

# Optimal partial boundary condition for degenerate parabolic equations <sup>☆</sup>

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## Abstract

For the stability of the non-Newtonian fluid equation

$$\frac{\partial u}{\partial t} - \operatorname{div} \left( a(x) |\nabla u|^{p-2} \nabla u \right) - \sum_{i=1}^N b_i(x) D_i u + c(x, t) u = f(x, t),$$

where  $a(x)|_{x \in \Omega} > 0$ ,  $a(x)|_{x \in \partial\Omega} = 0$  and  $b_i(x) \in C^1(\overline{\Omega})$ , we know that the degeneracy of  $a(x)$  may make the usual Dirichlet boundary value condition overdetermined and only a partial boundary value condition is expected. How to depict the geometric characteristic of the partial boundary value condition has been a long-time standing open problem. In this study, an optimal partial boundary value condition has been proposed, and the stability of weak solutions based on this partial boundary value condition is established. When the rate of the diffusion coefficient decays to zero, we explore how it affects the stability of weak solutions.

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## 1. Introduction

The nonlinear parabolic equation

$$\frac{\partial u}{\partial t} - \operatorname{div} \left( a(x) |\nabla u|^{p-2} \nabla u \right) - \sum_{i=1}^N b_i(x) D_i u + c(x, t) u = f(x, t), \quad (1.1)$$

$$(x, t) \in Q_T = \Omega \times (0, T),$$

arises from the theory of non-Newtonian fluid [6,22], where  $p > 1$ ,  $D_i = \frac{\partial}{\partial x_i}$ ,  $0 \leq a(x) \in C(\overline{\Omega})$ ,  $b_i(x) \in C^1(\overline{\Omega})$ ,  $c(x, t)$  and  $f(x, t)$  are continuous functions on  $Q_T$ , and  $\Omega \subset \mathbb{R}^N$  is a bounded domain with the  $C^2$  smooth boundary  $\partial\Omega$ . When  $a(x) = 1$ , the well-posedness problem, the regularity and the long-time behavior of weak solution have been well-established [1,6,8,14,22,28]. Antontsev-Shmarev [4,5] considered the well-posedness problem of the equation with the variable exponent

$$\frac{\partial u}{\partial t} = \operatorname{div} \left( a(x, t) |\nabla u|^{p(x)-2} \nabla u \right) + f(u, x, t),$$

where  $0 < a(x, t) \in C(\overline{Q_T})$ ,  $p(x) \in C(\overline{\Omega})$  and  $p(x) > 1$ . Let  $d(x) = \operatorname{dist}(x, \partial\Omega)$  be the distance function from the boundary. When  $a(x) = d(x)^\alpha$ , Yin-Wang [23,24] had proved the uniqueness of weak solution independent of the boundary value condition when  $\alpha > p - 1$ . This result has provided us an insight and inspiration that, if

$$a(x)|_{x \in \Omega} > 0 \text{ and } a(x)|_{x \in \partial\Omega} = 0, \quad (1.2)$$

with the initial value condition

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.3)$$

then only a partial boundary value condition

$$u(x, t) = 0, \quad (x, t) \in \Sigma_p \times (0, T), \quad (1.4)$$

may be sufficient to ensure the well-posedness of weak solution of equation (1.1), where  $\Sigma_p$  is a relatively open subset of  $\partial\Omega$ . The primary difficulty lies in the fact that, since the equation is nonlinear, the partial boundary  $\Sigma_p$  can not be expressed by the Fichera function as we usually do in the linear case [20]. In recent years, we have made an effort to work on the special case of  $\Sigma_p = \emptyset$  and on the stability of weak solution depending on the initial conditions [26,27]. The main feature which distinguishes this paper from other related works, e.g. [21–24,26,27], lies in the fact that we establish the stability of weak solution based on the partial boundary value condition (1.4) and present the expression of  $\Sigma_p$  explicitly in terms of the convection functions  $b_i(x)$ . Moreover, by comparing with the linear case, we will rigorously show that such a partial boundary is optimal.

**Definition 1.1.** A function  $u(x, t)$  is said to be a weak solution of the initial-boundary value problem of equation (1.1), if

$$u \in L^\infty(Q_T), \quad u_t \in L^r(Q_T), \quad a(x)|\nabla u|^p \in L^1(Q_T), \quad (1.5)$$

with  $r = 2$  when  $p \geq 2$ ,  $r = \frac{p}{p-1}$  when  $1 < p < 2$ , and for any function  $g(s) \in C^1(\mathbb{R})$  with  $g(0) = 0$ ,  $\varphi_1 \in C_0^1(\Omega)$  and  $\varphi_2 \in L^\infty(0, T; W_{loc}^{1,p}(\Omega))$ , there holds

$$\begin{aligned} & \iint_{Q_T} \left[ u_t g(\varphi_1 \varphi_2) + a(x) |\nabla u|^{p-2} \nabla u \cdot \nabla g(\varphi_1 \varphi_2) \right] dx dt \\ & + \sum_{i=1}^N \iint_{Q_T} u \left[ b_{ix_i}(x) g(\varphi_1 \varphi_2) + b_i(x) g_{x_i}(\varphi_1 \varphi_2) \right] dx dt \\ & + \sum_{i=1}^N \iint_{Q_T} [c(x, t) u g(\varphi_1 \varphi_2) - f(x, t) g(\varphi_1 \varphi_2)] dx dt \\ & = 0. \end{aligned} \quad (1.6)$$

The initial value is satisfied in the sense of

$$\lim_{t \rightarrow 0} \int_{\Omega} |u(x, t) - u_0(x)| dx = 0. \quad (1.7)$$

Moreover, the partial boundary value condition (1.4) is satisfied in the sense of the trace.

If  $p > 2$  and  $\int_{\Omega} a(x)^{-\frac{2}{p-2}} dx < \infty$ , proceeding in a manner analogous to that for the evolutionary  $p$ -Laplacian equations [22], we can prove that there is a weak solution  $u(x, t)$  of equation (1.1) with the initial value condition (1.3). One can refer to [27] for more details. In addition, if

$$\int_{\Omega} a(x)^{-\frac{1}{p-1}} dx < \infty, \quad (1.8)$$

then

$$\int_{\Omega} |\nabla u| dx < \infty$$

and so  $u(x, t) \in BV(Q_T) = \{f(x, t) : \iint_{Q_T} |u_t| < \infty, \iint_{Q_T} |\nabla u| < \infty\}$ , where the general derivatives  $u_t$  and  $u_{x_i}$ ,  $i = 1, 2, \dots, N$  are regular measures. Since  $C_0^\infty(Q_T)$  is dense in  $BV(Q_T)$ , there is a sequence  $u_n \in C_0^\infty(Q_T)$  such that

$$u_n \rightarrow u, \text{ in } BV(Q_T).$$

The trace of  $u(x, t) = 0$  on the boundary is defined by [22]

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T). \quad (1.9)$$

In particular, when  $a(x) = d(x)^\alpha$ , condition (1.8) is equivalent to  $0 < \alpha < p - 1$  [23]. It is notable that condition (1.8) is the sufficient condition to assure that the trace of  $u(x, t)$  is well-defined. If

$$\int_{\Omega} a(x)^{-\frac{1}{p-1}} dx = \infty, \quad (1.10)$$

then

$$\int_{\Omega} a(x) |\nabla u|^p dx < \infty.$$

So the trace of  $u(x, t) = 0$  on the boundary can not be defined by a classical way as (1.9). Yin-Wang [24] made a simple generalization of the trace of  $u(x, t) \in BV(Q_T)$  to the Banach space  $\mathbf{B}$  which is the closure of the set  $C_0^\infty(Q_T)$  with the norm

$$\|u\|_{\mathbf{B}} = \iint_{Q_T} a(x) (|u(x, t)|^p + |\nabla u(x, t)|^p) dx dt, \quad u \in \mathbf{B}.$$

If  $u(x, t) \in \mathbf{B}$ , the trace  $u(x, t) = 0$  on the boundary  $\partial\Omega$  is defined by

$$\begin{aligned} & \int_{\partial\Omega} u^2 \sum_{i=1}^N b_i(x) n_i(x) d\sigma \\ &= \text{ess} \lim_{\lambda \rightarrow 0} \int_{\left\{x \in \partial\Omega_\lambda : \sum_{i=1}^N b_i(x) n_i(x) < 0\right\}} u^2 \sum_{i=1}^N b_i(x) n_i(x) d\sigma \\ &= 0, \end{aligned} \quad (1.11)$$

where  $\lambda$  is small and positive,  $\text{ess} \lim_{\lambda \rightarrow 0} f(\lambda) = \inf_{\delta > 0} \{\text{ess} \sup\{f(\lambda) : |\lambda| < \delta\}\}$  denotes the super limit, and  $\vec{n} = \{n_i(x)\}$  is the inner normal vector on  $\partial\Omega_\lambda$ . As [24, Remark 2.2] states, the trace of  $u(x, t) = 0$  defined by (1.9) satisfies (1.11) too.

