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Well-posed and stable transmission problems

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ABSTRACT

We introduce the notion of a transmission problem to describe a general class of problems where different dynamics are coupled in time. Well-posedness and stability are analysed for continuous and discrete problems using both strong and weak formulations, and a general transmission condition is obtained. The theory is applied to the coupling of fluid-acoustic models, multi-grid implementations, adaptive mesh refinements, multi-block formulations and numerical filtering.

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1. Introduction and motivation

In this paper we will introduce a general class of initial-boundary value problems coupled in time, which we refer to as *transmission problems*. This class includes any setting described by the following schematic:

1. The solution u is governed by the dynamics $\mathcal{D}_1$ from time t_1 to time t_2,
2. At t_2, the solution is subject to an operation $v = \mathcal{X}(u)$,
3. At later times, the solution v is governed by the (possibly different) dynamics $\mathcal{D}_2$.

Fig. 1 illustrates the above schematic. Central to this class of problems is the transmission operator $\mathcal{X}$, which we assume admits a matrix representation, but is otherwise left completely general.

Note that we consider a coupling procedure in time rather than space. Spatially coupled problems have been considered e.g. in [1] in the context of multi-physics problems and in [2–4] in the context of general conforming and non-conforming grids, and forms an integral part of finite element, discontinuous Galerkin and flux reconstruction algorithms. Spatially coupled problems typically must obey well-posedness or stability conditions that are strongly dependent on the nature of $\mathcal{D}_1$ and $\mathcal{D}_2$. In this paper, we will show that the temporal coupling involved in the transmission problem is in some sense independent of the dynamics involved, as long as $\mathcal{D}_{1,2}$ define two well-posed problems.

The formulation of the transmission problem is very general and consequently there are many practical applications that fit the framework. Examples considered in this paper include, with continuous time, the coupling in a fluid-acoustics problem, multi-grid techniques and adaptive mesh refinement. With discrete time, we exemplify with multi-block formulations for adaptive time-stepping and numerical filtering.

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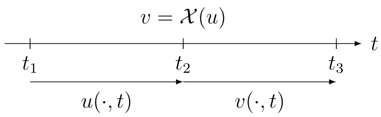

Fig. 1. Schematic of the transmission problem.

The aim of this paper is to obtain conditions for $\mathcal{X}$ under which the solution to the transmission problem is bounded by available data, in particular initial data available at time t_1; a prerequisite for well-posedness. However, boundedness of a solution depends on the norm in which it is estimated. This necessitates certain assumptions on the operators $\mathcal{D}_{1,2}$, and we will therefore confine the analysis to operators that are *semi-bounded* in a generalised L^2-norm [5,6]. In essence, this means that the transmission problem is amenable to analysis via the energy method.

We will consider continuous transmission problems where initial, boundary and coupling conditions are imposed either strongly, or weakly through so called lifting operators [7,8]. It will be shown that energy boundedness is equivalent to a certain condition relating the operator $\mathcal{X}$ and the norms in which the solution is estimated before and after the transmission time t_2. This *transmission condition* turns out to be independent of whether a strong or weak imposition is used. We will also discuss the implications of weighted norms on the transmission conditions and energy estimates.

Semi-discrete and fully discrete transmission problems will also be considered. In the latter cases, we utilise the theory of Summation-by-Parts (SBP) operators, first introduced in [9,10] to provide a means of obtaining stable finite difference procedures for the spatial discretisation of initial-boundary value problems. Since then, the SBP framework has been extended to methods outside the finite difference paradigm, including finite volume methods [11–13], spectral collocation, Galerkin and element methods [14–16], correction procedures via reconstruction [17] and temporal discretisations [18–21]. This opens the door to utilising energy arguments to analyse the stability of fully discrete numerical schemes, if initial, boundary and transmission conditions are imposed weakly [22,23].

It will be shown that all fully discrete transmission problems considered must satisfy a certain condition to obtain an energy estimate. This condition is completely analogous to the one obtained in the continuous setting. Throughout the paper we aim to keep the governing equations – continuous or discrete – as general as possible. Thus, the conditions derived will apply to a wide range of problems and numerical schemes, and may serve as an a priori test for the availability of an energy estimate in a given norm.

The remainder of the paper is structured as follows: In section 2 we introduce the notation and definitions required henceforth. The transmission problem is formally introduced in section 3 and a necessary and sufficient condition for energy boundedness is derived for both strong and weak formulations. In section 4, a discrete transmission problem is presented, and a stability analysis using a weak formulation is performed. We return to the transmission condition in section 5, show how to find weighted norms such that it is satisfied, and discuss its implications. A selection of applications illustrating the preceding theory are presented in section 6. Finally, conclusions are drawn in section 7.

2. Preliminaries

Before proceeding we introduce the necessary notation and definitions.

2.1. Semi-boundedness and well-posedness

Consider the initial-boundary value problem (IBVP)

$$\begin{aligned} u_t + \mathcal{D}(u) &= F, \quad t > 0, \quad x \in \Omega, \\ \mathcal{B}(u) &= g, \quad t \geq 0, \quad x \in \Gamma, \\ u &= f, \quad t = 0, \quad x \in \Omega \cup \Gamma, \end{aligned} \tag{1}$$

where Ω is an open d-dimensional region and $\mathcal{D}$ is a differential operator. The operator $\mathcal{B}$ defines a set of boundary conditions on the boundary Γ of Ω, and the functions F, g and f are given forcing, boundary and initial data.

For two functions u and v defined on Ω, we introduce the inner product and norm

$$(u, v)_P = \int_\Omega u^\top P v \mathrm{d}x, \quad \|u\|_P = (u, u)_P^{1/2}, \tag{2}$$

where P is a positive definite matrix whose dimension matches the number of variables contained in the vectors u and v. We will need the following two definitions [5,6].

Definition 1. Let $\mathbb{V}$ be the space of differentiable functions satisfying the boundary conditions $\mathcal{B}(v) = 0$. The differential operator $\mathcal{D}$ is *semi-bounded* if for all $v \in \mathbb{V}$, $\mathcal{D}(v)$ satisfies the inequality

$$(v, \mathcal{D}(v))_P \geq 0. \tag{3}$$

Definition 2. The differential operator $\mathcal{D}$ is *maximally semi-bounded* if it is semi-bounded in the function space $\mathbb{V}$ but not in any space with fewer boundary conditions.

Note that if $\mathcal{D}$ is semi-bounded (or maximally semi-bounded) in (1), and $F = g = 0$, then we can estimate the solution u since

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_P^2 = 2(u, u_t)_P = -2(u, \mathcal{D}(u))_P \leq 0,$$

which after integration gives

$$\|u(x,t)\|_P \leq \|f(x)\|_P. \tag{4}$$

Definition 3. The IBVP (1) with $F = g = 0$ is *well-posed* if for every sufficiently smooth f that vanishes in a neighbourhood of Γ, and every finite time interval $0 \leq t \leq \tau$, it has a unique smooth solution satisfying the estimate

$$\|u\|_P \leq K_c e^{\alpha_c t}\|f\|_P, \quad 0 \leq t \leq \tau. \tag{5}$$

In (5), K_c and α_c are constants independent of f.

Here, the notation *sufficiently smooth* refers to a function that is smooth enough for (1) to be well defined. Comparing (4) and (5) it is clear that if $\mathcal{D}$ in (1) is maximally semi-bounded, and hence a solution exists, well-posedness follows. In general terms, we say that the solution u *satisfies an energy estimate* if it is bounded in terms of data such as in (5), or in some other way.

Remark 1. The condition for semi-boundedness in Definition 1 may be relaxed to $(v, \mathcal{D}(v))_P \geq -\alpha\|v\|_P^2$ for non-zero data F and g. However, there is no chance of obtaining a semi-bounded problem for non-zero data with $\alpha > 0$ (disregarding zero order terms) unless one is already available for zero data with $\alpha = 0$ [5,6,24]. We therefore adopt Definition 1 in the remainder.

