Journal of Computational Physics 299 (2015) 196–214

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Spectral direction splitting methods for two-dimensional space fractional diffusion equations ☆

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ARTICLE INFO

Article history:
Received 20 January 2014
Received in revised form 3 May 2015
Accepted 1 July 2015
Available online 13 July 2015

Keywords:
Fractional diffusion equation
Spectral method
Direction splitting method

ABSTRACT

A numerical method for a kind of time-dependent two-dimensional two-sided space fractional diffusion equations is developed in this paper. The proposed method combines a time scheme based on direction splitting approaches and a spectral method for the spatial discretization. The direction splitting approach renders the underlying two-dimensional equation into a set of one-dimensional space fractional diffusion equations at each time step. Then these one-dimensional equations are solved by using the spectral method based on weak formulations. A time error estimate is derived for the semi-discrete solution, and the unconditional stability of the fully discretized scheme is proved. Some numerical examples are presented to validate the proposed method.

1. Introduction

The fractional partial differential equations are now winning more and more scientific applications across a variety of fields including control theory, biology, electrochemical processes, porous media, viscoelastic materials, polymer, finance, etc. The universality of anomalous diffusion phenomenon in various experiments has led to an intensive investigation of these equations in recent years. The fractional diffusion equation considered in this paper is of interest not only in its own right, but also in that it constitutes the principal part in solving many other more general fractional differential equations. We refer, e.g., to [19,20] for modeling chaotic dynamics charge transport in amorphous semiconductors, [18] for nuclear magnetic resonance diffusometry in disordered materials, and [15] for modeling the propagation of mechanical diffusive wave in viscoelastic media.

There have been a number of numerical methods constructed for the time-fractional diffusion equations; see, e.g., [13] for a finite difference scheme in time and spectral method in space, [34] for a particle tracking approach, [10] for a time–space spectral method, [35] for an alternating direction implicit scheme, [22] for finite difference schemes for a variable-order equation, [9] for a finite element method, and [32] for a spectral method using Jacobi polyfractonomials for fractional ODEs.

On the other hand, the space-fractional diffusion equations have also been a subject of many investigations. Among the existing numerical methods for this kind of fractional diffusion equations, we mention the finite difference methods based on the shifted Grüwald formulae in [16,17,23], spline approximations [21], the finite difference methods for Riesz fractional derivatives [14,31], the spectral method based on weak formulation [11], a finite element method for the space and

☆ This research is partially supported by NSF of China (Grant numbers 11471274, 11421110001, and 91130002).
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http://dx.doi.org/10.1016/j.jcp.2015.07.011
0021-9991/

time fractional Fokker–Planck equation [4], a Runge–Kutta discontinuous Galerkin methods for one- and two-dimensional fractional diffusion equations [8], a finite difference/element method for a two-dimensional modified fractional diffusion equation [33], and a method combining the alternating direction implicit method and the Crank–Nicolson scheme [3]. Wang and Wang [26] developed, without giving a stability analysis, an alternating direction implicit finite difference method for space-fractional diffusion equations.

In this paper we aim at designing an efficient method for solving the space fractional diffusion equation. The proposed method combines a stable direction splitting scheme with a spectral discretization in space that allows for efficient implementation. This work was motivated by the attempt to take double advantages of the spectral method and the direction splitting approach. Firstly, the fractional diffusion equation is featured by the presence of non-local operators involved in the definition of fractional derivatives. These non-local operators make any approximation, either low or high order methods, into non-sparse linear system. This nature obviously reduces the advantage of low order methods in term of computational complexity, and favours the use of high order methods if the solution to be approximated is smooth enough. It is well known that, as compared to low order methods, higher order methods like spectral methods require less degrees of freedom to achieve the same accuracy. This consideration has inspired a recent series of papers [10–12,32], which focused on developing spectral methods for some time/space fractional differential equations. It is worthy to mention that Wang et al. [28,29] showed that a fractional equation with smooth data can have non-smooth solutions. Hence, how to guarantee the smoothness of the solution is a difficult issue. Secondly, despite of the efficiency of the spectral method, the numerical solution of the fractional diffusion equation in high dimension requires more numerical techniques. Direction splitting methods are considered as powerful techniques which allow to split the underlying high dimensional problem into a set of one-dimensional sub-problems, thus can considerably reduce the computational complexity for some traditional equations; see, e.g., [2,7]. Note that Wang et al. [27,25] constructed and analyzed finite difference/ADI methods for fractional diffusion equations with variable coefficients. Their methods have also been shown to be fast with efficient storage.

The main purpose of this paper is to develop a stable direction splitting scheme in time with a spectral discretization in space for the space fractional diffusion equation. The stability of the overall scheme is rigorously established. Although such a combination has been constructed and analysed for a number of traditional equations, it's extension to problems with fractional operators does not seem to be trivial.

The outline of this paper is as follows. In the next section we describe the underlying problem, and construct the direction splitting scheme. A splitting error estimate is derived. In Section 3, we propose the full discrete scheme by using a spectral method for the spatial discretization of the fractional differential operators, and carry out a detailed analysis for the stability of the proposed scheme. The unconditional stability is proved under an assumption on the diffusion coefficients. We give in Section 4 some implementation details and present the numerical results to verify the stability and accuracy of the method. Finally, we give some concluding remarks in Section 5.

2. Direction splitting scheme

We consider the following two-dimensional space fractional diffusion equation:

$$\frac{\partial u(x,y,t)}{\partial t} = Lu(x,y,t) + f(x,y,t), \tag{2.1}$$

where $t \in (0,T]$, $(x,y) \in \Omega = \Lambda^2$, $\Lambda = (-1,1)$, $f(x,y,t)$ is a source function. L is the fractional operator defined by

$$Lu(x,y,t) = p\big(D_x^{\alpha} u(x,y,t) + {}_xD^{\alpha} u(x,y,t)\big) + q\big(D_y^{\beta} u(x,y,t) + {}_yD^{\beta} u(x,y,t)\big), \tag{2.2}$$

with p and q being positive diffusion coefficients, and the fractional derivatives of order α or β with $1 < \alpha, \beta < 2$ being defined in the Riemann–Liouville sense as follows:

$$D_x^{\gamma}\varphi(x) = \frac{1}{\Gamma(2-\gamma)}\frac{d^2}{dx^2}\int_{-1}^{x}\frac{\varphi(\xi)d\xi}{(x-\xi)^{\gamma-1}}, \ \forall x \in \Lambda, \gamma = \alpha, \beta, \tag{2.3}$$

$${}_xD^{\gamma}\varphi(x) = \frac{1}{\Gamma(2-\gamma)}\frac{d^2}{dx^2}\int_{x}^{1}\frac{\varphi(\xi)d\xi}{(\xi-x)^{\gamma-1}}, \ \forall x \in \Lambda, \gamma = \alpha, \beta. \tag{2.4}$$

Usually D_x^{γ} is called the left-sided fractional derivative, and ${}_xD^{\gamma}$ the right-sided fractional derivative of order γ.

The equation (2.1) is subject to the following initial and boundary conditions:

$$u(x,y,0) = u_0(x,y), \ \forall (x,y) \in \Omega, \tag{2.5}$$

$$u(x,y,t)|_{\partial\Omega} = 0, \ \forall t \in (0,T]. \tag{2.6}$$

The sum of terms $D_x^{\alpha} u(x,y,t) + {}_xD^{\alpha} u(x,y,t)$ in (2.2) is sometimes denoted by $D_{|x|}^{\alpha} u(x,y,t)$, called symmetrized fractional derivative. It has been shown in [11] that the existence and uniqueness of a solution to (2.1)–(2.5)–(2.6) can be guaranteed by keeping only one derivative term in x-direction and one derivative term in y-direction in the right-hand side of (2.2). However in the present work we are only able to construct a stable scheme for the equation with symmetrized fractional operators. In other words, the stability analysis carried out in this paper works only for the symmetrized operators.

2.1. The direction splitting scheme

We propose the following direction splitting scheme.

Let L be the number of the time steps, $\Delta t = T/L$. Set $u^0 = u_0(x,y)$. For $n = 0, 1, \dots, L-1$, we look for u^{n+1} as follows:

- Predictor for u^{n+1}. We compute ξ^{n+1} by

$$\frac{\xi^{n+1} - u^n}{\Delta t} = pD^{\alpha}_{|x|}u^n + qD^{\beta}_{|y|}u^n + f^{n+\frac{1}{2}}, \tag{2.7}$$

where $f^{n+\frac{1}{2}} = f(t_{n+\frac{1}{2}})$. Hereafter, for simplifying the notation, we use $D^{\alpha}_{|z|}\varphi$ to denote $D^{\alpha}_{z}\varphi + {}_zD^{\alpha}\varphi$, with z being x or y.

- Direction splitting. u^{n+1} is obtained by solving two sets of 1D-problems:

$$\begin{aligned} &\frac{\eta^{n+1} - \xi^{n+1}}{\frac{1}{2}\Delta t} = pD^{\alpha}_{|x|}(\eta^{n+1} - u^n), \ \ \eta^{n+1}|_{x=\pm 1} = 0, \ \ \forall y \in \Lambda, \\ &\frac{u^{n+1} - \eta^{n+1}}{\frac{1}{2}\Delta t} = qD^{\beta}_{|y|}(u^{n+1} - u^n), \ \ u^{n+1}|_{y=\pm 1} = 0, \ \ \forall x \in \Lambda. \end{aligned} \tag{2.8}$$

It is observed that each of the substeps in (2.8) consists of a set of one-dimensional fractional diffusion problems, which can be solved by any existing efficient method. The next subsection will be devoted to derive an error estimate for the splitting scheme (2.7)–(2.8). The spatial discretization of the above problems will be addressed in the next section.

2.2. Splitting error analysis

Note that the splitting scheme (2.7)–(2.8) is a straightforward extension of its counterpart for the traditional diffusion equation [5,30]. Now it is interesting to see whether the traditional analysis method equally applies to the fractional equation. In fact we are going to see that a similar error estimate can be equally derived for the fractional equation by using some newly established properties.

To carry out the error analysis for (2.7)–(2.8), we denote the splitting error by e^n:

$$e^n := u(\cdot,\cdot,t_n) - u^n,$$

where $u(\cdot,\cdot,t_n)$ is the exact solution of the problem (2.1) at $t = t_n$, and u^n is the solution of the semi-discrete problem (2.7)–(2.8).

We first introduce some notation. Let $\mathcal{D}$ be a domain in $\mathbb{R}$ or $\mathbb{R}^2$, we denote by $L^2(\mathcal{D})$, $H^{\gamma}(\mathcal{D})$, and $H^{\gamma}_0(\mathcal{D})$ the usual Sobolev spaces, where γ is any positive real number. The inner product of $L^2(\mathcal{D})$ will be denoted by $(\cdot,\cdot)_{\mathcal{D}}$. We will also make use of the following results, which can be found in [10].

