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Persistence of preys in a diffusive three species predator-prey system with a pair of strong-weak competing preys

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Abstract

We investigate the traveling wave solutions of a three-species system involving a single predator and a pair of strong-weak competing preys. Our results show how the predation may affect this dynamics. More precisely, we describe several situations where the environment is initially inhabited by the predator and by either one of the two preys. When the weak competing prey is an aboriginal species, we show that there exist traveling waves where the strong prey invades the environment and either replaces its weak counterpart, or more surprisingly the three species eventually co-exist. Furthermore, depending on the parameters, we can also construct traveling waves where the weaker prey actually invades the environment initially inhabited by its strong competitor and the predator. In all those situations, we find the infimum of the set of admissible wave speeds; these results are sharp at least when the three species diffusive at the same speed.
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1. Introduction

In this paper we consider the following system

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u - kv - b_1 w), x \in \mathbb{R},\ t > 0, \\ v_t = d_2 v_{xx} + r_2 v(1 - hu - v - b_2 w), x \in \mathbb{R},\ t > 0, \\ w_t = d_3 w_{xx} + r_3 w(-1 + au + av - w), x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.1}$$

in which $u(x,t)$ and $v(x,t)$ stand for the densities of two preys and $w(x,t)$ is the density of the predator at (x,t), d_i and r_i, $i = 1, 2, 3$, are their diffusion coefficients and intrinsic growth rates, respectively, h and k denote the interspecific competition coefficients of two preys, the carrying capacities of two preys are assumed to be 1, b_1 and b_2 are predation rates of u and v, respectively, and the conversion rates for both preys are assumed to be a.

Throughout this paper, we always assume that the parameters d_i, r_i $(i = 1, 2, 3)$, h, k, a, b_1 and b_2 are all positive such that

$$a > 1, \quad h < 1 < k. \tag{1.2}$$

In particular, the single predator w cannot survive without feeding on the preys, yet it can live together with either of those two preys. Moreover, the preys are competing and, in the absence of the predator w, the prey v is the strong competitor and the prey u is the weak competitor. However, both preys undergo a priori different predation rates, and therefore the presence of the predator may invert their roles and lead to new dynamics, as we shall show in our main results.

Throughout this work, we shall consider situations when the single predator w is an aboriginal species, and one of the two preys u and v is aboriginal while the other is alien. Our aim is to see the role of the predator in the ecological system (1.1) and in particular on the persistence of these two preys. It turns out that the following scenarios can happen, depending on the parameters:

(1) three species can co-exist, no matter which prey is alien;
(2) the alien strong competitor v can replace the aboriginal weak competitor u to co-exist with the predator w;
(3) more surprisingly, it seems that the alien weak competitor u may replace the aboriginal strong competitor v to co-exist with the predator w if it is more resistant to predation.

1.1. The ODE system

Let us start with some preliminary study of solutions of the diffusionless ODE system. Aside from the trivial steady state $(0,0,0)$, and since $a > 1$, we always have the existence of the *semi-co-existence* states $E_* = (0, v_*, w_*)$ and $E^* = (u^*, 0, w^*)$, where

$$u^* := \frac{1 + b_1}{1 + ab_1}, \quad w^* := \frac{a - 1}{1 + ab_1}; \tag{1.3}$$

$$v_* := \frac{1 + b_2}{1 + ab_2}, \quad w_* := \frac{a - 1}{1 + ab_2}. \tag{1.4}$$

Let us briefly describe the stability of these two constant states. To do so, we define

$$\beta_* := 1 - kv_* - b_1 w_*,\ \beta^* := 1 - hu^* - b_2 w^*,$$

or equivalently

$$\beta_* = \frac{-b_1(a-1) + b_2(a-k) - (k-1)}{1 + ab_2},\ \beta^* = \frac{b_1(a-h) - b_2(a-1) + (1-h)}{1 + ab_1}. \tag{1.5}$$

It is then easy to see that $\beta^* > 0$ (resp. $\beta_* > 0$) implies that E^* (resp. E_*) is unstable in the ODE sense. On the other hand, if $\beta^* < 0$ (resp. $\beta_* < 0$) then E^* (resp. E_*) is stable instead, again in the ODE sense. From (1.5), one can check that $\beta^* > 0$ if and only if

$$b_2 < \frac{a-h}{a-1} b_1 + \frac{1-h}{a-1}. \tag{1.6}$$

Also, $\beta_* > 0$ if and only if

$$a > k \text{ and } b_2 > \frac{a-1}{a-k} b_1 + \frac{k-1}{a-k}. \tag{1.7}$$

Finally, there exists at most one more (positive) constant state, in which the three species co-exist. This co-existence state only exists in some parameter range, in particular, one needs $\triangle \neq 0$, where

$$\triangle := 1 - hk + ab_1(1-h) - ab_2(k-1).$$

By Cramer's rule, when $\triangle \neq 0$ and if the co-existence state $E_c := (u_c, v_c, w_c)$ exists, then

$$u_c = \frac{\triangle_u}{\triangle} > 0,\ v_c = \frac{\triangle_v}{\triangle} > 0,\ w_c = \frac{\triangle_w}{\triangle} > 0, \tag{1.8}$$

where

$$\begin{aligned} &\triangle_u := -b_1(a-1) + b_2(a-k) - (k-1), \quad \triangle_v := b_1(a-h) - b_2(a-1) + (1-h), \\ &\triangle_w := a(2-h-k) - (1-hk). \end{aligned}$$

Assume the existence of the (unique) positive co-existence state E_c. Since the characteristic equation associated with the linearized system around E_c is given by $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$, where

$$\begin{aligned} a_2 &:= r_1 u_c + r_2 v_c + r_3 w_c, \\ a_1 &:= r_1 r_2 u_c v_c (1 - hk) + r_1 r_3 u_c w_c (1 + ab_1) + r_2 r_3 v_c w_c (1 + ab_2), \\ a_0 &:= (r_1 r_2 r_3 u_c v_c w_c)\triangle, \end{aligned}$$

it is clear that E_c is unstable, if $\triangle < 0$. Moreover, one can also check that E_c is stable, if $\triangle > 0$ and $hk < 1$. We point out that if either E^* or E_* is stable, then E_c (if exists) must be unstable in the ODE sense. Indeed, if E^* is stable so that $\beta^* < 0$, then $\triangle_v < 0$ and so $\triangle < 0$ (if E_c exists) which implies that E_c is unstable.

Let us mention a special case when E_c is stable, whose interest is that a particular Lyapunov function then exists, which we shall use to classify entire in time solutions of (1.1); see Lemma 4.7 below. Precisely, if E_c exists, i.e., (1.8) holds, and if moreover

$$k\sqrt{\frac{b_2}{b_1}}+h\sqrt{\frac{b_1}{b_2}}<2, \tag{1.9}$$

then E_c is stable while E^* and E_* are unstable. Indeed, (1.9) can be rewritten as $hk<1$ and

$$\frac{2-hk-2\sqrt{1-hk}}{k^2}b_1<b_2<\frac{2-hk+2\sqrt{1-hk}}{k^2}b_1.$$

We first show that E_c is stable, i.e., $\triangle>0$. We assume by contradiction that $\triangle<0$. Then, for E_c to exist, it must hold that $\triangle_u<0$, $\triangle_v<0$, and $\triangle_w<0$. To have $\triangle<0$, we need

$$b_2>b_1\frac{1-h}{k-1}.$$

Recalling the previous inequality, it follows that

$$b_1\frac{1-h}{k-1}<\frac{2-hk+2\sqrt{1-hk}}{k^2}b_1,$$

i.e.,

$$g(k):=\frac{2-hk+2\sqrt{1-hk}}{k^2}\cdot\frac{k-1}{1-h}>1.$$

The derivative of $g(k)$ has the same sign as

$$(2-k)(2-hk+2\sqrt{1-hk})-h(k^2-k)(1+\frac{1}{\sqrt{1-hk}}),$$

which is negative for $k\in(2,1/h)$, and decreasing with respect to $k\in(1,2)$. Also, for $k=2-h$, it is equal to 0. It follows that $g(k)$ is increasing with respect to $k\in[1,2-h]$, then decreasing. Furthermore, $g(2-h)=1$. Since $g(2-h)$ is the maximum of g, we have reached a contradiction. Hence $\triangle>0$ and E_c is stable. The existence of E_c also implies that $\triangle_u$, $\triangle_v$ and $\triangle_w$ are positive. It follows that (1.6) and (1.7) hold, and in particular E_* and E^* are unstable.

1.2. The notion of a traveling wave

To see the persistence of preys, our approach is to study the traveling wave solutions connecting two appropriate constant states of system (1.1). A solution of (1.1) is called a *traveling wave* solution with speed s, if there exist positive functions $\{\phi_1,\phi_2,\phi_3\}$ defined on $\mathbb{R}$ such that $u(x,t)=\phi_1(x+st)$, $v(x,t)=\phi_2(x+st)$ and $w(x,t)=\phi_3(x+st)$; here ϕ_j, $j=1,2,3$, are the wave profiles and are assumed to converge at $\pm\infty$ to constant states to be specified below.

Letting $z:=x+st$ and substituting $(u,v,w)(x,t)=(\phi_1,\phi_2,\phi_3)(z)$ into (1.1), we get that (s,ϕ_1,ϕ_2,ϕ_3) must satisfy the following system of equations:

$$\begin{cases} d_1\phi_1''(z) - s\phi_1'(z) + r_1\phi_1(z)[1-\phi_1(z)-k\phi_2(z)-b_1\phi_3(z)] = 0, \ z\in\mathbb{R}, \\ d_2\phi_2''(z) - s\phi_2'(z) + r_2\phi_2(z)[1-h\phi_1(z)-\phi_2(z)-b_2\phi_3(z)] = 0, \ z\in\mathbb{R}, \\ d_3\phi_3''(z) - s\phi_3'(z) + r_3\phi_3(z)[-1+a\phi_1(z)+a\phi_2(z)-\phi_3(z)] = 0, \ z\in\mathbb{R}, \end{cases} \tag{1.10}$$

where the prime denotes the derivative with respect to z.

Throughout this work, we shall consider several types of traveling waves which differ from each other by their limits as $z\to\pm\infty$. Up to the symmetric change of variables $x\leftarrow -x$, we can always assume that

$$s \geq 0,$$

and therefore we shall refer to the limit of (ϕ_1,ϕ_2,ϕ_3) at $-\infty$ as the *invaded state* or *unstable tail*, and to the limit at ∞ as the *invading state* or *stable tail*.

As we mentioned before, we shall assume that the predator is aboriginal and consider the two cases where either of the two preys co-exists with the predator. Therefore we shall assume at the unstable tail that either

$$\lim_{z\to-\infty}(\phi_1(z),\phi_2(z),\phi_3(z)) = E^* := (u^*,0,w^*), \tag{1.11}$$

or

$$\lim_{z\to-\infty}(\phi_1(z),\phi_2(z),\phi_3(z)) = E_* := (0,v_*,w_*). \tag{1.12}$$

On the other hand, depending on the parameters we shall face two situations where either the three species eventually co-exist, or the aboriginal prey goes to extinction and is replaced by the alien prey. In the former case, we shall have the asymptotic boundary condition at the stable tail

$$\lim_{z\to\infty}(\phi_1(z),\phi_2(z),\phi_3(z)) = E_c := (u_c,v_c,w_c). \tag{1.13}$$

In the latter case, one must distinguish whether the aboriginal prey is the weak or the strong one; that is, we shall have either (1.11) together with

$$\lim_{z\to\infty}(\phi_1(z),\phi_2(z),\phi_3(z)) = E_* := (0,v_*,w_*), \tag{1.14}$$

or (1.12) together with

$$\lim_{z\to\infty}(\phi_1(z),\phi_2(z),\phi_3(z)) = E^* := (u^*,0,w^*), \tag{1.15}$$

In order to study the existence of traveling waves for the non-monotone system (1.1), we apply a two-fold method based on a construction of appropriate generalized upper-lower solutions, and on a Schauder's fixed point theorem (cf. [17,18,14,15]). This method has been proved to be very successful, and we refer the reader to [11,12,14,5,13,15,3,20] for 2-component systems as well as [10,16,19,2,9] for 3-component systems. However, the construction of a suitable set of upper and lower solutions depends heavily on the system at issue and therefore it is rather nontrivial, which is one difficulty in applying this method. These generalized upper-lower solutions serve as the upper and lower bounds of the domain for an appropriate integral operator deduced from (1.10). Their purpose is to ensure that the integral operator maps this domain into itself, and therefore that a fixed point exists by Schauder's fixed point theorem. Provided that it satisfies the appropriate asymptotic conditions at the tails, this fixed point provides a traveling wave solution of (1.1).

Another difficulty is precisely to check that the wave profile obtained above satisfies the wanted stable tail limit, which requires different approaches depending on the invading state. One of the classical approaches for deriving this limit is the method of contracting rectangles (cf. [10,3,9]). However, it turns out that this method is not directly applicable to our problem. To overcome this difficulty in the case of the waves connecting the semi-co-existence states, we introduce a new idea of dimension reduction (see Section 4 below), which is one of the main contributions of this work. In this new method, we only consider, instead of 3-d rectangles, a sequence of shrinking 2-d rectangles. Moreover, we need to derive a priori certain positive lower bounds on ϕ_2 and ϕ_3 at the stable tail, since the lower bounds of these two components of our constructed upper-lower solutions are not good enough to apply the method of contracting rectangles. On the other hand, to obtain the traveling waves connecting the co-existence state at the stable tail limit, another approach is needed and therefore we instead apply a Lyapunov argument.

The rest of this paper is organized as follows. First, our main results are described in Section 2. Next, the existence of solutions to (1.10) with either (1.11) or (1.12) is carried out in Section 3. Section 4 is devoted to the derivation of the stable tail limit. Then we deal with the non-existence of waves in Section 5. Finally, in Section 6, we provide the detailed verification of upper-lower solutions constructed in Section 3.

2. Main results

In this section, we shall present the main results obtained in this paper. We recall that

$$\beta_* = 1 - kv_* - b_1 w_*,\ \beta^* = 1 - hu^* - b_2 w^*,$$

or equivalently

$$\beta_* = \frac{-b_1(a-1) + b_2(a-k) - (k-1)}{1+ab_2},\ \beta^* = \frac{b_1(a-h) - b_2(a-1) + (1-h)}{1+ab_1}.$$

We also recall that the sign of β_* (resp. β^*) determines the stability of the semi-co-existence states $E_* = (0, v_*, w_*)$ (resp. $E^* = (u^*, 0, w^*)$) in the ODE sense. In particular, both states may be admissible as the unstable tail limit of the traveling wave solution. This leads us to introduce

$$s_* := 2\sqrt{d_1 r_1 \beta_*},\ s^* := 2\sqrt{d_2 r_2 \beta^*},$$

whenever they are well-defined. These may be understood as the linear invasion speeds into the respective states E_* and E^* whenever they are unstable. By an analogy with the well-understood Fisher-KPP scalar equation, one may also expect these values to be the infimum wave speeds, and this shall be confirmed by our results.

Our first result deals with the situation when the strong competitor prey is the alien species, and either replaces the weak aboriginal prey, or eventually co-exists with both the other species. With our notation, this means that the state $E^* = (u^*, 0, w^*)$ may be invaded by either $E_* = (0, v_*, w_*)$ or $E_c = (u_c, v_c, w_c)$.

Theorem 2.1. *Suppose that $\beta^* > 0$, i.e., (1.6) holds. Assume further that*

$$r_2\beta^* \geq r_1[k + b_1(2a-1)]. \tag{2.1}$$

Then system (1.10) has a bounded positive solution (ϕ_1, ϕ_2, ϕ_3) satisfying the boundary condition (1.11), for $s > s^*$ provided that

$$d_2 \geq \max\{d_1, d_3\}, \quad r_2\beta^* \geq r_3; \tag{2.2}$$

and for $s = s^*$ provided that

$$\frac{d_3}{2} < d_1 = d_2 \leq d_3, \quad r_2\left(2 - \frac{d_3}{d_2}\right)\beta^* \geq r_3. \tag{2.3}$$

Moreover, (ϕ_1, ϕ_2, ϕ_3) satisfies (1.14) if $\beta_* < 0$ and

$$a > \frac{1}{1-h}, \quad b_2 < \frac{a(1-h)-1}{a(2a-1)}; \tag{2.4}$$

while (ϕ_1, ϕ_2, ϕ_3) satisfies (1.13) if (1.8) and (1.9) are enforced.