Remark 2. Henceforth, we will be concerned with the coupling of problems of the form (1) at some given time $t = t_2$. Well-posedness of such couplings are independent of F and g, whence for notational brevity, we set $F = g = 0$ in the remainder. We also assume that the boundary operator $\mathcal{B}$ contains a minimal set of boundary conditions such that $\mathcal{D}$ is maximally semi-bounded.

Remark 3. The results in this paper are valid also for non-linear operators $\mathcal{D}(v)$ satisfying condition (3) (the coupling is linear in time). However, for simplicity of presentation, we discuss the problem in the linear setting.

2.2. Strong and weak formulations

Throughout the paper, an IBVP formulated as (1) indicates a strong imposition of the boundary conditions, where the operator $\mathcal{B}$ is defined only on the domain boundary Γ. However, we may also define $\mathcal{B}$ on the whole domain and consider weak formulations of the form

$$\begin{aligned} u_t + \mathcal{D}(u) &= \mathcal{L}(\Sigma_\Gamma \mathcal{B}(u)), && t > 0, \quad x \in \Omega \cup \Gamma, \\ u &= f, && t = 0, \quad x \in \Omega \cup \Gamma. \end{aligned} \tag{6}$$

Here, $\mathcal{L}$ is a *lifting operator* [7,8], which imposes the boundary conditions in a weak sense, and is defined through the relation

$$(u, \mathcal{L}(v))_P = \oint_\Gamma u^\top v \mathrm{d}s.$$

The *penalty matrix* Σ_Γ in (6) will be chosen in such a way that an energy-bound is obtained.

We will sometimes abbreviate the operator $\mathcal{D}(u) - \mathcal{L}(\Sigma_\Gamma \mathcal{B}(u))$ with $\mathcal{D}_\mathcal{L}(u)$. When handling weak boundary conditions, maximal semi-boundedness will refer to $\mathcal{D}_\mathcal{L}(u)$ rather than $\mathcal{D}(u)$. The *energy method* applied to (1) or (6) is

$$\frac{\partial}{\partial t}\|u\|_P^2 = (u, u_t)_P + (u_t, u)_P = -(u, \mathcal{D})_P - (\mathcal{D}, u)_P,$$

followed by integration by parts. Here, $\mathcal{D}$ refers to either $\mathcal{D}(u)$ (strong formulation) or $\mathcal{D}_\mathcal{L}(u)$. The integration by parts procedure produces certain boundary terms. If they are negative semi-definite, $\mathcal{D}$ is maximally semi-bounded and well-posedness follows. If they are positive, there is in general no way to estimate them in terms of $\|u\|_P$ and to obtain well-posedness.

The initial condition may be imposed weakly in the same way as the boundary conditions, by adding another lifting operator to (6). Details of such a problem will be presented in section 3.2.

2.3. Stability

A semi-discretisation of (1) or (6) with $F = g = 0$ is formally given by

$$\begin{aligned} \mathbf{u}_t + D(\mathbf{u}) &= 0, \quad t > 0, \\ \mathbf{u} &= \mathbf{f}, \quad t = 0, \end{aligned} \tag{7}$$

where $\mathbf{u}$ is the solution defined on the d-dimensional grid $\mathbf{x}$. The precise nature of the grid is arbitrary. For convenience, we assume that $\mathbf{x}$ may be defined in terms of the characteristic grid spacing h.

The spatial discretisation $D(\mathbf{u})$ approximates the differential operator $\mathcal{D}$ augmented with the boundary operator $\mathcal{B}$. The grid vector $\mathbf{f}$ is obtained by projecting the data f onto the spatial grid.

We introduce the discrete inner product and norm

$$(\mathbf{u}, \mathbf{v})_h = \mathbf{u}^\top P_x \mathbf{v}, \quad \|\mathbf{u}\|_h = (\mathbf{u}, \mathbf{u})_h^{1/2}, \tag{8}$$

where the positive definite matrix P_x is such that (8) approximates the continuous inner product and norm in (2) on the grid $\mathbf{x}$.

Definition 4. The semi-discretisation (7) is *stable* if for every sufficiently smooth grid vector $\mathbf{f}$, and for every finite time interval $0 \leq t \leq \tau$, the estimate

$$\|\mathbf{u}\|_h \leq K_d e^{\alpha_d t} \|\mathbf{f}\|_h, \quad 0 \leq t \leq \tau \tag{9}$$

holds for each h small enough. The constants K_d and α_d are independent of $\mathbf{f}$ and h.

Here, *smooth grid vector* refers to the projection of a smooth function onto the grid. The grid vector is sufficiently smooth if, for a given discretisation method, (7) can be solved to design order accuracy. Note that if $D(\mathbf{u})$ in (7) is semi-bounded in the inner product (8), then an energy estimate and stability follows in the same way as for the maximally semi-bounded continuous problem.

Remark 4. Since existence is not an issue for discrete problems (consistency and stability suffice), semi-boundedness is the relevant concept.

Finally, for a fully discrete scheme defined on a $(d+1)$-dimensional spatio-temporal grid we introduce a fully discrete analogue of Definition 4:

Definition 5. A full discretisation of the IBVP (1) or (6) is *stable* if the estimate (9) holds at the final time.

3. Well-posedness of transmission problems

In this section we formally define the transmission problem and derive a necessary and sufficient condition for an energy estimate.

3.1. The strong formulation

Consider the following general coupled model problem:

$$\begin{aligned} u_t + \mathcal{D}_1(u) &= 0, & t_1 < t < t_2, \quad & x \in \Omega, \\ v_t + \mathcal{D}_2(v) &= 0, & t_2 < t < t_3, \quad & x \in \Omega, \\ \mathcal{B}_1(u) &= 0, & t_1 \leq t \leq t_2, \quad & x \in \Gamma, \\ \mathcal{B}_2(v) &= 0, & t_2 \leq t \leq t_3, \quad & x \in \Gamma, \\ u &= f_1, & t = t_1, \quad & x \in \Omega \cup \Gamma, \\ v &= \mathcal{X}(u), & t = t_2, \quad & x \in \Omega \cup \Gamma. \end{aligned} \tag{10}$$

We assume that (10) offers a *complete and well-posed description of the underlying dynamics*. By this we mean that there are positive definite matrices $P_{1,2}$ such that the operators $\mathcal{D}_{1,2}$ are maximally semi-bounded in the inner products induced by $P_{1,2}$. Hence, in the case $\mathcal{X}(u) = f_2$, where f_2 is solution-independent data, (10) is well-posed in the sense of Definition 3.

We refer to (10) as the *strong transmission problem*. This terminology is motivated by the fact that (10) describes any scenario where the dynamics governing the solution u is interrupted at time $t = t_2$; u is subject of the operation $\mathcal{X}$; and the resulting information is transmitted to v, after which the governing dynamics may have changed.

Note that the operator $\mathcal{X}$ accepts the vector argument u and must return a vector of the same dimension as v for (10) to make sense. We will generally let $\mathcal{X}$ admit a matrix representation X, such that we may write

$$\mathcal{X}(u) = Xu.$$

Of course, X may still implicitly depend on u through its matrix elements.

Our goal is to investigate when (10) is well-posed, and in particular when sharp energy-bounds can be obtained. The energy method gives $(u, u_t)_{P_1} \leq 0$ and $(v, v_t)_{P_2} \leq 0$. Summing the two inequalities and integrating in time yields

$$\|v(x,t_3)\|^2_{P_2} \leq \|f_1(x)\|^2_{P_1} - \int_{\Omega} u^\top \left\{ P_1 - X^\top P_2 X \right\} u \mathrm{d}x \Big|_{t=t_2}. \tag{11}$$

From (11) we immediately have

Proposition 1. *An energy estimate can be obtained for the strong transmission problem (10) if and only if the transmission condition*

$$P_1 - X^\top P_2 X \geq 0 \tag{12}$$

is satisfied at time $t = t_2$.

