$ by dominated convergence and confirms (2.14).

Since ϕ tends to zero as $|x| \to \infty$, the quadratic nonlinearity guarantees that for every $\delta > 0$ there exists a constant $R_\delta > 1$ such that

$$|\phi^2(x)| \le \delta|\phi(x)| \qquad \text{for} \quad |x| \ge R_\delta.$$

Since

$$\|h_\varepsilon\|^q_{L_q(\mathbb{R})} = \int_{\mathbb{R}} |h_\varepsilon(x)|^q \,\mathrm{d}x \le C + \int_{|x|\ge R_\delta} |h_\varepsilon(x)|^q \,\mathrm{d}x, \tag{2.17}$$

where $C = C(R_\delta) > 0$ is a constant independent of ε, it suffices to study the last integral on the right-hand side of (2.17).

Let $r \in (0, q)$. By (2.15) and Hölder's inequality, we have

$$\begin{aligned}
\int_{|x|\ge R_\delta} |h_\varepsilon(x)|^q \mathrm{d}x &\le \int_{|x|\ge R_\delta} |h_\varepsilon(x)|^{q-r} \left(\frac{e^{l|x|}}{(1+\epsilon e^{|x|})^\alpha}\right)^r |\phi(x)|^r \mathrm{d}x \\
&\le \int_{|x|\ge R_\delta} |h_\varepsilon(x)|^{q-r} \left(\frac{e^{l|x|}}{(1+\epsilon e^{|x|})^\alpha}\right)^r C^r_{\alpha,p}\left(\int_{\mathbb{R}} \frac{|\phi^2(y)|^q}{e^{\alpha q|x-y|}}\mathrm{d}y\right)^{\frac{r}{q}} \mathrm{d}x \\
&\le C^r_{\alpha,p}\left[\int_{|x|\ge R_\delta} |h_\varepsilon(x)|^q \mathrm{d}x\right]^{\frac{q-r}{q}} \left[\int_{|x|\ge R_\delta} \frac{e^{lq|x|}}{(1+\epsilon e^{|x|})^{\alpha q}} \left(\int_{\mathbb{R}} \frac{|\phi^2(y)|^q}{e^{\alpha q|x-y|}}\mathrm{d}y\right) \mathrm{d}x\right]^{\frac{r}{q}}.
\end{aligned}$$

Dividing both sides of the inequality by $\left[\int_{|x|\ge R_\delta} |h_\varepsilon(x)|^q \mathrm{d}x\right]^{\frac{q-r}{q}}$, we find that[3]

$$\int_{|x|\ge R_\delta} |h_\varepsilon(x)|^q \mathrm{d}x \le C^q_{\alpha,p} \int_{|x|\ge R_\delta} \frac{e^{lq|x|}}{(1+\epsilon e^{|x|})^{\alpha}} \left(\int_{\mathbb{R}} \frac{|\phi^2(y)|^q}{e^{\alpha q|x-y|}}\mathrm{d}y\right) \mathrm{d}x =: C^q_{\alpha,p} T. \tag{2.18}$$

By Fubini's theorem and Lemma 2.6, we obtain that

[3] Note that the term we are dividing by vanishes if and only if $\phi = 0$ everywhere in $\{|x| \ge R_\delta\}$, in which case the lemma is obviously true.

$$T = \int_{\mathbb{R}} |\phi^2(y)|^q \Bigg[\int_{|x|\geq R_\delta} \frac{e^{lq|x|}}{(1+\epsilon e^{|x|})^{\alpha q} e^{\alpha q|x-y|}} \mathrm{d}x \Bigg] \mathrm{d}y$$

$$\leq \int_{|y|\geq R_\delta} |\phi^2(y)|^q \frac{B e^{lq|y|}}{(1+\epsilon e^{|y|})^{\alpha q}} \mathrm{d}y + \int_{|y|<R_\delta} |\phi^2(y)|^q \int_{|x|\geq R_\delta} \frac{e^{lq|x|}}{(1+\epsilon e^{|x|})^{\alpha q} e^{\alpha q|x-y|}} \mathrm{d}x\mathrm{d}y, \tag{2.19}$$

where $B = B(l, q, \alpha) > 0$ does not depend on ε. Since $0 < l < \alpha$, the last integral in (2.19) is bounded by a constant C_1 which depends on $l, \alpha, q, \|\phi\|_\infty$ and R_δ but independent of ε. Combining (2.18), (2.19) and in view that $|\phi^2(y)| < \delta|\phi(y)|$ for all $|y| \geq R_\delta$, we have

$$\int_{|x|\geq R_\delta} |h_\varepsilon(x)|^q \mathrm{d}x \leq C_{\alpha,p}^q \Bigg[\delta^q B \int_{|x|\geq R_\delta} |h_\varepsilon(x)|^q \mathrm{d}x + C_1 \Bigg]. \tag{2.20}$$