Some of the conditions in Theorem 2.1 appear to be mostly technical. Setting aside such assumptions, Theorem 2.1 roughly states that, when the semi-co-existence state E^* is unstable, then there exists a traveling wave for any speed larger than s^* connecting E^* to a stable tail limit, which must also be a stable state of the ODE system. In particular, depending on the sign of β_*, the stable tail limit shall be either E_* or E_c.

Remark 2.1. Let us point out that all the situations in Theorem 2.1 can be encountered in some parameter ranges. Indeed, notice first that (2.1), (2.2) and (2.3) are the only conditions on the intrinsic growth rates and the diffusivities. These are clearly achievable and we focus on the choice of coupling parameters a, h, k, b_1 and b_2.

Consider first the case of a traveling wave satisfying (1.11) and (1.14), i.e., connecting the two semi-co-existence states. All conditions $\beta^* > 0$, $\beta_* < 0$ and the second inequality in (2.4) rewrite as upper bounds on b_2. The only remaining condition is $a > \frac{1}{1-h}$, which raises no compatibility issue. Therefore, a traveling wave connecting these two semi-co-existence states clearly exists in some parameter range, typically when b_2 is small.

The other case, i.e., of a traveling wave connecting E^* to E_c, is a bit more complicated, because the corresponding assumptions involve both upper and lower bounds on b_2. First, one may choose a, h and k so that $hk < 1$, $h + k < 2$ and $a > \frac{1-hk}{2-h-k}$ hold. It follows that $\triangle_w > 0$, and (1.8) rewrites as

$$\triangle > 0, \ \triangle_u > 0, \ \triangle_v > 0.$$

The positivity of $\triangle_v$ also implies that $\beta^* > 0$. On the other hand, the assumption $k\sqrt{\frac{b_2}{b_1}} + h\sqrt{\frac{b_1}{b_2}} < 2$ rewrites as

$$\frac{2 - hk - 2\sqrt{1-hk}}{k^2} b_1 < b_2 < \frac{2 - hk + 2\sqrt{1-hk}}{k^2} b_1.$$

From the definitions of $\triangle$, $\triangle_u$ and $\triangle_v$ in Subsection 1.1, one can find b_2 such that a traveling wave connecting E^* to E_c exists if

$$\max\left\{\frac{a-1}{a-k}b_1+\frac{k-1}{a-k}, \frac{2-hk-2\sqrt{1-hk}}{k^2}b_1\right\}$$
$$<\min\left\{\frac{1-h}{k-1}b_1+\frac{1-hk}{a(k-1)}, \frac{a-h}{a-1}b_1+\frac{1-h}{a-1}, \frac{2-hk+2\sqrt{1-hk}}{k^2}b_1\right\}.$$

This is achievable, for instance, if k is close to 1.

Now we turn to the more surprising case when the strong competitor is the aboriginal prey, i.e., the unstable tail limit of the traveling wave is the semi-co-existence state E_*. It turns out that, due to the predation, it is possible that the weak alien prey invades the environment with positive speed.

Theorem 2.2. *Suppose that $\beta_* > 0$, i.e., (1.7) holds. Assume further that*

$$r_1\beta_* \geq r_2[h+b_2(2a-1)]. \tag{2.5}$$

Then system (1.10) has a bounded positive solution (ϕ_1, ϕ_2, ϕ_3) satisfying the boundary condition (1.12), for $s > s_*$ provided that

$$d_1 \geq \max\{d_2, d_3\}, \quad r_1\beta_* \geq r_3; \tag{2.6}$$

and for $s = s_*$ provided that

$$\frac{d_3}{2} < d_1 = d_2 \leq d_3, \quad r_1\left(2-\frac{d_3}{d_1}\right)\beta^* \geq r_3. \tag{2.7}$$

Moreover, we have that

$$\liminf_{z\to+\infty}\phi_i(z) > 0 \ \textit{for}\ i = 1, 3.$$

Furthermore, if (1.8) and (1.9) are enforced, then (ϕ_1, ϕ_2, ϕ_3) satisfies (1.13).

Theorem 2.2 tells us that a weak intruding prey can invade an environment inhabited by the strong competing prey, thanks to the effect of predation. In the co-existence case, the three even species converge together to a positive equilibrium. In the case when the co-existence state E_c is unstable, we expect that the weak prey completely replaces the strong one, i.e., that (ϕ_1, ϕ_2, ϕ_3) satisfies (1.15). Unfortunately we have not been able to prove this rigorously and we leave it as an open issue for future work.

Remark 2.2. Let us again point out that the assumptions in Theorem 2.2 can indeed be satisfied. The argument is the same as in Remark 2.1.

Remark 2.3. We note from Theorem 2.1 that traveling waves exist for all speeds $s \geq s^*$ only when both conditions (2.2) and (2.3) hold. In particular, three species must diffuse at the same speed, i.e., $d_1 = d_2 = d_3$. The same limitation holds for the traveling waves obtained in Theorem 2.2.

Our last main result shows that the wave speeds s^* and s_* exhibited above are truly the infimum wave speeds of traveling wave solutions. More precisely:

Theorem 2.3. *The following statements hold:*

(1) Assume that $\beta^* > 0$, hence $s^* > 0$. Then no positive solutions of (1.10), (1.11) *and either* (1.13) *or* (1.14) *exist for $s < s^*$.*
(2) Assume that $\beta_* > 0$, hence $s_* > 0$. Then no positive solutions of (1.10), (1.12) *and either* (1.13) *or* (1.15) *exist for $s < s_*$.*

As we shall observe in Section 5, in the above non-existence theorem the stable tail limits can actually be replaced by the positivity of the infimum limit at ∞ of the alien species component. In particular, either prey invading the environment must do so at least at the corresponding speed s^* or s_*.

3. Existence of solutions to (1.10)

First, we give the definition of generalized upper-lower solutions of (1.10) as follows.

Definition 3.1. Nonnegative and continuous functions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ are called a pair of generalized upper and lower solutions of (1.10) if $\overline{\phi}_i''$, $\underline{\phi}_i''$, $\overline{\phi}_i'$, $\underline{\phi}_i'$, $i = 1, 2, 3$, are bounded functions and satisfy the following inequalities

$$\mathcal{U}_1(z) := d_1\overline{\phi}_1''(z) - s\overline{\phi}_1'(z) + r_1\overline{\phi}_1(z)[1 - \overline{\phi}_1(z) - k\underline{\phi}_2(z) - b_1\underline{\phi}_3(z)] \leq 0, \tag{3.1}$$

$$\mathcal{U}_2(z) := d_2\overline{\phi}_2''(z) - s\overline{\phi}_2'(z) + r_2\overline{\phi}_2(z)[1 - h\underline{\phi}_1(z) - \overline{\phi}_2(z) - b_2\underline{\phi}_3(z)] \leq 0, \tag{3.2}$$

$$\mathcal{U}_3(z) := d_3\overline{\phi}_3''(z) - s\overline{\phi}_3'(z) + r_3\overline{\phi}_3(z)[-1 + a\overline{\phi}_1(z) + a\overline{\phi}_2(z) - \overline{\phi}_3(z)] \leq 0, \tag{3.3}$$

$$\mathcal{L}_1(z) := d_1\underline{\phi}_1''(z) - s\underline{\phi}_1'(z) + r_1\underline{\phi}_1(z)[1 - \underline{\phi}_1(z) - k\overline{\phi}_2(z) - b_1\overline{\phi}_3(z)] \geq 0, \tag{3.4}$$

$$\mathcal{L}_2(z) := d_2\underline{\phi}_2''(z) - s\underline{\phi}_2'(z) + r_1\underline{\phi}_2(z)[1 - h\overline{\phi}_1(z) - \underline{\phi}_2(z) - b_2\overline{\phi}_3(z)] \geq 0, \tag{3.5}$$

$$\mathcal{L}_3(z) := d_3\underline{\phi}_3''(z) - s\underline{\phi}_3'(z) + r_3\underline{\phi}_3(z)[-1 + a\underline{\phi}_1(z) + a\underline{\phi}_2(z) - \underline{\phi}_3(z)] \geq 0, \tag{3.6}$$

for $z \in \mathbb{R}\backslash E$ with some finite set $E = \{z_1, z_2, \ldots, z_m\}$.

With this notion of generalized upper-lower solutions, we have the following existence theorem for system (1.10).

Proposition 3.2. *Given $s > 0$. Suppose that system* (1.10) *has a pair of generalized upper-lower solutions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ such that*

$$\underline{\phi}_i(z) \le \overline{\phi}_i(z),\ \forall z \in \mathbb{R},\ i = 1, 2, 3, \tag{3.7}$$

$$\lim_{z\to z_j^+} \overline{\phi}_i'(z) \le \lim_{z\to z_j^-} \overline{\phi}_i'(z),\ \lim_{z\to z_j^-} \underline{\phi}_i'(z) \le \lim_{z\to z_j^+} \underline{\phi}_i'(z),\ \forall z_j \in E,\ i = 1, 2, 3. \tag{3.8}$$

Then system (1.10) *has a solution* (ϕ_1, ϕ_2, ϕ_3) *such that* $\underline{\phi}_i \le \phi_i \le \overline{\phi}_i$, $i = 1, 2, 3$.

The proof of Proposition 3.2 can be done by a standard argument as that in, e.g., [17,18,10]), and thus we omit it here.

3.1. Upper-lower solutions for waves invading $E^* = (u^*, 0, w^*)$

In this subsection we shall construct generalized upper-lower solutions of (1.10) with boundary condition (1.11) at the unstable tail. Hence we assume that $\beta^* > 0$ so that E^* is unstable. Also, we impose condition (2.1) from Theorem 2.1.

3.1.1. Case $s > s^*$

We fix here $s > s^*$ and further assume that (2.2) is enforced. Let λ_1 and λ_2 be the two positive roots of

$$G(x) := d_2 x^2 - sx + r_2\beta^*,$$

which are given by

$$\lambda_1 := \frac{s - \sqrt{s^2 - 4d_2r_2\beta^*}}{2d_2}, \quad \lambda_2 := \frac{s + \sqrt{s^2 - 4d_2r_2\beta^*}}{2d_2}. \tag{3.9}$$

It follows from the first inequality in (2.2) that

$$0 < \lambda_1 \le \min\left\{\frac{s}{2d_1}, \frac{s}{2d_3}\right\}. \tag{3.10}$$

Moreover, by (2.1) and (2.2), we have that

$$d_1\lambda_1^2 - s\lambda_1 + r_1[k + b_1(2a-1)] \le d_2\lambda_1^2 - s\lambda_1 + r_2\beta^* = 0,$$

hence

$$0 < R := \frac{r_1[k + b_1(2a-1)]}{-(d_1\lambda_1^2 - s\lambda_1)} \le 1. \tag{3.11}$$

Now we introduce the following continuous functions

$$\overline{\phi}_1(z) = \begin{cases} u^* + b_1 w^* e^{\lambda_1 z}, & z < 0, \\ 1, & z > 0, \end{cases} \tag{3.12}$$

$$\underline{\phi}_1(z) = \begin{cases} u^*(1 - p_1 e^{\lambda_1 z}), & z < z_1, \\ 0, & z > z_1, \end{cases} \tag{3.13}$$

$$\overline{\phi}_2(z) = \begin{cases} e^{\lambda_1 z}, & z < 0, \\ 1, & z > 0, \end{cases} \tag{3.14}$$

$$\underline{\phi}_2(z) = \begin{cases} e^{\lambda_1 z} - q e^{\mu \lambda_1 z}, & z < z_2, \\ 0, & z > z_2, \end{cases} \tag{3.15}$$

$$\overline{\phi}_3(z) = \begin{cases} w^* + A e^{\lambda_1 z}, & z < 0, \\ 2a - 1, & z > 0, \end{cases} \tag{3.16}$$

$$\underline{\phi}_3(z) = \begin{cases} w^*(1 - e^{\lambda_1 z}), & z < 0, \\ 0, & z > 0, \end{cases} \tag{3.17}$$

where constants A, p_1, μ and q are defined in sequence as follows:

$$A = (2a - 1) - w^* > 0; \tag{3.18}$$

$$R \le p_1 \le 1; \tag{3.19}$$

$$1 < \mu < \min\{2, \lambda_2/\lambda_1\}; \tag{3.20}$$

$$q > \max\left\{1, \frac{r_2(hb_1 w^* + 1 + b_2 A)}{-G(\mu\lambda_1)}\right\}. \tag{3.21}$$

The points z_1 and z_2 are defined by

$$z_1 := \frac{-\ln p_1}{\lambda_1}, \; z_2 := \frac{-\ln(q)}{(\mu - 1)\lambda_1}.$$

By the choice of p_1 in (3.19), which is admissible due to (3.11), and because $q > 1$, we have that $z_2 < 0 \le z_1$. Note also $G(\mu\lambda_1) < 0$ so that q is well-defined.

Lemma 3.3. *Suppose that $\beta^* > 0$ and $s > s^*$. Let (2.1) and (2.2) be enforced. Then the functions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ defined in (3.12)-(3.17) are a pair of generalized upper and lower solutions of (1.10) in the sense of Definition 3.1, satisfy (3.7)-(3.8) and are such that boundary condition (1.11) holds at the unstable tail.*

The proof of Lemma 3.3 is given in Section 6.

3.1.2. Case $s = s^*$

When $s = s^*$, then $\lambda_1 = \lambda_2 = s/(2d_2)$. Here we impose condition (2.3). Then we introduce the following continuous functions

$$\overline{\phi}_1(z) = \begin{cases} u^* + L^* b_1 w^*(-z) e^{\lambda_1 z}, & z < -2/\lambda_1, \\ 1, & z > -2/\lambda_1, \end{cases} \tag{3.22}$$

$$\underline{\phi}_1(z) = \begin{cases} u^*[1 - p_1 L^*(-z) e^{\lambda_1 z}], & z < z_1, \\ 0, & z > z_1, \end{cases} \tag{3.23}$$

$$\overline{\phi}_2(z) = \begin{cases} L^*(-z) e^{\lambda_1 z}, & z < -2/\lambda_1, \\ 1, & z > -2/\lambda_1, \end{cases} \tag{3.24}$$

$$\underline{\phi}_2(z) = \begin{cases} [L^*(-z) - q(-z)^{1/2}]e^{\lambda_1 z}, & z < z_2, \\ 0, & z > z_2, \end{cases} \tag{3.25}$$

$$\overline{\phi}_3(z) = \begin{cases} w^* + L^* A(-z)e^{\lambda_1 z}, & z < -2/\lambda_1, \\ 2a-1, & z > -2/\lambda_1, \end{cases} \tag{3.26}$$

$$\underline{\phi}_3(z) = \begin{cases} w^*[1 - L^*(-z)e^{\lambda_1 z}], & z < -2/\lambda_1, \\ 0, & z > -2/\lambda_1, \end{cases} \tag{3.27}$$

where $L^* := \lambda_1 e^2/2$, $A = 2a - 1 - w^*$, and the constants p_1, q are chosen in sequence such that

$$\max\{R, 2e^{-1}\} \le p_1 \le 1 \tag{3.28}$$

(notice that $R \le 1$ still holds thanks to (2.1) and $d_1 = d_2$), and

$$q > \max\left\{ \frac{4r_2(L^*)^2 M(hb_1 w^* + 1 + b_2 A)}{d_2}, L^* \sqrt{\frac{2}{\lambda_1}} \right\}, \quad \text{with } M := \left(\frac{7}{2\lambda_1 e}\right)^{7/2}. \tag{3.29}$$

It is easy to check that the function $z \in (-\infty, 0] \mapsto p_1 L^*(-z)e^{\lambda_1 z}$ reaches its maximum at $-1/\lambda_1$, where it takes the value $p_1 e/2 \ge 1$. Thus we can define z_1 by

$$p_1 L^*(-z_1)e^{\lambda_1 z_1} = 1, \quad z_1 \in \left[-\frac{2}{\lambda_1}, -\frac{1}{\lambda_1}\right]. \tag{3.30}$$

We also define

$$z_2 := -\left(\frac{q}{L^*}\right)^2.$$

Note that the choice of q in (3.29) ensures that $z_2 < -2/\lambda_1$.