Let us summarize our main results on the stability of weak solution of equation (1.1). Throughout the whole paper, we assume that  $\Omega_\lambda = \{x \in \Omega : d(x) > \lambda\}$  and  $\lambda$  is a small positive number. For  $f(x) \in W_{loc}^{1,p}(\Omega)$ , it means that for any compact set  $\Omega_0 \subset \Omega$ ,  $f(x) \in W^{1,p}(\Omega_0)$ . The letter  $c$  may represent different values at different lines, and  $c(T)$  denotes a constant depending on  $T$ .

**Theorem 1.2.** *Let  $u(x, t)$  and  $v(x, t)$  be two weak solutions of the initial-boundary value problem of equation (1.1). If  $p > 1$ ,  $a(x)$  satisfies (1.2), and there are constants  $r, \delta, 0 < \delta < 1, r \geq p$  such that*

$$c_1 d(x)^r \leq a(x) \leq c_2 d(x)^p, \quad x \in \Omega \setminus \Omega_\lambda \quad (1.12)$$

and

$$|b_i(x)| \leq cd(x)^{\frac{r-(p-1)\delta}{p}}, \quad i = 1, 2, \dots, N, \quad (1.13)$$

then the partial boundary in (1.4) can be expressed by

$$\Sigma_p = \left\{ x \in \partial\Omega : \sum_{i=1}^N b_i(x)n_i(x) < 0 \right\}, \quad (1.14)$$

and the stability in the sense of

$$\int_{\Omega} |u(x, t) - v(x, t)| dx \leq c(T) \int_{\Omega} |u_0(x) - v_0(x)| dx, \quad t \in [0, T), \quad (1.15)$$

is true, where  $\vec{n} = \{n_i(x)\}$  is the inner normal vector of  $\partial\Omega$ .

**Theorem 1.3.** Let  $u(x, t)$  and  $v(x, t)$  be two weak solutions of the initial-boundary value problem of equation (1.1). Suppose that the partial boundary  $\Sigma_p$  in (1.4) is given by (1.14),  $p > 1$ ,  $a(x)$  satisfies (1.2) and

$$\int_{\Omega} a(x)d(x)^{-p} dx < \infty. \quad (1.16)$$

If  $\int_{\Omega} |u_0(x) - v_0(x)| dx > 0$ , then either

$$\int_{\Omega} |u(x, t) - v(x, t)|^{\frac{p}{p-1}} dx \leq \int_{\Omega} |u_0(x) - v_0(x)|^{\frac{p}{p-1}} dx, \quad t \in [0, T), \quad (1.17)$$

or there exists a  $T_1 \in (0, T)$  such that the stability is true in the sense of

$$\int_{\Omega} |u(x, t) - v(x, t)|^2 dx \leq c(T_1) \int_{\Omega} |u_0(x) - v_0(x)|^2 dx, \quad t \in [0, T_1). \quad (1.18)$$

If we consider an additional condition

$$b_i(x)a(x)^{-\frac{1}{p}} \leq c, \quad i = 1, 2, \dots, N, \quad (1.19)$$

we have the following theorem.

**Theorem 1.4.** Let  $u(x, t)$  and  $v(x, t)$  be two weak solutions of the initial-boundary value problem of equation (1.1). Suppose that the partial boundary  $\Sigma_p$  in (1.4) is given by (1.14),  $p > 1$ , and  $a(x)$  satisfies (1.2), (1.16) and (1.19). Then we have

$$\int_{\Omega} |u(x, t) - v(x, t)|^2 dx \leq c(T) \int_{\Omega} |u_0(x) - v_0(x)|^2 dx, \quad t \in [0, T). \quad (1.20)$$

Moreover, we can also prove the following theorem which shows the stability of weak solution based on another partial boundary value condition.

**Theorem 1.5.** *Let  $u(x, t)$  and  $v(x, t)$  be two weak solutions of the initial-boundary value problem of equation (1.1). If  $p > 1$ ,  $a(x)$  satisfies (1.2) and*

$$\int_{\Omega \setminus \Omega_{1\lambda}} a(x)^{1-p} |\nabla a|^p < \infty, \int_{\Omega} \left| b_i(x) a(x)^{-\frac{1}{p}} \right|^{\frac{p}{p-1}} dx < \infty, \quad i = 1, 2, \dots, N, \quad (1.21)$$

then the partial boundary in (1.4) can be expressed by

$$\Sigma_p = \left\{ x \in \partial\Omega : \sum_{i=1}^N b_i(x) a_{x_i}(x) < 0 \right\}, \quad (1.22)$$

and the stability (1.15) is true, where  $\Omega_{1\lambda} = \{x \in \Omega : a(x) > \lambda\}$  and  $\lambda$  is a small positive constant. Moreover, if

$$|b_i(x)| \leq ca(x), \quad i = 1, 2, \dots, N; \quad x \in \Omega,$$

then (1.15) is true without any boundary value condition.

In the present study, in Theorems 1.2–1.4 the trace  $u(x, t) = 0$  on the boundary is understood in the sense of (1.11). In Theorem 1.5, the trace  $u(x, t) = 0$  on the boundary  $\partial\Omega$  is defined by

$$\begin{aligned} & \int_{\partial\Omega} u^2 \sum_{i=1}^N b_i(x) a_{x_i}(x) d\sigma \\ &= \text{ess} \lim_{\lambda \rightarrow 0} \int_{\left\{ x \in \partial\Omega_{1\lambda} : \sum_{i=1}^N b_i(x) a_{x_i}(x) < 0 \right\}} u^2 \sum_{i=1}^N b_i(x) a_{x_i}(x) d\sigma \\ &= 0. \end{aligned} \quad (1.23)$$

For our convenience, the notation  $\lim_{\lambda \rightarrow 0}$  used in the proofs of Theorems 1.2 and 1.5 and  $\lim_{\varepsilon \rightarrow 0}$  used in the proofs of Theorems 1.3 and 1.4 stands for the super limit.

At the end of this paper, we will provide an explanation why the partial boundary (1.14) is optimal but the partial boundary (1.22) is not. Note that main results in [23] and the above Theorem 1.2 are achieved based on  $a(x) = d(x)^\alpha$ , which means that the diffusion rate decays as a power function. In addition, there are many functions satisfying condition (1.16). One typical example is  $a(x) = e^{-\frac{1}{d(x)}}$ , which indicates that the diffusion rate may decay exponentially and satisfies

$$\int_{\Omega} a(x)^{-\frac{1}{p-1}} dx = \infty. \quad (1.24)$$

Since the equation involving the  $p$ -Laplacian is one kind of the most interesting parabolic equations, there are a great number of issues to study various subjects such as the existence and uniqueness, stability, Harnack inequality, long time behaviors and extinction etc, see [9,12,21,26,27]. For studying the well-posedness problem of degenerate parabolic equations, apart from the boundary value conditions being imposed in the sense of the trace, there are quite a few innovative methods to deal with this problem, and we refer the reader to [2–4,13,15–17,24–26] and the references therein.

The rest of the paper is organized as follows. We prove Theorems 1.2 and 1.5 in Section 2, and present the proofs of Theorems 1.3 and 1.4 in Section 3. In Section 4, we will give a rigorous explanation on the partial boundary value condition (1.4) to show why the partial boundary (1.14) is optimal. In Section 5, we are concerned with the stability in the case of  $\int_{\Omega} a(x)^{-\frac{1}{p-1}} dx = \infty$  without any boundary value condition.