For δ small enough so that $C_{\alpha,p}^q \delta^q B < \frac{1}{2}$, (2.20) implies that

$$\int_{|x|\geq R_\delta} |h_\varepsilon(x)|^q dx \leq C_2,$$

where $C_2 = C_2(l, \alpha, p, \|\phi\|_\infty, R_\delta) > 0$ is a constant which does not rely on ε.

Hence, we have shown that

$$\int_{\mathbb{R}} |h_\varepsilon(x)|^q dx \lesssim 1.$$

Letting $\varepsilon \to 0$, the dominated convergence theorem ensures that

$$\int_{\mathbb{R}} e^{lq|x|} |\phi(x)|^q dx \lesssim 1,$$

which implies in particular $x \mapsto e^{l|x|} f(x) \in L_q(\mathbb{R})$ for $q = \frac{p}{p-1}$ and $l \in [0, \alpha)$, and therefore confirms (2.14).

We now prove that solitary waves decay exponentially by using (2.14) and Young's inequality in the steady DP equation (1.7). Note that

$$e^{\alpha|x|}\phi(x) \lesssim \frac{3}{2c-M} \left[\left(e^{\alpha|\cdot|} K(\cdot) \right) * \left(e^{\alpha|\cdot|} \phi^2(\cdot) \right) \right](x) \in L^\infty(\mathbb{R}) \tag{2.21}$$

for any $\alpha \in (0, 1)$. With this decay estimate, we can use the structure of the DP equation to improve the decay rate to cover the case $\alpha = 1$ so that ϕ decays at least as good as the kernel K. In fact, we have

$$e^{|x|}\phi(x) \leq \frac{1}{2c-M} \int_{\mathbb{R}} K(x-y) e^{|x-y|} \left(\phi(y) e^{\frac{|y|}{2}} \right)^2 dy \leq \|e^{|\cdot|} K(\cdot)\|_{L^\infty} \|\phi e^{\frac{|\cdot|}{2}}\|_{L^2}^2 < \infty.$$

The above shows that $\phi(x)$ decays as fast as $e^{-|x|}$ at infinity. The fact that the image $e^{|x|}\phi(x)$ forms a simply connected set follows from the continuity of ϕ. $\square$

Remark 2.8. This argument with exponential type convolution estimate is expected to work also for the Whitham and other dispersive equations in proper nonlocal formulation, where a kernel with exponential decay is convoluted with a superlinear nonlinearity.

3. Symmetry and one-crest structure of solitary waves

With the decay estimates above, we are ready to prove the symmetry for solitary waves to (1.2). Note that Arnesen recently studied in [4] the Degasperis-Procesi equation in the nonlocal formulation and proved that traveling waves had wave speed c as the upper bounded. In the following, we will prove that both solitary waves with height smaller than c and solitary waves of the maximum height c are symmetric, and have a unique crest, which in particular imply that a peaked solitary wave with decay only has one peak. For symmetry of solitary waves with height smaller than the wave speed c, our proof follows the idea in [7] for the Whitham equation, where the key observation is that the nonlocal operator L behaves as an elliptic operator and there exists a touching lemma on half-plane. This touching lemma plays the role as the maximum principle for elliptic equations. It is worth to mention that the way to remove the monotonicity assumption on solitary waves in [7] when using the method of moving planes is inspired by the work of Chen, Li and Ou [8] for symmetry of solutions to a class of integral equations induced by fractional Laplacian, although the idea of using Kelvin type transform in the latter fails to work in [7] due to inhomogeneity of the kernel function for the steady Whitham equation.