Then we have the following lemma, whose proof is given in Section 6.

Lemma 3.4. *Suppose that $\beta^* > 0$ and $s = s^*$. Let* (2.1) *and* (2.3) *be enforced. Then the functions* $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ *and* $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ *defined in* (3.22)-(3.27) *are a pair of generalized upper and lower solutions of* (1.10) *in the sense of Definition 3.1, satisfy* (3.7)-(3.8) *and are such that boundary condition* (1.11) *holds at the unstable tail.*

With Lemmas 3.3 and 3.4 in hand, the first part of Theorem 2.1 is proved by applying Proposition 3.2.

3.2. Upper-lower solutions for waves invading $E_* = (0, v_*, w_*)$

In this subsection we shall construct generalized upper-lower solutions of (1.10) with boundary condition (1.12). Hence we assume that $\beta_* > 0$ and E_* is unstable, and we impose condition (2.5).

3.2.1. Case $s > s_*$

Here we also impose condition (2.6). Let σ_1 and σ_2 be the two positive roots of $H(x) := d_1x^2 - sx + r_1\beta_*$, that is,

$$\sigma_1 := \frac{s - \sqrt{s^2 - 4d_1r_1\beta_*}}{2d_1}, \quad \sigma_2 := \frac{s + \sqrt{s^2 - 4d_1r_1\beta_*}}{2d_1}. \tag{3.31}$$

By (2.6), we have

$$0 < \sigma_1 \le \min\left\{\frac{s}{2d_2}, \frac{s}{2d_3}\right\}. \tag{3.32}$$

Moreover, by the same reasoning as that for (3.11), using (2.5) and (2.6), we have

$$0 < S := \frac{r_2[h + b_2(2a-1)]}{-(d_2\sigma_1^2 - s\sigma_1)} \le 1. \tag{3.33}$$

Let the constants B, p_2, μ, q be defined in sequence as follows

$$B = (2a-1) - w_*; \tag{3.34}$$

$$S \le p_2 \le 1; \tag{3.35}$$

$$1 < \mu < \min\{2, \sigma_2/\sigma_1\}; \tag{3.36}$$

$$q > \max\left\{1, \frac{r_1(1 + kb_2w_* + b_1B)}{-H(\mu\sigma_1)}\right\}; \tag{3.37}$$

as well as $z_0 := -\ln(q)/[(\mu-1)\sigma_1]$, $z_2 := -\ln(p_2)/\sigma_1$. Note that $H(\mu\sigma_1) < 0$ and so q is well-defined. Also, we have $z_0 < 0 \le z_2$.

With these parameters, we introduce the following continuous functions

$$\overline{\phi}_1(z) = \begin{cases} e^{\sigma_1 z}, & z < 0, \\ 1, & z > 0, \end{cases} \tag{3.38}$$

$$\underline{\phi}_1(z) = \begin{cases} e^{\sigma_1 z} - qe^{\mu\sigma_1 z}, & z < z_0, \\ 0, & z > z_0, \end{cases} \tag{3.39}$$

$$\overline{\phi}_2(z) = \begin{cases} v_* + b_2w_*e^{\sigma_1 z}, & z < 0, \\ 1, & z > 0, \end{cases} \tag{3.40}$$

$$\underline{\phi}_2(z) = \begin{cases} v_*(1 - p_2e^{\sigma_1 z}), & z < z_2, \\ 0, & z > z_2, \end{cases} \tag{3.41}$$

$$\overline{\phi}_3(z) = \begin{cases} w_* + Be^{\sigma_1 z}, & z < 0, \\ 2a-1, & z > 0, \end{cases} \tag{3.42}$$

$$\underline{\phi}_3(z) = \begin{cases} w_*(1 - e^{\sigma_1 z}), & z < 0, \\ 0, & z > 0. \end{cases} \tag{3.43}$$

Then we have the following lemma, whose proof is given in Section 6.

Lemma 3.5. *Suppose that $\beta_* > 0$ and $s > s_*$. Let (2.5) and (2.6) be enforced. Then the functions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ defined in (3.38)-(3.43) are a pair of generalized upper and lower solutions of (1.10) in the sense of Definition 3.1, satisfy (3.7)-(3.8) and are such that boundary condition (1.12) holds at the unstable tail.*

3.2.2. Case $s = s_*$

Here we impose (2.7). When $s = s_*$, then $\sigma_1 = \sigma_2 = s/(2d_1) = s/(2d_2)$. Then we introduce the following continuous functions $\overline{\phi}_j(z)$ and $\underline{\phi}_j(z)$ for $j = 1, 2, 3$.

$$\overline{\phi}_1(z) = \begin{cases} L_*(-z)e^{\sigma_1 z}, & z < -2/\sigma_1, \\ 1, & z > -2/\sigma_1, \end{cases} \tag{3.44}$$

$$\underline{\phi}_1(z) = \begin{cases} [L_*(-z) - q(-z)^{1/2}]e^{\sigma_1 z}, & z < z_0, \\ 0, & z > z_0, \end{cases} \tag{3.45}$$

$$\overline{\phi}_2(z) = \begin{cases} v_* + L_* b_2 w_*(-z)e^{\sigma_1 z}, & z < -2/\sigma_1, \\ 1, & z > -2/\sigma_1, \end{cases} \tag{3.46}$$

$$\underline{\phi}_2(z) = \begin{cases} v_*[1 - p_2 L_*(-z)e^{\sigma_1 z}], & z < z_2, \\ 0, & z > z_2, \end{cases} \tag{3.47}$$

$$\overline{\phi}_3(z) = \begin{cases} w_* + L_* B(-z)e^{\sigma_1 z}, & z < -2/\sigma_1, \\ 2a - 1, & z > -2/\sigma_1, \end{cases} \tag{3.48}$$

$$\underline{\phi}_3(z) = \begin{cases} w_*[1 - L_*(-z)e^{\sigma_1 z}], & z < -2/\sigma_1, \\ 0, & z > -2/\sigma_1, \end{cases} \tag{3.49}$$

where $L_* := \sigma_1 e^2/2$, $B = (2a-1) - w_*$, p_2 satisfies $\max\{S, 2e^{-1}\} \le p_2 \le 1$, and

$$q \ge \max\left\{ \frac{4r_1 L_*^2 M(1 + kb_2 w_* + b_1 B)}{d_1}, L_*\sqrt{\frac{2}{\sigma_1}} \right\}, \quad \text{with } M := \left(\frac{7}{2\sigma_1 e}\right)^{7/2}. \tag{3.50}$$

Moreover, $z_0 := -(q/L_*)^2$ and $z_2 \in [-2/\sigma_1, -1/\sigma_1]$ is defined (uniquely) by

$$p_2 L_*(-z_2)e^{\sigma_1 z_2} = 1.$$

Also, the choice of q in (3.50) implies that $z_0 \le -2/\sigma_1$. We shall obtain the following lemma:

Lemma 3.6. *Suppose that $\beta_* > 0$ and $s = s_*$. Let (2.5) and (2.7) be enforced. Then the functions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ defined in (3.44)-(3.49) are a pair of generalized upper and lower solutions of (1.10) in the sense of Definition 3.1, satisfy (3.7)-(3.8) and are such that boundary condition (1.12) holds at the unstable tail.*

The proof of Lemma 3.6 is also given in Section 6. Then, similarly as before, the first part of Theorem 2.2 is proved, by applying Proposition 3.2 together with Lemmas 3.5 and 3.6

4. Asymptotic behavior of stable tail

This section is devoted to the proof of the second parts of Theorems 2.1 and 2.2. Throughout this section, we shall denote

$$\phi_j^+ := \limsup_{z\to\infty} \phi_j(z),\ \phi_j^- := \liminf_{z\to\infty} \phi_j(z),\ j = 1, 2, 3,$$

where (ϕ_1, ϕ_2, ϕ_3) denotes any of the traveling wave solutions obtained in Subsections 3.1 and 3.2. We recall that, by construction,

$$0 < \phi_1, \phi_2(z) \le 1,\ 0 < \phi_3(z) \le 2a - 1, \quad \forall z \in \mathbb{R}. \tag{4.1}$$

4.1. Preliminaries: the ODE system

Let us first introduce the 6-dimensional first order ODE system corresponding to (1.10):

$$\begin{cases} \phi_1' = \psi_1, \\ \psi_1' = \frac{1}{d_1}[s\psi_1 - r_1\phi_1(1 - \phi_1 - k\phi_2 - b_1\phi_3)], \\ \phi_2' = \psi_2, \\ \psi_2' = \frac{1}{d_2}[s\psi_2 - r_2\phi_2(1 - h\phi_1 - \phi_2 - b_2\phi_3)], \\ \phi_3' = \psi_3, \\ \psi_3' = \frac{1}{d_3}[s\psi_3 - r_3\phi_3(-1 + a\phi_1 + a\phi_2 - \phi_3)]. \end{cases} \tag{4.2}$$

For convenience, we shall denote by $\Psi = (\phi_1, \psi_1, \phi_2, \psi_2, \phi_3, \psi_3)$ any solution of (4.2), and rewrite (4.2) as $\Psi' = F(\Psi)$. In this subsection we state several lemmas related to the stable manifolds of various equilibria of (4.2). For convenience, we write explicitly its Jacobian matrix:

$$J_F(\Psi) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{r_1}{d_1}(1 - 2\phi_1 - k\phi_2 - b_1\phi_3) & \frac{s}{d_1} & \frac{r_1}{d_1}k\phi_1 & 0 & \frac{r_1}{d_1}b_1\phi_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{r_2}{d_2}h\phi_2 & 0 & -\frac{r_2}{d_2}(1 - 2\phi_2 - h\phi_1 - b_2\phi_3) & \frac{s}{d_2} & \frac{r_2}{d_2}b_2\phi_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{r_3}{d_3}a\phi_3 & 0 & -\frac{r_3}{d_3}a\phi_3 & 0 & -\frac{r_3}{d_3}(-1 + a\phi_1 + a\phi_2 - 2\phi_3) & \frac{s}{d_3} \end{pmatrix}.$$

Lemma 4.1. *There exists an open neighborhood W_0 of the steady state $(0, 0, 0, 0, 0, 0)$ such that any solution Ψ of (4.2) satisfying $\Psi(z) \in W_0$ for all $z \ge 0$ must also satisfy $\Psi(z) \in \{\phi_1 = 0, \phi_2 = 0\}$ for all z.*

Proof. The matrix of the linearized system of (4.2) around $(0, 0, 0, 0, 0, 0)$ is

$$J_F(0) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{r_1}{d_1} & \frac{s}{d_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{r_2}{d_2} & \frac{s}{d_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{r_3}{d_3} & \frac{s}{d_3} \end{pmatrix},$$

which has one negative real eigenvalue and five eigenvalues with positive real parts. By standard perturbation theory, this means that the steady state $(0, 0, 0, 0, 0, 0)$ of (4.2) has a 1-dimensional stable manifold $\mathcal{S}$. Furthermore, there is a neighborhood W_0 of $(0, 0, 0, 0, 0, 0)$ such that, for any $\Psi(0) \in W_0 \setminus \mathcal{S}$, there exists $z > 0$ such that $\Psi(z) \notin W_0$.

Now notice that $\{\phi_1 = \psi_1 = 0, \phi_2 = \psi_2 = 0\}$ is an invariant set for (4.2). Repeating the same standard stability analysis, one finds that $(0, 0, 0, 0, 0, 0)$ also admits a 1-dimensional stable manifold in the subset $\{\phi_1 = \psi_1 = 0, \phi_2 = \psi_2 = 0\}$. This implies that $\mathcal{S}$ is actually included in $\{\phi_1 = \psi_1 = 0, \phi_2 = \psi_2 = 0\}$, and the lemma follows. □

The next two lemmas can be proved in the same way.

Lemma 4.2. *There exists an open neighborhood $W_{1,u}$ of the steady state $(1, 0, 0, 0, 0, 0)$ such that any solution Ψ of (4.2) satisfying $\Psi(z) \in W_{1,u}$ for all $z \geq 0$ must also satisfy $\Psi(z) \in \{\phi_2 = 0, \phi_3 = 0\}$ for all z.*

There exists an open neighborhood $W_{1,v}$ of the steady state $(0, 0, 1, 0, 0, 0)$ such that any solution Ψ of (4.2) satisfying $\Psi(z) \in W_{1,v}$ for all $z \geq 0$ must also satisfy $\Psi(z) \in \{\phi_3 = 0\}$ for all z.

Lemma 4.3. *Assume that $\beta_* > 0$. There exists an open neighborhood W_* of the steady state $(0, 0, v_*, 0, w_*, 0)$ such that any solution Ψ of (4.2) satisfying $\Psi(z) \in W_*$ for all $z \geq 0$ must also satisfy $\Psi(z) \in \{\phi_1 = 0\}$ for all z.*

Assume that $\beta^* > 0$. There exists an open neighborhood W^* of the steady state $(u^*, 0, 0, 0, w^*, 0)$ such that any solution Ψ of (4.2) satisfying $\Psi(z) \in W^*$ for all $z \geq 0$ must also satisfy $\Psi(z) \in \{\phi_2 = 0\}$ for all z.

4.2. Some general estimates

For the sake of conciseness, we state here some lemmas that hold in both cases of a strong alien and a weak alien competitor prey. In particular, throughout this subsection (ϕ_1, ϕ_2, ϕ_3) shall still denote any of the traveling wave solution constructed in either Subsections 3.1 or 3.2.

The first lemmas state that some components of these traveling wave solutions cannot go simultaneously to 0. They rely on a sequential argument inspired by persistence theory in dynamical systems, which has also been used in the context of spreading behavior in predator-prey systems [6,7].

Lemma 4.4. *It holds that*

$$\liminf_{z \to \infty} [\phi_1 + \phi_2](z) > 0.$$

In particular we can define

$$\delta_{12} := \inf_{z \geq 0} [\phi_1 + \phi_2](z) > 0.$$

Proof. We proceed by contradiction and assume that there exists a sequence $\{z_n\}_{n \in \mathbb{N}}$ such that $z_n \to \infty$ and $\phi_1(z_n) + \phi_2(z_n) \to 0$ as $n \to \infty$. Then we let any $\varepsilon > 0$ arbitrarily small, and we define another sequence

$$z_n' := \inf\{z \leq z_n \mid \forall z' \in (z, z_n),\ [\phi_1 + \phi_2](z') \leq \varepsilon\},$$

so that $[\phi_1 + \phi_2](z_n') = \varepsilon$ for all n. Furthermore, we know from elliptic estimates that $(\phi_1, \phi_2, \phi_3)(\cdot + z_n)$ converges up to extraction of a subsequence to a solution $(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$ of (1.10), and that $\bar{\phi}_1 \equiv \bar{\phi}_2 \equiv 0$ by the strong maximum principle. Hence $z_n - z_n' \to \infty$ as $n \to \infty$.

Passing to the limit as $n \to \infty$, we find that $(\phi_1, \phi_2, \phi_3)(\cdot + z_n')$ converges to another solution of (1.10), which we denote by $(\hat{\phi}_1^\varepsilon, \hat{\phi}_2^\varepsilon, \hat{\phi}_3^\varepsilon)$. Furthermore, by construction we have that $\hat{\phi}_1^\varepsilon(0) + \hat{\phi}_2^\varepsilon(0) = \varepsilon$ and $0 \leq \hat{\phi}_1^\varepsilon, \hat{\phi}_2^\varepsilon \leq \varepsilon$ for all $z \geq 0$.