Lemma 2.1. *(See [10].) If $1 < \gamma < 2$, $w, v \in H^{\gamma}_0(\Lambda)$, then*

$$\left(D^{\gamma}_z w(z),\ v(z)\right)_{\Lambda} = \left(D^{\frac{\gamma}{2}}_z w(z),\ {}_zD^{\frac{\gamma}{2}} v(z)\right)_{\Lambda}, \tag{2.9}$$

$$\left({}_zD^{\gamma} w(z),\ v(z)\right)_{\Lambda} = \left({}_zD^{\frac{\gamma}{2}} w(z),\ D^{\frac{\gamma}{2}}_z v(z)\right)_{\Lambda}. \tag{2.10}$$

Lemma 2.2. *(See [10].) For any real $s > 0$, $s \neq n + \frac{1}{2}$ with n being a non-negative integer, and for all $v \in H^s_0(\Lambda)$, we have*

$$\frac{(D^s_z v(z), {}_zD^s v(z))_{\Lambda}}{\cos(s\pi)} \cong \|D^s_z v\|^2_{L^2(\Lambda)} \cong \|{}_zD^s v\|^2_{L^2(\Lambda)}. \tag{2.11}$$

Next we extend some results obtained in one dimension — e.g. see [6,10] to the two-dimensional case. These extensions will be key to the proof of the stability of the direction splitting scheme. For any positive real numbers s and σ, we define the space

$$H^{s,\sigma}_{ll}(\Omega) := \{v;\ \|v\|_{H^{s,\sigma}_{ll}(\Omega)} < \infty\},$$

equipped with the norm

$$\|v\|_{H^{s,\sigma}_{ll}(\Omega)} := \left(\|v\|^2_{L^2(\Omega)} + |v|^2_{H^{s,\sigma}_{ll}(\Omega)}\right)^{1/2}, \quad |v|_{H^{s,\sigma}_{ll}(\Omega)} := \|D^s_x D^{\sigma}_y v\|_{L^2(\Omega)},$$

and

$$H^{s,\sigma}_{lr}(\Omega) := \{v; \|v\|_{H^{s,\sigma}_{lr}(\Omega)} < \infty\},$$

equipped with

$$\|v\|_{H^{s,\sigma}_{lr}(\Omega)} := \left(\|v\|^2_{L^2(\Omega)} + |v|^2_{H^{s,\sigma}_{lr}(\Omega)}\right)^{\frac{1}{2}}, \quad |v|_{H^{s,\sigma}_{lr}(\Omega)} := \|D^s_x\, {}_yD^\sigma v\|_{L^2(\Omega)}.$$

Similarly we define

$$H^{s,\sigma}_{rl}(\Omega) := \{v; \|v\|_{H^{s,\sigma}_{rl}(\Omega)} < \infty\},$$

equipped with the norm

$$\|v\|_{H^{s,\sigma}_{rl}(\Omega)} := \left(\|v\|^2_{L^2(\Omega)} + |v|^2_{H^{s,\sigma}_{rl}(\Omega)}\right)^{1/2}, \quad |v|_{H^{s,\sigma}_{rl}(\Omega)} := \|{}_xD^s D^\sigma_y v\|_{L^2(\Omega)},$$

and

$$H^{s,\sigma}_{rr}(\Omega) := \{v; \|v\|_{H^{s,\sigma}_{rr}(\Omega)} < \infty\},$$

equipped with

$$\|v\|_{H^{s,\sigma}_{rr}(\Omega)} := \left(\|v\|^2_{L^2(\Omega)} + |v|^2_{H^{s,\sigma}_{rr}(\Omega)}\right)^{1/2}, \quad |v|_{H^{s,\sigma}_{rr}(\Omega)} := \|{}_xD^s\, {}_yD^\sigma v\|_{L^2(\Omega)}.$$

In the above space definitions, the subscript "l" or "r" has the implication that the "left" or "right" derivative has been used in the norm definitions. Let $H^{s,\sigma}_{ll,0}(\Omega)$, $H^{s,\sigma}_{lr,0}(\Omega)$, $H^{s,\sigma}_{rl,0}(\Omega)$, and $H^{s,\sigma}_{rr,0}(\Omega)$ be the closures of $C^\infty_0(\Omega)$ with respect to the norms $\|\cdot\|_{H^{s,\sigma}_{ll}(\Omega)}$, $\|\cdot\|_{H^{s,\sigma}_{lr}(\Omega)}$, $\|\cdot\|_{H^{s,\sigma}_{rl}(\Omega)}$ and $\|\cdot\|_{H^{s,\sigma}_{rr}(\Omega)}$ respectively. Now we define the semi-norm

$$|v|_{H^{s,\sigma}(\Omega)} := \left\|(\mathrm{i}\omega)^s(\mathrm{i}\eta)^\sigma \mathcal{F}(v)\right\|_{L^2(\mathbb{R}^2)},$$

and norm

$$\|v\|_{H^{s,\sigma}(\Omega)} := \left(\|v\|^2_{L^2(\Omega)} + |v|^2_{H^{s,\sigma}(\Omega)}\right)^{1/2},$$

where i is such that $\mathrm{i}^2 = -1$, $\mathcal{F}(v)$ denotes the Fourier transform of $\tilde{v}$, which is the extension of v by zero outside of Ω, with Fourier variables ω and η. Also, let $H^{s,\sigma}_0(\Omega)$ denote the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_{H^{s,\sigma}(\Omega)}$.

Lemma 2.3. *For $s, \sigma > 0$, $s \neq n + 1/2$, $\sigma \neq m + 1/2$ with m and n being integers, the semi-norms $|\cdot|_{H^{s,\sigma}_{ll,0}(\Omega)}$, $|\cdot|_{H^{s,\sigma}_{lr,0}(\Omega)}$, $|\cdot|_{H^{s,\sigma}_{rl,0}(\Omega)}$, and $|\cdot|_{H^{s,\sigma}_{rr,0}(\Omega)}$ are all equivalent in the space $C^\infty_0(\Omega)$.*

Proof. For all $v \in C^\infty_0(\Omega)$, we have

$$D^\sigma_y v(\cdot, y) \in H^s_0(\Lambda), \; {}_yD^\sigma v(\cdot, y) \in H^s_0(\Lambda), \; \forall y \in \Lambda. \tag{2.12}$$

Hence from Lemma 2.2 we obtain

$$\|D^s_x D^\sigma_y v(x, y)\|_{L^2(\Lambda)} \cong \|{}_xD^s D^\sigma_y v(x, y)\|_{L^2(\Lambda)}, \; \forall y \in \Lambda, \tag{2.13}$$

$$\|D^s_x\, {}_yD^\sigma v(x, y)\|_{L^2(\Lambda)} \cong \|{}_xD^s\, {}_yD^\sigma v(x, y)\|_{L^2(\Lambda)}, \; \forall y \in \Lambda. \tag{2.14}$$

Furthermore integrating (2.13) and (2.14) in the y-direction leads to

$$\|D^s_x D^\sigma_y v(x, y)\|_{L^2(\Omega)} \cong \|{}_xD^s D^\sigma_y v(x, y)\|_{L^2(\Omega)}, \tag{2.15}$$

$$\|D^s_{x\,y} D^\sigma v(x, y)\|_{L^2(\Omega)} \cong \|{}_xD^s\, {}_yD^\sigma v(x, y)\|_{L^2(\Omega)}. \tag{2.16}$$

This proves that the semi-norm $|\cdot|_{H^{s,\sigma}_{ll,0}(\Omega)}$ is equivalent to $|\cdot|_{H^{s,\sigma}_{rl,0}(\Omega)}$, and the norm $|\cdot|_{H^{s,\sigma}_{lr,0}(\Omega)}$ is equivalent to $|\cdot|_{H^{s,\sigma}_{rr,0}(\Omega)}$ in the space $C^\infty_0(\Omega)$.

Similarly it can be proved that the semi-norms $|\cdot|_{H^{s,\sigma}_{ll,0}(\Omega)}$ and $|\cdot|_{H^{s,\sigma}_{lr,0}(\Omega)}$ are equivalent.

This completes the proof. □

Next we establish some useful results related to the whole plan. For the sake of simplification, we still use the notations to $D^\gamma_x v(x)$ and ${}_xD^\gamma v(x)$ to denote ${}_{-\infty}D^\gamma_x v(x)$ and ${}_xD^\gamma_\infty v(x)$ respectively.

Lemma 2.4. *If* $\frac{1}{2} < s, \sigma < 1$, *and* $s+\sigma > \frac{3}{2}$, $v \in C_0^\infty(\mathbb{R}^2)$, *then it holds*

$$\left(D_x^s D_y^\sigma v, {}_xD^s{}_yD^\sigma v\right)_{\mathbb{R}^2} \cong \|D_x^s D_y^\sigma v\|^2_{L^2(\mathbb{R}^2)}, \tag{2.17}$$

$$\left(D_x^s{}_yD^\sigma v, {}_xD^s D_y^\sigma v\right)_{\mathbb{R}^2} \cong \|D_x^s{}_yD^\sigma v\|^2_{L^2(\mathbb{R}^2)}, \tag{2.18}$$

$$\|v\|_{H_{ll}^{s,\sigma}(\mathbb{R}^2)} \cong \|v\|_{H_{lr}^{s,\sigma}(\mathbb{R}^2)} \cong \|v\|_{H_{rr}^{s,\sigma}(\mathbb{R}^2)} \cong \|v\|_{H^{s,\sigma}(\mathbb{R}^2)}. \tag{2.19}$$

Proof. First the following equality is well known:

$$\int_{\mathbb{R}^2} uv\,dxdy = \int_{\mathbb{R}^2} \hat{u}\bar{\hat{v}}\,d\omega d\eta, \ \forall u, v \in C_0^\infty(\mathbb{R}^2), \tag{2.20}$$

where $\hat{v}$ is the Fourier transform of v, $\bar{v}$ denotes the complex conjugate of v. Furthermore we have (see, e.g., [6]):

$$\overline{(\mathrm{i}\omega)^s} = \begin{cases} \exp(-\mathrm{i}\pi s)\overline{(-\mathrm{i}\omega)^s} & \text{if } \omega \ge 0, \\ \exp(\mathrm{i}\pi s)\overline{(-\mathrm{i}\omega)^s} & \text{if } \omega < 0. \end{cases} \tag{2.21}$$

Then by using (2.20) and (2.21), we obtain

$$\begin{aligned} &\left(D_x^s D_y^\sigma v, {}_xD^s{}_yD^\sigma v\right)_{\mathbb{R}^2} = \left((\mathrm{i}\omega)^s(\mathrm{i}\eta)^\sigma \hat{v}, \overline{(-\mathrm{i}\omega)^s(-\mathrm{i}\eta)^\sigma \hat{v}}\right)_{\mathbb{R}^2} \\ = &\int_{-\infty}^{0}\int_{-\infty}^{0} (\mathrm{i}\omega)^s(\mathrm{i}\eta)^\sigma \hat{v}\overline{(-\mathrm{i}\omega)^s(-\mathrm{i}\eta)^\sigma \hat{v}}d\omega d\eta + \int_{-\infty}^{0}\int_{0}^{\infty} (\mathrm{i}\omega)^s(\mathrm{i}\eta)^\sigma \hat{v}\overline{(-\mathrm{i}\omega)^s(-\mathrm{i}\eta)^\sigma \hat{v}}d\omega d\eta \\ &+ \int_{0}^{\infty}\int_{-\infty}^{0} (\mathrm{i}\omega)^s(\mathrm{i}\eta)^\sigma \hat{v}\overline{(-\mathrm{i}\omega)^s(-\mathrm{i}\eta)^\sigma \hat{v}}d\omega d\eta + \int_{0}^{\infty}\int_{0}^{\infty} (\mathrm{i}\omega)^s(\mathrm{i}\eta)^\sigma \hat{v}\overline{(-\mathrm{i}\omega)^s(-\mathrm{i}\eta)^\sigma \hat{v}}d\omega d\eta. \end{aligned}$$