## 2. Proofs of Theorems 1.2 and 1.5

For small  $\eta > 0$ , let  $S_{\eta}(s) = \int_0^s h_{\eta}(\tau) d\tau$  and

$$h_{\eta}(s) = \frac{2}{\eta} \left( 1 - \frac{|s|}{\eta} \right)_+ = \begin{cases} \frac{2}{\eta} \left( 1 - \frac{|s|}{\eta} \right), & \text{if } |s| < \eta, \\ 0, & \text{if } |s| \geq \eta. \end{cases} \quad (2.1)$$

Obviously,  $h_{\eta}(s) \in C(\mathbb{R})$ , and

$$\lim_{\eta \rightarrow 0} S_{\eta}(s) = \text{sgn}s, \quad \lim_{\eta \rightarrow 0} sS'_{\eta}(s) = 0. \quad (2.2)$$

**Lemma 2.1.** *Let  $u(x, t)$  and  $v(x, t)$  be two weak solutions of the initial-boundary value problem of equation (1.1). If  $p > 1$ ,  $a(x)$  satisfies (1.2), and*

$$a(x) \leq cd(x)^p, \quad x \in \Omega \setminus \Omega_{\lambda} \quad (2.3)$$

$$\int_{\Omega} \left| b_i(x) a(x)^{-\frac{1}{p}} \right|^{\frac{p}{p-1}} dx < \infty, \quad i = 1, 2, \dots, N, \quad (2.4)$$

*then the partial boundary in (1.4) can be expressed by (1.14) and the stability is true in the sense of (1.15). Moreover, if  $b_i(x)$  satisfies*

$$|b_i(x)| \leq cd(x), \quad i = 1, 2, \dots, N; \quad x \in \Omega, \quad (2.5)$$

*then (1.15) is true without any boundary value condition.*

**Proof.** We define

$$\phi_{\lambda}(x) = \begin{cases} 1, & \text{if } x \in \Omega_{\lambda}, \\ \frac{1}{\lambda} d(x), & \text{if } x \in \Omega \setminus \Omega_{\lambda}. \end{cases}$$

By a process of limit, choosing  $S_{\eta}(\phi_{\lambda}(x)(u - v))$  as the test function in (1.6) leads to

$$\begin{aligned}
& \int_{\Omega} S_{\eta}(\phi_{\lambda}(x)(u-v)) \frac{\partial(u-v)}{\partial t} dx \\
& + \int_{\Omega} a(x) \phi_{\lambda}(x) \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla(u-v) S'_{\eta}(\phi_{\lambda}(u-v)) dx \\
& + \int_{\Omega} a(x) \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla \phi_{\lambda}(u-v) S'_{\eta}(\phi_{\lambda}(u-v)) dx \\
& + \sum_{i=1}^N \int_{\Omega} D_i(b_i(x))(u-v) S_{\eta}(\phi_{\lambda}(u-v)) + \phi_{\lambda} b_i(x)(u-v) \cdot (u-v)_{x_i} S'_{\eta}(\phi_{\lambda}(u-v)) dx \\
& + \sum_{i=1}^N \int_{\Omega} b_i(x)(u-v) \cdot \phi_{\lambda x_i}(u-v) S'_{\eta}(\phi_{\lambda}(u-v)) dx \\
& + \int_{\Omega} c(x, t)(u-v) S_{\eta}(\phi_{\lambda}(u-v)) dx \\
& = 0.
\end{aligned} \tag{2.6}$$

Then, we further have

$$\lim_{\eta \rightarrow 0} \lim_{\lambda \rightarrow 0} \int_{\Omega} S_{\eta}(\phi_{\lambda}(u-v)) \frac{\partial(u-v)}{\partial t} dx = \frac{d}{dt} \int_{\Omega} |u(x, t) - v(x, t)| dx, \tag{2.7}$$

$$\int_{\Omega} a(x) \phi_{\lambda}(x) \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla(u-v) S'_{\eta}(\phi_{\lambda}(u-v)) dx \geq 0, \tag{2.8}$$

and

$$\begin{aligned}
& \left| \int_{\Omega} a(x)(u-v) S'_{\eta}(\phi_{\lambda}(u-v)) (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \nabla \phi_{\lambda} dx \right| \\
& = \left| \int_{\{\Omega: \phi_{\lambda}|u-v|<\eta\}} a(x)^{-\frac{p-1}{p}} a(x)(u-v) S'_{\eta}(\phi_{\lambda}(u-v)) a^{\frac{p-1}{p}} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \nabla \phi_{\lambda} dx \right| \\
& \leq \left( \int_{\{\Omega: \phi_{\lambda}|u-v|<\eta\}} \left| a(x)^{\frac{1}{p}} (u-v) S'_{\eta}(\phi_{\lambda}(u-v)) \nabla \phi_{\lambda} \right|^p dx \right)^{\frac{1}{p}} \\
& \quad \cdot \left( \int_{\{\Omega: \phi_{\lambda}|u-v|<\eta\}} a(x) (|\nabla u|^p + |\nabla v|^p) dx \right)^{\frac{p-1}{p}}.
\end{aligned} \tag{2.9}$$

If  $\{x \in \Omega : u - v = 0\}$  has measure zero, due to  $|\nabla d| = 1$  a.e. in  $\Omega$  and  $a(x) \leq c_2 d(x)^p$ , we get



$$\begin{aligned}
& \int_{\Omega} a(x) \left| \frac{\nabla \phi_{\lambda}}{\phi_{\lambda}} \right|^p dx = \int_{\Omega \setminus \Omega_{\lambda}} \frac{a(x)}{d^p} dx < c\lambda, \\
& \left| \int_{\{\Omega: \phi_{\lambda}|u-v|<\eta\}} a(x)^{\frac{1}{p}} \nabla \phi_{\lambda}(u-v) S'_{\eta}(\phi_{\lambda}(u-v))^p dx \right| \\
&= \left| \int_{\{\Omega: \phi_{\lambda}|u-v|<\eta\}} a(x)^{\frac{1}{p}} \frac{\nabla \phi_{\lambda}}{\phi_{\lambda}} \phi_{\lambda}(u-v) S'_{\eta}(\phi_{\lambda}(u-v))^p dx \right| \quad (2.10) \\
&\leq \int_{\{\Omega: \phi_{\lambda}|u-v|<\eta\}} \left| a(x)^{\frac{1}{p}} \frac{\nabla \phi_{\lambda}}{\phi_{\lambda}} \right|^p dx \\
&\leq c\lambda,
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{\lambda \rightarrow 0} \left( \int_{\{\Omega: \phi_{\lambda}|u-v|<\eta\}} a(x) (|\nabla u|^p + |\nabla v|^p) dx \right)^{\frac{p-1}{p}} \\
&= \left( \int_{\{\Omega: |u-v|<\eta\}} a(x) (|\nabla u|^p + |\nabla v|^p) dx \right)^{\frac{p-1}{p}} \quad (2.11) \\
&\leq c.
\end{aligned}$$

If  $\{x \in \Omega : u - v = 0\}$  has a positive measure, it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}
& \lim_{\eta \rightarrow 0} \lim_{\lambda \rightarrow 0} \left( \int_{\{\Omega: \phi_{\lambda}|u-v|<\eta\}} \left| a(x)^{\frac{1}{p}} \frac{\nabla \phi_{\lambda}}{\phi_{\lambda}} \phi_{\lambda}(u-v) S'_{\eta}(\phi_{\lambda}(u-v))^p dx \right|^{\frac{1}{p}} \right) \\
&= \lim_{\eta \rightarrow 0} \lim_{\lambda \rightarrow 0} \left( \int_{\{\Omega: |u-v|<\eta\} \cap \Omega \setminus \Omega_{\lambda}} a(x) \left| \frac{\nabla \phi_{\lambda}}{\phi_{\lambda}} \right|^p \left| \phi_{\lambda}(u-v) S'_{\eta}((u-v)\phi_{\lambda}) \right|^p dx \right)^{\frac{1}{p}} \\
&\leq c \lim_{\eta \rightarrow 0} \lim_{\lambda \rightarrow 0} \left( \int_{\{\Omega: |u-v|<\eta\}} \left| \phi_{\lambda}(u-v) S'_{\eta}((u-v)\phi_{\lambda}) \right|^p dx \right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\eta \rightarrow 0} \left( \int_{\{\Omega: |u-v| < \eta\}} |(u-v)S'_\eta(u-v)|^p dx \right)^{\frac{1}{p}} \\
&= 0.
\end{aligned} \tag{2.12}$$

By (2.2), in both cases we have

$$\lim_{\eta \rightarrow 0} \lim_{\lambda \rightarrow 0} \left| \int_{\Omega} a(x) \phi_\lambda(u-v) S'_\eta(\phi_\lambda(u-v)) \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \nabla \phi_\lambda dx \right| = 0. \tag{2.13}$$

If we denote

$$\Omega_1 = \left\{ x \in \Omega : - \sum_{i=1}^N b_i(x) d_{x_i}(x) > 0 \right\},$$

in the sense of Painlevé-Kuratowski (through inner and outer limits), see e.g. [7], using the partial boundary value condition (1.4) (with the expression (1.14)), due to  $|d_{x_i}| \leq |\nabla d| = 1$  a.e. in  $\Omega$  and

$$\lim_{\lambda \rightarrow 0} (\Omega \setminus \Omega_\lambda) \cap \Omega_1 = \Sigma_p,$$

we deduce that

$$\begin{aligned}
& - \lim_{\lambda \rightarrow 0} \int_{\Omega} b_i(x) (u-v) \phi_{\lambda x_i}(u-v) S'_\eta(\phi_\lambda(u-v)) dx \\
&= - \lim_{\lambda \rightarrow 0} \int_{\Omega \setminus \Omega_\lambda} b_i(x) (u-v) \frac{d_{x_i}}{\lambda} (u-v) S'_\eta(\phi_\lambda(u-v)) dx \\
&\leq - \lim_{\lambda \rightarrow 0} \int_{\Omega_1 \cap (\Omega \setminus \Omega_\lambda)} b_i(x) (u-v) \frac{d_{x_i}}{\lambda} (u-v) S'_\eta(\phi_\lambda(u-v)) dx \\
&= - \lim_{\lambda \rightarrow 0} \int_{\Omega_1 \cap (\Omega \setminus \Omega_\lambda)} b_i(x) (u-v) \frac{d_{x_i}}{\lambda} (u-v) \left[ S'_\eta(\phi_\lambda(u-v)) - S'_\eta(u-v) \right] dx \\
&\quad - \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\Omega_1 \cap (\Omega \setminus \Omega_\lambda)} b_i(x) d_{x_i} (u-v)^2 S'_\eta(u-v) dx.
\end{aligned} \tag{2.14}$$

Hereafter, the double indices of  $i$  represent the summation from 1 to  $N$ .