We now claim that there exists $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$|\hat{\phi}_3^\varepsilon(z)| \leq \delta(\varepsilon),\ \forall z \geq 0. \tag{4.3}$$

We proceed by contradiction and assume that there exist sequences $\varepsilon_n \to 0$ and $y_n \geq 0$ such that $\lim_{n\to\infty} \hat{\phi}_3^{\varepsilon_n}(y_n) > 0$. By standard elliptic estimates, we find that $(\hat{\phi}_1^{\varepsilon_n}, \hat{\phi}_2^{\varepsilon_n})(y_n + z) \to (0, 0)$, and then that $\hat{\phi}_3^{\varepsilon_n}(y_n + x + st)$ converges as $n \to \infty$ to an entire in time solution $w(x, t)$ of

$$w_t = d_3 w_{xx} + r_3 w(-1 - w),$$

which is also bounded from above by $2a - 1$. Thus this limit must be identical to 0, a contradiction. Claim (4.3) is proved.

Next, as $\varepsilon \to 0$, we get that $(\hat{\phi}_1^\varepsilon, \hat{\phi}_2^\varepsilon, \hat{\phi}_3^\varepsilon) \to (0, 0, 0)$ uniformly in $[0, +\infty)$. By standard elliptic estimates, it also follows that $((\hat{\phi}_1^\varepsilon)', (\hat{\phi}_2^\varepsilon)', (\hat{\phi}_3^\varepsilon)') \to (0, 0, 0)$ in $[0, +\infty)$. Therefore, we can find ε small enough so that

$$(\hat{\phi}_1^\varepsilon, (\hat{\phi}_1^\varepsilon)', \hat{\phi}_2^\varepsilon, (\hat{\phi}_2^\varepsilon)', \hat{\phi}_3^\varepsilon, (\hat{\phi}_3^\varepsilon)') \in W_0,$$

for all $z \geq 0$. Applying Lemma 4.1, we infer that $\hat{\phi}_1^\varepsilon \equiv \hat{\phi}_2^\varepsilon \equiv 0$. However, by construction we have that $\hat{\phi}_1^\varepsilon(0) + \hat{\phi}_2^\varepsilon(0) = \varepsilon > 0$, thus we have reached a contradiction. The lemma is proved. □

Lemma 4.5. *It holds that*

$$\liminf_{z\to\infty} [\phi_2 + \phi_3](z) > 0.$$

In particular we can define

$$\delta_{23} := \inf_{z \geq 0} [\phi_2 + \phi_3](z) > 0.$$

Proof. We again proceed by contradiction and assume that there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ such that $z_n \to \infty$ and $\phi_2(z_n) + \phi_3(z_n) \to 0$ as $n \to \infty$. Then we let any $\varepsilon > 0$ arbitrarily small, and we define another sequence

$$z_n' := \inf\{z \leq z_n \mid \forall z' \in (z, z_n),\ [\phi_2 + \phi_3](z') \leq \varepsilon\},$$

so that $[\phi_2 + \phi_3](z_n') = \varepsilon$ for all n. As in the previous lemma, it follows from a limit argument and a strong maximum principle that $z_n - z_n' \to \infty$ as $n \to \infty$.

Passing to the limit as $n \to \infty$, we find that $(\phi_1, \phi_2, \phi_3)(\cdot + z'_n)$ converges to another solution of (1.10), which we denote by $(\hat{\phi}_1^\varepsilon, \hat{\phi}_2^\varepsilon, \hat{\phi}_3^\varepsilon)$. Furthermore, by construction we have that $\hat{\phi}_2^\varepsilon(0) + \hat{\phi}_3^\varepsilon(0) = \varepsilon$ and $0 \leq \hat{\phi}_2^\varepsilon, \hat{\phi}_3^\varepsilon \leq \varepsilon$ for all $z \geq 0$. Provided that $\varepsilon > 0$ is small, we also have by Lemma 4.4 that

$$\hat{\phi}_1^\varepsilon(z) \geq \frac{\delta_{12}}{2} > 0, \tag{4.4}$$

for any $z \geq 0$.

We now claim that there exist $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and z_ε such that

$$|1 - \hat{\phi}_1^\varepsilon(z)| \leq \delta(\varepsilon), \ \forall z \geq z_\varepsilon. \tag{4.5}$$

Indeed, we proceed by contradiction and assume that there exist sequences $\varepsilon_n \to 0$ and $y_n \to \infty$ such that $\lim_{n\to\infty} \hat{\phi}_1^{\varepsilon_n}(y_n) < 1$. By standard elliptic estimates, we find that $(\hat{\phi}_2^{\varepsilon_n}, \hat{\phi}_3^{\varepsilon_n})(y_n + z) \to (0, 0)$, and then that $\hat{\phi}_1^{\varepsilon_n}(y_n + x + st)$ converges as $n \to \infty$ to an entire in time solution $u(x,t)$ of

$$u_t = d_1 u_{xx} + r_1 u(1-u),$$

which is also bounded from below by $\frac{\delta_{12}}{2}$ (recall (4.4) and that $y_n \to \infty$). Thus this limit must be identical to 1. We have reached a contradiction and the claim (4.5) is proved.

Therefore, for ε small enough we have found a solution of (1.10) which is in $W_{1,u}$ for all $z \geq z_\varepsilon$, hence $\hat{\phi}_2^\varepsilon \equiv \hat{\phi}_3^\varepsilon \equiv 0$ by Lemma 4.2 (with a shift $z \to z - z_\varepsilon$). This contradicts the fact that $\hat{\phi}_2^\varepsilon(0) + \hat{\phi}_3^\varepsilon(0) = \varepsilon > 0$. The lemma is proved. □

It follows that, in all cases, the predator cannot go to extinction at the stable tail.

Lemma 4.6. *It holds that* $\phi_3^- = \liminf_{z\to\infty} \phi_3(z) > 0$.

Proof. The argument is again very similar. As in the proofs of Lemmas 4.4 and 4.5, we assume by contradiction that, for any small $\varepsilon > 0$, there exist sequences $\{z_n\}_{n\in\mathbb{N}}$ and $\{z'_n\}_{n\in\mathbb{N}}$ such that $z_n - z'_n \to \infty$ as $n \to \infty$, and

$$\phi_3(z'_n) = \varepsilon \geq \phi_3(z),$$

for all $z \in [z'_n, z_n]$. From elliptic estimates, $(\phi_1, \phi_2, \phi_3)(\cdot + z'_n)$ converges to another solution of (1.10), which we denote by $(\hat{\phi}_1^\varepsilon, \hat{\phi}_2^\varepsilon, \hat{\phi}_3^\varepsilon)$. Furthermore, we have that $\hat{\phi}_3^\varepsilon(0) = \varepsilon$ and $\hat{\phi}_3^\varepsilon \leq \varepsilon$ for all $z \geq 0$. Up to reducing $\varepsilon > 0$, we also have by Lemma 4.5 that

$$\hat{\phi}_2^\varepsilon(z) \geq \frac{\delta_{23}}{2} > 0$$

for any $z \geq 0$.

Next we claim that there exist $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and z_ε such that

$$|\hat{\phi}_1^\varepsilon(z)| + |1 - \hat{\phi}_2^\varepsilon(z)| \leq \delta(\varepsilon), \ \forall z \geq z_\varepsilon. \tag{4.6}$$

Indeed, taking any sequence $z_n \to \infty$ as $n \to \infty$, we have up to extraction of a subsequence that the pair $(\hat{\phi}_1^\varepsilon, \hat{\phi}_2^\varepsilon)(\cdot + z_n)$ converges to $(\tilde{\phi}_1, \tilde{\phi}_2)$ which satisfies

$$0 \le \tilde{\phi}_1 \le 1, \; \frac{\delta_{23}}{2} \le \tilde{\phi}_2 \le 1,$$

and

$$\begin{cases} d_1\tilde{\phi}_1''(z) - s\tilde{\phi}_1'(z) + r_1\tilde{\phi}_1(z)[1 - \tilde{\phi}_1(z) - k\tilde{\phi}_2(z)] \ge 0, \; z \in \mathbb{R}, \\ d_2\tilde{\phi}_2''(z) - s\tilde{\phi}_2'(z) + r_2\tilde{\phi}_2(z)[1 - h\tilde{\phi}_1(z) - \tilde{\phi}_2(z) - b_2\varepsilon] \le 0, \; z \in \mathbb{R}. \end{cases} \tag{4.7}$$

Letting $\overline{\psi}_1 = 1 - \tilde{\phi}_1$ and $\overline{\psi}_2 = \tilde{\phi}_2$, we get that $(\overline{\psi}_1, \overline{\psi}_2)$ is a nonnegative supersolution of

$$\begin{cases} \partial_t \psi_1 - d_1(\psi_1)_{zz} + s(\psi_1)_z - r_1(1-\psi_1)(-\psi_1 + k\psi_2) = 0, \; z \in \mathbb{R}, \; t > 0, \\ \partial_t \psi_2 - d_2(\psi_2)_{zz} + s(\psi_2)_z - r_2\psi_2(1 - h + h\psi_1 - \psi_2 - b_2\varepsilon) = 0, \; z \in \mathbb{R}, \; t > 0, \end{cases} \tag{4.8}$$

which is a cooperative reaction-diffusion system and hence satisfies a comparison principle. In particular, we must have that $\overline{\psi}_1(z) \ge \underline{\psi}_1(t)$ and $\overline{\psi}_2(z) \ge \underline{\psi}_2(t)$ for all $t > 0$ and $z \in \mathbb{R}$, where $(\underline{\psi}_1, \underline{\psi}_2)$ solves (4.8) with the initial data $(0, \delta_{23}/2)$; notice that, due to the invariance by translation, $(\underline{\psi}_1, \underline{\psi}_2)$ does not depend on the spatial variable z.

Furthermore, provided that ε is small enough, then $(0, \delta_{23}/2)$ and $(1, 1)$ are respectively a sub and a supersolution of (4.8). By the comparison principle and parabolic estimates, it follows that $\underline{\psi}_1$ and $\underline{\psi}_2$ are nondecreasing in time and converge to a constant steady state (P, Q) of (4.8), with $0 \le P \le 1$ and $\delta_{23}/2 \le Q \le 1$. It is straightforward to check that, when $\varepsilon > 0$ is small enough, the only such steady state is $(1, 1 - b_2\varepsilon)$. Putting the above facts together, we find that $\tilde{\phi}_1 \equiv 0$ and $\tilde{\phi}_2 \in [1 - b_2\varepsilon, 1]$. Hence (4.6) is proved.

Then we get for ε small enough a solution of (1.10) which is in $W_{1,v}$ for all $z \ge z_\varepsilon$, hence $\hat{\phi}_3^\varepsilon \equiv 0$ by Lemma 4.2. This contradicts our construction and the lemma is proved. $\square$

We complete this subsection with a lemma which shall allow us to derive the stable tail limit, regardless of the alien species, in the co-existence case.

Lemma 4.7. *Assume that E_c exists, i.e., (1.8) holds, and if also*

$$k\sqrt{\frac{b_2}{b_1}} + h\sqrt{\frac{b_1}{b_2}} < 2, \tag{4.9}$$

let $(u, v, w) = (u, v, w)(x, t)$ be a bounded entire solution of (1.1) such that

$$m := \min\left(\inf_{(x,t)\in\mathbb{R}^2} u(x,t), \; \inf_{(x,t)\in\mathbb{R}^2} v(x,t), \; \inf_{(x,t)\in\mathbb{R}^2} w(x,t) \right) > 0. \tag{4.10}$$

Then $(u, v, w) \equiv (u_c, v_c, w_c)$.

Proof. First, we denote $M = \max\{\|u\|_\infty, \|v\|_\infty, \|w\|_\infty\}$ and define the functions $g(x) = x - \ln(x) - 1$ and $\Phi = \Phi(u, v, w)$ given by

$$\Phi(u, v, w) := \frac{r_3 a u_c}{b_1 r_1} g\left(\frac{u}{u_c}\right) + \frac{r_3 a v_c}{b_2 r_2} g\left(\frac{v}{v_c}\right) + w_c g\left(\frac{w}{w_c}\right).$$

Let us compute the Lie derivative of Φ, denoted by $L_X\Phi$, along the three dimensional vector field

$$X := (r_1 u(1 - u - kv - b_1 w, r_2 v(1 - hu - v - b_2 w), r_3 w(-1 + au + av - w))$$

associated to the kinetic part of (1.1). Then, using that (u_c, v_c, w_c) is a stationary state, we find

$$\begin{aligned} L_X\Phi(u, v, w) &= (\Phi_u, \Phi_v, \Phi_w) \cdot X \\ &= \frac{r_3 a}{b_1}(u - u_c)(1 - u - kv - b_1 w) + \frac{r_3 a}{b_2}(v - v_c)(1 - v - hu - b_2 w) \\ &\quad + r_3(w - w_c)(-1 + au + av - w) \\ &= -\frac{r_3 a}{b_1}(u - u_c)^2 - \frac{r_3 a}{b_2}(v - v_c)^2 - r_3(w - w_c)^2 \\ &\quad - \left(\frac{r_3 ak}{b_1} + \frac{r_3 ah}{b_2}\right)(u - u_c)(v - v_c). \end{aligned}$$

On the other hand, for any $(X_1, X_2) \in \mathbb{R}^2$, we have

$$X_1^2 + X_2^2 - \left(\frac{k}{b_1} + \frac{h}{b_2}\right)\sqrt{b_1 b_2}|X_1||X_2| \geq \left[1 - \frac{1}{2}\left(\frac{k}{b_1} + \frac{h}{b_2}\right)\sqrt{b_1 b_2}\right](X_1^2 + X_2^2).$$

Taking $X_1 = \sqrt{\frac{r_3 a}{b_1}}(u - u_c)$ and $X_2 = \sqrt{\frac{r_3 a}{b_2}}(v - v_c)$, we infer from (4.9) that

$$L_X\Phi(u, v, w) \leq -\alpha\left[(u - u_c)^2 + (v - v_c)^2 + (w - w_c)^2\right]$$

for some $\alpha > 0$ and any $u, v, w > 0$. Furthermore, recalling the definition of Φ above, there exists $\beta > 0$ such that

$$L_X\Phi(u, v, w) \leq -\beta\Phi(u, v, w), \ \forall (u, v, w) \in [m, M]^3.$$

From this inequality, the proof of Lemma 4.7 follows from the same arguments as that in [6, Lemma 4.1]. $\square$

We point out that similar results hold for the two species predator-prey system:

Lemma 4.8. *Any entire in time solution of*

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u - b_1 w), \\ w_t = d_3 w_{xx} + r_3 w(-1 + au - w), \end{cases} \tag{4.11}$$

such that

$$0 < \min\{\inf_{\mathbb{R}^2} u, \inf_{\mathbb{R}^2} w\} < \max\{\sup_{\mathbb{R}^2} u, \sup_{\mathbb{R}^2} w\} < +\infty,$$

must satisfy that $u \equiv u^*$ *and* $w \equiv w^*$.

Similarly, any entire in time solution of the subsystem derived from (1.1) *by letting* $u = 0$*, and satisfying*

$$0 < \min\{\inf_{\mathbb{R}^2} v, \inf_{\mathbb{R}^2} w\} < \max\{\sup_{\mathbb{R}^2} v, \sup_{\mathbb{R}^2} w\} < +\infty,$$

must satisfy that $v \equiv v_*$ *and* $w \equiv w_*$.

Proof. The proof of this lemma can be found in [6, Lemma 4.2] by a Lyapunov argument (see also [4,8]). We omit it here. □

4.3. The case of the alien strong competitor

Now we turn to the case when the strong competing prey is the alien species, and in particular we assume here that $\beta^* > 0$, i.e., (1.6) holds. Now (ϕ_1, ϕ_2, ϕ_3) denotes the traveling wave solution constructed in Subsection 3.1. In order to apply either method of contracting rectangles (see [9]) or Lyapunov function (see Lemma 4.7 above), positive lower bounds on the traveling wave solutions are typically required.

We already know by Lemma 4.6 that $\phi_3^- > 0$. We therefore continue the proof of the stable tail limit by obtaining some better lower bounds for ϕ_2. Because ϕ_1 only persists when the invading state is E_c, it shall be considered separately in Subsection 4.3.2.

Lemma 4.9. *It holds that* $\phi_2^- = \liminf_{z\to\infty} \phi_2(z) > 0$.