It follows from (2.21) that

$$\begin{aligned} &\left(D_x^s D_y^\sigma v, {}_xD^s{}_yD^\sigma v\right)_{\mathbb{R}^2} \\ = &\int_{-\infty}^{0}\int_{-\infty}^{0} \exp(-\mathrm{i}s\pi)\exp(-\mathrm{i}\sigma\pi) f(\omega,\eta,\hat{v})d\omega d\eta + \int_{-\infty}^{0}\int_{0}^{\infty} \exp(\mathrm{i}s\pi)\exp(-\mathrm{i}\sigma\pi) f(\omega,\eta,\hat{v})d\omega d\eta \\ &+ \int_{0}^{\infty}\int_{-\infty}^{0} \exp(-\mathrm{i}s\pi)\exp(\mathrm{i}\sigma\pi) f(\omega,\eta,\hat{v})d\omega d\eta + \int_{0}^{\infty}\int_{0}^{\infty} \exp(\mathrm{i}\sigma\pi)\exp(\mathrm{i}s\pi) f(\omega,\eta,\hat{v})d\omega d\eta \\ = &\ I_r + \mathrm{i}I_i, \end{aligned} \tag{2.22}$$

where

$$f(\omega,\eta,\hat{v}) = (\mathrm{i}\omega)^s(\mathrm{i}\eta)^\sigma \hat{v}\ \overline{(\mathrm{i}\omega)^s(\mathrm{i}\eta)^\sigma \hat{v}},$$

$$\begin{aligned} I_r = &\cos((s+\sigma)\pi)\int_{-\infty}^{0}\int_{-\infty}^{0} f(\omega,\eta,\hat{v})d\omega d\eta + \cos((s-\sigma)\pi)\int_{-\infty}^{0}\int_{0}^{\infty} f(\omega,\eta,\hat{v})d\omega d\eta \\ &+ \cos((s-\sigma)\pi)\int_{0}^{\infty}\int_{-\infty}^{0} f(\omega,\eta,\hat{v})d\omega d\eta + \cos((s+\sigma)\pi)\int_{0}^{\infty}\int_{0}^{\infty} f(\omega,\eta,\hat{v})d\omega d\eta, \end{aligned}$$

and

$$\begin{aligned} I_i = &-\sin((s+\sigma)\pi)\int_{-\infty}^{0}\int_{-\infty}^{0} f(\omega,\eta,\hat{v})d\omega d\eta + \sin((s-\sigma)\pi)\int_{-\infty}^{0}\int_{0}^{\infty} f(\omega,\eta,\hat{v})d\omega d\eta \\ &+ \sin((\sigma-s)\pi)\int_{0}^{\infty}\int_{-\infty}^{0} f(\omega,\eta,\hat{v})d\omega d\eta + \sin((s+\sigma)\pi)\int_{0}^{\infty}\int_{0}^{\infty} f(\omega,\eta,\hat{v})d\omega d\eta. \end{aligned}$$

For a real function $v(x, y)$, it is known that $\overline{\hat{v}(\omega, \eta)} = \hat{v}(-\omega, -\eta)$, and thus we have

$$\int_{-\infty}^{0}\int_{-\infty}^{0} f(\omega,\eta,\hat{v})d\omega d\eta = \int_{0}^{\infty}\int_{0}^{\infty} f(\omega,\eta,\hat{v})d\omega d\eta, \tag{2.23}$$

$$\int_{-\infty}^{0}\int_{0}^{\infty} f(\omega,\eta,\hat{v})d\omega d\eta = \int_{0}^{\infty}\int_{-\infty}^{0} f(\omega,\eta,\hat{v})d\omega d\eta. \tag{2.24}$$

Furthermore, under the conditions $\frac{3}{2} < s+\sigma < 2$ and $0 < |s-\sigma| < \frac{1}{2}$ we have $\cos((s+\sigma)\pi) > 0$, $\cos((s-\sigma)\pi) > 0$. Therefore we obtain

$$c_1\left(D_x^s D_y^\sigma v, D_x^s D_y^\sigma v\right)_{\mathbb{R}^2} \le I_r \le c_2\left(D_x^s D_y^\sigma v, D_x^s D_y^\sigma v\right)_{\mathbb{R}^2}, \quad I_i = 0,$$

where $c_1 = \min\{\cos((s+\sigma)\pi), \cos((s-\sigma)\pi)\}$ and $c_2 = \max\{\cos((s+\sigma)\pi), \cos((s-\sigma)\pi)\}$. This proves (2.17). Similarly we can prove (2.18). The equivalence (2.19) is a direction extension of the one-dimensional result of Theorem 2.1 in [6]. □

The fractional derivative $D_x^s D_y^\sigma v$ can be generalized for all v in $L^2(\Omega)$ in the following way: for $v \in L^2(\Omega)$, we define the linear functional, denoted still by $D_x^s D_y^\sigma v : C_0^\infty(\Omega) \to \mathbb{R}$, through

$$D_x^s D_y^\sigma v(\phi) := \int_\Omega v\, {}_x D^s{}_y D^\sigma \phi dxdy, \ \forall \phi \in C_0^\infty(\Omega). \tag{2.25}$$

Then it can be verified that $D_x^s D_y^\sigma v(\phi)$ is continuous in $C_0^\infty(\Omega)$. In fact, for all $\phi_j \in C_0^\infty(\Omega)$, such that

$$\left\|\partial_x^n \partial_y^m \phi_j\right\|_{L^\infty(\Omega)} \to 0, \ \ \forall n, m \in Z, \ \text{as } j \to \infty,$$

we have

$$\begin{aligned}
|D_x^s D_y^\sigma v(\phi_j)| &= \left|\int_{-1}^{1}\int_{-1}^{1} v\, {}_x D^s{}_y D^\sigma \phi_j dxdy\right| \le \|v\|_{L^2(\Omega)} \left\|{}_x D^s{}_y D^\sigma \phi_j\right\|_{L^2(\Omega)} \\
&= \|v\|_{L^2(\Omega)} \left\| \frac{1}{\Gamma(n-s)\Gamma(m-\sigma)} \partial_x^n \partial_y^m \int_x^1\int_y^1 \phi_j(\tau,\eta) \frac{1}{(\tau-x)^{s-n+1}(\eta-y)^{\sigma-m+1}} d\tau d\eta \right\|_{L^2(\Omega)} \\
&= \|v\|_{L^2(\Omega)} \left\| \frac{1}{\Gamma(n-s)\Gamma(m-\sigma)} \int_x^1\int_y^1 \partial_\tau^n \partial_\eta^m \phi_j(\tau,\eta) \frac{1}{(\tau-x)^{s-n+1}(\eta-y)^{\sigma-m+1}} d\tau d\eta \right\|_{L^2(\Omega)} \\
&\lesssim \|v\|_{L^2(\Omega)} \left\|\partial_x^n \partial_y^m \phi_j(x,y)\right\|_\infty \left\| \int_x^1\int_y^1 \frac{1}{(\tau-x)^{s-n+1}(\eta-y)^{\sigma-m+1}} d\tau d\eta \right\|_{L^2(\Omega)} \\
&\lesssim \|v\|_{L^2(\Omega)} \left\|\partial_x^n \partial_y^m \phi_j(x,y)\right\|_\infty \|(1-x)^{n-s}(1-y)^{m-\sigma}\|_{L^2(\Omega)} \to 0, \ \text{as } j \to \infty,
\end{aligned}$$

where n and m are the integers such that $n-1 \le s < n$, $m-1 \le \sigma < m$. Thus $D_x^s D_y^\sigma(\cdot)$ defined in (2.25) is a distribution, which coincides with the composite left-sided Riemann–Liouville derivative if v belongs to $H_{ll}^{s,\sigma}(\Omega)$. The operators ${}_x D^s{}_y D^\sigma(\cdot)$, ${}_x D^s D_y^\sigma(\cdot)$, etc., can be generalized in a similar way to the space $L^2(\Omega)$. In what follows by default the fractional derivative is always defined in the distribution sense.

Lemma 2.5. *For all positive real numbers s and σ, the spaces $H_{ll}^{s,\sigma}(\Omega)$, $H_{lr}^{s,\sigma}(\Omega)$, $H_{rl}^{s,\sigma}(\Omega)$ and $H_{rr}^{s,\sigma}(\Omega)$ are complete.*

Proof. We only give a proof for $H_{ll}^{s,\sigma}(\Omega)$, the same lines apply to the other three spaces. Let v_n be a Cauchy sequence under norm $\|\cdot\|_{H_{ll}^{s,\sigma}(\Omega)}$. Then, by the completeness of the space $L^2(\Omega)$, there exist $v \in L^2(\Omega)$ and $w \in L^2(\Omega)$ such that

$$v_n \to v, \ \ D_x^s D_y^\sigma v_n \to w \ \text{ in } L^2(\Omega). \tag{2.26}$$

Next we want to prove that $D_x^s D_y^\sigma v = w$.

On one hand, by (2.26), we have

$$\int_\Omega D_x^s D_y^\sigma v_n \phi dxdy \to \int_\Omega w\phi dxdy, \ \forall \phi \in C_0^\infty(\Omega). \tag{2.27}$$

On the other hand, from Lemma 2.4 in [11] and (2.26) we have

$$\int\limits_{\Omega} D_x^s D_y^\sigma v_n \phi dxdy = \int\limits_{\Omega} v_{n\,x}D^s{}_y D^\sigma \phi dxdy \to \int\limits_{\Omega} v_{x}D^s{}_y D^\sigma \phi dxdy,\ \forall \phi \in C_0^\infty(\Omega). \tag{2.28}$$

Furthermore from (2.25) we obtain

$$\int\limits_{\Omega} D_x^s D_y^\sigma v_n \phi dxdy \to D_x^s D_y^\sigma v(\phi),\ \forall \phi \in C_0^\infty(\Omega). \tag{2.29}$$

Then by combining (2.27) and (2.29), we get $D_x^s D_y^\sigma v = w$. The proof is completed. □

Lemma 2.6. *Let s and σ be real number such that $\frac{1}{2} < s, \sigma < 1$, $s + \sigma > \frac{3}{2}$. Then for all $v \in H_0^{s,\sigma}(\Omega)$, it holds*

$$(D_x^s D_y^\sigma v, {}_xD^s{}_y D^\sigma v)_\Omega \cong \|D_x^s D_y^\sigma v\|^2_{L^2(\Omega)}, \tag{2.30}$$

$$(D_x^s{}_y D^\sigma v, {}_xD^s D_y^\sigma v)_\Omega \cong \|D_x^s{}_y D^\sigma v\|^2_{L^2(\Omega)}. \tag{2.31}$$

Proof. We only prove (2.30). By following a standard density argument it suffices to prove it for all $v \in C_0^\infty(\Omega)$. For all $v \in C_0^\infty(\Omega)$, let $\tilde{v}$ be the extension of v by zero outside Ω. Then we have

$$\operatorname{supp}\left(D_x^s D_y^\sigma \tilde{v}\ {}_xD^s{}_y D^\sigma \tilde{v}\right) \subseteq \Omega.$$

In fact, it can be directly verified that

$$\operatorname{supp} D_x^s D_y^\sigma \tilde{v} \subseteq (-1,\infty)^2, \quad \operatorname{supp}{}_xD^s{}_y D^\sigma \tilde{v} \subseteq (-\infty, 1)^2.$$

Thus

$$\operatorname{supp} D_x^s D_y^\sigma \tilde{v} \cap \operatorname{supp}{}_xD^s{}_y D^\sigma \tilde{v} \subseteq \Omega,$$

and hence

$$(D_x^s D_y^\sigma v, {}_xD^s{}_y D^\sigma v)_\Omega = (D_x^s D_y^\sigma \tilde{v}, {}_xD^s{}_y D^\sigma \tilde{v})_{\mathbb{R}^2}.$$

Using the above identity, together with Lemma 2.4 and the Hölder inequality, we obtain

$$|v|^2_{H_{ll}^{s,\sigma}(\Omega)} \le |\tilde{v}|^2_{H_{ll}^{s,\sigma}(\mathbb{R}^2)} \cong \left(D_x^s D_y^\sigma \tilde{v}, {}_xD^s{}_y D^\sigma \tilde{v}\right)_{\mathbb{R}^2} = (D_x^s D_y^\sigma v, {}_xD^s{}_y D^\sigma v)_\Omega \le |v|_{H_{ll}^{s,\sigma}(\Omega)} |v|_{H_{rr}^{s,\sigma}(\Omega)}.$$

From Lemma 2.3, we know that the semi-norms $|\cdot|_{H_{ll}^{s,\sigma}(\Omega)}$ and $|\cdot|_{H_{rr}^{s,\sigma}(\Omega)}$ are equivalent. Thus we have

$$(D_x^s D_y^\sigma v, {}_xD^s{}_y D^\sigma v)_\Omega \cong \|D_x^s D_y^r v\|^2_{L^2(\Omega)}.$$

This proves (2.30). □

We are now in a position to derive the splitting error estimate.