For solitary waves of the maximum height (see [18]), i.e., $\sup_{x\in\mathbb{R}}\phi(x)=c$, the argument in [7] unfortunately fails to confirm the symmetry. It seems that there exists no argument for confirming the symmetry of a solitary wave with wave speed equal to its height up to now for the Degasperis-Procesi equation,[4] so we put forward an argument here for waves of the maximum height and expect it to be effective also for symmetry issues of highest waves of other equations, like the Whitham equation in [7]. We first introduce the notion of supersolution and subsolution of the steady DP equation. A solution ϕ to the steady Degasperis-Procesi equation (1.7) is called a *supersolution* if

$$\frac{\phi}{3}(2c-\phi) \geq K * \phi^2$$

and a *subsolution* if the inequality above is replaced by $\leq$. With the supersolution and subsolution, we can prove the following touching lemma, which can be intuitively explained as: if a supersolution stays above a subsolution on a half plane (λ, ∞) for some $\lambda \in \mathbb{R}$, then the supersolution never touches the subsolution at any finite point unless they are equal on the whole half plane (λ, ∞).

Lemma 3.1 *(Touching lemma on a half plane). Let ϕ_1 and ϕ_2 be a supersolution and a subsolution of the steady Degasperis-Procesi equation* (1.7) *on a subset $[\lambda, \infty) \subset \mathbb{R}$, respectively, such that $\phi_1 \geq \phi_2$ on $[\lambda, \infty)$ and $(\phi_1^2 - \phi_2^2)(x) = -(\phi_1^2 - \phi_2^2)(2\lambda - x)$. Then either*

[4] Note that the peaked or cusped waves defined and found in [22] are only locally symmetric near a peaked or cusped point.

- $\phi_1 = \phi_2$ *in* $[\lambda, \infty)$*, or*
- $\phi_1 > \phi_2$ *with* $\phi_1 + \phi_2 < 2c$ *in* (λ, ∞)*.*

Proof. In view of its symmetry and monotonicity, K acts as a positive convolution operator on functions which are odd with respect to λ and do not change sign on the half line $[\lambda, \infty)$. In fact, let $f \geq 0$ on $[\lambda, \infty)$, and $f(x) = -f(2\lambda - x)$. Then

$$\begin{aligned} K * f(x) &= \int_\lambda^\infty K(y) f(x-y)\mathrm{d}y + \int_{-\infty}^\lambda K(x-y) f(y)\mathrm{d}y \\ &= \int_\lambda^\infty K(x-y) f(y)\mathrm{d}y + \int_\lambda^\infty K(x+y-2\lambda) f(2\lambda - y)\mathrm{d}y \\ &= \int_\lambda^\infty (K(x-y) - K(x+y-2\lambda)) f(y)\mathrm{d}y, \end{aligned}$$

where the last equality holds due to f being odd with respect to λ. For $x, y > \lambda$, we have

$$(x + y - 2\lambda) - |x - y| = 2\min\{x - \lambda, y - \lambda\} > 0. \tag{3.1}$$

Therefore, in view of K being an even function and monotonically decreasing on $(0, \infty)$, we have

$$K(x-y) - K(x+y-2\lambda) > 0 \tag{3.2}$$

so that

$$K * f(x) \geq 0 \qquad \text{for all} \quad x \geq \lambda.$$

In particular, the strict positivity of K implies that either $K * f > 0$ or $f \equiv 0$ on (λ, ∞). As a consequence, for the supersolution ϕ_1 and subsolution and ϕ_2 in this lemma, we have

$$(2c - (\phi_1 + \phi_2))(\phi_1 - \phi_2) \geq 3K * (\phi_1^2 - \phi_2^2) > 0$$

for all $x > \lambda$ unless $\phi_1 = \phi_2$ on $[\lambda, \infty)$. The lemma then follows directly. □

We now use the method of moving planes to prove the symmetry and one-crest structure of the wave profile. The first step is to prove that solitary waves $\phi(x)$ satisfy the following *strict overlay property* in Lemma 3.2 below, which means that there exists $\lambda \in \mathbb{R}$ so that for each $x > \lambda$ the reflection of $\phi(x)$ with respect to λ stays strictly above the part of ϕ at the reflection point $2\lambda - x$, i.e., $\phi(x) > \phi(2\lambda - x)$. For convenience, we define the open sets

$$\Sigma_\lambda := \{x \in \mathbb{R} \mid x > \lambda\} \quad \text{and} \quad \Sigma_\lambda^- := \{x \in \Sigma_\lambda \mid \phi(x) < \phi_\lambda(x)\},$$

where $\phi_\lambda(\cdot) := \phi(2\lambda - \cdot)$ is the reflection of ϕ about the axis $x = \lambda$.