Proof. We again proceed by contradiction and, as in the proof of Lemma 4.6, for any small $\varepsilon > 0$ we find sequences $\{z_n\}_{n\in\mathbb{N}}$ and $\{z'_n\}_{n\in\mathbb{N}}$ such that $z_n - z'_n \to \infty$, $\phi_2(z'_n) = \varepsilon$ and $\phi_2 \leq \varepsilon$ in $[z'_n, z_n]$.

By standard elliptic estimates, we find that $(\phi_1, \phi_2, \phi_3)(\cdot + z'_n)$ converges to another solution of (1.10), which we denote by $(\hat{\phi}_1^\varepsilon, \hat{\phi}_2^\varepsilon, \hat{\phi}_3^\varepsilon)$. Furthermore, by construction we have that $\hat{\phi}_2^\varepsilon(0) = \varepsilon$ and $\hat{\phi}_2^\varepsilon \leq \varepsilon$ for all $z \geq 0$. Also, due to Lemmas 4.4 and 4.5, there exist some $\delta_0 > 0$ independent of ε such that $\hat{\phi}_1^\varepsilon(z), \hat{\phi}_3^\varepsilon(z) \geq \delta_0$ for all $z \geq 0$.

We then claim that there exist z_ε and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$|\hat{\phi}_1^\varepsilon(z) - u^*| + |\hat{\phi}_3^\varepsilon(z) - w^*| \leq \delta(\varepsilon),\ \forall z \geq z_\varepsilon. \tag{4.12}$$

Indeed, proceed by contradiction and assume that there exists $z^\varepsilon \to \infty$ such that for instance $\liminf |\hat{\phi}_1^\varepsilon(z^\varepsilon) - u^*| > 0$ when (some subsequence of) $\varepsilon \to 0$. By parabolic estimates and up to extraction of another subsequence, we get that $(\hat{\phi}_1^\varepsilon, \hat{\phi}_2^\varepsilon, \hat{\phi}_3^\varepsilon)(\cdot + z^\varepsilon)$ converges to a solution $(\hat{\phi}_1^0, \hat{\phi}_2^0, \hat{\phi}_3^0)$ of (1.10). Moreover, we have by construction that $\hat{\phi}_2^0 \equiv 0$, and therefore $(\hat{\phi}_1^0, \hat{\phi}_3^0)(x + st)$ is an entire in time solution of (4.11). It also follows from our construction that

$0 < \delta_0 \le \hat{\phi}_1^0 \le 1$ and $0 < \delta_0 \le \hat{\phi}_3^0 \le 2a-1$, hence from Lemma 4.8 that $\hat{\phi}_1^0 \equiv u^*$ and $\hat{\phi}_3^0 \equiv w^*$. We have reached a contradiction and proved the claim (4.12).

Therefore, we can choose $\varepsilon > 0$ small enough so that $(\hat{\phi}_1^\varepsilon, (\hat{\phi}_1^\varepsilon)', \hat{\phi}_2^\varepsilon, (\hat{\phi}_2^\varepsilon)', \hat{\phi}_3^\varepsilon, (\hat{\phi}_3^\varepsilon)') \in W^*$, for all $z \ge z_\varepsilon$. Since $\beta^* > 0$ and applying Lemma 4.3, we conclude that $\hat{\phi}_2^\varepsilon \equiv 0$. This contradicts the fact that $\hat{\phi}_2^\varepsilon(0) = \varepsilon > 0$, and ends the proof of the lemma. □

4.3.1. Semi-co-existence case

This subsection is devoted to showing that the traveling wave solution obtained in Subsection 3.1 approaches $E_* = (0, v_*, w_*)$ as $z \to \infty$, if $\beta_* < 0$ and (2.4) holds.

Recall that

$$\phi_j^+ := \limsup_{z\to\infty} \phi_j(z),\ \ \phi_j^- := \liminf_{z\to\infty} \phi_j(z),\ \ j = 1, 2, 3,$$

Unfortunately, the implicit lower bound in Lemma 4.9 on ϕ_2^- is not enough for our purpose. We immediately improve it:

Lemma 4.10. *It holds that $\phi_2^- \ge 1 - h - b_2(2a-1) := \gamma_2$.*

Proof. The proof is the same as that of Lemma 4.1 in [9]. Take any sequence $z_n \to \infty$, and by elliptic estimates assume up to extraction of a subsequence that $\phi_2(\cdot + z_n) \to \tilde{\phi}_2$ as $n \to \infty$. Then $\tilde{\phi}_2 \ge \phi_2^- > 0$ and, by (4.1), it satisfies

$$d_2\tilde{\phi}_2'' - s\tilde{\phi}_2' + r_2\tilde{\phi}_2[1 - h - \tilde{\phi}_2 - b_2(2a-1)] \le 0.$$

On the other hand, the solution of the ODE

$$\partial_t \underline{u} = r_2\underline{u}[1 - h - \underline{u} - b_2(2a-1)],$$

with initial condition

$$\underline{u}(t=0) = \phi_2^- > 0,$$

converges to γ_2 as $t \to +\infty$. Since $\tilde{\phi}_2 \ge \phi_2^-$, we can apply the parabolic comparison principle on the whole real line, and we find that $\tilde{\phi}_2 \ge \gamma_2$. Due to the arbitrary choice of the sequence z_n, we conclude as wanted that $\phi_2^- \ge \gamma_2$. □

Now define

$$\begin{cases} m_2(\theta) := (1-\theta)(\gamma_2 - \varepsilon) + \theta v_*, & M_2(\theta) := (1-\theta)(1+\varepsilon^2) + \theta v_*, \\ m_3(\theta) := (1-\theta)(\delta_3 - \varepsilon) + \theta w_*, & M_3(\theta) := (1-\theta)(2a - 1 + \varepsilon) + \theta w_*, \end{cases}$$

where $\gamma_2 = 1 - h - b_2(2a-1) > 0$, $\delta_3 := \min\{w_*/2, (a\gamma_2 - 1)/2, \phi_3^-\} > 0$, and ε satisfies

$$0 < \varepsilon < \min\left\{\gamma_2, \delta_3, \frac{hk\gamma_2 + hb_1\delta_3}{hk + hb_1 + b_2}, \frac{ak\gamma_2 + ab_1\delta_3}{ak + ab_1 + 1}, \frac{a\gamma_2 - \delta_3 - 1}{a}\right\}. \tag{4.13}$$

Notice that $a\gamma_2 - 1 > 0$ follows from (2.4), which in turn ensures together with Lemma 4.6 that such constants δ_3 and ε indeed exist. Also, due to $a > 1$ and the definition of δ_3, we have that $0 < \gamma_2 < v_* < 1$ and $\delta_3 < w_* < 2a - 1$. Hence $m_j(\theta)$ is increasing and $M_j(\theta)$ is decreasing in $\theta \in [0, 1]$ for $j = 2, 3$.

Due to the lack of a positive lower bound for ϕ_1, instead of considering 3-d rectangles, we consider the following 2-d contracting rectangles:

$$Q(\theta) := [m_2(\theta), M_2(\theta)] \times [m_3(\theta), M_3(\theta)] \subset (0, \infty)^2, \quad \theta \in [0, 1]. \tag{4.14}$$

Also, we consider the set

$$\mathcal{A} := \{\theta \in [0, 1) \mid m_k(\theta) < \phi_k^- \le \phi_k^+ < M_k(\theta),\ k = 2, 3\}. \tag{4.15}$$

Obviously, by (4.1) and Lemma 4.10, we have

$$\begin{aligned} m_2(0) &= \gamma_2 - \varepsilon < \gamma_2 \le \phi_2^- \le \phi_2^+ \le 1 < 1 + \varepsilon^2 = M_2(0),\\ m_3(0) &= \delta_3 - \varepsilon < \delta_3 \le \phi_3^- \le \phi_3^+ \le 2a - 1 < 2a - 1 + \varepsilon = M_3(0). \end{aligned}$$

Hence $0 \in \mathcal{A}$ and also the quantity $\theta_0 := \sup \mathcal{A}$ is well-defined such that $\theta_0 \in (0, 1]$.

By passing to the limit, we have

$$m_j(\theta_0) \le \phi_j^- \le \phi_j^+ \le M_j(\theta_0), \quad j = 2, 3. \tag{4.16}$$

To proceed further, we derive a better upper bound for ϕ_1 as follows.

Lemma 4.11. *Under the condition* (4.16)*, it holds that*

$$\phi_1^+ \le M_1(\theta_0) := \max\{0, 1 - km_2(\theta_0) - b_1 m_3(\theta_0)\}.$$

Proof. Taking any sequence $\{z_n\}_{n\in\mathbb{N}}$ tending to ∞, up to extraction of a subsequence, we have $\phi_1(\cdot + z_n) \to \hat{\phi}_1$ as $n \to \infty$. It follows from (4.16) that

$$d_1\hat{\phi}_1'' - s\hat{\phi}_1' + r_1\hat{\phi}_1[1 - \hat{\phi}_1 - km_2(\theta_0) - b_1 m_3(\theta_0)] \ge 0,$$

on the whole real line. Recalling also that $\phi_1 \le 1$ and applying the parabolic comparison principle, we get that $\hat{\phi}_1(z) \le \overline{u}(t)$ for any $z \in \mathbb{R}$ and $t > 0$, where $\overline{u}$ solves

$$\partial_t \overline{u} = r_1\overline{u}[1 - \overline{u} - km_2(\theta_0) - b_1 m_3(\theta_0)],$$

with initial condition

$$\overline{u}(t = 0) = 1.$$

One may check that $\overline{u}(t) \to M_1(\theta_0)$ as $t \to +\infty$, hence $\hat{\phi}_1 \le M_1(\theta_0)$. Due to the arbitrary choice of the sequence z_n, we reach the wanted conclusion. □

Next, we prove that $\theta_0 = 1$. We assume by contradiction that $\theta_0 \in (0, 1)$. In particular, one of the following equalities must hold:

$$\phi_j^- = m_j(\theta_0),\ \phi_j^+ = M_j(\theta_0),\ j = 2, 3. \tag{4.17}$$

To reach a contradiction with (4.17), we introduce and compute

$$\begin{aligned}
\alpha_2 &:= 1 - hM_1(\theta_0) - m_2(\theta_0) - b_2M_3(\theta_0),\\
\omega_2 &:= 1 - M_2(\theta_0) - b_2m_3(\theta_0) = -(1-\theta_0)[\varepsilon^2 + b_2(\delta_3 - \varepsilon)] < 0,\\
\alpha_3 &:= -1 + am_2(\theta_0) - m_3(\theta_0) = (1-\theta_0)[a\gamma_2 - 1 - \delta_3 - \varepsilon(a-1)] > 0,\\
\omega_3 &:= -1 + aM_1(\theta_0) + aM_2(\theta_0) - M_3(\theta_0).
\end{aligned}$$

We also compute that

$$\alpha_2 > 0,$$

and to do so we distinguish the two cases when $M_1(\theta_0) > 0$ or $M_1(\theta_0) = 0$. In the former, we have

$$\begin{aligned}
\alpha_2 &= (1-h) + (1-\theta_0)[-(1-hk)\gamma_2 + hb_1\delta_3 - b_2(2a-1)]\\
&\quad + \theta_0[(hk-1)v_* + hb_1w_* - b_2w_*] - (1-\theta_0)\varepsilon[-(1-hk) + hb_1 + b_2]\\
&= (1-h) + (1-\theta_0)[-(1-hk)\gamma_2 + hb_1\delta_3 - b_2(2a-1)]\\
&\quad + \theta_0[h(1-\beta_*) - 1] - (1-\theta_0)\varepsilon[-(1-hk) + hb_1 + b_2]\\
&= (1-\theta_0)\{(hk\gamma_2 + hb_1\delta_3) - \varepsilon(-1 + hk + hb_1 + b_2)\} - \theta_0 h\beta_*,
\end{aligned}$$

using $\gamma_2 = 1 - h - b_2(2a-1)$, $v_* + b_2w_* = 1$ and $\beta_* = 1 - kv_* - b_1w_*$. Since $\beta_* < 0$, it easily follows from (4.13) that $\alpha_2 > 0$ in that case. In the other case when $M_1(\theta_0) = 0$, we have

$$\alpha_2 = (1-\theta)[h + (1-b_2)\varepsilon] > 0.$$

Similarly, when $M_1(\theta_0) > 0$, we have

$$\omega_3 = -(1-\theta_0)[(ak\gamma_2 + ab_1\delta_3) - \varepsilon(ak + ab_1 + 1 - a\varepsilon)] + a\theta_0\beta_* < 0,$$

using $\beta_* < 0$ and (4.13). On the other hand, $\omega_3 = -(1-\theta_0)(a + \varepsilon - a\varepsilon^2) < 0$ when $M_1(\theta_0) = 0$.

From these inequalities, we can get a contradiction following an argument given in [9]. For instance, if $\phi_2^- = m_2(\theta_0)$ in (4.17), then there exists a sequence $z_n \to \infty$ such that $(\phi_1, \phi_2, \phi_3)(\cdot + z_n)$ converges to a solution $(\phi_{1,\infty}, \phi_{2,\infty}, \phi_{3,\infty})$ of (1.10) such that

$$\begin{aligned}
&0 \le \phi_{1,\infty} \le M_1(\theta_0), \quad m_3(\theta_0) \le \phi_{3,\infty} \le M_3(\theta_0),\\
&\phi_{2,\infty} \ge \phi_{2,\infty}(0) = m_2(\theta_0) > 0.
\end{aligned}$$

Evaluating the equation for $\phi_{2,\infty}$ at 0, one finds a contradiction with the fact that $\alpha_2 > 0$. Other cases can be dealt with similarly.

Hence $\theta_0 = 1$. This implies that $\phi_2^- = \phi_2^+ = v_*$ and $\phi_3^- = \phi_3^+ = w_*$, i.e.,

$$\lim_{z\to\infty} \phi_2(z) = v_*, \quad \lim_{z\to\infty} \phi_3(z) = w_*. \tag{4.18}$$

Finally, applying again Lemma 4.11 with $\theta_0 = 1$ and recalling that $1 - kv_* - b_1 w_* < 0$, we also conclude that $\phi_1^- = \phi_1^+ = 0$. The proof is now completed.

4.3.2. Co-existence case

In this subsection, we show that the traveling wave solutions obtained in Subsection 3.1 converge to E_c as $z \to \infty$, if (1.8) and (1.9) hold. Recall that in particular $\beta_* > 0$; see the discussion in Subsection 1.1.

Since the method of contracting rectangles is not applicable in this case, we switch to a Lyapunov argument. However, in applying this method, we still need to derive some positive lower bounds for all components. The positive lower bound of ϕ_1 is derived by a similar argument as that in Lemma 4.9.

Lemma 4.12. *It holds that* $\phi_1^- = \liminf_{z\to\infty} \phi_1(z) > 0$.

Proof. By contradiction, we find for any small $\varepsilon > 0$ a solution $(\hat{\phi}_1^\varepsilon, \hat{\phi}_2^\varepsilon, \hat{\phi}_3^\varepsilon)$ of (1.10), such that $\hat{\phi}_1^\varepsilon(0) = \varepsilon$ and $\hat{\phi}_1^\varepsilon \le \varepsilon$ for all $z \ge 0$. Furthermore, by Lemmas 4.6 and 4.9, there exists $\delta > 0$ such that $\hat{\phi}_2^\varepsilon, \hat{\phi}_3^\varepsilon \ge \delta$ on $[0, +\infty)$.

As in the proof of Lemma 4.9 and more specifically of (4.12), one can then use Lemma 4.8 to infer that

$$|\hat{\phi}_2^\varepsilon(z) - v_*| + |\hat{\phi}_3^\varepsilon(z) - w_*| \le \delta(\varepsilon), \ \forall z \ge z_\varepsilon,$$

where $z_\varepsilon \ge 0$ and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Finally, due to $\beta_* > 0$ and applying Lemma 4.3, we conclude that $\hat{\phi}_1^\varepsilon \equiv 0$, a contradiction. The lemma is proved. □

Finally, by elliptic estimates, for any sequence $z_n \to \infty$ and any limit $(\phi_{1,\infty}, \phi_{2,\infty}, \phi_{3,\infty})$ of $(\phi_1, \phi_2, \phi_3)(\cdot + z_n)$, then $(u, v, w)(x,t) = (\phi_{1,\infty}, \phi_{2,\infty}, \phi_{3,\infty})(x + st)$ is an entire solution of (1.1). Moreover, by Lemmas 4.6, 4.9 and 4.12, it satisfies the assumptions of Lemma 4.7. It follows that the stable tail limit is the desired state E_c.