Before proceeding it is appropriate to remark that by assumption, $\mathcal{D}_1(u)$ is maximally semi-bounded, whence u satisfies the estimate

$$\|u(x,t_2)\|^2_{P_1} \leq \|f_1(x)\|^2_{P_1}.$$

Consequently we could define new data $f_2 = Xu|_{t=t_2}$ and treat the problem for v in (10) as a stand-alone IBVP. Since $\mathcal{D}_2(v)$ is also maximally semi-bounded, v would satisfy the estimate

$$\|v(x,t_3)\|^2_{P_2} \leq \|f_2(x)\|^2_{P_2}, \tag{13}$$

ultimately yielding a non-sharp estimate for $\|v(x,t_3)\|_{P_2}$.

However, there are several reasons for why this approach is undesirable: Firstly, f_2 is not available a priori and hence it is unknown whether $\|f_2\|_{P_2}$ is large. This is particularly evident for problems with non-zero boundary data g and forcing F, where u will satisfy an estimate [5] of the form

$$\|u(x,t_2)\|^2_{P_1} \leq K e^{\alpha(t_2-t_1)} \left(\|f_1(x)\|^2_{P_1} + \int_{t_1}^{t_2} \left(\|F\|^2_{\tilde{P}_1} + \|g\|^2_{\Gamma} \right) \mathrm{d}t \right). \tag{14}$$

In (14), $\|\cdot\|_{\Gamma}$ is a norm defined on the boundary of the spatial domain. Thus, $\|u(x,t_2)\|_{P_1}$ may be large compared to $\|f_1\|_{P_1}$.

Note also that from Definitions 3, 4 and 5 of well-posedness and stability that the estimates of the solutions are formulated in terms of *initially available data*. Hence, we only label estimates that are obtained in terms of f_1 as *energy estimates* in the remainder of this paper.

Furthermore, introducing f_2 at time t_2 enforces u and v to be obtained sequentially from (10). This property is inherited by any discretisation of (10), which renders parallel implementations impossible. Thus, even if it is possible to compute a numerical solution using f_2, the effect on the efficiency would be detrimental. In light of these considerations, we will not discuss the above formally uncoupled approach further.

3.2. The weak formulation

A weak formulation of (10) is given by

$$\begin{aligned} u_t + \mathcal{D}_{\mathcal{L}_1}(u) &= \mathcal{L}_{t_1}(\Sigma_{t_1}(u - f_1)), \quad t_1 \leq t < t_2, \quad x \in \Omega \cup \Gamma, \\ v_t + \mathcal{D}_{\mathcal{L}_2}(v) &= \mathcal{L}_{t_2}(\Sigma_{t_2}(v - Xu)), \quad t_2 \leq t < t_3, \quad x \in \Omega \cup \Gamma. \end{aligned} \tag{15}$$

We refer to (15) as the *weak transmission problem*. In the event that $\mathcal{X}(u) = Xu = f_2$ is solution-independent data, we assume (in concert with the strong formulation) that (15) forms a well-posed problem. The lifting operators $\mathcal{L}_{t_1}$ and $\mathcal{L}_{t_2}$ impose the initial and transmission conditions weakly and are defined through the relations

$$\int_{t_{1,2}}^{t_{2,3}} (u, \mathcal{L}_{t_{1,2}}(v))_{P_{1,2}} \mathrm{d}t = (u, v)_{P_{1,2}}|_{t=t_{1,2}}.$$

Our goal is to find the conditions under which there exists a penalty matrix Σ_{t_2} such that the solution v of (15) satisfies an energy estimate.

The energy method applied to (15), followed by integration in time, gives

$$\begin{aligned}\|u(x,t_2)\|^2_{P_1} &\le \left\{\|u\|^2_{P_1} + (u, \Sigma_{t_1}(u-f_1))_{P_1} + (\Sigma_{t_1}(u-f_1), u)_{P_1}\right\}_{t=t_1}\\ \|v(x,t_3)\|^2_{P_2} &\le \left\{\|v\|^2_{P_2} + (v, \Sigma_{t_2}(v-Xu))_{P_2} + (\Sigma_{t_2}(v-Xu), v)_{P_2}\right\}_{t=t_2}.\end{aligned} \tag{16}$$

To bound the terms evaluated at $t=t_1$ in (16), we choose $\Sigma_{t_1} = -I_1$, where I_1 is the identity matrix of the same dimension as P_1. Adding and subtracting $\|f_1\|^2_{P_1}$ to these terms gives

$$\begin{aligned}&\left\{\|u\|^2_{P_1} - (u, u-f_1)_{P_1} - (u-f_1, u)_{P_1} \pm \|f_1\|^2_{P_1}\right\}_{t=t_1}\\ &\quad= \int_\Omega \left\{f_1^\top P_1 f_1 - (u-f_1)^\top P_1(u-f_1)\right\}_{t=t_1} \mathrm{d}x\\ &\quad= \left\{\|f_1\|^2_{P_1} - \|(u-f_1)\|^2_{P_1}\right\}_{t=t_1},\end{aligned} \tag{17}$$

which clearly is bounded.

Inserting (17) into (16) and adding the two inequalities results in

$$\begin{aligned}\|v(x,t_3)\|^2_{P_2} \le &\left\{\|f_1\|^2_{P_1} - \|(u-f_1)\|^2_{P_1}\right\}_{t=t_1} + \left\{-\|u\|^2_{P_1} + \|v\|^2_{P_2}\right.\\ &\left. + (v, \Sigma_{t_2}(v-Xu))_{P_2} + (\Sigma_{t_2}(v-Xu))_{P_2}, v\right\}_{t=t_2}.\end{aligned} \tag{18}$$

To obtain an energy estimate for v, we must find a penalty matrix Σ_{t_2} such that the terms evaluated at time $t=t_2$ in (18) are negative semi-definite. The following proposition states when such a matrix can be found.

Proposition 2. *An energy estimate can be obtained for the weak transmission problem (15) if and only if the transmission condition (12) is satisfied at time $t=t_2$.*

Proof. Let T denote the terms in (18) that are evaluated at time $t=t_2$. Then T may be written

$$T = \begin{pmatrix} u \\ v\end{pmatrix}^\top \underbrace{\begin{pmatrix} -P_1 & -\left(P_2\Sigma_{t_2}X\right)^\top \\ -\left(P_2\Sigma_{t_2}X\right) & \left(P_2\Sigma_{t_2}\right) + \left(P_2\Sigma_{t_2}\right)^\top + P_2\end{pmatrix}}_{M} \begin{pmatrix} u \\ v\end{pmatrix}.$$

We begin by proving that (12) is a necessary condition. Thus, assume that $T \le 0$, which implies that the symmetric matrix M is negative semi-definite. Let $S = \left(I_1, X^\top\right)$. By Sylvester's theorem,

$$SMS^\top = -P_1 + X^\top P_2 X \le 0$$

must hold, which is equivalent to (12); hence the condition is necessary.

Next, we show that (12) is a sufficient condition. Assume that (12) is satisfied. We add and subtract $(Xu)^\top P_2(Xu)$ from T and choose $\Sigma_{t_2} = -I_2$, where I_2 is the identity matrix of the same dimension as P_2, to obtain

$$T = \begin{pmatrix} Xu \\ v\end{pmatrix}^\top \begin{pmatrix} -P_2 & P_2 \\ P_2 & -P_2\end{pmatrix} \begin{pmatrix} Xu \\ v\end{pmatrix} - u^\top \left(P_1 - X^T P_2 X\right) u.$$

The rightmost term is negative semi-definite by condition (12). The matrix in the left term can be written

$$\begin{pmatrix} -P_2 & P_2 \\ P_2 & -P_2\end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1\end{pmatrix} \otimes P_2 \equiv B \otimes P_2,$$

where $\otimes$ denotes the Kronecker product. Since P_2 is positive definite and B has eigenvalues $\{0, -2\}$, T is negative semi-definite. Hence (18) is bounded by data and an energy estimate for (15) is obtained. $\square$

Remark 5. The proof of Proposition 2 shows that if (12) is satisfied, the choice $\Sigma_{t_2} = -I_2$ leads to an energy estimate. However, other choices of Σ_{t_2} are also possible.