Lemma 3.2 *(Strict overlay property). There exists $N > 0$ sufficiently large such that*

$$\phi(x) > \phi_\lambda(x), \quad x > \lambda, \tag{3.3}$$

for any $\lambda \leq -N$. In other words, $\Sigma_\lambda^- = \emptyset$ for any $\lambda \leq -N$.

Proof. Note that $\phi_\lambda(x)$ is also a solution to the steady Degasperis-Procesi equation in nonlocal formulation (1.7) if $\phi(x)$ does. Therefore, we deduce from (1.7) that

$$\begin{aligned} &2c(\phi_\lambda(x) - \phi(x)) \\ &= 3\left(\int\limits_{\Sigma_\lambda \setminus \Sigma_\lambda^-} + \int\limits_{\Sigma_\lambda^-} \right) \big(K(x-y) - K(2\lambda - x - y)\big)(\phi_\lambda^2(y) - \phi^2(y))\mathrm{d}y \\ &\quad + \phi_\lambda^2(x) - \phi^2(x). \end{aligned} \tag{3.4}$$

For $x \in \Sigma_\lambda^-$, we use (3.1) and find that the integral over $\Sigma_\lambda \setminus \Sigma_\lambda^-$ on the right side of (3.4) is negative so that

$$\begin{aligned} &2c(\phi_\lambda(x) - \phi(x)) \\ &\leq 3 \int\limits_{\Sigma_\lambda^-} \big(K(x-y) - K(2\lambda - x - y)\big)(\phi_\lambda^2(y) - \phi^2(y))\mathrm{d}y + \phi_\lambda^2(x) - \phi^2(x). \end{aligned} \tag{3.5}$$

Moreover, Theorem 2.7 implies that for any small $\epsilon > 0$, we can choose sufficiently large N such that

$$\phi(x) < \phi_\lambda(x) < \epsilon, \quad x \in \Sigma_\lambda^- \tag{3.6}$$

for any $\lambda < -N$. Then by taking the L^∞-norm on both sides of (3.4) over Σ_λ^- and using Lemma 3.1, we have

$$\begin{aligned} \|\phi_\lambda - \phi\|_{L^\infty(\Sigma_\lambda^-)} &\leq \frac{3}{2c} \|\phi + \phi_\lambda\|_{L^\infty(\Sigma_\lambda^-)} \left(\|K\|_{L_1(\mathbb{R})} + 1\right) \|\phi_\lambda - \phi\|_{L_\infty(\Sigma_\lambda^-)} \\ &\leq \frac{3\epsilon}{c} \left(\|K\|_{L_1(\mathbb{R})} + 1\right) \|\phi_\lambda - \phi\|_{L_\infty(\Sigma_\lambda^-)}, \end{aligned} \tag{3.7}$$

where $\left(\Sigma_\lambda^-\right)^*$ is the reflection of Σ_λ^- about the plane $x = \lambda$. By choosing $\epsilon < \frac{c}{6(\|K\|_{L_1(\mathbb{R})}+1)}$, we get a contradiction in (3.7) unless $\|\phi - \phi_\lambda\|_{L_\infty(\Sigma_\lambda^-)} = 0$ for $\lambda \leq -N$. As a consequence, Σ_λ^- must be of measure zero. Since Σ_λ^- is open, we deduce that Σ_λ^- is empty for $\lambda \leq -N$. $\square$

3.1. Solitary waves with height strictly smaller than wave speed c

We are now ready to prove that solitary waves are symmetric and have exactly one crest at the symmetric axis. The method is similar as that for the Whitham equation in [7] but we give full details for the proof here and write it in a way to better indicate the obstacle for the case of highest solitary waves.