4.4. The case of the alien weak competitor

Next we turn to the case when the aboriginal prey is the strong competitor. Here we only manage to establish the stable tail when it is the co-existence state. Still, the following result states that the alien weak competitor always invades the environment when (1.7) and (2.5) hold, as we assume throughout this subsection.

Lemma 4.13. *It holds that* $\phi_1^- = \liminf_{z\to\infty} \phi_1(z) > 0$.

The proof is exactly the same as that of Lemma 4.12, and therefore we omit it. It relies on the fact that, in Theorem 2.2, we make the assumption that $\beta_* > 0$.

Let us now make the additional assumptions that (1.8) and (1.9) hold. Then we show that (ϕ_1, ϕ_2, ϕ_3) satisfies (1.13). Thanks to Lemma 4.7, it is enough to show that $\phi_i^- > 0$ for $i = 1, 2, 3$. We already dealt with $i = 1, 3$, so that it only remains to prove the following lemma.

Lemma 4.14. *It holds that* $\phi_2^- = \liminf_{z\to\infty} \phi_2(z) > 0$.

Again, this result has actually already been proved above. Indeed, it is the same as Lemma 4.9 thanks to the fact that $\beta^* > 0$, which is itself a consequence of (1.8) and (1.9). This concludes the proof of Theorem 2.2.

5. Non-existence of traveling waves

The first statement of Theorem 2.3 immediately follows from the next result.

Theorem 5.1. *Assume that* $\beta^* > 0$. *For* $s < s^*$, *there is no positive solution* (ϕ_1, ϕ_2, ϕ_3) *of* (1.10) *and* (1.11) *with*

$$\liminf_{z\to\infty} \phi_2(z) > 0.$$

Proof. First, suppose that for some $s \le 0$, there exists a positive solution (ϕ_1, ϕ_2, ϕ_3) of (1.10) satisfying the boundary condition (1.11). We choose $N > 1$ large enough so that if $y < -N$ then

$$1 - h\phi_1(y) - \phi_2(y) - b_2\phi_3(y) > \frac{\beta^*}{2}.$$

Now we integrate the second equation in (1.10) in y from $-\infty$ to $z \le -N$ and in z from $-\infty$ to $-N$. Then we obtain the following contradiction

$$0 < \frac{r_2\beta^*}{2} \int_{-\infty}^{-N} \int_{-\infty}^{z} \phi_2(y)dydz < -\int_{-\infty}^{-N} d_2\phi_2'(z)dz = -d_2\phi_2(-N) < 0.$$

Now suppose that there exists such a traveling wave solution for some $s \in (0, s^*)$. Then we pick ε small enough so that $0 < s < 2\sqrt{d_2r_2[\beta^* - (h + b_2)\varepsilon]}$. By the positivity and continuity of (ϕ_1, ϕ_2, ϕ_3), and the fact that

$$\lim_{z\to-\infty} (\phi_1, \phi_2, \phi_3)(z) = E^* \quad \text{and} \quad \liminf_{z\to\infty} \phi_2)(z) > 0,$$

there are nonnegative constants c_1 and c_2 such that

$$\phi_1(z) - c_1\phi_2(z) < u^* + \varepsilon, \ \ \forall z \in \mathbb{R}, \tag{5.1}$$

$$\phi_3(z) - c_2\phi_2(z) < w^* + \varepsilon, \ \ \forall z \in \mathbb{R}. \tag{5.2}$$

Using the notation $(u, v, w)(x, t) = (\phi_1, \phi_2, \phi_3)(x + st)$ and plugging (5.1) and (5.2) into the second equation of (1.1), we get

$$v_t \ge d_2 v_{xx} + r_2 v[\beta^* - (h + b_2)\varepsilon - (1 + hc_1 + b_2c_2)v], \;\; x \in \mathbb{R}, \; t > 0.$$

The spreading theory of [1] gives

$$\liminf_{t\to\infty} v(ct, t) \ge \frac{\beta^* - (h + b_2)\varepsilon}{1 + hc_1 + b_2c_2} > 0,$$

for any $|c| < 2\sqrt{d_2r_2[\beta^* - (h + b_2)\varepsilon]}$. This in particular holds true with

$$c := -\frac{s + 2\sqrt{d_2r_2[\beta^* - (h + b_2)\varepsilon]}}{2}.$$

On the other hand $ct + st = (s - 2\sqrt{d_2r_2[\beta - (h + b_2)\varepsilon]})t/2 \to -\infty$ as $t \to \infty$. This implies that $v(ct, t) = \phi_2(ct + st) \to 0$ as $t \to \infty$, a contradiction. □

When $\beta_* > 0$, one can check by the same argument that there is no positive traveling wave solution going to E_* at the unstable tail limit and such that the infimum limit at ∞ of the first component is positive. This concludes the proof of Theorem 2.3.

6. Verification of upper-lower-solutions

6.1. *Proof of Lemma 3.3*

It is easy to check that (3.7) and (3.8) hold, as well as the unstable tail limit. Therefore we focus only on the differential inequalities.

(1) $\mathcal{U}_1(z) \le 0$ for $z \ne 0$. Recall that for $z > 0$,

$$\overline{\phi}_1(z) = 1, \; \underline{\phi}_2(z) = 0, \; \underline{\phi}_3(z) = 0.$$

It immediately follows that

$$\mathcal{U}_1(z) = 0 \quad \text{for } z > 0.$$

On the other hand, for $z < 0$, we have

$$\overline{\phi}_1(z) = u^* + b_1 w^* e^{\lambda_1 z}, \; \underline{\phi}_2(z) \ge 0, \; \underline{\phi}_3(z) = w^*(1 - e^{\lambda_1 z}).$$

Then, using $u^* + b_1 w^* = 1$, we obtain

$$\begin{aligned}\mathcal{U}_1(z) &\le b_1 w^*(d_1\lambda_1^2 - s\lambda_1)e^{\lambda_1 z} + r_1(u^* + b_1 w^* e^{\lambda_1 z})(-b_1 w^* e^{\lambda_1 z} + b_1 w^* e^{\lambda_1 z}) \\ &= b_1 w^*(d_1\lambda_1^2 - s\lambda_1)e^{\lambda_1 z} \le 0,\end{aligned}$$

for $z < 0$, where the last inequality holds thanks to (3.10).

(2) $\mathcal{U}_2(z) \le 0$ for $z \ne 0$. For $z > 0$, we have

$$\underline{\phi}_1(z) \ge 0,\ \overline{\phi}_2(z) = 1,\ \underline{\phi}_3(z) = 0.$$

Therefore $\mathcal{U}_2(z) \le 0$ for $z > 0$. For $z < 0$,

$$\underline{\phi}_1(z) = u^*(1 - p_1 e^{\lambda_1 z}),\ \overline{\phi}_2(z) = e^{\lambda_1 z},\ \underline{\phi}_3(z) = w^*(1 - e^{\lambda_1 z}),$$

and then

$$\begin{aligned}\mathcal{U}_2(z) &= (d_2\lambda_1^2 - s\lambda_1)e^{\lambda_1 z} + r_2 e^{\lambda_1 z}[1 - hu^* + hu^* p_1 e^{\lambda_1 z} - e^{\lambda_1 z} - b_2 w^* + b_2 w^* e^{\lambda_1 z}] \\ &= G(\lambda_1)e^{\lambda_1 z} + r_2 e^{2\lambda_1 z}(hu^* p_1 + b_2 w^* - 1) \\ &\le r_2 e^{2\lambda_1 z}(hu^* + b_2 w^* - 1) \le 0.\end{aligned}$$

Here we used $G(\lambda_1) = 0$, $p_1 \le 1$ from (3.19) and $\beta^* > 0$.

(3) $\mathcal{U}_3(z) \le 0$ for $z \ne 0$. For $z > 0$, we have

$$\overline{\phi}_1(z) = 1,\ \overline{\phi}_2(z) = 1,\ \overline{\phi}_3(z) = 2a - 1,$$

hence

$$\mathcal{U}_3(z) = r_3(2a - 1)[-1 + a + a - (2a - 1)] = 0.$$

Furthermore, for $z < 0$,

$$\overline{\phi}_1(z) = u^* + b_1 w^* e^{\lambda_1 z},\ \overline{\phi}_2(z) = e^{\lambda_1 z},\ \overline{\phi}_3(z) = w^* + Ae^{\lambda_1 z}.$$

Using $-1 + au^* - w^* = 0$ and (3.18), we get

$$\begin{aligned}\mathcal{U}_3(z) &= A(d_3\lambda_1^2 - s\lambda_1)e^{\lambda_1 z} + r_3(w^* + Ae^{\lambda_1 z})e^{\lambda_1 z}(ab_1 w^* + a - A) \\ &= A(d_3\lambda_1^2 - s\lambda_1)e^{\lambda_1 z} + r_3(w^* + Ae^{\lambda_1 z})e^{\lambda_1 z}[(1 + ab_1)w^* - a + 1] \\ &= A(d_3\lambda_1^2 - s\lambda_1)e^{\lambda_1 z} \le 0,\end{aligned}$$

for $z < 0$, thanks to $(1 + ab_1)w^* = a - 1$ and (3.10).

(4) $\mathcal{L}_1(z) \ge 0$ for $z \notin \{0, z_1\}$. For $z > z_1 \ge 0$, we get

$$\underline{\phi}_1(z) = 0,\ \overline{\phi}_2(z) = 1,\ \overline{\phi}_3(z) = 2a - 1,$$

and thus $\mathcal{L}_1(z) = 0$.

For $0 < z < z_1$, we have

$$\underline{\phi}_1(z) = u^*(1 - p_1 e^{\lambda_1 z}),\ \overline{\phi}_2(z) = 1,\ \overline{\phi}_3(z) = 2a - 1,$$

and, by (3.10) and the choice of p_1 in (3.19), it follows that

$$\begin{aligned}\mathcal{L}_1(z) &= -u^* p_1(d_1\lambda_1^2 - s\lambda_1)e^{\lambda_1 z} + r_1 u^*(1-p_1 e^{\lambda_1 z})[1-u^*(1-p_1 e^{\lambda_1 z}) - k - b_1(2a-1)] \\ &\geq -u^* p_1(d_1\lambda_1^2 - s\lambda_1)e^{\lambda_1 z} - r_1 u^*[k + b_1(2a-1)] \\ &\geq u^* \left\{ -p_1(d_1\lambda_1^2 - s\lambda_1) - r_1[k + b_1(2a-1)] \right\} \geq 0.\end{aligned}$$

Lastly, for $z < 0$, recall that

$$\underline{\phi}_1(z) = u^*(1-p_1 e^{\lambda_1 z}),\ \overline{\phi}_2(z) = e^{\lambda_1 z},\ \overline{\phi}_3(z) = w^* + Ae^{\lambda_1 z}.$$

Then, using $u^* + b_1 w^* = 1$ and again the choice of p_1 in (3.19), we obtain

$$\begin{aligned}\mathcal{L}_1(z) &= -u^* p_1(d_1\lambda_1^2 - s\lambda_1)e^{\lambda_1 z} + r_1 u^*(1-p_1 e^{\lambda_1 z})e^{\lambda_1 z}[u^* p_1 - k - b_1 A] \\ &\geq -u^* p_1(d_1\lambda_1^2 - s\lambda_1)e^{\lambda_1 z} + r_1 u^*(1-p_1 e^{\lambda_1 z})e^{\lambda_1 z}[-k - b_1(2a-1)] \\ &\geq u^* e^{\lambda_1 z} \left\{ -p_1(d_1\lambda_1^2 - s\lambda_1) - r_1[k + b_1(2a-1)] \right\} \geq 0,\end{aligned}$$

for $z < 0$.

(5) $\mathcal{L}_2(z) \geq 0$ for $z \neq z_2$. For $z > z_2$, we have $\underline{\phi}_2(z) = 0$ and thus $\mathcal{L}_2(z) = 0$.

Then, for $z < z_2 < 0$,

$$\overline{\phi}_1(z) = u^* + b_1 w^* e^{\lambda_1 z},\ \underline{\phi}_2(z) = e^{\lambda_1 z} - qe^{\mu\lambda_1 z},\ \overline{\phi}_3(z) = w^* + Ae^{\lambda_1 z}.$$

Using $G(\lambda_1) = 0$ and $\beta^* > 0$, we get

$$\begin{aligned}\mathcal{L}_2(z) &\geq -qG(\mu\lambda_1)e^{\mu\lambda_1 z} + r_2(e^{\lambda_1 z} - qe^{\mu\lambda_1 z})(-hb_1 w^* e^{\lambda_1 z} - e^{\lambda_1 z} - b_2 A e^{\lambda_1 z}) \\ &\geq -qG(\mu\lambda_1)e^{\mu\lambda_1 z} - r_2 e^{2\lambda_1 z}(hb_1 w^* + 1 + b_2 A) \\ &\geq e^{\mu\lambda_1 z}[-qG(\mu\lambda_1) - r_2 e^{(2-\mu)\lambda_1 z}(hb_1 w^* + 1 + b_2 A)] \\ &\geq e^{\mu\lambda_1 z}[-qG(\mu\lambda_1) - r_2(hb_1 w^* + 1 + b_2 A)] \geq 0,\end{aligned}$$

for $z < z_2$, by the choice of μ in (3.20), which ensures that $2 - \mu > 0$ and $G(\mu\lambda_1) < 0$, and the choice of q in (3.21).

(6) $\mathcal{L}_3(z) \geq 0$ for $z \neq 0$. For $z > 0$, $\underline{\phi}_3(z) = 0$ gives $\mathcal{L}_3(z) = 0$.

For $z < 0$, we have

$$\underline{\phi}_1(z) = u^*(1-p_1 e^{\lambda_1 z}),\ \underline{\phi}_2(z) \geq 0,\ \underline{\phi}_3(z) = w^*(1-e^{\lambda_1 z}) \leq w^*.$$

Then, using $p_1 \leq 1$, $-1 + au^* - w^* = 0$ and (2.2), we get

$$\begin{aligned}\mathcal{L}_3(z) &\geq -w^*(d_3\lambda_1^2 - s\lambda_1)e^{\lambda_1 z} + r_3\underline{\phi}_3(z)e^{\lambda_1 z}(-au^* p_1 + w^*) \\ &\geq -w^*(d_2\lambda_1^2 - s\lambda_1)e^{\lambda_1 z} - r_3\underline{\phi}_3(z)e^{\lambda_1 z} \\ &\geq w^* e^{\lambda_1 z}(r_2\beta^* - r_3) \geq 0,\end{aligned}$$

for $z < 0$. This completes the proof of Lemma 3.3.

6.2. *Proof of Lemma 3.4*

Here $s = s^*$. As before, we only deal with the differential inequalities.

(1) $\mathcal{U}_1(z) \le 0$ for $z \ne -2/\lambda_1$. For $z > -2/\lambda_1$, then $\overline{\phi}_1(z) = 1$, $\underline{\phi}_2(z) \ge 0$ and $\underline{\phi}_3(z) = 0$, so that $\mathcal{U}_1(z) \le 0$.

For $z < -2/\lambda_1$, we have that

$$\overline{\phi}_1(z) = u^* + L^* b_1 w^*(-z)e^{\lambda_1 z},\ \underline{\phi}_2(z) \ge 0,\ \underline{\phi}_3(z) = w^*[1 - L^*(-z)e^{\lambda_1 z}].$$

Then

$$\begin{aligned}\mathcal{U}_1(z) &\le L^* b_1 w^*(-2d_1\lambda_1 + s)e^{\lambda_1 z} + L^* b_1 w^*(d_1\lambda_1^2 - s\lambda_1)(-z)e^{\lambda_1 z}\\ &= -r_2 L^* b_1 w^* \beta^*(-z)e^{\lambda_1 z} \le 0,\end{aligned}$$

for $z < -2/\lambda_1$, by using the first part of (2.3), (3.9) with $s = s^*$ and $\beta^* > 0$.