Theorem 2.1. *If $1 < \alpha, \beta < 2$ and $\alpha + \beta > 3$, and the exact solution u is smooth enough, then we have the following error estimate:*

$$\|e^n\|_{L^2(\Omega)} \le c\Delta t^2,\ n = 1, \cdots, L, \tag{2.32}$$

where c is a constant depending only on T, p, q, and the exact solution u.

Proof. By combining the different steps in (2.7)–(2.8), we obtain, for all $n = 0, 1, \ldots, L-1$,

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}\left(pD^\alpha_{|x|} + qD^\beta_{|y|}\right)(u^{n+1} + u^n) - \frac{pq}{4}\Delta t(D^\alpha_{|x|} D^\beta_{|y|})(u^{n+1} - u^n) + f^{n+\frac{1}{2}}. \tag{2.33}$$

Applying the Taylor expansion to the equation (2.1) yields

$$\frac{u(t_{n+1}) - u(t_n)}{\Delta t} = \frac{1}{2}\left(pD^\alpha_{|x|} + qD^\beta_{|y|}\right)\left(u(t_{n+1}) + u(t_n)\right) + f(t_{n+\frac{1}{2}}) + O(\Delta t^2). \tag{2.34}$$

Then we deduce from subtracting (2.33) from (2.34)

$$\begin{aligned} \frac{e^{n+1} - e^n}{\Delta t} &= \frac{1}{2}\left(pD^\alpha_{|x|} + qD^\beta_{|y|}\right)\left(e^{n+1} + e^n\right) - \frac{pq}{4}\Delta t\left(D^\alpha_{|x|} D^\beta_{|y|}\right)\left(e^{n+1} - e^n\right) \\ &\quad + \frac{pq}{4}\Delta t\left(D^\alpha_{|x|} D^\beta_{|y|}\right)(u(t_{n+1}) - u(t_n)) + O(\Delta t^2). \end{aligned}$$

Noticing that the third term in the right-hand side is of order $O(\Delta t^2)$, we obtain

$$e^{n+1}-e^n=\frac{1}{2}\Delta t\big(pD^{\alpha}_{|x|}+qD^{\beta}_{|y|}\big)\big(e^{n+1}+e^n\big)-\frac{pq}{4}\Delta t^2\big(D^{\alpha}_{|x|}D^{\beta}_{|y|}\big)\big(e^{n+1}-e^n\big)+O(\Delta t^3). \tag{2.35}$$

Multiplying both sides of (2.35) by $2e^{n+1}$ and integrating the resulting equation, we get

$$\|e^{n+1}-e^n\|^2_{L^2(\Omega)}+\|e^{n+1}\|^2_{L^2(\Omega)}-\|e^n\|^2_{L^2(\Omega)}=\Delta tR_1-\frac{pq}{2}\Delta t^2R_2+(O(\Delta t^3),e^{n+1})_\Omega, \tag{2.36}$$

where

$$R_1=\big((pD^{\alpha}_{|x|}+qD^{\beta}_{|y|})(e^{n+1}+e^n),e^{n+1}\big)_\Omega,$$

$$R_2=\big((D^{\alpha}_{|x|}D^{\beta}_{|y|})(e^{n+1}-e^n),e^{n+1}\big)_\Omega.$$

By using Lemma 2.1, we obtain

$$\begin{aligned}R_1=&\,p\big[\big(D^{\frac{\alpha}{2}}_x e^{n+1},{}_xD^{\frac{\alpha}{2}}e^{n+1}\big)_\Omega-\big(D^{\frac{\alpha}{2}}_x e^n,{}_xD^{\frac{\alpha}{2}}e^n\big)_\Omega+\big(D^{\frac{\alpha}{2}}_x(e^{n+1}+e^n),{}_xD^{\frac{\alpha}{2}}(e^{n+1}+e^n)\big)_\Omega\big]\\&+q\big[\big(D^{\frac{\beta}{2}}_y e^{n+1},{}_yD^{\frac{\beta}{2}}e^{n+1}\big)_\Omega-\big(D^{\frac{\beta}{2}}_y e^n,{}_yD^{\frac{\beta}{2}}e^n\big)_\Omega+\big(D^{\frac{\beta}{2}}_y(e^{n+1}+e^n),{}_yD^{\frac{\beta}{2}}(e^{n+1}+e^n)\big)_\Omega\big].\end{aligned} \tag{2.37}$$

Similarly we have

$$\begin{aligned}R_2=&\,\big(D^{\frac{\alpha}{2}}_x\,{}_yD^{\frac{\beta}{2}}(e^{n+1}-e^n),{}_xD^{\frac{\alpha}{2}}D^{\frac{\beta}{2}}_y(e^{n+1}-e^n)\big)_\Omega\\&+\big(D^{\frac{\alpha}{2}}_xD^{\frac{\beta}{2}}_y(e^{n+1}-e^n),{}_xD^{\frac{\alpha}{2}}\,{}_yD^{\frac{\beta}{2}}(e^{n+1}-e^n)\big)_\Omega\\&+\big(D^{\frac{\alpha}{2}}_x\,{}_yD^{\frac{\beta}{2}}e^{n+1},{}_xD^{\frac{\alpha}{2}}D^{\frac{\beta}{2}}_ye^{n+1}\big)_\Omega-\big(D^{\frac{\alpha}{2}}_x\,{}_yD^{\frac{\beta}{2}}e^{n},{}_xD^{\frac{\alpha}{2}}D^{\frac{\beta}{2}}_ye^{n}\big)_\Omega\\&+\big(D^{\frac{\alpha}{2}}_xD^{\frac{\beta}{2}}_ye^{n+1},{}_xD^{\frac{\alpha}{2}}\,{}_yD^{\frac{\beta}{2}}e^{n+1}\big)_\Omega-\big(D^{\frac{\alpha}{2}}_xD^{\frac{\beta}{2}}_ye^{n},{}_xD^{\frac{\alpha}{2}}\,{}_yD^{\frac{\beta}{2}}e^{n}\big)_\Omega.\end{aligned} \tag{2.38}$$

In virtue of Lemma 2.2 and Lemma 2.6, we observe that the terms $\big(D^{\frac{\alpha}{2}}_x(e^{n+1}+e^n),{}_xD^{\frac{\alpha}{2}}(e^{n+1}+e^n)\big)_\Omega$ and $\big(D^{\frac{\beta}{2}}_y(e^{n+1}+e^n),{}_yD^{\frac{\beta}{2}}(e^{n+1}+e^n)\big)_\Omega$ are non-positive, and the terms $\big(D^{\frac{\alpha}{2}}_x\,{}_yD^{\frac{\beta}{2}}(e^{n+1}-e^n),{}_xD^{\frac{\alpha}{2}}D^{\frac{\beta}{2}}_y(e^{n+1}-e^n)\big)_\Omega$ and $\big(D^{\frac{\alpha}{2}}_xD^{\frac{\beta}{2}}_y(e^{n+1}-e^n),{}_xD^{\frac{\alpha}{2}}\,{}_yD^{\frac{\beta}{2}}(e^{n+1}-e^n)\big)_\Omega$ are non-negative for $1<\alpha,\beta<2$ and $\alpha+\beta>3$. Dropping these terms from (2.37) and (2.38), then bringing the above expressions into (2.36), we obtain, for all $n=0,1,\dots,L-1$,

$$\begin{aligned}&\|e^{n+1}\|^2_{L^2(\Omega)}-\Delta t\big[p(D^{\frac{\alpha}{2}}_xe^{n+1},{}_xD^{\frac{\alpha}{2}}e^{n+1})_\Omega+q(D^{\frac{\beta}{2}}_ye^{n+1},{}_yD^{\frac{\beta}{2}}e^{n+1})_\Omega\big]\\&+\frac{pq}{2}\Delta t^2\big[(D^{\frac{\alpha}{2}}_x\,{}_yD^{\frac{\beta}{2}}e^{n+1},{}_xD^{\frac{\alpha}{2}}D^{\frac{\beta}{2}}_ye^{n+1})_\Omega+(D^{\frac{\alpha}{2}}_xD^{\frac{\beta}{2}}_ye^{n+1},{}_xD^{\frac{\alpha}{2}}\,{}_yD^{\frac{\beta}{2}}e^{n+1})_\Omega\big]\\\le&\;\|e^{n}\|^2_{L^2(\Omega)}-\Delta t\big[p(D^{\frac{\alpha}{2}}_xe^{n},{}_xD^{\frac{\alpha}{2}}e^{n})_\Omega+q(D^{\frac{\beta}{2}}_ye^{n},{}_yD^{\frac{\beta}{2}}e^{n})_\Omega\big]\\&+\frac{pq}{2}\Delta t^2\big[(D^{\frac{\alpha}{2}}_x\,{}_yD^{\frac{\beta}{2}}e^{n},{}_xD^{\frac{\alpha}{2}}D^{\frac{\beta}{2}}_ye^{n})_\Omega+(D^{\frac{\alpha}{2}}_xD^{\frac{\beta}{2}}_ye^{n},{}_xD^{\frac{\alpha}{2}}\,{}_yD^{\frac{\beta}{2}}e^{n})_\Omega\big]+O(\Delta t^3)\|e^{n+1}\|_{L^2(\Omega)}.\end{aligned}$$

Repeating the above inequality from step n up to the first step, we get

$$\begin{aligned}&\|e^{n+1}\|^2_{L^2(\Omega)}-\Delta t\Big[p(D^{\frac{\alpha}{2}}_xe^{n+1},{}_xD^{\frac{\alpha}{2}}e^{n+1})_\Omega+q(D^{\frac{\beta}{2}}_ye^{n+1},{}_yD^{\frac{\beta}{2}}e^{n+1})_\Omega\Big]\\&+\frac{pq}{2}\Delta t^2\Big[(D^{\frac{\alpha}{2}}_x\,{}_yD^{\frac{\beta}{2}}e^{n+1},{}_xD^{\frac{\alpha}{2}}D^{\frac{\beta}{2}}_ye^{n+1})_\Omega+(D^{\frac{\alpha}{2}}_xD^{\frac{\beta}{2}}_ye^{n+1},{}_xD^{\frac{\alpha}{2}}\,{}_yD^{\frac{\beta}{2}}e^{n+1})_\Omega\Big]\\\le&\;\|e^{0}\|^2_{L^2(\Omega)}-\Delta t\Big[p(D^{\frac{\alpha}{2}}_xe^{0},{}_xD^{\frac{\alpha}{2}}e^{0})_\Omega+q(D^{\frac{\beta}{2}}_ye^{0},{}_yD^{\frac{\beta}{2}}e^{0})_\Omega\Big]\\&+\frac{pq}{2}\Delta t^2\Big[(D^{\frac{\alpha}{2}}_x\,{}_yD^{\frac{\beta}{2}}e^{0},{}_xD^{\frac{\alpha}{2}}D^{\frac{\beta}{2}}_ye^{0})_\Omega+(D^{\frac{\alpha}{2}}_xD^{\frac{\beta}{2}}_ye^{0},{}_xD^{\frac{\alpha}{2}}\,{}_yD^{\frac{\beta}{2}}e^{0})_\Omega\Big]+O(\Delta t^3)\sum_{k=0}^{n}\|e^{k+1}\|_{L^2(\Omega)}.\end{aligned}$$