If  $|\phi_\lambda(u-v)| < \eta$ , then

$$\begin{aligned}
S'_\eta(\phi_\lambda(u-v)) - S'_\eta(u-v) &= \left( 1 - \frac{\phi_\lambda |u-v|}{\eta} \right)_+ - \left( 1 - \frac{|u-v|}{\eta} \right)_+, \\
&= \frac{|u-v|}{\eta} (1 - \phi_\lambda).
\end{aligned}$$

If  $\frac{\eta}{\phi_\lambda} > |\phi_\lambda(u - v)| \geq \eta$ , then

$$S'_\eta(\phi_\lambda(u - v)) - S'_\eta(u - v) = 1 - \frac{|u - v|}{\eta}.$$

From (1.11) we have

$$\begin{aligned} & - \lim_{\lambda \rightarrow 0} \int_{\Omega_1 \cap (\Omega \setminus \Omega_\lambda)} b_i(x)(u - v) \frac{d_{x_i}}{\lambda}(u - v) \left[ S'_\eta(\phi_\lambda(u - v)) - S'_\eta(u - v) \right] dx \\ & \leq - \lim_{\lambda \rightarrow 0} \int_{\Omega_1 \cap (\Omega \setminus \Omega_\lambda)} b_i(x)(u - v) \frac{d_{x_i}}{\lambda}(u - v) \frac{|u - v|}{\eta} (1 - \phi_\lambda) dx \\ & \quad - \lim_{\lambda \rightarrow 0} \int_{\Omega_1 \cap (\Omega \setminus \Omega_\lambda)} b_i(x)(u - v) \frac{d_{x_i}}{\lambda}(u - v) \left( 1 - \frac{|u - v|}{\eta} \right)_+ dx \\ & \leq - \frac{1}{\eta} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\Omega_1 \cap (\Omega \setminus \Omega_\lambda)} b_i(x) d_{x_i} |u - v|^3 dx \\ & \quad - \frac{1}{\eta} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\Omega_1 \cap (\Omega \setminus \Omega_\lambda)} b_i(x) d_{x_i} (u - v)^2 (\eta - |u - v|)_+ dx \\ & = - \frac{1}{\eta} \int_{\Sigma_p} b_i(x) n_i(x) \left[ |u - v|^3 + (u - v)^2 (\eta - |u - v|) \right] d\Sigma = 0. \end{aligned} \tag{2.15}$$

Meanwhile, a straightforward calculation gives

$$\begin{aligned} & - \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\Omega_1 \cap (\Omega \setminus \Omega_\lambda)} b_i(x) d_{x_i} (u - v)^2 S'_\eta(u - v) dx \\ & = - \int_{\Sigma_p} b_i(x) n_i(x) |u - v|^2 S'_\eta(u - v) d\Sigma \\ & = 0. \end{aligned} \tag{2.16}$$

From (2.14)-(2.16), we derive that

$$- \lim_{\lambda \rightarrow 0} \int_{\Omega} b_i(x)(u - v) \phi_{\lambda x_i}(u - v) S'_\eta(\phi_\lambda(u - v)) dx = 0. \tag{2.17}$$

Moreover, since  $b_i(x)$  and  $a(x)$  satisfy (2.4), and

$$\int_{\Omega} \left| b_i(x) a(x)^{-\frac{1}{p}} \right|^{\frac{p}{p-1}} dx \leq c,$$

it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}
 & \lim_{\eta \rightarrow 0} \lim_{\lambda \rightarrow 0} \left| \int_{\Omega} b_i(x) \phi_{\lambda}(x) (u-v) S_{\eta}'(\phi_{\lambda}(u-v)) (u-v)_{x_i} dx \right| \\
 & \leq c \lim_{\eta \rightarrow 0} \lim_{\lambda \rightarrow 0} \left( \int_{\Omega} a(x) (|\nabla u|^p + |\nabla v|^p) dx \right)^{\frac{1}{p}} \\
 & \quad \cdot \left( \int_{\Omega} \left| b_i(x) a(x)^{-\frac{1}{p}} \phi_{\lambda}(x) (u-v) S_{\eta}'(\phi_{\lambda}(u-v)) \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\
 & \leq c \lim_{\eta \rightarrow 0} \left( \int_{\Omega} a(x) (|\nabla u|^p + |\nabla v|^p) dx \right)^{\frac{1}{p}} \\
 & \quad \cdot \left( \int_{\Omega} \left| b_i(x) a(x)^{-\frac{1}{p}} (u-v) S_{\eta}'(u-v) \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} = 0.
 \end{aligned} \tag{2.18}$$

Note that  $b_i(x) \in C^1(\overline{\Omega})$  and  $c(x, t)$  is a continuous function on  $Q_T$ , so we have

$$\begin{aligned}
 & \lim_{\eta \rightarrow 0} \lim_{\lambda \rightarrow 0} \left| \int_{\Omega} D_i(b_i(x)) (u-v) S_{\eta}(\phi_{\lambda}(u-v)) + \int_{\Omega} c(x, t) (u-v) S_{\eta}(\phi_{\lambda}(u-v)) dx \right| \\
 & \leq c \int_{\Omega} |u(x, t) - v(x, t)| dx.
 \end{aligned} \tag{2.19}$$

We now let  $\eta \rightarrow 0$  and  $\lambda \rightarrow 0$  in (2.6), and then obtain

$$\frac{d}{dt} \int_{\Omega} |u(x, t) - v(x, t)| dx \leq \int_{\Omega} |u(x, t) - v(x, t)| dx.$$

Applying Gronwall's inequality directly leads to

$$\int_{\Omega} |u(x, t) - v(x, t)| dx \leq c(T) \int_{\Omega} |u_0(x) - v_0(x)| dx$$

for  $t \in [0, T)$ .

From the above discussions, it is clear to see that the partial boundary value condition (1.4) (with the expression (1.14)) is used only for the derivation of (2.17).

If  $b_i(x)$  satisfies (2.5), then

$$\begin{aligned}
 & - \int_{\Omega} b_i(x)(u-v) \cdot \phi_{\lambda x_i}(u-v) S'_\eta(\phi_\lambda(u-v)) dx \\
 &= - \int_{\Omega \setminus \Omega_\lambda} \frac{b_i(x) d_{x_i}}{d(x)} (u-v) \phi_\lambda(u-v) S'_\eta(\phi_\lambda(u-v)) dx \\
 &\leq - \int_{(\Omega \setminus \Omega_\lambda) \cap \Omega_1} \frac{b_i(x) d_{x_i}}{d(x)} (u-v) \phi_\lambda(u-v) S'_\eta(\phi_\lambda(u-v)) dx \\
 &\leq - \int_{(\Omega \setminus \Omega_\lambda) \cap \Omega_1} \frac{b_i(x) d_{x_i}}{d(x)} \left| (u-v) \phi_\lambda(u-v) S'_\eta(\phi_\lambda(u-v)) \right| dx \\
 &\leq -c \int_{(\Omega \setminus \Omega_\lambda) \cap \Omega_1} \frac{b_i(x) d_{x_i}}{d(x)} |u-v| dx.
 \end{aligned} \tag{2.20}$$

Letting  $\lambda \rightarrow 0$ , we have

$$\begin{aligned}
 & - \lim_{\lambda \rightarrow 0} \int_{\Omega} b_i(x)(u-v) \phi_{\lambda x_i}(u-v) S'_\eta(\phi_\lambda(u-v)) dx \\
 &\leq -c \lim_{\lambda \rightarrow 0} \int_{(\Omega \setminus \Omega_\lambda) \cap \Omega_1} \frac{b_i(x) d_{x_i}}{d(x)} |u-v| dx = 0.
 \end{aligned} \tag{2.21}$$

Consequently, without the partial boundary value condition (1.4) (with the expression (1.14)), according to (2.20)-(2.21), we can arrive at (1.15).  $\square$

**Proof of Theorem 1.2.** If there are constants  $r, \delta$ ,  $0 < \delta < 1$  and  $r \geq p$  such that  $a(x)$  and  $b_i(x)$  satisfies conditions (1.12)-(1.13), then conditions (2.3)-(2.4) are true. Thus, Theorem 1.2 follows from Lemma 2.1 immediately.  $\square$

**Proof of Theorem 1.5.** Let  $u(x, t)$  and  $v(x, t)$  be two weak solutions of the initial-boundary value problem of equation (1.1), and let  $\Omega_{1\lambda} = \{x \in \Omega : a(x) > \lambda\}$ .