Theorem 3.3. *Let ϕ be a solitary solution to the steady Degasperis-Procesi equation (1.7) with $\phi(x) < c$. Then, there exists a unique $\lambda_0 \in \mathbb{R}$ such that ϕ is symmetric about $x = \lambda_0$ and ϕ is strictly monotonic on each side of the symmetric axis $x = \lambda_0$.*

Proof. According to Lemma 3.2, there exists $N > 0$ such that Σ_λ^- is empty for all $\lambda < -N$. We now move the axis $x = \lambda$ from $\lambda = -N$ to the right and it is clear that Σ_λ^- remains empty unless $x = \lambda$ reaches a local maximum of ϕ, or there exists $x_0 > \lambda$ such that the reflection image of ϕ on the left side of $x = \lambda$ touches the wave profile on the right side of $x = \lambda$ at x_0, namely $\phi(2\lambda - x_0) = \phi(x_0)$. However, Lemma 3.1 will exclude the latter case. In fact, suppose that the latter case happens and the procedure stops at $x = \lambda_0$ so that $\phi(x) \geq \phi_{\lambda_0}(x)$ with the equality holding for the first time at $x = x_0 > \lambda_0$, and that $\phi(x)$ do not match $\phi_{\lambda_0}(x)$ exactly for $x > \lambda_0$. By taking ϕ and ϕ_λ as the supersolution and subsolution, respectively, and using Lemma 3.1, we find that $\phi(x) > \phi_{\lambda_0}(x)$ for all $x > \lambda_0$ so that a contradiction appears. So, the above process only stops at $x = \lambda_0$, where ϕ reaches its local maximum for the first time.

We now show that ϕ is symmetric with respect to $x = \lambda_0$ so that this local maximum of ϕ at λ_0 is just the unique crest. We now assume ϕ to be asymmetric with respect to $x = \lambda_0$, and seek a contradiction. First of all, the touching Lemma 3.1 excludes the possibility for $\phi(x) \equiv \phi(\lambda_0)$ to hold on $[\lambda_0, \lambda_0 + \delta]$ for any small $\delta > 0$. Also, the above process indicates that ϕ is strictly increasing on $(-\infty, \lambda_0)$. Then, for any $\epsilon > 0$, we can choose $\delta > 0$ sufficiently small such that Σ_λ^- is simply connected and its size $|\Sigma_\lambda^-| < \epsilon$ for $\lambda \in (\lambda_0, \lambda_0 + \delta)$. For a fixed $\lambda \in (\lambda_0, \lambda_0 + \delta)$, it is clear that $2\lambda - \lambda_0 \in \Sigma_\lambda^-$. Since ϕ has height strictly smaller than c, we have

$$\phi(x) \leq \|\phi\|_{L^\infty(\mathbb{R})} < c \tag{3.8}$$

for $x \in \Sigma_\lambda^-$ and

$$c_\lambda := \sup_{x \in \Sigma_\lambda^-} [\phi(x) + \phi_\lambda(x)] < 2\|\phi\|_{L^\infty(\mathbb{R})} < 2c. \tag{3.9}$$

Then, by simple connectedness of Σ_λ^-, we restrict (3.4) on Σ_λ^- and get

$$(2c - c_\lambda)(\phi_\lambda - \phi)(x) \leq 3 \int\limits_{\Sigma_\lambda^-} [K(x-y) - K(2\lambda - x - y)]\,(\phi_\lambda^2(y) - \phi^2(y))\mathrm{d}y. \tag{3.10}$$

In view of (3.9), we take L^∞-norm over Σ_λ^- on both sides of (3.10), and get

$$\begin{aligned}\|\phi_\lambda-\phi\|_{L^\infty(\Sigma_\lambda^-)} &< \frac{3|\Sigma_\lambda^-|}{2c-c_\lambda}\|K\|_{L^\infty(\Sigma_\lambda^-)}\|\phi_\lambda+\phi\|_{L_\infty(\Sigma_\lambda^-)}\|\phi_\lambda-\phi\|_{L^\infty(\Sigma_\lambda^-)} \\ &< \frac{3\epsilon\|\phi\|_{L^\infty(\mathbb{R})}}{c-\|\phi\|_{L^\infty(\mathbb{R})}}\|\phi_\lambda-\phi\|_{L^\infty(\Sigma_\lambda^-)},\end{aligned} \tag{3.11}$$

which leads to a contradiction if we choose $\epsilon < \frac{c-\|\phi\|_{L^\infty(\mathbb{R})}}{6\|\phi\|_{L^\infty(\mathbb{R})}}$. Therefore $\phi(x)$ matches $\phi_{\lambda_0}(x)$ for all $x \in \Sigma_{\lambda_0}$, i.e., ϕ is symmetric with respect to $x = \lambda_0$. In addition, the above process of moving $x = \lambda$ from far left to $x = \lambda_0$ guarantees that ϕ has a unique crest located at $x = \lambda_0$ and is monotone on each side of this symmetry axis. □