If condition (12) is satisfied, then the strong and weak implementation yields the same energy estimate, $\|v\|_{P_2} \le \|f_1\|_{P_1}$, since $\|u(x,t_1) - f_1\|^2_{P_1} = 0$ in (18). Thus, (12) is a completely general condition that must be satisfied independent of the way in which the transmission condition is implemented.

4. Stability of discrete transmission problems

In this section we turn our attention to discrete transmission problems. Before proceeding with the stability analyses of semi-discrete and fully discrete problems, we introduce relevant details about SBP operators, which we will subsequently utilise for temporal discretisations.

4.1. SBP in time

An SBP operator may be defined as follows:

Definition 6. A matrix $D = P^{-1}Q$ is an SBP operator of order q if

1. $D\mathbf{x}^m = m\mathbf{x}^{m-1}, \quad m = 0, \dots, q,$
2. $P = P^T > 0,$
3. $Q + Q^\top = \mathrm{diag}(-1, 0, \dots, 0, 1).$

Recently, the SBP framework was extended to more general approximations [14,25] as well as temporal discretisation [18]. As an example of the temporal procedure, consider as a special case of (1), the initial-value problem

$$\begin{aligned} u_t + \lambda u &= 0, \quad 0 < t < T, \\ u &= f, \quad t = 0, \end{aligned} \tag{19}$$

where λ is a complex constant with $\mathrm{Re}(\lambda) \geq 0$.

The energy method applied to (19) (multiplying by the complex conjugated solution $\bar{u}$ and integrating in time) leads to the bound

$$u^2(T) + 2\,\mathrm{Re}(\lambda)\|u\|^2 = f^2, \tag{20}$$

where $\|u\|^2 = \int_0^T |u|^2 \mathrm{d}t$.

Applying SBP in time to (19) yields

$$P^{-1}Q\,\mathbf{u} + \lambda\mathbf{u} = \sigma P^{-1}(u_0 - f)\mathbf{e}_0, \tag{21}$$

where $\mathbf{e}_0 = (1, 0, \dots, 0)^\top$ and σ is a scalar penalty parameter yet to be determined. Here, $\mathbf{u}$ approximates the continuous solution u at each grid point in time. The right-hand side contains the Simultaneous Approximation Term (SAT) [22] that weakly imposes the initial condition. The SAT term is an example of a discrete lifting operator.

Choosing $\sigma = -1$ and applying the *discrete energy method* to (21) (multiplying from the left by $\mathbf{u}^*P$, adding the conjugate transpose and using Definition 6) leads to

$$|u_n|^2 + 2\,\mathrm{Re}(\lambda)\|\mathbf{u}\|_h^2 = |f|^2 - |u_0 - f|^2. \tag{22}$$

In (22), $\|\mathbf{u}\|_h^2 = \mathbf{u}^*P\mathbf{u}$, where $\mathbf{u}^*$ denotes the conjugate transpose of the vector $\mathbf{u}$. Note that (22) mimics (20) up to the small dissipative term $|u_0 - f|^2$, which vanishes with grid refinements.

Clearly, (22) implies $|u_n|^2 \leq |f|^2$, and hence the SBP discretisation (21) is stable in the sense of Definition 5. For further reading about SBP in time, see [19–21]. For comprehensive reviews of the SBP-SAT technique and examples of its use, see [26,27].

4.2. Discrete transmission problems

Energy estimates for semi-discrete transmission problems are essentially covered by the analysis presented in section 3. All that remains is to replace $\mathcal{D}_{1,2}$ by $D_{1,2}$, u and v by $\mathbf{u}$ and $\mathbf{v}$, and f by $\mathbf{f}$ as appropriate. We therefore immediately have

Proposition 3. *An energy estimate can be obtained for the semi-discrete transmission problems (10) and (15) if and only if the transmission condition (12) is satisfied at time $t = t_2$.*

Next, we focus our attention on fully discrete schemes discretised using SBP in time. The corresponding transmission problem is given by

$$\begin{aligned} \left(P_{t,(u)}^{-1}Q_{t,(u)} \otimes I_{x,(u)}\right)\mathbf{u} + \left(I_{t,(u)} \otimes D_1\right)\mathbf{u} &= \left(P_{t,(u)}^{-1} \otimes \Sigma_{h,t_1}\right)\left(\mathbf{e}_{t,(u)} \otimes (\mathbf{u}_0 - \mathbf{f}_1)\right), \\ \left(P_{t,(v)}^{-1}Q_{t,(v)} \otimes I_{x,(v)}\right)\mathbf{v} + \left(I_{t,(v)} \otimes D_2\right)\mathbf{v} &= \left(P_{t,(v)}^{-1} \otimes \Sigma_{h,t_2}\right)\left(\mathbf{e}_{t,(v)} \otimes (\mathbf{v}_0 - X\mathbf{u}_n)\right). \end{aligned} \tag{23}$$

For simplicity of notation, we have in (23) used a notation consistent with linear problems. With a slight abuse of notation, $\mathbf{u}$ and $\mathbf{v}$ denote grid vectors on a $d+1$ dimensional spatio-temporal grid in a similar fashion to (7). We assume that $D_1(\mathbf{u})$ and $D_2(\mathbf{v})$ are semi-bounded in inner products induced by the symmetric positive definite matrices $P_{x,(u,v)}$, such that when $X\mathbf{u}_n = \mathbf{f}_2$ is solution-independent data, (23) is stable in the sense of Definition 5.

In (23), the vectors $\mathbf{e}_{t,(u,v)}$ denote the first column of the corresponding identity matrices $I_{t,(u,v)}$. The right-hand sides contain penalty terms, i.e. discrete lifting operators that weakly enforce the initial and transmission conditions in a way analogous to the right-hand sides of (15). The vectors $\mathbf{u}_n$ and $\mathbf{v}_0$ respectively denote the numerical solution at time t_2 before and after application of X. Finally, Σ_{h,t_1} and Σ_{h,t_2} are penalty matrices yet to be determined.

Multiplying (23) from the left by $\mathbf{u}^\top \left(P_{t,(u)} \otimes P_{x,(u)}\right)$ and $\mathbf{v}^\top \left(P_{t,(v)} \otimes P_{x,(v)}\right)$ as appropriate, adding the transpose of the result, applying Definition 6 and adding the two equations, leads to

$$\begin{aligned}\|\mathbf{v}_m\|^2_{P_{x,(v)}} &= \|\mathbf{f}_1\|^2_{P_{x,(u)}} - \|\mathbf{u}_0 - \mathbf{f}_1\|^2_{P_{x,(u)}} - \|\mathbf{u}_n\|^2_{P_{x,(u)}} + \|\mathbf{v}_0\|^2_{P_{x,(v)}} \\ &\quad + (\mathbf{v}_0, \Sigma_{h,t_2}(\mathbf{v}_0 - X\mathbf{u}_n))_{P_{x,(v)}} + (\Sigma_{h,t_2}(\mathbf{v}_0 - X\mathbf{u}_n), \mathbf{v}_0)_{P_{x,(v)}}.\end{aligned} \tag{24}$$

Here we have used $\Sigma_{h,t_1} = -I_{x,(u)}$ and performed a calculation similar to the one in (17) to obtain the first two terms.