Neglecting the initial error and taking into account the fact that the second and third terms in LHS are positive, we obtain

$$
\begin{aligned}
\|e^{n+1}\|^2_{L^2(\Omega)} &\le O(\Delta t^3)\sum_{k=0}^{n}\|e^{k+1}\|_{L^2(\Omega)} \\
&\le O(\Delta t^4)+\frac{\Delta t^2}{2T^2}(\sum_{k=0}^{n}\|e^{k+1}\|_{L^2(\Omega)})^2 \\
&\le O(\Delta t^4)+\frac{\Delta t^2}{2T^2}n\sum_{k=0}^{n}\|e^{k+1}\|^2_{L^2(\Omega)} \\
&\le O(\Delta t^4)+\frac{n}{2L^2}\sum_{k=0}^{n-1}\|e^{k+1}\|^2_{L^2(\Omega)}+\frac{1}{2}\|e^{n+1}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

This gives

$$\|e^{n+1}\|^2_{L^2(\Omega)} \le O(\Delta t^4)+\frac{n}{L^2}\sum_{k=0}^{n-1}\|e^{k+1}\|^2_{L^2(\Omega)}.$$

Finally by using the Gronwall's inequality, we have

$$\|e^{n+1}\|^2_{L^2(\Omega)} \le O(\Delta t^4)\exp(\frac{n^2}{L^2}) \le O(\Delta t^4).$$

The proof is completed. □

3. Spectral method for the spatial discretization

In this section we propose a spectral method for the spatial discretization, and carry out a stability analysis for the direction splitting spectral fully discrete scheme.

3.1. The spatial discretization

Let N be a non-negative integer, we denote by $\mathbb{P}_N(\Lambda)$ the set of all polynomials of degree less or equal to N defined in Λ, and set $\mathbb{P}^0_N(\Lambda) := \{\phi \in \mathbb{P}_N(\Lambda) : \phi(\pm 1) = 0\}$, $\mathbb{P}_N(\Omega) = \mathbb{P}_N(\Lambda) \otimes \mathbb{P}_N(\Lambda)$, $\mathbb{P}^0_N(\Omega) := \{\phi \in \mathbb{P}_N(\Omega) : \phi|_{\partial\Omega} = 0\}$. Let $S_N = \mathbb{P}^0_N(\Omega)$.

Let $\{x_i\}_{i=0}^N$ (also denoted, when the variable is y, by $\{y_j\}_{j=0}^N$ with $y_i = x_i$ for $0 \le i \le N$) and $\{\omega_i\}_{i=0}^N$ be the nodes and associated weights of the Legendre–Gauss–Lobatto quadrature in Λ, i.e., $\{x_i\}_{i=0}^N$ are zeros of $(1-x^2)L'_N(x)$, where L_N is the Legendre polynomial of degree N, and $\{\omega_i\}_{i=0}^N$ are such that

$$\int_{-1}^{1}\varphi(x)dx = \sum_{i=0}^{N}\varphi(x_i)\omega_i,\ \forall\varphi \in \mathbb{P}_{2N-1}(\Lambda).$$

Let Σ be the set of all collocation points, i.e., $\Sigma := \{(x_i, y_j) : 0 \le i, j \le N\}$, and Σ_I the set of all interior collocation points, i.e., $\Sigma_I := \{(x_i, y_j), 1 \le i, j \le N-1\}$. We denote by $\mathcal{I}_N$ the polynomial interpolation operator based on the set Σ, i.e., $\mathcal{I}_N : C(\bar{\Omega}) \to \mathbb{P}_N(\Omega)$, such that, for all $f \in C(\bar{\Omega})$,

$$\mathcal{I}_N f(x_i, y_j) = f(x_i, y_j),\ \forall (x_i, y_j) \in \Sigma. \tag{3.39}$$

We then define a number of discrete inner products:

$$(\varphi, \psi)_{N,\Lambda} := \sum_{i=0}^{N}\varphi(x_i)\psi(x_i)\omega_i,\ \forall\varphi, \psi \in C^0(\bar{\Lambda}), \tag{3.40}$$

$$(u, v)_{N,\Omega} := \sum_{i,j=0}^{N}u(x_i, y_j)v(x_i, y_j)\omega_i\omega_j,\ \forall u, v \in C^0(\bar{\Omega}), \tag{3.41}$$

$$(u, v)_{y,N,\Omega} = \sum_{j=0}^{N}\big(u(x, y_j), v(x, y_j)\big)_\Lambda\omega_j,\ \forall u, v \in C^0(\bar{\Omega}), \tag{3.42}$$

$$(u, v)_{x,N,\Omega} = \sum_{i=0}^{N}\big(u(x_i, y), v(x_i, y)\big)_\Lambda\omega_i,\ \forall u, v \in C^0(\bar{\Omega}), \tag{3.43}$$

and its associated norm

$$\|v\|_{N,\Omega} := (v,v)_{N,\Omega}^{1/2}, \quad \|v\|_{y,N,\Omega} = (v,v)_{y,N,\Omega}^{1/2}, \quad \|v\|_{x,N,\Omega} = (v,v)_{x,N,\Omega}^{1/2}.$$

It is well known (cf. [1]) that

$$\|v_N\|_{N,\Omega} \cong \|v_N\|_{L^2(\Omega)} \cong \|v_N\|_{x,N,\Omega} \cong \|v_N\|_{y,N,\Omega}, \ \forall v_N \in \mathbb{P}_N(\Omega). \tag{3.44}$$

In cases where no confusion would arise, Ω and Λ may be dropped from the notations.

Now we propose the following spectral method for (2.7)–(2.8) based on the weak formulation with Legendre–Gauss–Lobatto quadratures.

- Let $u_N^0 = \mathcal{I}_N u_0$.
- Predictor for u_N^{n+1}. We compute the predictor ξ_N^{n+1} by

$$\begin{aligned}\Big(\frac{\xi_N^{n+1}-u_N^n}{\Delta t}, v_N\Big)_{N,\Omega} = \; & p\big((D_x^{\frac{\alpha}{2}} u_N^n, {}_xD^{\frac{\alpha}{2}} v_N)_{y,N,\Omega} + ({}_xD^{\frac{\alpha}{2}} u_N^n, D_x^{\frac{\alpha}{2}} v_N)_{y,N,\Omega}\big) \\ & + q\big((D_y^{\frac{\beta}{2}} u_N^n, {}_yD^{\frac{\beta}{2}} v_N)_{x,N,\Omega} + ({}_yD^{\frac{\beta}{2}} u_N^n, D_y^{\frac{\beta}{2}} v_N)_{x,N,\Omega}\big) \\ & + (f_N^{n+\frac{1}{2}}, v_N)_{N,\Omega}, \quad \forall v_N \in \mathbb{P}_N^0(\Omega).\end{aligned} \tag{3.45}$$

- Direction splitting: First find $\eta_N^{n+1}(x, y_j) \in \mathbb{P}_N^0(\Lambda)$, $j = 0, \ldots, N$, such that, $\forall \varphi_N \in \mathbb{P}_N^0(\Lambda)$,

$$\begin{aligned}\Big(\frac{\eta_N^{n+1}-\xi_N^{n+1}}{\frac{1}{2}\Delta t}(x, y_j), \varphi_N(x)\Big)_{N,\Lambda} = \; & p\big(D_x^{\frac{\alpha}{2}}(\eta_N^{n+1}-u_N^n)(x, y_j), {}_xD^{\frac{\alpha}{2}}\varphi_N(x)\big)_\Lambda \\ & + p\big({}_xD^{\frac{\alpha}{2}}(\eta_N^{n+1}-u_N^n)(x, y_j), D_x^{\frac{\alpha}{2}}\varphi_N(x)\big)_\Lambda;\end{aligned} \tag{3.46}$$

Then find $u_N^{n+1}(x_i, y) \in \mathbb{P}_N^0(\Lambda)$, $i = 0, \ldots, N$, such that, $\forall \psi_N \in \mathbb{P}_N^0(\Lambda)$,

$$\begin{aligned}\Big(\frac{u_N^{n+1}-\eta_N^{n+1}}{\frac{1}{2}\Delta t}(x_i, y), \psi_N(y)\Big)_{N,\Lambda} = \; & q\big(D_y^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n)(x_i, y), {}_yD^{\frac{\beta}{2}}\psi_N(y)\big)_\Lambda \\ & + q\big({}_yD^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n)(x_i, y), D_y^{\frac{\beta}{2}}\psi_N(y)\big)_\Lambda.\end{aligned} \tag{3.47}$$

Remark 3.1. We remark that in the right-hand sides of the above discretization, numerical quadratures are only used in the direction in which non-fractional derivatives are applied. As it is known that the fractional derivative of a polynomial is no longer a polynomial, and naive applications of Legendre–Gauss–Lobatto quadratures would result in a lose of accuracy. In our implementation, we use an efficient way to evaluate the integrals in the right-hand sides in (3.45)–(3.47), which is described below. A direct calculation shows

$$D_x^s p_N(x) = \frac{1}{\Gamma(1-s)}\frac{d}{dx}\int_{-1}^{x}(x-\tau)^{-s}p_N(\tau)d\tau = (1+x)^{1-s}\varphi(p_N(x)),$$

$${}_xD^s p_N(x) = -\frac{1}{\Gamma(1-s)}\frac{d}{dx}\int_{x}^{1}(\tau-x)^{-s}p_N(\tau)d\tau = (1-x)^{1-s}\psi(p_N(x)),$$

where

$$\varphi(p_N(x)) = \frac{1}{\Gamma(1-s)}\int_{-1}^{1}(1-\theta)^{-s}p_N'\Big(\frac{x+1}{2}\theta + \frac{x-1}{2}\Big)d\theta,$$

$$\psi(p_N(x)) = -\frac{1}{\Gamma(1-s)}\int_{-1}^{1}(1+\theta)^{-s}p_N'\Big(\frac{1-x}{2}\theta + \frac{x+1}{2}\Big)d\theta.$$

Obviously, if $p_N(x) \in \mathbb{P}_N(\Lambda)$, then $\varphi(p_N(x)), \psi(p_N(x)) \in \mathbb{P}_{N-1}(\Lambda)$. Thus for $p_N, q_N \in \mathbb{P}_N(\Lambda)$, we have

$$\begin{aligned}(D_x^s p_N(x), {}_xD^s q_N(x))_\Lambda &= \int_\Lambda (1-x)^{1-s}(1+x)^{1-s}\varphi(p_N(x))\psi(q_N(x))dx \\ &= \sum_{i=0}^{N}\varphi(p_N(x_i^{1-s,1-s}))\psi(q_N(x_i^{1-s,1-s}))\omega_i^{1-s,1-s},\end{aligned} \tag{3.48}$$

where $\{x_i^{\gamma,\gamma}\}_{i=0}^N$ and $\{\omega_i^{\gamma,\gamma}\}_{i=0}^N$ are the Jacobi–Gauss–Lobatto points and weights associated with the weight function $(1-x)^\gamma(1+x)^\gamma$. The formula (3.48) will be used to evaluate the integrals in (3.45)–(3.47) in the direction where fractional derivatives appear.