3.2. The solitary wave of maximum height

For a solitary wave whose crest reaches the maximum height c, the term c_λ can be very close to $2c$ so that $2c - c_\lambda$ may be comparable with (or much smaller than) ϵ and makes (3.11) fail to lead to a contradiction. In order to get around this difficulty, we have to study the delicate structure of (3.10). We now explain the idea to get around this difficulty. Suppose that we push $x = \lambda$ from far left to the right on the real line and the set Σ_λ^- remains empty until $x = \lambda$ meets a crest of the wave profile at λ_0. For λ to be slightly larger than λ_0, the factor $2c - c_\lambda$ could be very small but $|\Sigma_\lambda^-|$ is also small. Then for $x, y \in \Sigma_\lambda^-$, a new and key observation is that the difference $|2\lambda - x - y| - |x - y|$ satisfies

$$\big||2\lambda - x - y| - |x - y|\big| = 2\min\{x - \lambda, y - \lambda\} \le 2|\Sigma_\lambda^-|, \tag{3.12}$$

and is also small. Therefore, the term $K(x - y) - K(2\lambda - x - y)$ contributes extra smallness which may be used to control the smallness from $2c - c_\lambda$.

In the idea above, the size of $2c - c_\lambda$ relies on the structure of the wave profile ϕ near the crest at λ_0. It is indicated by [22] and [4] that the wave profile ϕ will become non-smooth and a peak or cusp may form at the crest when wave height reach the wave speed c. So, it is reasonable to assume that a highest solitary wave ϕ is non-smooth at the crest, but we will give a method which work for different non-smooth structures (peak or cusp) near the crest. Without loss of generality, we assume that the crest for the highest solitary wave is located at $x = \lambda_0$ and its local structure is characterized by

$$c - \phi(x) \in [C_1|x - \lambda_0|^\alpha, C_2|x - \lambda_0|^\alpha] \tag{3.13}$$

for $\alpha \in (0, 1]$ and some constants $C_1, C_2 > 0$ when x is very close to λ_0. In this way, the argument below can be adapted to treat the symmetry issues of steady solutions with other Hölder regularity at the crest.[5]

Theorem 3.4. *There exists a finite $\lambda_0 \in \mathbb{R}$ such that the highest solitary solution ϕ to the steady Degasperis-Procesi equation is symmetric about $x = \lambda_0$ where the crest is located. Moreover, ϕ is strictly monotone on each side of the symmetric axis.*

[5] It is expected that the type of non-smoothness at the crest of the solitary wave will be the same as that for the convolutional kernel $K(\cdot)$, see the peaked solitary wave for DP in [22] and cusped periodic waves for the Whitham in [18].

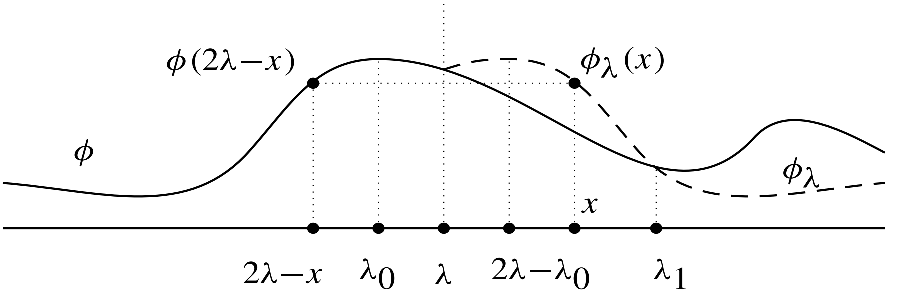

Fig. 1. Assume that the procedure of pushing $x = \lambda$ from left to right stops for the first time at a local maximum of ϕ at $x = \lambda_0$. Then, the reflection axis $x = \lambda$ is pushed slightly to the right side of $x = \lambda_0$, which generates a non-empty set Σ_λ^- denoted by the interval (λ, λ_1). ϕ_λ as the reflection of ϕ is partially depicted by dashed lines. For a point $x \in \Sigma_\lambda^-$, its reflection $2\lambda - x$ is also depicted.