For (23) to be energy stable, we need the right-hand side of (24) to be bounded. This requires finding a penalty matrix Σ_{h,t_2} such that the transmission terms involving $\mathbf{u}_n$ and $\mathbf{v}_0$ are negative semidefinite. The following proposition establishes the conditions under which this is possible:

Proposition 4. *A penalty matrix Σ_{h,t_2} exists such that (23) is stable if and only if the transmission condition (12) holds with $P_1 = P_{x,(u)}$ and $P_2 = P_{x,(v)}$.*

Proof. The estimate (24) is term for term analogous to the continuous energy estimate (18). The proof is therefore identical to that of Proposition 2. □

5. Scaled norms

The transmission condition (12) is completely general and applies to any problem of the form (10), (15) or (23). In this section we show that norm-inducing matrices $P_{1,2}$ may always be found such that (12) is satisfied. We also discuss the implications of this fact.

Let $\kappa > 0$ be a real constant. Our starting point is the observation that if the solution to an IBVP satisfies an energy estimate in the norm $\|\cdot\|_P$, then it also satisfies an estimate in the *scaled norm* $\|\cdot\|_{\kappa P} \equiv \sqrt{\kappa}\|\cdot\|_P$, which is clear from (5). We may thus redo the energy analysis for the strong transmission problem (10) using the scaled norm $\|u\|_{\kappa P_1}$ to obtain the energy rate

$$\|v(x,t_3)\|^2_{P_2} \le \kappa \|f_1(x)\|^2_{P_1} - \int\limits_{\Omega} u^\top \left\{\kappa P_1 - X^\top P_2 X\right\} u \mathrm{d}x\Big|_{t=t_2}. \tag{25}$$

Analogous energy rates hold for the weak and discrete transmission problems. Clearly, the transmission condition (12) is now replaced by the *scaled transmission condition*

$$\kappa P_1 - X^\top P_2 X \ge 0. \tag{26}$$

Next, we investigate if we can find a κ such that (26) is satisfied.

Let $\lambda_{max}(H) = \max_{j\in\{1,\dots,n\}} |\lambda_j(H)|$ denote the spectral radius of an $n\times n$ matrix H. Let $\lambda_{min}(H)$ be defined similarly.

Proposition 5. *The scaled transmission condition (26) is satisfied if*

$$\kappa \ge \frac{\lambda_{max}(X^\top P_2 X)}{\lambda_{min}(P_1)}. \tag{27}$$

The proof is found in Appendix A. Proposition 5 reveals that *as long as the governing equations for the solutions u and v satisfy energy estimates, then so does the corresponding transmission problem*, if the norms $P_{1,2}$ are scaled appropriately. In other words, the temporal coupling of two well-posed (or stable) problems preserves the well-posedness (stability).

However, there might be drawbacks with the scaling procedure described above. Replacing the transmission condition (12) with the scaled condition (26) changes the energy estimate (11) for the strong transmission problem to

$$\|v(x,t_3)\|^2_{P_2} \le \|f_1(x)\|^2_{\kappa P_1} = \kappa\|f_1(x)\|^2_{P_1}. \tag{28}$$

Analogous results hold for the weak and discrete transmission problems. If $\kappa > 1$, the scaled estimate (28) is clearly weaker than the unscaled estimate (11). In a problem involving m transmissions, each requiring scaled norms satisfying

$$\kappa_j P_j - X_j^\top P_{j+1} X_j \geq 0, \quad j = 1, \dots m,$$

at time $t = t_{j+1}$, the final energy estimate becomes

$$\|v(x,t_m)\|^2_{P_{m+1}} \leq \|f_1(x)\|^2_{P_1} \prod_{j=1}^{m} \kappa_j. \tag{29}$$

If each $\kappa_j > 1$, the estimate (29) may of course be very weak. This is an obvious disadvantage in situations where $v(x,t)$ represents an error or a small disturbance, and hence it is generally desirable to choose X such that κ_j becomes as small as possible.

Remark 6. The choice of κ in (27) is not minimal, however it is often simple to calculate. As the proof of Proposition 5 suggests, $\kappa = \lambda_{max}(H)$, where H is given in (A.1) in Appendix A, is an optimal choice. However, calculating $\lambda_{max}(H)$ requires that a spectral decomposition of P_1 is available.

We make a separate note of the important special case where $P_1 = P_2$. In this case, we have the following necessary condition:

Proposition 6. *If $P_1 = P_2$, a necessary condition for the scaled transmission condition (26) to be satisfied is that*

$$\kappa \geq \lambda^2_{max}(X). \tag{30}$$

The proof is found in Appendix B. Verifying (30) is clearly simpler than (27), but it might not be sufficient. In section 6.2.2 we will consider a problem where the choice of κ has a significant impact on the energy estimate.

6. Applications

In this section we describe a selection of applications that can be modelled as transmission problems. In some of these, X can systematically be constructed such that the transmission condition (12) is satisfied, while in others this poses challenges. The examples are chosen so that a minimal number of assumptions on $\mathcal{D}_{1,2}$ or $D_{1,2}$ are made. In what follows, we let the penalty matrix for the initial data be $-I$ such that we do not have to consider the initial conditions further.

6.1. Continuous transmission problems

We start by considering problems in continuous time, and exemplify with a derivation of an energy estimate for a coupled fluid-acoustics problem followed by a multi-grid application and an adaptive mesh refinement problem.

6.1.1. Fluid-acoustics coupling

Consider the linearised one-dimensional Euler equations,

$$\mathbf{u}_t + A\mathbf{u}_x = 0, \quad A = \begin{pmatrix} \bar{u} & \bar{\rho} & 0 \\ 0 & \bar{u} & \frac{1}{\bar{\rho}} \\ 0 & \gamma\bar{p} & \bar{u} \end{pmatrix} \tag{31}$$

augmented with initial and boundary conditions. Here, the elements of the solution vector $\mathbf{u} = (\rho, u, p)^\top$ denote small perturbations of density, velocity and pressure around a constant background flow $(\bar{\rho}, \bar{u}, \bar{p})^\top$ and γ is the ratio of specific heats.

Under the assumption of an irrotational flow, the Euler equations may be simplified to the acoustic wave equation,

$$\mathbf{v}_t + B\mathbf{v}_x = 0, \quad B = \begin{pmatrix} 0 & \bar{c} \\ \bar{c} & 0 \end{pmatrix} \tag{32}$$

where $\mathbf{v} = (p/\bar{\rho}, \bar{c}v)^\top$ and $\bar{c} = \sqrt{\gamma\bar{p}/\bar{\rho}}$ is the speed of sound in the fluid.

Consider a situation where we solve (31) up to time $t = t_2$, after which we switch to solve the simpler problem (32). Such a situation may arise when computational resources are limited, and where it is known that the solution approaches an irrotational state at $t = t_2$.

It is reasonable to impose that the pressure and velocity should not change during the switch. This corresponds to the transmission condition

$$\mathbf{v} = X\mathbf{u}, \quad X = \begin{pmatrix} 0 & 0 & \frac{1}{\bar{\rho}} \\ 0 & \bar{c} & 0 \end{pmatrix}. \tag{33}$$

It remains to find norms in which (31) and (32) can be estimated. If we can find a matrix S such that upon multiplying (31) by S, the resulting system

$$(S\mathbf{u})_t + SAS^{-1}(S\mathbf{u})_x = 0$$

is symmetric, then the corresponding matrix inducing the proper norm is $P = S^\top S$. An analogous argument holds for (32).