3.2. A proof of stability

We shall prove that the scheme (3.45)–(3.47) is unconditionally stable with respect to the initial data. To this end, we denote, for any $\boldsymbol{\varphi} := \{\varphi^n\}_{n=0}^L$ with $\varphi^n \in L^2(\Omega)$,

$$\|\boldsymbol{\varphi}\|^2_{l^2(L^2(\Omega))} = \Delta t \sum_{n=1}^{L} \|\varphi^n\|^2_{L^2(\Omega)}, \ \|\boldsymbol{\varphi}\|^2_{l^\infty(L^2(\Omega))} = \max_{0\le n\le L} \|\varphi^n\|^2_{L^2(\Omega)}.$$

Theorem 3.1. *If $\boldsymbol{u}_N := \{u_N^n\}_{n=0}^L$ is the solution of (3.45)–(3.47) with $f=0$, and $\alpha+\beta>3$, then $\boldsymbol{u}_N$ satisfies the following stability inequality:*

$$\begin{aligned}
&\|\boldsymbol{u}_N\|^2_{l^2(L^2(\Omega))} + \Delta t c_\alpha \|D_x^{\frac{\alpha}{2}} \boldsymbol{u}_N\|^2_{l^2(L^2(\Omega))} + \Delta t c_\beta \|{}_yD^{\frac{\beta}{2}} \boldsymbol{u}_N\|^2_{l^2(L^2(\Omega))} \\
&\quad + \Delta t^2 c_{\alpha,\beta}\Big(\|D_x^{\frac{\alpha}{2}}\, {}_yD^{\frac{\beta}{2}} \boldsymbol{u}_N\|^2_{l^2(L^2(\Omega))} + \|{}_xD^{\frac{\alpha}{2}} D_y^{\frac{\beta}{2}} \boldsymbol{u}_N\|^2_{l^2(L^2(\Omega))}\Big) \\
&\le c\Big(\|u_0\|^2_{L^2(\Omega)} + \Delta t c_\alpha \|D_x^{\frac{\alpha}{2}} u_0\|^2_{L^2(\Omega)} + \Delta t c_\beta \|{}_yD^{\frac{\beta}{2}} u_0\|^2_{L^2(\Omega)} \\
&\quad + \Delta t^2 c_{\alpha,\beta}\Big(\|D_x^{\frac{\alpha}{2}}\, {}_yD^{\frac{\beta}{2}} u_0\|^2_{L^2(\Omega)} + \|{}_xD^{\frac{\alpha}{2}} D_y^{\frac{\beta}{2}} u_0\|^2_{L^2(\Omega)}\Big)\Big),
\end{aligned} \tag{3.49}$$

where c is a constant independent of the discretization parameters, $c_\alpha = -p\cos\dfrac{\alpha\pi}{2}$, $c_\beta = -q\cos\dfrac{\beta\pi}{2}$, and $c_{\alpha,\beta}$ depends on $\cos(\frac{\alpha+\beta}{2}\pi)$ and $\cos(\frac{\alpha-\beta}{2}\pi)$.

Proof. By performing the Legendre–Gauss–Lobatto quadrature in the y-direction in (3.46), we obtain: $\forall v_N \in \mathbb{P}_N^0(\Omega)$,

$$\begin{aligned}
\Big(\frac{\eta_N^{n+1} - \xi_N^{n+1}}{\frac{1}{2}\Delta t}, v_N\Big)_{N,\Omega} = p\Big[&\big(D_x^{\frac{\alpha}{2}}(\eta_N^{n+1} - u_N^n), {}_xD^{\frac{\alpha}{2}} v_N\big)_{y,N,\Omega} \\
&+ \big({}_xD^{\frac{\alpha}{2}}(\eta_N^{n+1} - u_N^n), D_x^{\frac{\alpha}{2}} v_N\big)_{y,N,\Omega}\Big].
\end{aligned} \tag{3.50}$$

Similarly, by performing the Legendre–Gauss–Lobatto quadrature in the x-direction in (3.47), we get, $\forall v_N \in \mathbb{P}_N^0(\Omega)$,

$$\begin{aligned}
\Big(\frac{u_N^{n+1} - \eta_N^{n+1}}{\frac{1}{2}\Delta t}, v_N\Big)_{N,\Omega} = q\Big[&\big(D_y^{\frac{\beta}{2}}(u_N^{n+1} - u_N^n), {}_yD^{\frac{\beta}{2}} v_N\big)_{x,N,\Omega} \\
&+ \big({}_yD^{\frac{\beta}{2}}(u_N^{n+1} - u_N^n), D_y^{\frac{\beta}{2}} v_N\big)_{x,N,\Omega}\Big].
\end{aligned} \tag{3.51}$$

On the other hand, observing that the functions u_N^{n+1}, η_N^{n+1}, $D_y^{\frac{\beta}{2}}(u_N^{n+1} - u_N^n)$, and ${}_yD^{\frac{\alpha}{2}}(u_N^{n+1} - u_N^n)$ are all polynomials of degree less than or equal to N with respect to x, we derive from (3.47):

$$\begin{aligned}
\big(\eta_N^{n+1}(x,y), \psi_N(y)\big)_{N,\Lambda} = -\frac{q\Delta t}{2}\Big[&\big(D_y^{\frac{\beta}{2}}(u_N^{n+1} - u_N^n)(x,y), {}_yD^{\frac{\beta}{2}}\psi_N(y)\big)_\Lambda \\
&+ \big({}_yD^{\frac{\beta}{2}}(u_N^{n+1} - u_N^n)(x,y), D_y^{\frac{\beta}{2}}\psi_N(y)\big)_\Lambda\Big] \\
&+ \big(u_N^{n+1}(x,y), \psi_N(y)\big)_{N,\Lambda}, \ \forall x \in \Lambda, \psi_N \in \mathbb{P}_N^0(\Lambda),
\end{aligned} \tag{3.52}$$

where $(\cdot,\cdot)_{N,\Lambda}$ acts in the y-variable. Taking respectively the left-sided and the right-sided α-order derivative of both sides of (3.52) with respect to x, we have

$$\begin{aligned}
\big(D_x^\alpha \eta_N^{n+1}(x,y), \psi_N(y)\big)_{N,\Lambda} = -\frac{q\Delta t}{2}\Big[&\big(D_x^\alpha D_y^{\frac{\beta}{2}}(u_N^{n+1} - u_N^n)(x,y), {}_yD^{\frac{\beta}{2}}\psi_N(y)\big)_\Lambda \\
&+ \big(D_x^\alpha\, {}_yD^{\frac{\beta}{2}}(u_N^{n+1} - u_N^n)(x,y), D_y^{\frac{\beta}{2}}\psi_N(y)\big)_\Lambda\Big] \\
&+ \big(D_x^\alpha u_N^{n+1}(x,y), \psi_N(y)\big)_{N,\Lambda}, \ \forall x \in \Lambda, \psi_N \in \mathbb{P}_N^0(\Lambda),
\end{aligned}$$

$$\begin{aligned}\left({}_xD^{\alpha}\eta_N^{n+1}(x,y),\psi_N(y)\right)_{N,\Lambda} &= -\frac{q\Delta t}{2}\Big[\left({}_xD^{\alpha}D_y^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n)(x,y),{}_yD^{\frac{\beta}{2}}\psi_N(y)\right)_{\Lambda}\\&+\left({}_xD^{\alpha}{}_yD^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n)(x,y),D_y^{\frac{\beta}{2}}\psi_N(y)\right)_{\Lambda}\Big]\\&+\left({}_xD^{\alpha}u_N^{n+1}(x,y),\psi_N(y)\right)_{N,\Lambda},\ \forall x\in\Lambda,\ \psi_N\in\mathbb{P}_N^0(\Lambda).\end{aligned}$$

By using Lemme 2.1 to the above two equations with respect to the x-variable, we get the weak forms as follows: $\forall v_N\in\mathbb{P}_N^0(\Omega)$,

$$\begin{aligned}(D_x^{\frac{\alpha}{2}}\eta_N^{n+1},{}_xD^{\frac{\alpha}{2}}v_N)_{y,N,\Omega} &= -\frac{q\Delta t}{2}\Big[(D_x^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n),{}_xD^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}v_N)_{\Omega}\\&+(D_x^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n),{}_xD^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}v_N)_{\Omega}\Big]\\&+(D_x^{\frac{\alpha}{2}}u_N^{n+1},{}_xD^{\frac{\alpha}{2}}v_N)_{y,N,\Omega},\end{aligned}\tag{3.53}$$

$$\begin{aligned}({}_xD^{\frac{\alpha}{2}}\eta_N^{n+1},D_x^{\frac{\alpha}{2}}v_N)_{y,N,\Omega} &= -\frac{q\Delta t}{2}\Big[({}_xD^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n),D_x^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}v_N)_{\Omega}\\&+({}_xD^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n),D_x^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}v_N)_{\Omega}\Big]\\&+({}_xD^{\frac{\alpha}{2}}u_N^{n+1},D_x^{\frac{\alpha}{2}}v_N)_{y,N,\Omega}.\end{aligned}\tag{3.54}$$

From putting (3.45), (3.50), (3.51), (3.53), and (3.54) together, and taking $v_N=2\Delta t u_N^{n+1}$, we obtain

$$\begin{aligned}&(u_N^{n+1}-u_N^n,2u_N^{n+1})_{N,\Omega}\\&=\Delta t\Big[p\big(D_x^{\frac{\alpha}{2}}(u_N^{n+1}+u_N^n),{}_xD^{\frac{\alpha}{2}}u_N^{n+1}\big)_{y,N,\Omega}+p\big({}_xD^{\frac{\alpha}{2}}(u_N^{n+1}+u_N^n),D_x^{\frac{\alpha}{2}}u_N^{n+1}\big)_{y,N,\Omega}\\&+q\big(D_y^{\frac{\beta}{2}}(u_N^{n+1}+u_N^n),{}_yD^{\frac{\beta}{2}}u_N^{n+1}\big)_{x,N,\Omega}+q\big({}_yD^{\frac{\beta}{2}}(u_N^{n+1}+u_N^n),D_y^{\frac{\beta}{2}}u_N^{n+1}\big)_{x,N,\Omega}\Big]\\&-\frac{pq\Delta t^2}{2}\Big[\big(D_x^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n),{}_xD^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}u_N^{n+1}\big)_{\Omega}+\big({}_xD^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n),D_x^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}u_N^{n+1}\big)_{\Omega}\\&+\big(D_x^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n),{}_xD^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}u_N^{n+1}\big)_{\Omega}+\big({}_xD^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n),D_x^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}u_N^{n+1}\big)_{\Omega}\Big].\end{aligned}\tag{3.55}$$