Proof. First of all, we can prove similarly as in Theorem 3.3 that we are able to push $x = \lambda$ from far left to right until it stops at $x = \lambda_0$ where a crest of ϕ is located. If $\phi(\lambda_0) < c$, then the proof reduces to the case for waves with height strictly smaller than c, as in Theorem 3.3. So, we can assume that $\phi(\lambda_0) = c$. As before, the touching Lemma 3.1 excludes the possibility for $\phi(x) \equiv \phi(\lambda_0)$ to hold on $[\lambda_0, \lambda_0 + \delta]$ for any small $\delta > 0$. Also, for any $\epsilon > 0$, we can choose $\delta > 0$ sufficiently small such that Σ_λ^- will be simply connected and its size satisfies $|\Sigma_\lambda^-| < \epsilon$ for $\lambda \in (\lambda_0, \lambda_0 + \delta)$. As in Fig. 1, for a fixed $\lambda \in (\lambda_0, \lambda_0 + \delta)$, we denote Σ_λ^- by (λ, λ_1) and define

$$\delta_1 := \lambda - \lambda_0, \quad \delta_2 := \lambda_1 - \lambda. \tag{3.14}$$

Note that δ_2 can be very small if δ_1 is chosen small enough, and in particular, δ_2 approaches 0 as δ_1 does. Then, from (3.12) and the property of kernel K, we have

$$0 < K(x - y) - K(2\lambda - x - y) \le 2(x - \lambda), \quad x, y \in \Sigma_\lambda^-. \tag{3.15}$$

The key observation is for estimate of the term $2c - \phi(x) - \phi_\lambda(x)$, $x \in \Sigma_\lambda^-$ as follows: For sufficiently small δ_1 and any $x \in \Sigma_\lambda^-$, we use (3.13) and get

$$\begin{aligned} 2c - (\phi(x) + \phi_\lambda(x)) &= \phi(\lambda_0) - \phi(x) + \phi(\lambda_0) - \phi(2\lambda - x) \\ &\ge C_1 \left[(x - \lambda_0)^\alpha + [(2\lambda - \lambda_0) - x]^\alpha \right] \\ &\ge C_1 |x - \lambda|^\alpha, \end{aligned} \tag{3.16}$$

where the first inequality can be well-illustrated by

$$c - \phi_\lambda(x) = \phi(\lambda_0) - \phi(2\lambda - x) = \phi_\lambda(2\lambda - \lambda_0) - \phi_\lambda(x)$$

and the fact where the distance between x and $2\lambda - \lambda_0$ is the same as the distance between $2\lambda - x$ and λ_0 (as illustrated in Fig. 1). Therefore, for any $x \in \Sigma_\lambda^-$, we use (3.15) and (3.16) to get

$$
\begin{aligned}
&\phi_\lambda(x)-\phi(x)\\
&=\frac{3}{2c-(\phi+\phi_\lambda)}\int\limits_{\Sigma_\lambda^-}\big(K(x-y)-K(2\lambda-x-y)\big)(\phi_\lambda^2(y)-\phi^2(y))\mathrm{d}y\\
&\le\frac{3}{C_1|x-\lambda|^\alpha}\left[2(x-\lambda)|\Sigma_\lambda^-|\|\phi+\phi_\lambda\|_{L^\infty_{\Sigma_\lambda^-}}\|\phi_\lambda-\phi\|_{L^\infty_{\Sigma_\lambda^-}}\right]\\
&\le 12cC_1^{-1}\delta_2|x-\lambda|^{1-\alpha}\|\phi_\lambda-\phi\|_{L^\infty_{\Sigma_\lambda^-}}.
\end{aligned}
\tag{3.17}
$$

From (3.17), we see clearly that $|x-\lambda|^{1-\alpha}$ is a small quantity with non-negative power $1-\alpha$, which shows that the smallness of $K(x-y)-K(2\lambda-x-y)$ balances the singularity caused by the term $2c-(\phi+\phi_\lambda)$ on Σ_λ^-. Then, by choosing δ_1 sufficiently small, we can make $\delta_2<\epsilon<(\frac{C_1}{24})^{\frac{1}{2-\alpha}}$ so that

$$12cC_1^{-1}\delta_2|x-\lambda|^{1-\alpha}\le 12cC_1^{-1}\delta_2^{2-\alpha}<\frac{1}{2}.$$

Therefore, we get a contradiction by taking the $L^\infty_{\Sigma_\lambda^-}$ norm on the left side of (3.17), and the lemma is proved. □

Remark 3.5. In the proof for Theorem 3.4, we used the boundedness of the kernel function. For unbounded kernel which may appear in other equations like the Whitham equation, it is expected that proper L^p-norms instead of L^∞-norm should be used for (3.17).