(2) $\mathcal{U}_2(z) \le 0$ for $z \ne -2/\lambda_1$. For $z > -2/\lambda_1$, we have that $\underline{\phi}_1(z) \ge 0$, $\overline{\phi}_2(z) = 1$ and $\underline{\phi}_3(z) = 0$, hence $\mathcal{U}_2(z) \le 0$.

For $z < -2/\lambda_1$, due to $z_1 > -2/\lambda_1$, we have

$$\underline{\phi}_1(z) = u^*[1 - p_1 L^*(-z)e^{\lambda_1 z}],\ \overline{\phi}_2(z) = L^*(-z)e^{\lambda_1 z},\ \underline{\phi}_3(z) = w^*[1 - L^*(-z)e^{\lambda_1 z}].$$

Then

$$\begin{aligned}\mathcal{U}_2(z) &= L^*(-2d_2\lambda_1 + s)e^{\lambda_1 z} + L^*(d_2\lambda_1^2 - s\lambda_1)(-z)e^{\lambda_1 z}\\ &\quad + r_2 L^*(-z)e^{\lambda_1 z}[\beta^* + hu^* p_1 L^*(-z)e^{\lambda_1 z} - L^*(-z)e^{\lambda_1 z} + b_2 w^* L^*(-z)e^{\lambda_1 z}]\\ &= r_2[L^*(-z)e^{\lambda_1 z}]^2(-1 + hu^* p_1 + b_2 w^*)\\ &\le -r_2[L^*(-z)e^{\lambda_1 z}]^2(1 - hu^* - b_2 w^*) = -r_2[L^*(-z)e^{\lambda_1 z}]^2\beta^* \le 0,\end{aligned}$$

for $z < -2/\lambda_1$, by using $p_1 \le 1$ and $\beta^* > 0$.

(3) $\mathcal{U}_3(z) \le 0$ for $z \ne -2/\lambda_1$. For $z > -2/\lambda_1$, then $\overline{\phi}_1(z) = 1$, $\overline{\phi}_2(z) = 1$ and $\overline{\phi}_3(z) = 2a - 1$ and hence $\mathcal{U}_3(z) = 0$.

For $z < -2/\lambda_1$,

$$\overline{\phi}_1(z) = u^* + L^* b_1 w^*(-z)e^{\lambda_1 z},\ \overline{\phi}_2(z) = L^*(-z)e^{\lambda_1 z},\ \overline{\phi}_3(z) = w^* + L^* A(-z)e^{\lambda_1 z}.$$

Then

$$\begin{aligned}\mathcal{U}_3(z) &= L^* A(-2d_3\lambda_1 + s)e^{\lambda_1 z} + L^* A(d_3\lambda_1^2 - s\lambda_1)(-z)e^{\lambda_1 z}\\ &\quad + r_3\overline{\phi}_3(z)[aL^* b_1 w^*(-z)e^{\lambda_1 z} + aL^*(-z)e^{\lambda_1 z} - L^* A(-z)e^{\lambda_1 z}]\\ &= L^* A(-2d_3\lambda_1 + s)e^{\lambda_1 z} + L^* A(d_3\lambda_1^2 - s\lambda_1)(-z)e^{\lambda_1 z}\\ &\quad + r_3\overline{\phi}_3(z)[(1 + ab_1)w^* - (a-1)]L^*(-z)e^{\lambda_1 z}\\ &= L^* A(-2d_3\lambda_1 + s)e^{\lambda_1 z} + L^* A(d_3\lambda_1^2 - s\lambda_1)(-z)e^{\lambda_1 z} \le 0,\end{aligned}$$

for $z < -2/\lambda_1$, using $A = 2a - 1 - w^* > 0$ and since $-2d_3\lambda_1 + s \le 0$ and $d_3\lambda_1^2 - s\lambda_1 \le 0$, by the first part of (2.3).

(4) $\mathcal{L}_1(z) \ge 0$ for $z \notin \{-2/\lambda_1, z_1\}$. For $z > z_1$, $\mathcal{L}_1(z) = 0$ by $\underline{\phi}_1 = 0$. Next, for $-2/\lambda_1 < z < z_1$, we have

$$\underline{\phi}_1(z) = u^*[1 - p_1 L^*(-z)e^{\lambda_1 z}],\ \overline{\phi}_2(z) = 1,\ \overline{\phi}_3(z) = 2a - 1.$$

Then

$$\begin{aligned}\mathcal{L}_1(z) &= -u^* p_1 L^*(-2d_1\lambda_1 + s)e^{\lambda_1 z} - u^* p_1 L^*(d_1\lambda_1^2 - s\lambda)(-z)e^{\lambda_1 z}\\ &\quad + r_1\underline{\phi}_1(z)[1 - \underline{\phi}_1(z) - k - b_1(2a-1)]\\ &\ge -u^* p_1(d_1\lambda_1^2 - s\lambda_1) - r_1 u^*[k + b_1(2a-1)]\\ &= u^*\{-p_1(d_1\lambda_1^2 - s\lambda_1) - r_1[k + b_1(2a-1)]\} \ge 0,\end{aligned}$$

by the first part of (2.3), $L^*(-z)e^{\lambda_1 z} \ge 1$ for $z \in (-2/\lambda_1, z_1)$, $d_1\lambda_1^2 - s\lambda < 0$ and the choice of p_1 in (3.28).

Lastly, for $z < -2/\lambda_1$,

$$\underline{\phi}_1(z) = u^*[1 - p_1 L^*(-z)e^{\lambda_1 z}],\ \overline{\phi}_2(z) = L^*(-z)e^{\lambda_1 z},\ \overline{\phi}_3(z) = w^* + L^* A(-z)e^{\lambda_1 z},$$

and thus

$$\begin{aligned}\mathcal{L}_1(z) &= -u^* p_1 L^*(-2d_1\lambda_1 + s)e^{\lambda_1 z} - u^* p_1 L^*(d_1\lambda_1^2 - s\lambda_1)(-z)e^{\lambda_1 z}\\ &\quad + r_1 u^*[1 - p_1 L^*(-z)e^{\lambda_1 z}][u^* p_1 L^*(-z)e^{\lambda_1 z} - kL^*(-z)e^{\lambda_1 z} - b_1 L^* A(-z)e^{\lambda_1 z}]\\ &= -u^* p_1 L^*(d_1\lambda_1^2 - s\lambda_1)(-z)e^{\lambda_1 z}\\ &\quad + r_1 u^*[1 - p_1 L^*(-z)e^{\lambda_1 z}][u^* p_1 + b_1 w^* - k - b_1(2a-1)]L^*(-z)e^{\lambda_1 z}\\ &\ge -u^* p_1 L^*(d_1\lambda_1^2 - s\lambda_1)(-z)e^{\lambda_1 z} + r_1 u^*[-k - b_1(2a-1)]L^*(-z)e^{\lambda_1 z}\\ &= u^*\{-p_1(d_1\lambda_1^2 - s\lambda_1) - r_1[k + b_1(2a-1)]\}L^*(-z)e^{\lambda_1 z} \ge 0,\end{aligned}$$

where we again used the first part of (2.3) and (3.28), and in particular the fact that $u^* p_1 + b_1 w^* - k \le 1 - k < 0$.

(5) $\mathcal{L}_2(z) \ge 0$ for $z \ne z_2$. For $z > z_2$, then $\underline{\phi}_2(z) = 0$ and $\mathcal{L}_2(z) = 0$.

For $z < z_2$, and since $z_2 < -2/\lambda_1$, we have $\overline{\phi}_1(z) = u^* + L^* b_1 w^*(-z)e^{\lambda_1 z}$, as well as

$$\underline{\phi}_2(z) = [L^*(-z) - q(-z)^{1/2}]e^{\lambda_1 z},\ \overline{\phi}_3(z) = w^* + L^* A(-z)e^{\lambda_1 z}.$$

It follows that, for $z < z_2$,

$$\begin{aligned}\mathcal{L}_2(z) &= q\frac{d_2}{4}(-z)^{-3/2}e^{\lambda_1 z} + \underline{\phi}_2(z)(d_2\lambda_1^2 - s\lambda_1)\\ &\quad + r_2\underline{\phi}_2(z)[\beta^* - hL^* b_1 w^*(-z)e^{\lambda_1 z} - L^*(-z)e^{\lambda_1 z} + q(-z)^{1/2}e^{\lambda_1 z} - b_2 L^* A(-z)e^{\lambda_1 z}]\\ &\ge q\frac{d_2}{4}(-z)^{-3/2}e^{\lambda_1 z} + r_2[L^*(-z)e^{\lambda_1 z}]^2(-hb_1 w^* - 1 - b_2 A)\end{aligned}$$

$$= \frac{d_2}{4}(-z)^{-3/2}e^{\lambda_1 z}\left[q - \frac{4}{d_2}r_2(L^*)^2(-z)^{7/2}e^{\lambda_1 z}(hb_1w^* + 1 + b_2A)\right]$$
$$\geq \frac{d_2}{4}(-z)^{-3/2}e^{\lambda_1 z}\left[q - \frac{4}{d_2}r_2(L^*)^2 M(hb_1w^* + 1 + b_2A)\right] \geq 0,$$

by the choice of q in (3.29), where we have used

$$(-z)^{7/2}e^{\lambda_1 z} \leq M := \left(\frac{7}{2\lambda_1 e}\right)^{7/2}, \ \forall z < 0.$$

(6) $\mathcal{L}_3(z) \geq 0$ for $z \neq -2/\lambda_1$. For $z > -2/\lambda_1$, due to $\underline{\phi}_3(z) = 0$, we immediately get that $\mathcal{L}_3(z) = 0$.

For $z < -2/\lambda_1$, we also have $z < z_1$ and

$$\underline{\phi}_1(z) = u^*[1 - p_1L^*(-z)e^{\lambda_1 z}],\ \underline{\phi}_2(z) \geq 0,\ \underline{\phi}_3(z) = w^*[1 - L^*(-z)e^{\lambda_1 z}].$$

Then

$$\begin{aligned}\mathcal{L}_3(z) &\geq -w^*L^*(-2d_3\lambda_1 + s)e^{\lambda_1 z} - w^*L^*(d_3\lambda_1^2 - s\lambda_1)(-z)e^{\lambda_1 z} \\ &\quad + r_3\underline{\phi}_3(z)[-au^*p_1L^*(-z)e^{\lambda_1 z} + w^*L^*(-z)e^{\lambda_1 z}] \\ &\geq [-(d_3\lambda_1^2 - s\lambda_1) - r_3au^*p_1 + r_3w^*]w^*L^*(-z)e^{\lambda_1 z} \\ &\geq [-(d_3\lambda_1^2 - s\lambda_1) - r_3]w^*L^*(-z)e^{\lambda_1 z},\end{aligned}$$

using $s = 2d_2\lambda_1 \leq 2d_3\lambda_1$, $p_1 \leq 1$ and $au^* - w^* = 1$. Now notice that

$$d_3\lambda_1^2 - s\lambda_1 = \left(\frac{d_3}{d_2} - 2\right)r_2\beta^* \leq -r_3,$$

due to the second part of (2.3). It follows that $\mathcal{L}_3(z) \geq 0$ for $z < -2/\lambda_1$. This completes the proof of this lemma.

6.3. *Proof of Lemma 3.5*

We now turn to the case when the invaded state is E_*, first when $s > s_*$.

(1) $\mathcal{U}_1(z) \leq 0$ for $z \neq 0$. For $z > 0$, then $\overline{\phi}_1(z) = 1$, $\underline{\phi}_2(z) \geq 0$, $\underline{\phi}_3(z) = 0$ and it follows that $\mathcal{U}_1(z) \leq 0$.

For $z < 0$,

$$\overline{\phi}_1(z) = e^{\sigma_1 z},\ \underline{\phi}_2(z) = v_*(1 - p_2e^{\sigma_1 z}),\ \underline{\phi}_3(z) = w_*(1 - e^{\sigma_1 z}).$$

In that case,

$$\begin{aligned}\mathcal{U}_1(z) &= (d_1\sigma_1^2 - s\sigma_1)e^{\sigma_1 z} + r_1e^{\sigma_1 z}[1 - e^{\sigma_1 z} - kv_*(1 - p_2e^{\sigma_1 z}) - b_1w_*(1 - e^{\sigma_1 z})] \\ &= H(\sigma_1)e^{\sigma_1 z} + r_1e^{2\sigma_1 z}(-1 + kv_*p_2 + b_1w_*) \\ &\leq r_1e^{2\sigma_1 z}(-1 + kv_* + b_1w_*) = -r_1e^{2\sigma_1 z}\beta_* \leq 0,\end{aligned}$$

by $p_2 \le 1$ and $\beta_* > 0$.

(2) $\mathcal{U}_2(z) \le 0$ for $z \ne 0$. For $z > 0$, $\underline{\phi}_1(z) = 0$, $\overline{\phi}_2(z) = 1$, $\underline{\phi}_3(z) = 0$ and so $\mathcal{U}_2(z) = 0$.

For $z < 0$,

$$\underline{\phi}_1(z) \ge 0,\ \overline{\phi}_2(z) = v_* + b_2 w_* e^{\sigma_1 z},\ \underline{\phi}_3(z) = w_*(1 - e^{\sigma_1 z}).$$

Using $1 - v_* - b_2 w_* = 0$ and (3.32), we get

$$\mathcal{U}_2(z) \le b_2 w_*(d_2\sigma_1^2 - s\sigma_1)e^{\sigma_1 z} \le 0 \text{ for } z < 0.$$

(3) $\mathcal{U}_3(z) \le 0$ for $z \ne 0$. For $z > 0$, $\overline{\phi}_1(z) = 1$, $\overline{\phi}_2(z) = 1$, $\overline{\phi}_3(z) = 2a - 1$ and hence $\mathcal{U}_3(z) = 0$.

For $z < 0$, we have

$$\overline{\phi}_1(z) = e^{\sigma_1 z},\ \overline{\phi}_2(z) = v_* + b_2 w_* e^{\sigma_1 z},\ \overline{\phi}_3(z) = w_* + Be^{\sigma_1 z}.$$

Using $-1 + av_* - w_* = 0$ and again (3.32), we obtain

$$\begin{aligned}\mathcal{U}_3(z) &= B(d_3\sigma_1^2 - s\sigma_1)e^{\sigma_1 z} + r_3\overline{\phi}_3(z)[ae^{\sigma_1 z} + ab_2 w_* e^{\sigma_1 z} - Be^{\sigma_1 z}]\\ &\le r_3\overline{\phi}_3(z)[ae^{\sigma_1 z} + ab_2 w_* e^{\sigma_1 z} - Be^{\sigma_1 z}].\end{aligned}$$

Now note that

$$ae^{\sigma_1 z} + ab_2 w_* e^{\sigma_1 z} - Be^{\sigma_1 z} = e^{\sigma_1 z}[a + (1 + ab_2)w_* - (2a - 1)] = 0,$$

since $B = (2a - 1) - w_*$ and $(1 + ab_2)w_* = a - 1$. Hence we deduce that $\mathcal{U}_3(z) \le 0$ for $z < 0$.

(4) $\mathcal{L}_1(z) \ge 0$ for $z \ne z_0$. For $z > z_0$, we have $\underline{\phi}_1(z) = 0$, $\overline{\phi}_2(z) \le 1$, $\overline{\phi}_3(z) \le 2a - 1$, therefore $\mathcal{L}_1(z) = 0$.