Applying similar techniques as in the proof of Theorem 2.1 to (3.55) yields:

$$\begin{aligned}&\|u_N^{n+1}\|_{N,\Omega}^2+\|u_N^{n+1}-u_N^n\|_{N,\Omega}^2-\|u_N^n\|_{N,\Omega}^2\\&-\Delta t\Big[p(D_x^{\frac{\alpha}{2}}u_N^{n+1},{}_xD^{\frac{\alpha}{2}}u_N^{n+1})_{y,N,\Omega}+q(D_y^{\frac{\beta}{2}}u_N^{n+1},{}_yD^{\frac{\beta}{2}}u_N^{n+1})_{x,N,\Omega}\Big]\\&+\frac{pq\Delta t^2}{2}\Big[(D_x^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}u_N^{n+1},{}_xD^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}u_N^{n+1})_{\Omega}+(D_x^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}u_N^{n+1},{}_xD^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}u_N^{n+1})_{\Omega}\Big]\\&=-\Delta t\Big[p(D_x^{\frac{\alpha}{2}}u_N^n,{}_xD^{\frac{\alpha}{2}}u_N^n)_{y,N,\Omega}+q(D_y^{\frac{\beta}{2}}u_N^n,{}_yD^{\frac{\beta}{2}}u_N^n)_{x,N\Omega}\Big]\\&+\frac{pq\Delta t^2}{2}\Big[(D_x^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}u_N^n,{}_xD^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}u_N^n)_{\Omega}+(D_x^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}u_N^n,{}_xD^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}u_N^n)_{\Omega}\Big]+F_n,\end{aligned}\tag{3.56}$$

where

$$\begin{aligned}F_n=&\ \Delta t\Big[p\big(D_x^{\frac{\alpha}{2}}(u_N^{n+1}+u_N^n),{}_xD^{\frac{\alpha}{2}}(u_N^{n+1}+u_N^n)\big)_{y,N,\Omega}\\&+q\big(D_y^{\frac{\alpha}{2}}(u_N^{n+1}+u_N^n),{}_yD^{\frac{\alpha}{2}}(u_N^{n+1}+u_N^n)\big)_{x,N,\Omega}\Big]\\&-\frac{pq\Delta t^2}{2}\Big[\big(D_x^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n),{}_xD^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n)\big)_{\Omega}\\&+\big(D_x^{\frac{\alpha}{2}}D_y^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n),{}_xD^{\frac{\alpha}{2}}{}_yD^{\frac{\beta}{2}}(u_N^{n+1}-u_N^n)\big)\Big]_{\Omega}.\end{aligned}$$

In virtue of Lemma 2.2 and Lemma 2.6, F_n is non-positive, so that we can drop F_n and $\|u_N^{n+1} - u_N^n\|_{N,\Omega}^2$ from (3.56) to yield

$$\begin{aligned}
&\|u_N^{n+1}\|_{N,\Omega}^2 + \Delta t\Big(c_\alpha \|D_x^{\frac{\alpha}{2}} u_N^{n+1}\|_{y,N,\Omega}^2 + c_\beta \|D_y^{\frac{\beta}{2}} u_N^{n+1}\|_{x,N,\Omega}^2\Big)\\
&+ \Delta t^2 c_{\alpha,\beta}\Big(\|D_x^{\frac{\alpha}{2}}\, {}_yD^{\frac{\beta}{2}} u_N^{n+1}\|_{L^2(\Omega)}^2 + \|D_x^{\frac{\alpha}{2}} D_y^{\frac{\beta}{2}} u_N^{n+1}\|_{L^2(\Omega)}^2\Big)\\
\le\ &\|u_N^{n}\|_{N,\Omega}^2 + \Delta t\Big(c_\alpha \|D_x^{\frac{\alpha}{2}} u_N^{n}\|_{y,N,\Omega}^2 + c_\beta \|D_y^{\frac{\beta}{2}} u_N^{n}\|_{x,N,\Omega}^2\Big)\\
&+ \Delta t^2 c_{\alpha,\beta}\Big(\|D_x^{\frac{\alpha}{2}}\, {}_yD^{\frac{\beta}{2}} u_N^{n}\|_{L^2(\Omega)}^2 + \|D_x^{\frac{\alpha}{2}} D_y^{\frac{\beta}{2}} u_N^{n}\|_{L^2(\Omega)}^2\Big),
\end{aligned} \tag{3.57}$$

where $c_\alpha = -p\cos\frac{\alpha\pi}{2}$, $c_\beta = -q\cos\frac{\beta\pi}{2}$, and $c_{\alpha,\beta} = \frac{pq}{4}\max\left\{\cos\frac{(\alpha+\beta)\pi}{2}, \cos\frac{(\alpha-\beta)\pi}{2}\right\}$, which are all positive for $1 < \alpha, \beta < 2$, $\alpha+\beta > 3$, $p > 0$, $q > 0$.

Finally, the desired result is obtained by summing up the above estimate from $n = 0$ to $n = M$ for $1 \le M \le L-1$, and using the norm equivalence in (3.44). $\square$

Remark 3.2. A direction splitting scheme similar to (3.45)–(3.47) can be constructed for the equation (2.1) with L defined by

$$Lu = D_x^{\frac{\alpha}{2}} p(x,y)\,{}_xD^{\frac{\alpha}{2}} u + D_y^{\frac{\alpha}{2}} q(x,y)\,{}_yD^{\frac{\alpha}{2}} u, \tag{3.58}$$

$$Lu = D_x^{\frac{\alpha}{2}} p(x,y)\,{}_xD^{\frac{\alpha}{2}} u + {}_yD^{\frac{\alpha}{2}} q(x,y) D_y^{\frac{\alpha}{2}} u, \tag{3.59}$$

where p and q can be variable coefficients. Similar techniques as in the proof of Theorem 3.1 can be applied to establish the unconditional stability. In fact, stable direction splitting schemes can be designed and analyzed as far as the underlying operator is self-adjoint (it is an easy matter to verify that both operators defined in (3.58) and (3.59) are such operators).

Remark 3.3. For the time being, we are unable to prove the stability of the scheme for non-self-adjoint fractional operators although the numerical experiment performed in the last section tends to predict the actual stability.

4. Numerical results

4.1. Implementation

We start with some implementation details of the scheme (3.45)–(3.47). First, we choose to use the Lagrangian polynomials as basis, which it is extendable to more general problems, e.g., variable coefficients and/or deformed domains. Let $\{h_i : i = 0, 1, \dots, N\}$ be the Lagrangian polynomials associated with the Legendre–Gauss–Lobatto points $\{x_i\}_{0\le i\le N}$. Then

$$\mathbb{P}_N^0(\Lambda) = \text{span}\{h_i;\ i = 1, 2, \cdots, N-1\},$$

$$\mathbb{P}_N^0(\Omega) = \text{span}\{h_i(x)h_j(y);\ i, j = 1, 2, \cdots, N-1\},$$

and u_N can be expressed under this basis as follows:

$$u_N(x,y) = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} u_{ij} h_i(x) h_j(y), \quad \text{with } u_{ij} = u_N(x_i, y_j).$$

At each step, let the test functions v_N, φ_N, and ψ_N go through all basis functions in $\mathbb{P}_N^0(\Omega)$ and $\mathbb{P}_N^0(\Lambda)$ respectively, we arrive at the matrix equations

$$B(\boldsymbol{\xi}^{n+1} - \boldsymbol{u}^n)B = \Delta t(A_\alpha \boldsymbol{u}^n B + B\boldsymbol{u}^n A_\beta) + B\boldsymbol{f}^{n+\frac{1}{2}}B, \tag{4.1}$$

$$B(\boldsymbol{\eta}^{n+1} - \boldsymbol{\xi}^{n+1}) = \frac{\Delta t}{2} A_\alpha(\boldsymbol{\eta}^{n+1} - \boldsymbol{u}^n), \tag{4.2}$$

$$(\boldsymbol{u}^{n+1} - \boldsymbol{\eta}^{n+1})B = \frac{\Delta t}{2}(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n)A_\beta, \tag{4.3}$$

where the boldface lower letters mean matrices of the nodal values, e.g., $\boldsymbol{u}^n = (u_{ij}^n)_{i,j=1,\cdots N-1}$ with $u_{ij}^n = u_N^n(x_i, y_j)$, and so on. In (4.1)–(4.3), B is the diagonal mass matrix $B := \text{diag}(\omega_1, \omega_2, \dots, \omega_{N-1})$, A_s is the matrix form of the s-order fractional derivative, i.e.,

$$(A_s)_{ij} = (D_z^{\frac{s}{2}} h_j(z), {}_zD^{\frac{s}{2}} h_i(z))_\Lambda + ({}_zD^{\frac{s}{2}} h_j(z), D_z^{\frac{s}{2}} h_i(z))_\Lambda,\ i, j = 1, 2, \dots, N-1. \tag{4.4}$$

Table 1
Errors in L^2-norm and convergence orders for the numerical solution obtained at $T=1$ and $N=50$ for several pair of α and β.

Δt	$\alpha=\beta=1.2$	rate	$\alpha=1.01, \beta=1.99$	rate	$\alpha=\beta=1.9$	rate
0.1	1.5297e-05	–	6.7742e-05	–	8.8618e-04	–
0.05	3.8200e-06	2.0016	1.6983e-05	1.9960	2.1439e-04	2.0474
0.02	6.1097e-07	2.0004	2.7178e-06	1.9998	3.4009e-05	2.0094
0.01	1.5270e-07	2.0004	6.7945e-07	2.0000	8.4920e-06	2.0017
0.005	3.8145e-08	2.0011	1.6986e-07	2.0000	2.1224e-06	2.0004
0.002	6.0824e-09	2.0037	2.7169e-08	2.0003	3.3958e-07	2.0000
0.001	1.5494e-09	1.9729	6.7848e-09	2.0016	8.4916e-08	1.9996

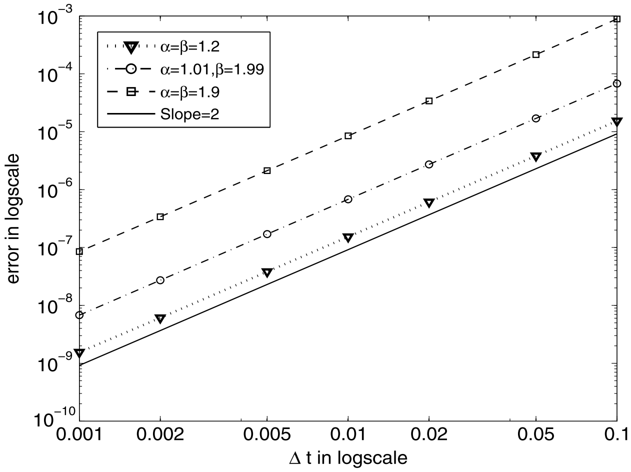

Fig. 1. Plot of the errors in L^2-norm as a function of Δt.

The entries of A_s are computed by using (3.48). It is assumed that the boundary conditions are already incorporated into the matrix operators.

Note that A_α and A_β are symmetric matrices, we will use the conjugate gradient iteration to solve the matrix equtions (4.2) and (4.3).

If no dimensional splitting was used, we will have a linear system as

$$B\boldsymbol{u}^{n+1}B - \frac{\Delta t}{2}(A_\alpha \boldsymbol{u}^{n+1}B + B\boldsymbol{u}^{n+1}A_\beta) = \boldsymbol{F}(\boldsymbol{u}^n, \boldsymbol{f}^{n+\frac{1}{2}}).$$

In this case, the computational cost will be $O(N^4)$ per iteration, and the storage cost will be $O(N^2)$ with N being the number of spatial grid points in one direction. This is comparable to the finite difference method using the same total grid number. However the fast solver proposed by Wang and Basu [24] in the finite difference framework is not extendable to the spectral method due to the lack of matrix structure.

4.2. Numerical results

In this subsection, we present some numerical results to verify the stability and accuracy of the proposed numerical method.