4. A new method for symmetric solutions to be traveling waves

It has been confirmed in [17] that classical symmetric solutions to the Degasperis-Procesi equation must be traveling waves. The idea for the proof in [17] is to construct a traveling wave solution $\bar{u}(t,x)$ which shares the same initial data with a symmetric solution $u(t,x)$, then the uniques of solutions implies that $\bar{u}(t,x)$ coincide with $u(t,x)$ so that symmetric solutions are traveling waves. However, we hope to understand how the symmetric structure of waves can be connected with the fixed shape and constant propagation speed, which can not be clearly seen from the constructive proof in [17]. With this goal, we check carefully the two constraint conditions (see (4.6)-(4.5) below) and find that they actually contain information for the shape of wave profile and the wave propagation speed, respectively. This new finding also leads to a new, more straightforward proof for symmetric solutions to be traveling waves as follows. For convenience, we work on the Degasperis-Procesi equation in nonlocal formulation (1.2).

Theorem 4.1. *Solutions to the Degasperis-Procesi equation with* a priori *spatial symmetry are steady solutions.*

Proof. Assume that $u(t,x)$ is a solution to the Degasperis-Procesi equation with symmetric axis $x=\lambda(t)$ for some function $\lambda(\cdot)\in C^1(\mathbb{R})$, i.e.,

$$u(t,x)=u(t,2\lambda(t)-x). \tag{4.1}$$

Then, the spatial and time derivatives of $u(t,x)$ satisfy

$$u_t|_{(t,x)} = (u_t + 2\dot{\lambda} u_x)|_{(t,2\lambda-x)}, \quad u_x|_{(t,x)} = -u_x|_{(t,2\lambda-x)}. \tag{4.2}$$

In addition, we have

$$\frac{1}{2}\partial_x L(u^2)\big|_{(t,x)} = -\int_{\mathbb{R}} K(y)[uu_x](t, 2\lambda - x - y)dy = -L(uu_x)\big|_{(t,2\lambda-x)}, \tag{4.3}$$

where in the second equality we used the evenness of the kernel $k(\cdot)$. Inserting (4.1)-(4.3) into (1.2) and in view of the arbitrariness[6] of t and x, we find that u satisfies the following equation

$$u_t + 2\dot{\lambda} u_x - uu_x - 3L(uu_x) = 0, \tag{4.4}$$

where $\dot{\lambda} := \dot{\lambda}(t)$ denotes the derivative of $\lambda(t)$ with respect to t. The comparison between (4.4) and (1.2) then leads to the following constraint conditions

$$u_t + \dot{\lambda} u_x = 0, \tag{4.5}$$

$$-\dot{\lambda} u_x + uu_x + 3L(uu_x) = 0. \tag{4.6}$$

A key observation is that (4.5) is a linear PDE of first order with coefficients relying only on the time variable so that $u(t,x)$ must take the form

$$u(t,x) = g(x - \lambda(t)) \tag{4.7}$$

for some function g, which implies that the shape of the solution will not change in later evolution and the solution propagates with speed $\dot{\lambda}(t)$. Inserting (4.7) into (4.6), we get the following differential equation

$$\left[-\dot{\lambda}(t)g' + gg' + 3L(gg')\right]\Big|_{x-\lambda(t)} = 0. \tag{4.8}$$

Choose arbitrarily two pairs $(t_1, x_1), (t_2, x_2) \in \mathbb{R}^+ \times \mathbb{R}$ (for which the solution exists and makes sense) such that

$$x_1 - \lambda(t_1) = x_2 - \lambda(t_2) =: X. \tag{4.9}$$

Evaluating (4.8) at these two pairs gives

$$(\dot{\lambda}(t_1) - \dot{\lambda}(t_2))g'(X) = 0.$$

Due to the arbitrariness of X, $\dot{\lambda}(t)$ has to be a constant so that the wave profile has a constant propagation speed. Therefore $u(t,x)$, with fixed shape and constant propagation speed, is a traveling wave solution. □

[6] The variable t should of course be chosen from an interval where solutions stay in the same function space as the initial datum does.

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