Define

$$\phi_{1\lambda}(x) = \begin{cases} 1, & \text{if } x \in \Omega_{1\lambda}, \\ \frac{1}{\lambda} a(x), & \text{if } x \in \Omega \setminus \Omega_{1\lambda}. \end{cases}$$

By a process of limit, we can choose  $S_\eta(\phi_{1\lambda}(x)(u-v))$  as the test function in (1.6). From the proof of Lemma 2.1, we find that (2.6)-(2.9) still hold.

If  $\{x \in \Omega : u-v=0\}$  has measure zero, using (1.21) we get

$$\int_{\Omega \setminus \Omega_{1\lambda}} a(x)^{1-p} |\nabla a|^p < \infty$$

and

$$\int_{\Omega} a(x) \left| \frac{\nabla \phi_{\lambda}}{\phi_{\lambda}} \right|^p dx = \int_{\Omega \setminus \Omega_{1\lambda}} \frac{a(x) |\nabla a|^p}{a(x)^p} dx < \infty.$$

This implies that formulas (2.10)-(2.13) hold too.

If we denote

$$\Omega_{11} = \left\{ x \in \Omega : - \sum_{i=1}^N b_i(x) a_{x_i}(x) > 0 \right\},$$

in the sense of Painlevé-Kuratowski (through inner and outer limits), see e.g. [7], using (1.23) and the partial boundary value condition (1.4) (with the expression (1.22)), due to  $|a_{x_i}| \leq c$  and

$$\lim_{\lambda \rightarrow 0} (\Omega \setminus \Omega_{1\lambda}) \cap \Omega_{11} = \Sigma_p,$$

as discussing for (2.17) we can obtain

$$\begin{aligned} & - \lim_{\lambda \rightarrow 0} \int_{\Omega} b_i(x) (u - v) \phi_{\lambda x_i} (u - v) S'_{\eta}(\phi_{\lambda}(u - v)) dx \\ &= - \lim_{\lambda \rightarrow 0} \int_{\Omega \setminus \Omega_{1\lambda}} b_i(x) (u - v) \frac{a_{x_i}}{\lambda} (u - v) S'_{\eta}(\phi_{\lambda}(u - v)) dx \\ &\leq - \lim_{\lambda \rightarrow 0} \int_{\Omega_{11} \cap (\Omega \setminus \Omega_{1\lambda})} b_i(x) (u - v) \frac{a_{x_i}}{\lambda} (u - v) S'_{\eta}(\phi_{\lambda}(u - v)) dx \\ &= - \lim_{\lambda \rightarrow 0} \int_{\Omega_{11} \cap (\Omega \setminus \Omega_{1\lambda})} b_i(x) (u - v) \frac{a_{x_i}}{\lambda} (u - v) \left[ S'_{\eta}(\phi_{\lambda}(u - v)) - S'_{\eta}(u - v) \right] dx \\ &\quad - \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\Omega_{11} \cap (\Omega \setminus \Omega_{1\lambda})} b_i(x) a_{x_i} (u - v)^2 S'_{\eta}(u - v) dx \tag{2.22} \\ &\leq - \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\Omega_{11} \cap (\Omega \setminus \Omega_{1\lambda})} b_i(x) a_{x_i} (u - v)^2 S'_{\eta}(u - v) dx \\ &= - \frac{1}{\eta} \int_{\Sigma_p} b_i(x) a_i(x) \left[ |u - v|^3 + (u - v)^2 (\eta - |u - v|) \right] d\Sigma \\ &\quad - \int_{\Sigma_p} b_i(x) a_{x_i} |u - v|^2 S'_{\eta}(u - v) d\Sigma \\ &= 0. \end{aligned}$$

In view of  $b_i(x)$  and  $a(x)$  satisfying

$$\int_{\Omega} \left| b_i(x) a(x)^{-\frac{1}{p}} \right|^{\frac{p}{p-1}} dx \leq c,$$

analogous to the proof of Lemma 2.1, we can see that (2.18)-(2.19) hold too.

Letting  $\eta \rightarrow 0$  and  $\lambda \rightarrow 0$  in (2.6), and applying Gronwall's inequality, we can arrive at the desired result (1.15). Note that the partial boundary value condition (1.4) (with the expression (1.22)) is used only for the derivation of (2.22).

If  $b_i(x)$  satisfies (1.22), then it follows that

$$\begin{aligned} & - \int_{\Omega} b_i(x)(u-v) \cdot \phi_{\lambda x_i}(u-v) S'_{\eta}(\phi_{\lambda}(u-v)) dx \\ &= - \int_{\Omega \setminus \Omega_{1\lambda}} \frac{b_i(x) a_{x_i}}{a(x)} (u-v) \phi_{\lambda}(u-v) S'_{\eta}(\phi_{\lambda}(u-v)) dx \\ &\leq - \int_{(\Omega \setminus \Omega_{1\lambda}) \cap \Omega_{11}} \frac{b_i(x) a_{x_i}}{a(x)} (u-v) \phi_{\lambda}(u-v) S'_{\eta}(\phi_{\lambda}(u-v)) dx \\ &\leq - \int_{(\Omega \setminus \Omega_{1\lambda}) \cap \Omega_{11}} \frac{b_i(x) a_{x_i}}{a(x)} \left| (u-v) \phi_{\lambda}(u-v) S'_{\eta}(\phi_{\lambda}(u-v)) \right| dx \\ &\leq -c \int_{(\Omega \setminus \Omega_{1\lambda}) \cap \Omega_{11}} \frac{b_i(x) a_{x_i}}{a(x)} |u-v| dx. \end{aligned}$$

Due to  $|b_i(x)| \leq ca(x)$  and

$$\lim_{\lambda \rightarrow 0} (\Omega \setminus \Omega_{1\lambda}) \cap \Omega_{11} = \Sigma_p,$$

by letting  $\lambda \rightarrow 0$  we have

$$\begin{aligned} & - \lim_{\lambda \rightarrow 0} \int_{\Omega} b_i(x)(u-v) \phi_{\lambda x_i}(u-v) S'_{\eta}(\phi_{\lambda}(u-v)) dx \\ &\leq -c \lim_{\lambda \rightarrow 0} \int_{(\Omega \setminus \Omega_{1\lambda}) \cap \Omega_{11}} \frac{b_i(x) a_{x_i}}{a(x)} |u-v| dx \\ &\leq c \lim_{\lambda \rightarrow 0} \int_{(\Omega \setminus \Omega_{1\lambda}) \cap \Omega_{11}} |u-v| dx \\ &= 0. \end{aligned}$$

Consequently, without the partial boundary value condition (1.4) (with the expression (1.22)), we can obtain (1.15) too.  $\square$

### 3. Proofs of Theorems 1.3 and 1.4

**Proof of Theorem 1.3.** From the definition of weak solution, if  $g(s) = s$ , for any  $\varphi_1 \in C_0^1(\Omega)$  and  $\varphi_2 \in L^\infty(0, T; W_{loc}^{1,p}(\Omega))$  we have

$$\begin{aligned} & \iint_{Q_T} \varphi_1 \varphi_2 \frac{\partial(u-v)}{\partial t} dx dt \\ &= - \iint_{Q_T} a(x) \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \nabla(\varphi_1 \varphi_2) dx dt \\ & \quad - \sum_{i=1}^N \iint_{Q_T} (u-v) [b_{ix_i}(\varphi_1 \varphi_2) + b_i(x)(\varphi_1 \varphi_2)_{x_i}] dx dt + \iint_{Q_T} (u-v)c(x, t)(\varphi_1 \varphi_2) dx dt. \end{aligned} \quad (3.1)$$

If

$$\int_{\Omega} |u(x, t) - v(x, t)|^{\frac{p}{p-1}} dx \leq \int_{\Omega} |u(x, 0) - v(x, 0)|^{\frac{p}{p-1}} dx, \quad t \in [0, T], \quad (3.2)$$

the desired result is obviously true. If (3.2) does not hold, then there is a  $T_1$ ,  $0 < T_1 \leq T$  such that

$$\int_{\Omega} |u(x, t) - v(x, t)|^{\frac{p}{p-1}} dx > \int_{\Omega} |u(x, 0) - v(x, 0)|^{\frac{p}{p-1}} dx, \quad t \in [0, T_1]. \quad (3.3)$$