A matrix that symmetries the Euler equations is [28]

$$S_1 = \begin{pmatrix} \frac{1}{\beta\bar{\rho}} & 0 & -\frac{1}{\beta\bar{\rho}\bar{c}^2} \\ 0 & \frac{1}{2\bar{c}} & \frac{1}{2\bar{\rho}\bar{c}^2} \\ 0 & -\frac{1}{2\bar{c}} & \frac{1}{2\bar{\rho}\bar{c}^2} \end{pmatrix} \Rightarrow P_1 = \begin{pmatrix} \frac{1}{(\beta\bar{\rho})^2} & 0 & -\frac{1}{(\beta\bar{\rho}\bar{c})^2} \\ 0 & \frac{1}{2\bar{c}^2} & 0 \\ -\frac{1}{(\beta\bar{\rho}\bar{c})^2} & 0 & \frac{2+\beta^2}{2(\beta\bar{\rho}\bar{c}^2)^2} \end{pmatrix},$$

where $\beta = \sqrt{2(\gamma-1)}$. Similarly, for the acoustic wave equation we use

$$S_2 = \frac{1}{2\bar{c}^2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow P_2 = \frac{1}{2\bar{c}^4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the operator X defined in (33), the transmission condition (12) becomes

$$0 \le P_1 - X^\top P_2 X = \begin{pmatrix} \frac{1}{(\beta\bar{\rho})^2} & 0 & -\frac{1}{(\beta\bar{\rho}\bar{c})^2} \\ 0 & 0 & 0 \\ -\frac{1}{(\beta\bar{\rho}\bar{c})^2} & 0 & \frac{1}{(\beta\bar{\rho}\bar{c}^2)^2} \end{pmatrix}.$$

The matrix on the right-hand side has the eigenvalues $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = (1+\bar{c}^4)/(\beta\bar{\rho}\bar{c}^2)^2 > 0$. Thus, the transmission condition (12) is satisfied and an energy estimate can be obtained.

6.1.2. Multi-grid iterations

We want to solve the following discretised boundary-value problem:

$$D_1\mathbf{w} = \mathbf{F},$$

augmented with suitable boundary conditions such that D_1 has eigenvalues with positive real parts. A possible approach is to introduce a pseudo-time derivative and solve the corresponding problem

$$\begin{aligned} \mathbf{u}_t + D_1\mathbf{u} &= \mathbf{F}, \quad t > 0, \\ \mathbf{u} &= \mathbf{f}, \quad t = 0, \end{aligned} \tag{34}$$

by marching in time until a steady state is reached. Here, $\mathbf{f}$ is arbitrary data.

Convergence in (34) may be fast initially, but often stagnates after some time, say t_2. To accelerate the convergence we may use a multi-grid technique as follows: We introduce the residual $\mathbf{r} = D_1^{-1}\mathbf{F} - \mathbf{u}$ and define a residual equation

$$D_2\mathbf{y} = I_r D_1 \mathbf{r},$$

where D_2 is a *coarse grid operator* obtained by restricting D_1 to a coarser mesh using the restriction operator I_r. Assuming that the residual equation can be solved exactly, we obtain $\mathbf{y} = D_2^{-1} I_r (\mathbf{F} - D_1\mathbf{u})$.

At this stage, we return to the fine grid problem (34) by applying a prolongation operator I_p to $\mathbf{y}$, and solve

$$\begin{aligned} \mathbf{v}_t + D_1\mathbf{v} &= \mathbf{F}, & t &> t_2, \\ \mathbf{v} &= \mathbf{u} + I_p D_2^{-1} I_r(\mathbf{F} - D_1\mathbf{u}), & t &= t_2. \end{aligned} \tag{35}$$

The above procedure may of course be performed in several cycles using multiple grid levels obtained by repeated applications of the restriction and prolongation operators.

If $\mathbf{F} = 0$, then (34) and (35) combine to form the following semi-discrete transmission problem:

$$\begin{aligned} \mathbf{u}_t + D_1\mathbf{u} &= -\mathcal{L}_{t_1}(\mathbf{u} - \mathbf{f}), \\ \mathbf{v}_t + D_1\mathbf{v} &= \mathcal{L}_{t_2}\left(\Sigma_{t_2}(\mathbf{v} - (I_x - Y)\mathbf{u})\right), \end{aligned} \tag{36}$$

where $Y = I_p D_2^{-1} I_r D_1$.

To obtain an energy estimate, condition (12) implies that we must have

$$0 \le P_x - (I_x - Y)^\top P_x (I_x - Y),$$

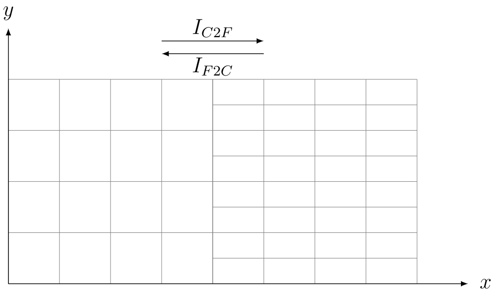

Fig. 2. Interpolation procedure on a pair of 2D non-conforming grids.

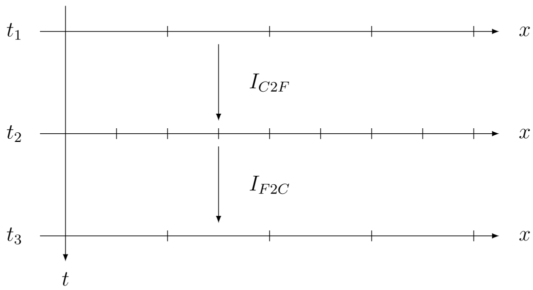

Fig. 3. 1D adaptive mesh refinement.

for some positive definite matrix P_x. Since Y is problem dependent, this condition has to be checked on a case to case basis. However, Proposition 6 gives the necessary condition $\kappa = 1 \geq \lambda^2_{max}(I_x - Y)$. Clearly this implies that the real part $\Re[\lambda_j(I_x - Y)] \leq 1$ for every j, from which it follows that

$$\Re[\lambda_j(Y)] \geq 0, \quad \forall j. \tag{37}$$

Consequently, (37) is necessary in order to obtain an energy estimate for the implementation (36).

Remark 7. If the so called *Galerkin condition* $D_2 = I_r D_1 I_p$ is satisfied it can be shown [29] that Y has eigenvalues equal to 0 or 1, and hence (37) is satisfied. For details about multi-grid methods in the SBP setting, see [30].

6.1.3. Adaptive mesh refinement

Adaptive mesh refinement [31] has become a standard technique in large scale scientific computing and consequently, the problem of interpolating between computational domains has attracted much attention. In [3], so called SBP preserving interpolation operators were introduced that successfully achieve stable and accurate interpolation between non-conforming grids. The interpolation procedure in [3] is not adaptive, but rather assumes that two fixed and non-conforming domains define the grid at all times. The setting is illustrated in 2D in Fig. 2 where I_{C2F} and I_{F2C} respectively denote interpolation operators from a coarser to a finer grid, and vice versa.

The interpolation operators are said to be SBP preserving if the following property holds:

$$P_C I_{F2C} = I_{C2F}^{\top} P_F. \tag{38}$$

Here, P_C and P_F are norm-inducing matrices defined on the coarse and fine grid respectively. Further, for hyperbolic or parabolic problems with characteristic interface conditions, it was shown in [3] that the following conditions must be satisfied together with (38) for stability:

$$P_C\,(I_C - I_{F2C} I_{C2F}) \geq 0, \quad P_F\,(I_F - I_{C2F} I_{F2C}) \geq 0. \tag{39}$$

In (39), I_C and I_F are identity matrices on the coarse and fine grids respectively.