For $z < z_0 < 0$,

$$\underline{\phi}_1(z) = e^{\sigma_1 z} - qe^{\mu\sigma_1 z},\ \overline{\phi}_2(z) = v_* + b_2 w_* e^{\sigma_1 z},\ \overline{\phi}_3(z) = w_* + Be^{\sigma_1 z}.$$

Then

$$\begin{aligned}\mathcal{L}_1(z) &= -qH(\mu\sigma_1)e^{\mu\sigma_1 z}\\ &\quad + r_1(e^{\sigma_1 z} - qe^{\mu\sigma_1 z})[-e^{\sigma_1 z} + qe^{\mu\sigma_1 z} - kb_2 w_* e^{\sigma_1 z} - b_1 Be^{\sigma_1 z}]\\ &\ge -qH(\mu\sigma_1)e^{\mu\sigma_1 z} + r_1 e^{\sigma_1 z}(-e^{\sigma_1 z} - kb_2 w_* e^{\sigma_1 z} - b_1 Be^{\sigma_1 z})\\ &= e^{\mu\sigma_1 z}[-qH(\mu\sigma_1) - r_1 e^{(2-\mu)\sigma_1 z}(1 + kb_2 w_* + b_1 B)]\\ &\ge e^{\mu\sigma_1 z}[-qH(\mu\sigma_1) - r_1(1 + kb_2 w_* + b_1 B)] \ge 0,\end{aligned}$$

by the choice of μ in (3.36) and q in (3.37).

(5) $\mathcal{L}_2(z) \ge 0$ for $z \notin \{0, z_2\}$. First, for $z > z_2 \ge 0$, we have $\mathcal{L}_2(z) = 0$ since $\underline{\phi}_2(z) = 0$.

Next, for any $0 < z < z_2$, we have $\overline{\phi}_1(z) = 1$, $\underline{\phi}_2(z) = v_*(1 - p_2 e^{\sigma_1 z}) < 1$, $\overline{\phi}_3(z) = 2a - 1$, and then

$$\begin{aligned}\mathcal{L}_2(z) &= -v_* p_2(d_2\sigma_1^2 - s\sigma_1)e^{\sigma_1 z} + r_2 v_*(1-p_2 e^{\sigma_1 z})[1-h-v_*(1-p_2e^{\sigma_1 z}) - b_2(2a-1)]\\ &\geq -v_* p_2(d_2\sigma_1^2 - s\sigma_1)e^{\sigma_1 z} + r_2 v_*(1-p_2 e^{\sigma_1 z})[-h-b_2(2a-1)]\\ &\geq -v_* p_2(d_2\sigma_1^2 - s\sigma_1) - r_2 v_*[h+b_2(2a-1)]\\ &= v_*\left\{-p_2(d_2\sigma_1^2 - s\sigma_1) - r_2[h+b_2(2a-1)]\right\} \geq 0,\end{aligned}$$

by (3.32) and the choice of p_2 in (3.35).

Lastly, for $z<0$, then $\overline{\phi}_1(z) = e^{\sigma_1 z}$, $\underline{\phi}_2(z) = v_*(1-p_2e^{\sigma_1 z})$, $\overline{\phi}_3(z) = w_* + Be^{\sigma_1 z}$. It follows that

$$\begin{aligned}\mathcal{L}_2(z) &= -v_* p_2(d_2\sigma_1^2 - s\sigma_1)e^{\sigma_1 z} + r_2 v_*(1-p_2 e^{\sigma_1 z})[-he^{\sigma_1 z} + v_* p_2 e^{\sigma_1 z} - b_2 B e^{\sigma_1 z}]\\ &\geq -v_* p_2(d_2\sigma_1^2 - s\sigma_1)e^{\sigma_1 z} + r_2 v_*(1-p_2 e^{\sigma_1 z})[-he^{\sigma_1 z} - b_2(2a-1)e^{\sigma_1 z}]\\ &\geq -v_* p_2(d_2\sigma_1^2 - s\sigma_1)e^{\sigma_1 z} - r_2 v_*[h+b_2(2a-1)]e^{\sigma_1 z}\\ &= v_* e^{\sigma_1 z}\left\{-p_2(d_2\sigma_1^2 - s\sigma_1) - r_2[h+b_2(2a-1)]\right\} \geq 0,\end{aligned}$$

where we used $1 - v_* - b_2 w_* = 0$, and again (3.32) and the choice of p_2 in (3.35).

(6) $\mathcal{L}_3(z) \geq 0$ for $z \neq 0$. For $z>0$, since $\underline{\phi}_1(z) = 0$, we immediately get that $\mathcal{L}_3(z) = 0$.

For $z<0$, then

$$\underline{\phi}_1(z) \geq 0, \underline{\phi}_2(z) = v_*(1-p_2e^{\sigma_1 z}),\ \underline{\phi}_3(z) = w_*(1-e^{\sigma_1 z}).$$

Using $-1 + av_* - w_* = 0$, $p_2 \leq 1$, $d_3 \leq d_1$ and (2.5), we obtain

$$\begin{aligned}\mathcal{L}_3(z) &\geq -w_*(d_3\sigma_1^2 - s\sigma_1)e^{\sigma_1 z} + r_3\underline{\phi}_3(-av_* p_2 + w_*)e^{\sigma_1 z}\\ &\geq -w_*(d_1\sigma_1^2 - s\sigma_1)e^{\sigma_1 z} - r_3\underline{\phi}_3 e^{\sigma_1 z}\\ &\geq w_*(r_1\beta_* - r_3)e^{\sigma_1 z} \geq 0,\end{aligned}$$

for $z<0$. This completes the proof of Lemma 3.5.

6.4. *Proof of Lemma 3.6*

Finally we consider the case when the invaded state is E_* and the speed $s = s_*$.

(1) $\mathcal{U}_1(z) \leq 0$ for $z \neq -2/\sigma_1$. For $z > -2/\sigma_1$,

$$\overline{\phi}_1(z) = 1,\ \underline{\phi}_2(z) \geq 0,\ \underline{\phi}_3(z) = 0,$$

hence $\mathcal{U}_1(z) \leq 0$.

For $z < -2/\sigma_1$,

$$\overline{\phi}_1(z) = L_*(-z)e^{\sigma_1 z},\ \underline{\phi}_2(z) = v_*[1 - p_2 L_*(-z)e^{\sigma_1 z}],\ \underline{\phi}_3(z) = w_*[1 - L_*(-z)e^{\sigma_1 z}].$$

Then

$$\begin{aligned}\mathcal{U}_1(z) &= L_*(-2d_1\sigma_1+s)e^{\sigma_1 z}+L_*(d_1\sigma_1^2-s\sigma_1)(-z)e^{\sigma_1 z}\\ &\quad +r_1L_*(-z)e^{\sigma_1 z}[\beta_*-L_*(-z)e^{\sigma_1 z}+kv_*p_2L_*(-z)e^{\sigma_1 z}+b_1L_*w_*(-z)e^{\sigma_1 z}]\\ &\le r_1L_*^2(-z)^2e^{2\sigma_1 z}(-1+kv_*+b_1w_*)\\ &= -r_1L_*^2(-z)^2e^{2\sigma_1 z}\beta_*<0,\end{aligned}$$

using $p_2\le 1$ and $\beta_*=1-kv_*-b_1w_*>0$.

(2) $\mathcal{U}_2(z)\le 0$ for $z\neq -2/\sigma_1$. For $z>-2/\sigma_1$,

$$\underline{\phi}_1(z)\ge 0,\ \overline{\phi}_2(z)=1,\ \underline{\phi}_3(z)=0,$$

and so $\mathcal{U}_2(z)\le 0$.

For $z<-2/\sigma_1$, we have

$$\underline{\phi}_1(z)\ge 0,\ \overline{\phi}_2(z)=v_*+L_*b_2w_*(-z)e^{\sigma_1 z},\ \underline{\phi}_3(z)=w_*[1-L_*(-z)e^{\sigma_1 z}].$$

Then, we get

$$\begin{aligned}\mathcal{U}_2(z)&\le L_*(-2d_2\sigma_1+s)e^{\sigma_1 z}+L_*(d_2\sigma_1^2-s\sigma_1)(-z)e^{\sigma_1 z}\\ &=-r_1L_*\beta_*(-z)e^{\sigma_1 z}\le 0,\end{aligned}$$

using $v_*+b_2w_*=1$, $d_1=d_2$ and again $\beta_*>0$.

(3) $\mathcal{U}_3(z)\le 0$ for $z\neq -2/\sigma_1$. For $z>-2/\sigma_1$, we have $\overline{\phi}_1(z)=1$, $\overline{\phi}_2(z)=1$, $\overline{\phi}_3(z)=2a-1$, and so $\mathcal{U}_3(z)=0$.

For $z<-2/\sigma_1$,

$$\overline{\phi}_1(z)=L_*(-z)e^{\sigma_1 z},\ \overline{\phi}_2(z)=v_*+L_*b_2w_*(-z)e^{\sigma_1 z},\ \overline{\phi}_3(z)=w_*+L_*B(-z)e^{\sigma_1 z}.$$

Then

$$\begin{aligned}\mathcal{U}_3(z) &= L_*B(-2d_3\sigma_1+s)e^{\sigma_1 z}+L_*B(d_3\sigma_1^2-s\sigma_1)(-z)e^{\sigma_1 z}\\ &\quad +r_3\overline{\phi}_3(z)[1-a+(1+ab_2)w_*]L_*(-z)e^{\sigma_1 z}\\ &= L_*B(-2d_3\sigma_1+s)e^{\sigma_1 z}+L_*B(d_3\sigma_1^2-s\sigma_1)(-z)e^{\sigma_1 z}\le 0,\end{aligned}$$

for $z<-2/\sigma_1$, using $B=(2a-1)-w_*$ and $(1+ab_2)w_*=a-1$, as well as $-2d_3\sigma_1+s\le 0$ and $d_3\sigma_1^2-s\sigma_1\le 0$ which are due to (2.7).

(4) $\mathcal{L}_1(z)\ge 0$ for $z\neq z_0$. For $z>z_0$, $\underline{\phi}_1(z)=0$ and so $\mathcal{L}_1(z)=0$.

For $z<z_0\le -2/\sigma_1$, there holds $\underline{\phi}_1(z)=[L_*(-z)-q(-z)^{1/2}]e^{\sigma_1 z}$ and

$$\overline{\phi}_2(z)=v_*+L_*b_2w_*(-z)e^{\sigma_1 z},\ \overline{\phi}_3(z)=w_*+L_*B(-z)e^{\sigma_1 z}.$$

It follows that

$$
\begin{aligned}
\mathcal{L}_1(z) &= \frac{d_1}{4} q(-z)^{-3/2} e^{\sigma_1 z} + \underline{\phi}_1(z)(d_1\sigma_1^2 - s\sigma_1) \\
&\quad + r_1 \underline{\phi}_1(z)[\beta_* - \underline{\phi}_1(z) - kL_* b_2 w_*(-z)e^{\sigma_1 z} - b_1 L_* B(-z)e^{\sigma_1 z}] \\
&\geq \frac{d_1}{4} q(-z)^{-3/2} e^{\sigma_1 z} \\
&\quad + r_1[L_*(-z) - q(-z)^{1/2}]e^{\sigma_1 z}[-L_*(-z)e^{\sigma_1 z} - kL_* b_2 w_*(-z)e^{\sigma_1 z} - b_1 L_* B(-z)e^{\sigma_1 z}] \\
&\geq \frac{d_1}{4} q(-z)^{-3/2} e^{\sigma_1 z} + r_1 L_*^2(-z)^2 e^{2\sigma_1 z}(-1 - kb_2 w_* - b_1 B) \\
&= (-z)^{-3/2} e^{\sigma_1 z}\left[q\frac{d_1}{4} - r_1 L_*^2(-z)^{7/2} e^{\sigma_1 z}(1 + kb_2 w_* + b_1 B)\right] \\
&\geq (-z)^{-3/2} e^{\sigma_1 z}\left[q\frac{d_1}{4} - r_1 M L_*^2(1 + kb_2 w_* + b_1 B)\right] \geq 0,
\end{aligned}
$$

for $z < z_0$, by the choice of q in (3.50) and $(-z)^{7/2} e^{\sigma_1 z} \leq M$ for all $z < 0$.

(5) $\mathcal{L}_2(z) \geq 0$ for $z \notin \{-2/\sigma_1, z_2\}$. For $z > z_2$, we have $\underline{\phi}_2(z) = 0$ and so $\mathcal{L}_2(z) = 0$. Then, for $-2/\sigma_1 < z < z_2$,

$$
\overline{\phi}_1(z) = 1,\ \underline{\phi}_2(z) = v_*[1 - p_2 L_*(-z)e^{\sigma_1 z}],\ \overline{\phi}_3(z) = 2a - 1,
$$

and, using $s = 2d_2\sigma_1$,

$$
\begin{aligned}
\mathcal{L}_2(z) &= -v_* p_2 L_*(d_2\sigma_1^2 - s\sigma_1)(-z)e^{\sigma_1 z} + r_2 \underline{\phi}_2(z)[1 - h - \underline{\phi}_2(z) - b_2(2a-1)] \\
&\geq -v_* p_2 L_*(d_2\sigma_1^2 - s\sigma_1)(-z)e^{\sigma_1 z} - r_2 v_*[h + b_2(2a-1)].
\end{aligned}
$$

Since $L_*(-z)e^{\sigma_1 z} > 1$ for all $-2/\sigma_1 < z < z_2$, we obtain that

$$
\mathcal{L}_2(z) \geq v_*\{-p_2(d_2\sigma_1^2 - s\sigma_1) - r_2[h + b_2(2a-1)]\} \geq 0,
$$

for $z \in (-2/\sigma_1, z_2)$, by our choice of p_2.

Next, for $z < -2/\sigma_1$,

$$
\overline{\phi}_1(z) = L_*(-z)e^{\sigma_1 z},\ \underline{\phi}_2(z) = v_*[1 - p_2 L_*(-z)e^{\sigma_1 z})],\ \overline{\phi}_3(z) = w_* + L_* B(-z)e^{\sigma_1 z}.
$$

Then we compute

$$
\begin{aligned}
\mathcal{L}_2(z) &= -v_* p_2 L_*(d_2\sigma_1^2 - s\sigma_1)(-z)e^{\sigma_1 z} \\
&\quad + r_2 v_*[1 - p_2 L_*(-z)e^{\sigma_1 z}][-hL_*(-z)e^{\sigma_1 z} + v_* p_2 L_*(-z)e^{\sigma_1 z} - b_2 L_* B(-z)e^{\sigma_1 z}] \\
&\geq -v_* p_2 L_*(d_2\sigma_1^2 - s\sigma_1)(-z)e^{\sigma_1 z} - r_2 v_*[h + b_2(2a-1)]L_*(-z)e^{\sigma_1 z} \\
&= v_*\{-p_2(d_2\sigma_1^2 - s\sigma_1) - r_2[h + b_2(2a-1)]\}L_*(-z)e^{\sigma_1 z} \geq 0,
\end{aligned}
$$

using $v_* + b_2 w_* = 1$, $s = 2d_2\sigma_1$, and again our choice of p_2.

(6) $\mathcal{L}_3(z) \geq 0$ for $z \neq -2/\sigma_1$. For $z > -2/\sigma_1$, we have $\underline{\phi}_3(z) = 0$ and so $\mathcal{L}_3(z) = 0$.

For $z < -2/\sigma_1$,

$$\underline{\phi}_1(z) \geq 0,\ \underline{\phi}_2(z) = v_*[1 - p_2 L_*(-z)e^{\sigma_1 z}],\ \underline{\phi}_3(z) = w_*[1 - L_*(-z)e^{\sigma_1 z}].$$

Then, using $av_* - w_* = 1$, $p_2 \leq 1$, and $s = 2d_1\sigma_1 \leq d_3\sigma_1$ by (2.7), we get

$$\begin{aligned}\mathcal{L}_3(z) \geq & -w_* L_*(-2d_3\sigma_1 + s)e^{\sigma_1 z} - w_* L_*(d_3\sigma_1^2 - s\sigma_1)(-z)e^{\sigma_1 z} \\ & + r_3\underline{\phi}_3(z)(-av_* + w_*)L_*(-z)e^{\sigma_1 z} \\ \geq & [-(d_3\sigma_1^2 - s\sigma_1) - r_3]w_* L_*(-z)e^{\sigma_1 z}.\end{aligned}$$

Due to the second part of (2.7), one may infer that $\mathcal{L}_3(z) \geq 0$ for $z < -2/\sigma_1$. The proof of this lemma is thus completed.

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