Denote  $\Omega_\varepsilon = \{x \in \Omega : d(x) > \varepsilon\}$ . Let  $\xi_\varepsilon \in C_0^\infty(\Omega_\varepsilon)$  such that  $\xi_\varepsilon = 0$  on  $\Omega \setminus \Omega_\varepsilon$ ,  $\xi_\varepsilon = 1$  on  $\Omega_{2\varepsilon}$ ,  $0 \leq \xi_\varepsilon \leq 1$  and  $|\nabla \xi_\varepsilon| \leq \frac{c}{\varepsilon}$ . We choose

$$\varphi_1 = \xi_\varepsilon \quad \text{and} \quad \varphi_2 = \chi_{[\tau, s]}(u - v)$$

in (3.1), where  $\chi_{[\tau, s]}$  is the characteristic function on  $[\tau, s] \subset (0, T_1)$ . Then

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} [u(x, s) - v(x, s)]^2 \xi_\varepsilon dx \\ &= \frac{1}{2} \int_{\Omega} [u(x, \tau) - v(x, \tau)]^2 \xi_\varepsilon dx \\ & \quad - \iint_{Q_{\tau s}} \xi_\varepsilon a(x) \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \nabla(u-v) dx dt \\ & \quad - \iint_{Q_{\tau s}} (u-v)a(x) \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \nabla \xi_\varepsilon dx dt \end{aligned}$$



$$\begin{aligned}
& - \sum_{i=1}^N \iint_{Q_{\tau s}} (u-v) [b_{ix_i}(u-v)\xi_\varepsilon + b_i(x)((u-v)\xi_\varepsilon)_{x_i}] dx dt \\
& + \iint_{Q_{\tau s}} (u-v)^2 c(x, t) \xi_\varepsilon dx dt,
\end{aligned} \tag{3.4}$$

where  $Q_{\tau s} = \Omega \times [\tau, s]$ .

A straightforward calculation leads to

$$\begin{aligned}
& \left| - \iint_{Q_{\tau s}} (u-v) a(x) (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \nabla \xi_\varepsilon dx dt \right| \\
& \leq \iint_{Q_{\tau s}} |u-v| a(x) (|\nabla u|^{p-1} + |\nabla v|^{p-1}) |\nabla \xi_\varepsilon| dx dt \\
& \leq c \int_{\tau}^s \int_{\Omega_\varepsilon \setminus \Omega_{2\varepsilon}} \left[ \frac{p-1}{p} a(x) (|\nabla u|^p + |\nabla v|^p) + \frac{1}{p} a(x) |\nabla \xi_\varepsilon|^p \right] dx dt \\
& \leq c \int_{\tau}^s \int_{\Omega_\varepsilon \setminus \Omega_{2\varepsilon}} \left[ \frac{p-1}{p} a(x) (|\nabla u|^p + |\nabla v|^p) + \frac{1}{p} a(x) \varepsilon^{-p} \right] dx dt \\
& \leq c \int_{\tau}^s \int_{\Omega_\varepsilon \setminus \Omega_{2\varepsilon}} \left[ \frac{p-1}{p} a(x) (|\nabla u|^p + |\nabla v|^p) + \frac{1}{p} a(x) d(x)^{-p} \right] dx dt.
\end{aligned} \tag{3.5}$$

Due to condition (1.16), it is not difficult to see that the right-hand side of (3.5) tends to zero as  $\varepsilon \rightarrow 0$ .

Note that

$$\begin{aligned}
& \iint_{Q_{\tau s}} (u-v) [b_{ix_i}(u-v)\xi_\varepsilon + b_i(x)((u-v)\xi_\varepsilon)_{x_i}] dx dt \\
& = \iint_{Q_{\tau s}} (u-v)^2 b_{ix_i} \xi_\varepsilon dx dt \\
& + \iint_{Q_{\tau s}} (u-v)^2 b_i(x) \xi_{\varepsilon x_i} dx dt + \iint_{Q_{st}} (u-v) b_i(x) \xi_\varepsilon (u-v)_{x_i} dx dt.
\end{aligned} \tag{3.6}$$

By a process of limit, we take the  $\xi_\varepsilon$  as

$$\xi_\varepsilon(x) = \begin{cases} 1, & \text{if } x \in \Omega_{2\varepsilon}, \\ \frac{1}{\varepsilon}(d(x) - \varepsilon), & \text{if } x \in \Omega_\varepsilon \setminus \Omega_{2\varepsilon}, \\ 0, & \text{if } x \in \Omega \setminus \Omega_\varepsilon. \end{cases}$$

Denote

$$\Omega_1 = \left\{ x \in \Omega : \sum_{i=1}^N b_i(x) d_{x_i} \leq 0 \right\} \text{ and } \Omega_2 = \left\{ x \in \Omega : \sum_{i=1}^N b_i(x) d_{x_i} > 0 \right\}.$$

Then, using the partial boundary value condition (1.4) (with the expression (1.14)), we have

$$\begin{aligned} & - \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (u - v)^2 b_i(x) \xi_{\varepsilon x_i} dx \\ &= - \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon \setminus \Omega_{2\varepsilon}} (u - v)^2 b_i(x) d_{x_i} \\ &\leq - \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{(\Omega_\varepsilon \setminus \Omega_{2\varepsilon}) \cap \Omega_1} (u - v)^2 b_i(x) d_{x_i} dx \\ &= \int_{\Sigma_p} (u - v)^2 b_i(x) n_i d\Sigma \\ &= 0. \end{aligned} \tag{3.7}$$

Meanwhile, we have

$$\begin{aligned} & \left| \int_{\Omega} (u - v) b_i(x) \xi_{\varepsilon} (u - v)_{x_i} dx \right| \\ &= \left| \int_{\Omega} (u - v) b_i(x) \xi_{\varepsilon} a(x)^{-\frac{1}{p}} a(x)^{\frac{1}{p}} (u - v)_{x_i} dx \right| \\ &\leq \left( \int_{\Omega} \left| (u - v) b_i(x) a(x)^{-\frac{1}{p}} \right|^{\frac{p}{p-1}} dx \right)^{\frac{p}{p-1}} \left( \int_{\Omega} a(x) (|\nabla u|^p + |\nabla v|^p) dx \right)^{\frac{1}{p}}. \end{aligned} \tag{3.8}$$

Since  $\int_{\Omega} |u_0(x) - v_0(x)| dx > 0$ , by (3.3) we can choose a large constant  $M$  such that

$$\begin{aligned} M &\geq \frac{\int_{\Omega} \left| (u - v) b_i(x) a(x)^{-\frac{1}{p}} \right|^{\frac{p}{p-1}} dx}{\int_{\Omega} |u_0(x) - v_0(x)|^{\frac{p}{p-1}} dx} \\ &\geq \frac{\int_{\Omega} \left| (u - v) b_i(x) a(x)^{-\frac{1}{p}} \right|^{\frac{p}{p-1}} dx}{\int_{\Omega} |u(x, t) - v(x, t)|^{\frac{p}{p-1}} dx}, \quad t \in [0, T_1]. \end{aligned} \tag{3.9}$$

By (3.8) and (3.9), we have

$$\left| \int_{\Omega} (u-v)b_i(x)\xi_{\varepsilon}(u-v)_{x_i} dx \right| \leq c \left( \int_{\Omega} |u(x,t) - v(x,t)|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}}, \quad t \leq T_1. \quad (3.10)$$

When  $\frac{p}{p-1} \geq 2$ , we find

$$\left( \int_{\tau}^s \int_{\Omega} |u-v|^{\frac{p}{p-1}} dx dt \right)^{\frac{p-1}{p}} \leq c \left( \int_{\tau}^s \int_{\Omega} |u-v|^2 dx dt \right)^{\frac{p-1}{p}}. \quad (3.11)$$

When  $1 < \frac{p}{p-1} < 2$ , by Hölder's inequality we get

$$\left( \int_{\tau}^s \int_{\Omega} |u-v|^{\frac{p}{p-1}} dx dt \right)^{\frac{p-1}{p}} \leq c \left( \int_{\tau}^s \int_{\Omega} |u-v|^2 dx dt \right)^{\frac{1}{2}}. \quad (3.12)$$

Note that

$$-\iint_{Q_{\tau s}} \xi_{\varepsilon} a(x) \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \nabla(u-v) dx dt \leq 0. \quad (3.13)$$

Using (3.4), (3.5) and (3.10)–(3.13), we have

$$\int_{\Omega} [u(x,s) - v(x,s)]^2 dx \leq \int_{\Omega} [u(x,\tau) - v(x,\tau)]^2 dx + c \left( \int_{\tau}^s \int_{\Omega} |u(x,t) - v(x,t)|^2 dx dt \right)^l,$$

where  $l \leq 1$ .