Here, we will use the SBP preserving interpolation operators defined by (38) and (39) and show that an energy-bounded transmission implementation can be obtained in the adaptive setting. For simplicity, we consider the 1D problem of interpolating from a coarse to a fine grid. Fig. 3 illustrates the setup together with the converse problem of interpolating back to the coarse grid. Let $\mathbf{u}$ be a discrete solution vector defined on a coarse grid and let $\mathbf{v}$ similarly be defined on a finer grid. The transmission problem corresponding to interpolation between the grids is given by

$$\begin{aligned} \mathbf{u}_t + D_1(\mathbf{u}) &= -\mathcal{L}_{t_1}(\mathbf{u} - \mathbf{f}), & t_1 < t < t_2, \\ \mathbf{v}_t + D_2(\mathbf{v}) &= \mathcal{L}_{t_2}(\Sigma_{t_2}(\mathbf{v} - I_{C2F}\mathbf{u})), & t_2 < t < t_3, \end{aligned} \tag{40}$$

where we have ignored boundary conditions for simplicity. The transmission condition (12) becomes

$$P_C - I_{C2F}^{\top} P_F I_{C2F} \geq 0. \tag{41}$$

Observe from (38) that $I_{F2C} = P_C^{-1} I_{C2F}^{\top} P_F$. Then, (41) can be rewritten

$$\begin{aligned} 0 \leq P_C - I_{C2F}^{\top} P_F I_{C2F} &= P_C \left(I_C - P_C^{-1} I_{C2F}^{\top} P_F I_{C2F} \right) \\ &= P_C \left(I_C - I_{F2C} I_{C2F} \right), \end{aligned}$$

which is precisely the first condition in (39). In the converse case, where we interpolate from a fine to a coarse grid, an analogous equivalence is obtained between the second condition in (39) and (41) with $\mathcal{I}_{C2F}$ replaced by $\mathcal{I}_{F2C}$. From Proposition 3 it follows that a penalty matrix Σ_{t_2} can be found such that (40) is energy stable for SBP preserving interpolation operators satisfying (38) and (39).

As an example of the temporal interpolation discussed above, consider a second order accurate case, where a coarse, uniform one-dimensional grid consisting of four grid points, and a similar but finer seven point grid is used. Let the step size be $h = 2$ on the coarse grid and $h = 1$ on the finer one. The second order norm-inducing matrices P_F and P_C are [9]

$$P_C = 2\,\mathrm{diag}(1/2, 1, 1, 1/2), \quad P_F = \mathrm{diag}(1/2, 1, 1, 1, 1, 1, 1/2),$$

and the SBP preserving interpolation operator becomes [30]

$$\mathcal{I}_{C2F} = \frac{1}{2} \begin{pmatrix} 2 & & & \\ 1 & 1 & & \\ & 2 & & \\ & 1 & 1 & \\ & & 2 & \\ & & 1 & 1 \\ & & & 2 \end{pmatrix}.$$

Then, the eigenvalues of $P_C - I_{C2F}^{\top} P_F I_{C2F}$ to four decimal places are given by $\{0.5955, 0.6047, 1.1545, 1.3953\}$, and hence (41) is clearly satisfied.

6.2. Discrete transmission problems

Next, we consider problems in discrete time, and exemplify with an adaptive time-stepping method and a filtering procedure.

6.2.1. Multi-block time-stepping

The simplest and most straightforward discrete transmission problem is that of a multi-block formulation in time. In such a formulation, the time interval is divided into several small blocks, which often is beneficial from a computational point of view [21,32,33].

Multi-block formulations correspond to the transmission problem (23) with $D_1 = D_2 \equiv D$ and $X = I_x$,

$$\begin{aligned} \left(P_{t,(u)}^{-1} Q_{t,(u)} \otimes I_x \right) \mathbf{u} + \left(I_{t,(u)} \otimes D \right) \mathbf{u} &= -\left(P_{t,(u)}^{-1} \otimes I_x \right) \left(\mathbf{e}_{t,(u)} \otimes (\mathbf{u}_0 - \mathbf{f}) \right), \\ \left(P_{t,(v)}^{-1} Q_{t,(v)} \otimes I_x \right) \mathbf{v} + \left(I_{t,(v)} \otimes D \right) \mathbf{v} &= \left(P_{t,(v)}^{-1} \otimes \Sigma_{h,t_2} \right) \left(\mathbf{e}_{t,(v)} \otimes (\mathbf{v}_0 - \mathbf{u}_n) \right). \end{aligned}$$

The transmission condition (12) simply reduces to $0 \leq P_x - P_x = 0$, i.e. it is trivially satisfied. Hence, a penalty matrix exists that leads to stability. Note that there is no requirement on the time-blocks to be of similar size. This technique thus allows for a provably stable adaptive time-stepping procedure.

6.2.2. Explicit filters

Errors are inevitably present in numerical simulations, even when the computations are well resolved, and are predominantly introduced at high wavenumbers. To see this, we may plot the *numerical* wavenumber ξ_n associated with a given derivative approximation, against the *analytic* wavenumber ξ associated with the actual derivative. This is done in Fig. 4 for central finite difference stencils of various orders. Evidently, the *dispersion error* $\xi - \xi_n$ grows as the wavenumber increases and, in fact, this growth is monotonic [34]. A plethora of finite difference stencils have been presented with reduced dispersion error for various ranges of wavenumbers; see [35] for a review and [36] for examples within the SBP-SAT framework. Yet, no difference stencil is accurate in the vicinity of $\xi = \pi$. Instead, filters designed to remove high wavenumbers from the computational domain may be applied.

Here, we will restrict our attention to the finite difference-type filters introduced in [37]. They take the form

$$F = \left(I_x + \alpha_F D_x^{(2n)} \right),$$

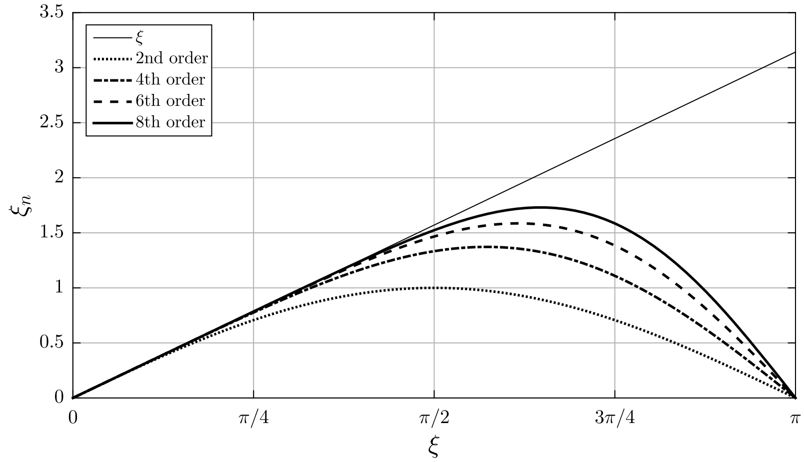

Fig. 4. Numerical wavenumbers associated with central difference stencils.

where $D_x^{(2n)}$ is an *undivided* (i.e. independent of the grid resolution) difference operator approximating the $2n$th derivative. The scalar $\alpha_F = (-1)^n 2^{-2n}$ ensures that the π-mode is removed. A candidate implementation of such a filter is

$$\begin{aligned} \left(P_t^{-1} Q_t \otimes I_x\right) \mathbf{u} + (I_t \otimes D)\,\mathbf{u} &= -\left(P_t^{-1} \otimes I_x\right)(\mathbf{e}_t \otimes (\mathbf{u}_0 - \mathbf{f}))\,, \\ \left(P_t^{-1} Q_t \otimes I_x\right) \mathbf{v} + (I_t \otimes D)\,\mathbf{v} &= \left(P_t^{-1} \otimes \Sigma_{h,t_2}\right)(\mathbf{e}_t \otimes (\mathbf{v}_0 - F\mathbf{u}_n))\,. \end{aligned} \tag{42}$$

Here, the filter is applied after a given number of time steps determined by the dimension of the matrix P_t. Naturally this process is repeated at regular intervals.