By virtue of Gronwall's Lemma, we are able to obtain

$$\int_{\Omega} [u(x,s) - v(x,s)]^2 dx \leq c(T_1) \int_{\Omega} [u(x,\tau) - v(x,\tau)]^2 dx.$$

Letting  $\tau \rightarrow 0$ , consequently, we arrive at the desired result.  $\square$

**Proof of Theorem 1.4.** Recalling the proof of Theorem 1.3, to prove Theorem 1.4, we only need to prove (3.10). Using condition (1.19) immediately gives

$$\begin{aligned} & \left| \int_{\Omega} (u-v)b_i(x)\xi_{\varepsilon}(u-v)_{x_i} dx \right| \\ &= \left| \int_{\Omega} (u-v)b_i(x)\xi_{\varepsilon} a(x)^{-\frac{1}{p}} a(x)^{\frac{1}{p}} (u-v)_{x_i} dx \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left( \int_{\Omega} \left| (u-v)b_i(x)a(x)^{-\frac{1}{p}} \right|^{\frac{p}{p-1}} dx \right)^{\frac{p}{p-1}} \left( \int_{\Omega} a(x)(|\nabla u|^p + |\nabla v|^p) dx \right)^{\frac{1}{p}} \\
&\leq \left( \int_{\Omega} |u-v|^{\frac{p}{p-1}} dx \right)^{\frac{p}{p-1}}. \quad \square
\end{aligned}$$

#### 4. Optimal partial boundary condition

In this section, we shall give an explanation on the expression (1.11). Let us briefly review the classical Fichera-Oleinik theory of the second-order linear equations with the nonnegative characteristic values [10,11,18,19]. For a linear degenerate elliptic equation [10,19]

$$\sum_{r,s=1}^{N+1} a^{rs}(x) \frac{\partial^2 u}{\partial x_r \partial x_s} + \sum_{r=1}^{N+1} b_r(x) \frac{\partial u}{\partial x_r} + c(x)u = f(x), \quad x \in \tilde{\Omega} \subset \mathbb{R}^{N+1}, \quad (4.1)$$

the symmetric matrix  $(a^{rs}(x))$  has nonnegative characteristic values. To study the well-posedness problem, we need and only need to give a proper partial boundary condition. Specifically, let  $\{n_s\}$  be the unit inner normal vector of  $\partial\tilde{\Omega}$  and denote

$$\begin{aligned}
\Sigma_2 &= \{x \in \partial\tilde{\Omega} : a^{rs}n_r n_s = 0, (b_r - a_{x_s}^{rs})n_r < 0\}, \\
\Sigma_3 &= \{x \in \partial\tilde{\Omega} : a^{rs}n_s n_r > 0\}.
\end{aligned}$$

Then, to ensure the well-posedness of equation (4.1), the Fichera-Oleinik theory tells us that an appropriate boundary condition is

$$u|_{\Sigma_2 \cup \Sigma_3} = g(x). \quad (4.2)$$

In particular, if the matrix  $((a^{rs}))$  is positive definite, condition (4.2) is just the usual Dirichlet boundary condition. Consider the classical parabolic equation

$$u_t = \sum_{i,j=1}^N a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x, t) \frac{\partial u}{\partial x_i} - c(x, t)u + f(x, t), \quad (4.3)$$

where the matrix  $((a^{ij}))$  is positive definite. In addition to the initial condition:

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (4.4)$$

only a parabolically boundary value condition

$$u(x, t) = g(x, t), \quad (x, t) \in \partial\Omega \times [0, T) \quad (4.5)$$

is imposed.

However, for the nonlinear degenerate parabolic equations, the Oleinik-Fichera theory is invalid. The problem that how to depict the partial boundary value condition (1.4) explicitly becomes more complicated and challenging in the past decades. Some mathematicians have tried to solve the problem by considering the boundary value condition (4.5) in the other sense, see [2,3,13,15–17,24] etc. In this section, we still deal with the boundary value condition (1.4) or (4.5) in the classical sense, i.e. in the trace sense. Next we compare our results with those in the linear case.

To show that the partial boundary value condition (1.4) with the expression (1.14) is optimal, we suppose that  $p = 2$ . Then equation (1.1) becomes

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(x)\nabla u) - \sum_{i=1}^N b_i(x)D_i u + c(x, t)u = f(x, t). \quad (4.6)$$

According to the Fichera-Oleinik theory [10,11,18,19], the optimal boundary value condition is

$$u(x, t) = 0, \quad (x, t) \in \Sigma \times [0, T),$$

with

$$\Sigma = \left\{ x \in \partial\Omega : \sum_{i=1}^N b_i(x)n_i(x) < 0 \right\}, \quad (4.7)$$

where  $\vec{n} = \{n_i\}$  is the inner normal vector of  $\partial\Omega$ . Note that (4.7) is identical to the expression (1.14). Thus, the partial boundary value condition (1.4) with the expression (1.14) is optimal. This fact also implies that the expression (1.22) in Theorem 1.5 may not be the optimal, and we will focus on this case in a subsequent work.

## 5. Stability without any boundary value condition

In this appendix, we will demonstrate how the diffusion coefficient  $a(x)$  and the convection functions  $b_i(x)$  take a joint effect on the solutions of the equation, and present some stability results in the case of  $\int_{\Omega} a(x)^{-\frac{1}{p-1}} dx = \infty$  without any boundary value condition.

When  $a(x) = d(x)^\alpha$ , consider the equation [23]

$$u_t = \operatorname{div}(d^\alpha |\nabla u|^{p-2} \nabla u). \quad (5.1)$$

Clearly, if  $0 < \alpha < p - 1$ , then

$$\int_{\Omega} d(x)^{-\frac{\alpha}{p-1}} dx < \infty. \quad (5.2)$$

The usual boundary value condition

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T) \quad (5.3)$$

can be imposed in the sense of the trace. If  $\alpha \geq p - 1$ , then

$$\int_{\Omega} d(x)^{-\frac{\alpha}{p-1}} dx = \infty.$$

In this case the problem whether we can impose the boundary value condition (5.3) in the sense of trace is still an open problem. Yin-Wang [23] had shown that the uniqueness of weak solution of equation (5.1) is true without the usual boundary value condition. This means that, when  $\int_{\Omega} d(x)^{-\frac{\alpha}{p-1}} dx = \infty$ , the condition  $d(x)^{\alpha}|_{x \in \partial\Omega} = 0$  can replace the usual boundary value condition (5.3).

Comparing with the existing results in the literature, we have three observations.

(i) If  $a(x) = d(x)^{\alpha}$ ,  $\alpha < p - 1$  and  $b_i(x) = 0 = c(x, t) = f(x, t)$ , then

$$\Sigma_p = \emptyset. \quad (5.4)$$

Since condition (1.12) is only used to deal with the transport term  $\sum_{i=1}^N b_i(x) D_i u$ , from Theorem 1.2 we have

**Theorem 5.1.** *Let  $u(x, t)$  and  $v(x, t)$  be two weak solutions of equation (5.1) with the initial value condition (1.3). If  $p > 1$  and  $\alpha < p - 1$ , then the stability in the sense of*

$$\int_{\Omega} |u(x, t) - v(x, t)| dx \leq c(T) \int_{\Omega} |u_0(x) - v_0(x)| dx, \quad t \in [0, T],$$

is true.

Theorem 5.1 shows that even if  $\alpha < p - 1$ , the boundary value condition (5.3) is superfluous. This result improves and generalizes the corresponding result [23, Theorem 1.1].

(ii) If  $a(x) = d(x)^{\alpha}$  and  $\alpha \geq p - 1$ , we have the following result for equation (1.1).

**Theorem 5.2.** *Let  $u(x, t)$  and  $v(x, t)$  be two weak solutions of equation (1.1) with the initial value condition (1.3). If  $p > 1$ ,  $a(x) = d(x)^{\alpha}$  with  $\alpha \geq p - 1$ , and  $b_i(x)$  satisfies*

$$\int_{\Omega} \left| b_i(x) d(x)^{-\frac{\alpha}{p}} \right|^{\frac{p}{p-1}} dx < \infty, \quad i = 1, 2, \dots, N, \quad (5.5)$$

then we have

$$\int_{\Omega} |u(x, t) - v(x, t)| dx \leq c(T) \int_{\Omega} |u_0(x) - v_0(x)| dx, \quad t \in [0, T].$$

As we can see, the transport term  $\sum_{i=1}^N b_i(x) D_i u$  has a crucial effect on the equation. If  $b_i(x) \geq c > 0$ , condition (5.5) means that

$$\int_{\Omega} d(x)^{-\frac{\alpha}{p-1}} dx < \infty,$$

which contradicts the assumption  $\alpha \geq p - 1$ . From condition (5.5) it indicates that there is a comprehensive effect on the diffusion process and the transport process. In other words, as we just mentioned, for the non-Newtonian fluid equation (5.1) when  $\int_{\Omega} a(x)^{-\frac{1}{p-1}} dx = \infty$ , the condition  $d(x)^{\alpha}|_{x \in \partial\Omega} = 0$  can replace the usual boundary value condition (5.3). However, for the non-Newtonian fluid equation with the transport term, whether the condition  $d(x)^{\alpha}|_{x \in \partial\Omega} = 0$  can replace the usual boundary value condition (5.3) or not is still an open and challenging problem. Theorem 5.2 partially provides us a hint to answer this problem.

**Proof of Theorem 5.2.** According to Definition 1.1, if  $g(s) = s$ , for any  $\varphi_1 \in C_0^1(\Omega)$  and  $\varphi_2 \in L^\infty(0, T; W_{loc}^{1,p}(\Omega))$ , we have

$$\begin{aligned} \iint_{Q_T} \varphi_1 \varphi_2 \frac{\partial(u-v)}{\partial t} dx dt &= - \iint_{Q_T} d(x)^\alpha \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \nabla(\varphi_1 \varphi_2) dx dt \\ &\quad - \sum_{i=1}^N \iint_{Q_T} (u-v) [b_{ix_i}(\varphi_1 \varphi_2) + b_i(x)(\varphi_1 \varphi_2)_{x_i}] dx dt \\ &\quad - \iint_{Q_T} (u-v) c(x, t) (\varphi_1 \varphi_2) dx dt. \end{aligned} \quad (5.6)$$

Let  $\Omega_\lambda = \{x \in \Omega : d(x) > \lambda\}$  and the function  $\phi_\lambda(x)$  be the same as given in Section 2. By a process of limit, we choose

$$\varphi_1 = \phi_\lambda(x) \text{ and } \varphi_2 = \chi_{[\tau, s]} S_\eta(u-v)$$

in (5.6). Then it follows that

$$\begin{aligned} &\int_{\Omega} [u(x, s) - v(x, s)] S_\eta(u-v) \phi_\lambda(x) dx \\ &= \int_{\Omega} [u(x, \tau) - v(x, \tau)] S_\eta(u-v) \phi_\lambda(x) dx \\ &\quad - \iint_{Q_{\tau s}} d(x)^\alpha \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \nabla(\phi_\lambda(x) S_\eta(u-v)) dx dt \\ &\quad - \sum_{i=1}^N \iint_{Q_{\tau s}} (u-v) [b_{ix_i} \phi_\lambda(x) S_\eta(u-v) + b_i(x) (\phi_{\lambda x_i} S_\eta(u-v) \\ &\quad \quad + \phi_{\lambda x_i} S'_\eta(u-v)(u-v)_{x_i})] dx dt \end{aligned}$$

$$- \iint_{Q_{\tau s}} (u-v)c(x,t)\phi_\lambda(x)S_\eta(u-v)dxdt, \quad (5.7)$$

where  $Q_{\tau s} = \Omega \times [\tau, s]$ .

A straightforward calculation gives

$$- \iint_{Q_{\tau s}} d(x)^\alpha (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla(u-v)S'_\eta(u-v)\phi_\lambda(x)dxdt \leq 0, \quad (5.8)$$

and

$$\begin{aligned} & \left| - \iint_{Q_{\tau s}} d(x)^\alpha (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla\phi_\lambda(x)S_\eta(u-v)dxdt \right| \\ & \leq \iint_{Q_{\tau s}} |u-v|d(x)^\alpha (|\nabla u|^{p-1} + |\nabla v|^{p-1})|\nabla\phi_\lambda(x)|dxdt \\ & \leq c \left( \int_\tau^s \int_{\Omega \setminus \Omega_\lambda} d(x)^\alpha (|\nabla u|^p + |\nabla v|^p)dxdt \right)^{\frac{p-1}{p}} \left( \int_\tau^s \int_{\Omega \setminus \Omega_\lambda} d(x)^\alpha |\nabla\phi_\lambda|^p dxdt \right)^{\frac{1}{p}} \\ & \leq c \left( \int_\tau^s \int_{\Omega \setminus \Omega_\lambda} d(x)^\alpha (|\nabla u|^p + |\nabla v|^p)dxdt \right)^{\frac{p-1}{p}} \left( \int_\tau^s \int_{\Omega \setminus \Omega_\lambda} d(x)^\alpha \lambda^{-p} dxdt \right)^{\frac{1}{p}}. \quad (5.9) \end{aligned}$$

Due to  $\alpha \geq p-1$ , it is clear that the right-hand side of the above inequality tends to zero as  $\lambda \rightarrow 0$ .

Since  $b_i(x) \in C^1(\overline{\Omega})$ , we get

$$\left| \iint_{Q_{\tau s}} S_\eta(u-v)(u-v)b_{ix_i}\phi_\lambda(x)dxdt \right| \leq \int_\tau^s \int_\Omega |u(x,t) - v(x,t)|dxdt. \quad (5.10)$$

In view of condition (5.5) and  $b_i(x) = 0$  when  $x \in \partial\Omega$ , we have

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \left| \iint_{Q_{\tau s}} S_\eta(u-v)(u-v)b_i(x)\phi_{\lambda x_i}(x)dxdt \right| \\ & \leq c \lim_{\lambda \rightarrow 0} \int_\tau^s \frac{1}{\lambda} \int_{\Omega \setminus \Omega_\lambda} |b_i(x)|dx \\ & = 0. \quad (5.11) \end{aligned}$$



By (2.2) and (5.5)–(5.11), using Lebesgue's dominated convergence theorem yields

$$\begin{aligned} & \lim_{\eta \rightarrow 0} \lim_{\lambda \rightarrow 0} \left| \iint_{Q_{\tau s}} S'_\eta(u-v)(u-v)b_i(x)\phi_\lambda(x)(u-v)_{x_i} dx dt \right| \\ & \leq \lim_{\eta \rightarrow 0} \left( \iint_{Q_{\tau s}} \left| S'_\eta(u-v)(u-v)b_i(x)d(x)^{-\frac{\alpha}{p}} \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left( \iint_{Q_{\tau s}} d(x)^\alpha (|\nabla u|^p + |\nabla v|^p) dx dt \right)^{\frac{1}{p}} \\ & = 0. \end{aligned}$$

Letting  $\eta \rightarrow 0$  and  $\lambda \rightarrow 0$  in (5.6), we derive

$$\int_{\Omega} |u(x, s) - v(x, s)| dx \leq \int_{\Omega} |u(x, \tau) - v(x, \tau)| dx + c \int_{\tau}^s \int_{\Omega} |u(x, t) - v(x, t)| dx dt.$$

It follows from Gronwall's Lemma that

$$\int_{\Omega} |u(x, s) - v(x, s)| dx \leq c(T) \int_{\Omega} |u(x, \tau) - v(x, \tau)| dx.$$

Letting  $\tau \rightarrow 0$ , consequently, we arrive at the desired result.  $\square$

**Corollary 5.3.** *Let  $u(x, t)$  and  $v(x, t)$  be two weak solutions of equation (5.1) with the initial value condition (1.3). If  $p > 1$  and  $\alpha \geq p - 1$ , then we have*

$$\int_{\Omega} |u(x, t) - v(x, t)| dx \leq c(T) \int_{\Omega} |u_0(x) - v_0(x)| dx, \quad t \in [0, T].$$

Corollary 5.3 is identical to [23, Theorem 1.2].

(iii) For the case of  $a(x)$  where  $\int_{\Omega} a(x)^{-\frac{1}{p-1}} dx = \infty$ , we have the following theorem.

**Theorem 5.4.** *Let  $u(x, t)$  and  $v(x, t)$  be two weak solutions of equation (1.1) with the initial value condition (1.3). If  $p > 1$  and*

$$\begin{aligned} & \sum_{i=1}^N a_{x_i}(x)b_i(x) = 0, \quad x \in \partial\Omega, \\ & \int_{\Omega} \left| b_i(x)a(x)^{-\frac{1}{p-1}} \right|^{\frac{p}{p-1}} dx < \infty, \quad i = 1, 2, \dots, N, \end{aligned}$$

then we have

$$\int_{\Omega} |u(x, t) - v(x, t)| dx \leq c(T) \int_{\Omega} |u_0(x) - v_0(x)| dx, \quad t \in [0, T).$$

**Proof of Theorem 5.4.** Let  $\Omega_{1\lambda} = \{x \in \Omega : a(x) > \lambda\}$  and

$$\phi_{1\lambda}(x) = \begin{cases} 1, & \text{if } x \in \Omega_{1\lambda}, \\ \frac{1}{\lambda}a(x), & \text{if } x \in \Omega \setminus \Omega_{1\lambda}. \end{cases}$$

By a process of limit, we can choose  $\phi_{1\lambda}(x)\chi_{[\tau, s]}S_{\eta}(u - v)$  as the test function. The rest of the proof is closely similar to the proof of Theorem 5.2, so we omit the details here.  $\square$

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