With the implementation (42), the condition (12) becomes

$$P_x - F^\top P_x F \geq 0. \tag{43}$$

It is easy to find examples of P_x and F where (43) is not satisfied. For simplicity, let the step size $h = 1$. Then, the common choices [9,37]

$$P_x = \begin{pmatrix} \frac{1}{2} & & & \\ & 1 & & \\ & & 1 & \\ & & & \frac{1}{2} \end{pmatrix}, \quad F = \frac{1}{4}\begin{pmatrix} 3 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 2 & 1 \\ & & 1 & 3 \end{pmatrix}$$

result in $P_x - F^\top P_x F$ having eigenvalues $\{0.9375, 0.5890, 0.1250, -0.0265\}$, and hence an energy estimate cannot be obtained. We must therefore rescale the norm used to estimate the first equation in (42), and from (26) we obtain the scaled transmission condition

$$\kappa P_x - F^\top P_x F \geq 0.$$

By Proposition 5 we may chose

$$\kappa = \frac{\lambda_{max}(F^\top P_x F)}{\lambda_{min}(P_x)} = 1.5531,$$

which yields eigenvalues $\{1.4826, 1.0813, 0.4095, 0.3108\}$ for $\kappa P_x - F^\top P_x F$, and an energy estimate is viable.

However, recall from (29) that if the filter is applied at m regular intervals, the resulting energy estimate becomes

$$\|\mathbf{v}\|_{P_x}^2 \leq \kappa^m \|\mathbf{f}\|_{P_x}^2.$$

Already for $m = 11$ we have $\kappa^m > 100$, which is much too weak for most applications. The minimal value of κ that yields an energy estimate is $\kappa = 1.0411$, for which $\kappa^m > 100$ with $m = 115$. For long-time simulations, this may still be too weak.

Remark 8. This example illustrates that successful filtering may include a delicate balance between the need to remove high frequency oscillations (filter often) and the need to avoid possible growth (filter seldom).

Remark 9. Artificial dissipation operators are akin to filters applied at each time step and thus become an integral part of the spatial discretisation. See [38] for artificial dissipation operators in the SBP-SAT framework.

7. Summary and conclusions

In this paper, we have introduced a general class of problems, referred to as transmission problems, describing the transmission of information between time-dependent problems governed by possibly different dynamics. No specific assumptions about the nature of the problems have been made, apart from them being maximally semi-bounded and well posed. A necessary and sufficient condition for energy boundedness has been obtained, which relates the transmission operator $\mathcal{X}$ to the norms in which the energy estimates are obtained before and after the time of transmission. This transmission condition is independent of whether a strong or weak formulation is used.

It has further been shown that the transmission condition can always be satisfied if scaled norms are used. However, the choice of norms has a non-negligible impact on the resulting energy estimate, and it is therefore desirable to chose optimal transmission operators.

Summation-by-Parts discretisations in time with a weak imposition of transmission conditions through SAT terms have been used to model discrete transmission problems. A necessary and sufficient condition for energy stability, analogous to the one obtained for the continuous problem, has been obtained. No assumptions about the spatial discretisations was made, apart from them being semi-bounded and stable. Thus, the presented results are general and applies to any problem with prescribed norms.

Transmission problems encompass many important problems as special cases; the list presented in this work is certainly not exhaustive. Here we have attempted to include examples with a wide spectrum of applications that, when possible, are independent of the underlying dynamics. Continuous transmission problems typically arise whenever one set of governing equations is replaced by another after some time. We have illustrated this by coupling the linearised Euler equations with the acoustic wave equation, and shown that with an appropriate choice of norms, an energy estimate is obtained.

Further, a multi-grid implementation in connection to dual time-stepping has been considered. We cannot make claims about the stability of such implementations without specific knowledge about the underlying problem and its stability properties. Nevertheless, we have obtained a necessary condition for an energy estimate, which depends only on the eigenvalues of the multi-grid update matrix, and is simple to verify.

By using SBP preserving interpolation, it is possible to find a penalty matrix and construct an energy-bounded adaptive mesh refinement procedure, which may have a significant impact on the performance of the scheme. It has also been shown that SBP preserving operators in combination with the transmission condition lead to the stability conditions required when imposing characteristic interface conditions at non-conforming interfaces.

Multi-block formulations in time trivially satisfy the condition necessary for a stable implementation. This may serve as a basis for an adaptive time-stepping method if SBP in time is used for temporal discretisations.

Finally, we have considered a numerical filter for which scaled norms must be used to obtain an energy estimate. It has been shown that even the sharpest possible energy bound becomes very weak as the number of filtrations grow. This indicates that successful filtering may include a delicate balance between the need to remove high frequency oscillations (filter often) and the need to avoid possible growth (filter seldom).

Acknowledgements

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Appendix A. Proof of Proposition 5

Proof. Let $R_\kappa = \kappa P_1 - X^\top P_2 X$ and let $\mathbf{y}$ be a real-valued vector. We must show that the quadratic form $\mathbf{y}^\top R_\kappa \mathbf{y} \geq 0$ for any choice of $\mathbf{y}$.

Recall that P_1 is symmetric positive definite. Hence, there is an orthogonal matrix U such that $P_1 = U^\top \Lambda U$, where Λ is diagonal positive definite. Further, we may uniquely define the matrix $\Lambda^{1/2}$ as the square root of Λ. Consequently, we have

$$\begin{aligned}
\mathbf{y}^\top R_\kappa \mathbf{y} &= \mathbf{y}^\top(\kappa P_1 - X^\top P_2 X)\mathbf{y} \\
&= (\Lambda^{1/2}U\mathbf{y})^\top(\kappa I_1 - \Lambda^{-1/2}UX^\top P_2 X U^\top \Lambda^{-1/2})(\Lambda^{1/2}U\mathbf{y}) \\
&= \mathbf{z}^\top(\kappa I_1 - \underbrace{\Lambda^{-1/2}UX^\top P_2 X U^\top \Lambda^{-1/2}}_{H})\mathbf{z},
\end{aligned} \tag{A.1}$$

where $\mathbf{z} = (\Lambda^{1/2}U\mathbf{y})$. Note that H is symmetric positive semi-definite, and hence $\lambda_{max}(H)$ coincides with the spectral norm of H;

$$\lambda_{max}(H) = \|H\|_2 = \sup_{\|\mathbf{x}\|=1} \|H\mathbf{x}\| = \sqrt{\lambda_{max}(H^\top H)},$$

where $\|\cdot\|$ without subscripts denotes the Euclidean vector norm. Clearly it suffices to choose $\kappa \geq \lambda_{max}(H)$ in order for the quadratic form (A.1) to be positive semi-definite. But

$$\lambda_{max}(H) = \|H\|_2 \le \|\Lambda^{-1/2}\|_2 \|U^\top\|_2 \|X^\top P_2 X\|_2 \|U\|_2 \|\Lambda^{-1/2}\|_2$$
$$= \frac{\|X^\top P_2 X\|_2}{\lambda_{min}(P_1)} = \frac{\lambda_{max}(X^\top P_2 X)}{\lambda_{min}(P_1)},$$

whence the proposition follows. □

Appendix B. Proof of Proposition 6

Proof. Let $P_1 = P_2 \equiv P$ and assume that (26) holds. Since P is positive definite it has a uniquely defined, positive definite square root, $P^{1/2}$. Multiplying (26) from the left and right by $P^{-1/2}$ gives by Sylvester's theorem

$$0 \le \kappa I - (P^{-1/2} X^\top P^{1/2})(P^{1/2} X P^{-1/2}) = \kappa I - \hat{X}^\top \hat{X}, \tag{B.1}$$

where $\hat{X} = P^{1/2} X P^{-1/2}$. Clearly (B.1) implies that $\kappa \ge \lambda_{max}(\hat{X}^\top \hat{X}) \ge \lambda_{max}^2(\hat{X})$ (see e.g. [39]). However, by similarity, the eigenvalues of X are the same as those of $\hat{X}$, whence the proposition follows. □

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