Example 4.1 *(Time and space accuracy).* We consider the 2D time-dependent space fractional diffusion equation (2.1) with the exact solution:

$$u(x, y, t) = e^{-ct}\sin^3\pi x \sin^3\pi y, \tag{4.5}$$

where c is constant. We take $p = q = \frac{1}{2}$.

We first investigate the time accuracy, i.e., the direction splitting error. We plot in Fig. 1 the L^2 errors at $T=1$ with respect to Δt in log–log scale for $c=1$ and a number pair of α and β. The polynomial degree for the spatial approximation is taken large enough, i.e., $N=50$, such that the spatial discretization error is negligible compared to the time discretization error. Detailed error values are listed in Table 1. The presented results clearly indicate that the proposed direction splitting scheme is of second order accuracy for any α and β ranging from 1 to 2.

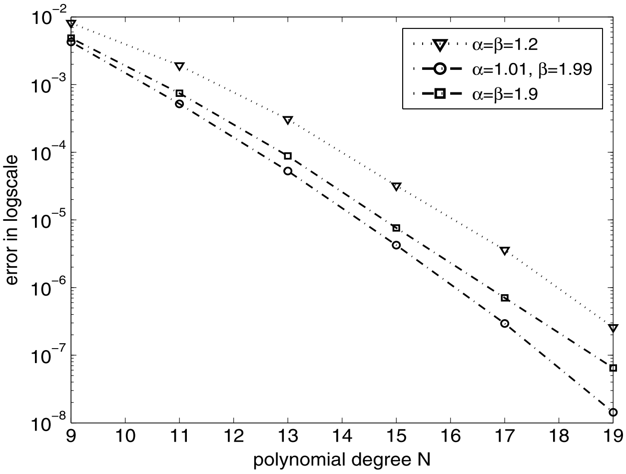

Fig. 2. Errors in L^2-norm as a function of N for several values of α and β.

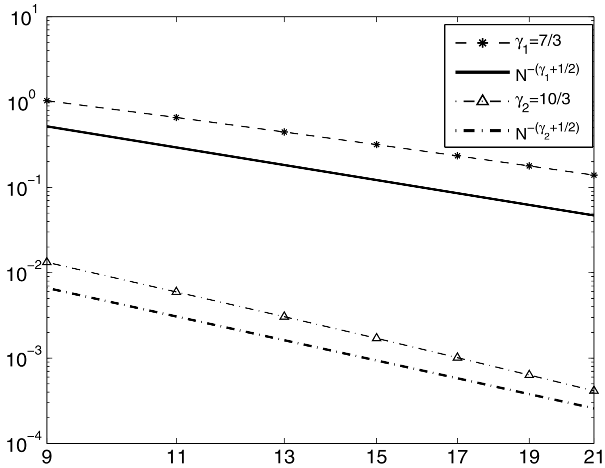

Fig. 3. Decay rates of the errors in L^2-norm as functions of N for solutions with limited regularity.

We then investigate the space accuracy by checking the convergence behavior of numerical solutions with respect to the polynomial degrees N. We plot in semi-log scale in Fig. 2 the L^2-errors versus N for $c = 0$, $\Delta t = 0.001$. It is observed that the error variations are linear versus the degrees of polynomial N, which means that the convergence is exponential since it is a semi-log plot.

Example 4.2 *(Solutions with limited regularity).* We consider an exact solution with limited regularity to examine the sharpness of the error estimate for the spatial approximation. Precisely, we test the scheme for the exact solution

$$u(x, y, t) = 1000e^{-t}(1 - x^2)^\gamma (1 - y^2)^\gamma, \tag{4.6}$$

where γ is a constant.

It can be verified that this solution belongs to $H^{\gamma+\frac{1}{2}}(\Omega)$ in space if γ is not an integer. We present the convergence result in Fig. 3, in which the errors versus N are plotted for the solution with $\gamma = \frac{10}{3}, \frac{7}{3}$, computed with $\Delta t = 0.001, \alpha = 1.99$, and $\beta = 1.01$. It is shown that the convergence rate slows down as γ, i.e., the regularity of the solution, decreases. A closer look at the two different error curves corresponding to two values of γ respectively predicts that the decay rate of the L^2-error is close to $N^{-(\gamma+\frac{1}{2})}$.

Example 4.3 *(Stability of the scheme for non-symmetric operators).* Consider the following fractional diffusion equation with non-symmetric fractional differential operators:

Table 2
L^2-errors and convergence rates for the non-symmetric equation with $\alpha = \beta = 1.9$ for different coefficients $p_i, q_j, i, j = 1, 2$.

Δt	$p_1 = 1, q_1 = 0$	rate	$p_1 = 0.6, q_1 = 0.4$	rate	$p_1 = 0.8, q_1 = 0.2$	rate
0.1	9.0324e-04		8.8687e-04		8.9233e-04	
0.05	2.1842e-04	2.0480	2.1455e-04	2.0474	2.1584e-04	2.0476
0.02	3.4644e-05	2.0095	3.4035e-05	2.0094	3.4238e-05	2.0094
0.01	8.6505e-06	2.0018	8.4984e-06	2.0018	8.5492e-06	2.0017
0.005	2.1620e-06	2.0004	2.1240e-06	2.0004	2.1367e-06	2.0004
0.002	3.4590e-07	2.0001	3.3983e-07	2.0000	3.4186e-07	2.0000
0.001	8.6486e-08	1.9998	8.4979e-08	1.9996	8.5483e-08	1.9997

Table 3
L^2-errors and convergence rates for the non-symmetric equation with $\alpha = \beta = 1.01$.

Δt	$p_1 = 0, q_1 = 1$	rate	$p_1 = 0.3, q_1 = 0.7$	rate	$p_1 = 0.2, q_1 = 0.8$	rate
0.1	1.8133e-03		4.6149e-04		8.3572e-04	
0.05	4.7782e-04	1.9241	1.1596e-04	1.9927	2.1159e-04	1.9817
0.02	7.7891e-05	1.9796	1.8580e-05	1.9984	3.3981e-05	1.9959
0.01	1.9528e-05	1.9959	4.6459e-06	1.9997	8.4999e-06	1.9992
0.005	4.8854e-06	1.9990	1.1615e-06	2.0000	2.1253e-06	1.9998
0.002	7.8183e-07	1.9998	1.8585e-07	1.9999	3.4006e-07	2.0000
0.001	1.9546e-07	2.0000	4.6461e-08	2.0000	8.5016e-08	2.0000

$$\frac{\partial u(x,y,t)}{\partial t} = p_1\, D_x^{\alpha} u(x,y,t) + p_2\, {}_xD^{\alpha} u(x,y,t) + q_1\, D_y^{\beta} u(x,y,t) + q_2\, {}_yD^{\beta} u(x,y,t) + f(x,y,t), \tag{4.7}$$

where $p_1 \neq p_2$, $q_1 \neq q_2$. The forcing function is given such that the exact solution is

$$u(x,y,t) = e^{-t} \sin^3 \pi x \sin^3 \pi y. \tag{4.8}$$

In this test we aim at the numerical investigation of the stability of the direction splitting schemes constructed for the above non-symmetric problem. The full discretization of the above problem results in a set of non-symmetric linear systems, which is then solved by applying the Bi-CG algorithm. Tables 2 and 3 list the errors and their decay rates with respect to the time step size for different coefficients $p_i, q_j, i, j = 1, 2$. No instability has been encountered in the calculation, even for relatively large time step sizes. Furthermore it is observed that the direction splitting scheme remains to be of second order for this non-symmetric equation.

5. Conclusions

We have presented a spectral direction splitting method for the time-dependent two-dimensional space fractional diffusion equation in a finite domain. The proposed method combines a direction splitting scheme in time with a mixed Galerkin-collocation spectral method for the spatial discretization. A rigorous analysis was carried out, showing that the overall scheme is unconditionally stable with a second order convergence in time. A number of numerical examples were provided to demonstrate the efficiency of the method.

Appendix A

Here we give some basic properties of the spaces related to fractional derivatives, which is a simplified version of more general results given in [11]. For the sake of simplification we only consider the 1D case.

For any positive integer n and $n-1 \le s < n$, the Caputo derivative of order s are defined as

$$\text{left Caputo derivative:} \quad {}^C D_x^s \varphi(x) = \frac{1}{\Gamma(n-s)} \int_{-1}^{x} \frac{v^{(n)}(\xi)d\xi}{(x-\xi)^{s-n+1}} \quad \forall x \in \Lambda, \tag{A.1}$$

$$\text{right Caputo derivative:} \quad {}_x^C D^s \varphi(x) = \frac{(-1)^n}{\Gamma(n-s)} \int_{x}^{1} \frac{\varphi^{(n)}(\xi)d\xi}{(\xi-x)^{s-n+1}} \quad \forall x \in \Lambda. \tag{A.2}$$

For any real $s \ge 0$, we define the spaces

$$^l H^s(\Lambda) := \left\{ v;\, \|v\|_{^l H^s(\Lambda)} < \infty \right\}, \tag{A.3}$$

with

$$\|v\|_{{}^lH^s(\Lambda)} := \left(\|v\|^2_{0,\Lambda} + |v|^2_{{}^lH^s(\Lambda)}\right)^{\frac{1}{2}}, \quad |v|_{{}^lH^s(\Lambda)} := \left\|D^s_x v\right\|_{0,\Lambda}, \tag{A.4}$$

and

$${}^rH^s(\Lambda) := \left\{v; \|v\|_{{}^rH^s(\Lambda)} < \infty\right\}, \tag{A.5}$$

with

$$\|v\|_{{}^rH^s(\Lambda)} := \left(\|v\|^2_{0,\Lambda} + |v|^2_{{}^rH^s(\Lambda)}\right)^{\frac{1}{2}}, \quad |v|_{{}^rH^s(\Lambda)} := \left\|{}_xD^s v\right\|_{0,\Lambda}. \tag{A.6}$$

Lemma A.1. *For real s,* $0 < s < 1$, *if* $w \in {}^lH^s(\Lambda) \cap H^s(\Lambda)$, $v \in C^\infty(\Lambda)$, *then*

$$\left({}^RD^s_x w(x),\ v(x)\right)_\Lambda = \left(w(x),\ {}^R_xD^s v(x)\right)_\Lambda. \tag{A.7}$$

Proof. By using integration by parts, we get

$${}^R_xD^s v(x) = \frac{v(1)}{\Gamma(1-s)(1-x)^s} + {}^C_xD^s v(x).$$

In fact, we have

$$\begin{aligned}
\text{LHS} &= \frac{-1}{\Gamma(1-s)} \frac{d}{dx} \int_x^1 \frac{v(\xi)}{(\xi - x)^s}\, d\xi \\
&= \frac{-1}{\Gamma(1-s)} \left\{ \frac{d}{dx} \left[\frac{v(\xi)(\xi - x)^{1-s}}{1-s}\Big|_x^1 - \frac{1}{1-s} \int_x^1 v'(\xi)(\xi - x)^{1-s}\, d\xi \right] \right\} \\
&= \frac{-1}{\Gamma(1-s)} \left\{ \frac{d}{dx} \left[\frac{v(1)(1-x)^{1-s}}{1-s} \right] - \frac{1}{1-s} \frac{d}{dx} \int_x^1 v'(\xi)(\xi - x)^{1-s}\, d\xi \right\} \\
&= \frac{-1}{\Gamma(1-s)} \left\{ \frac{-v(1)}{(1-x)^s} - \frac{1}{1-s} \int_x^1 \frac{d}{dx} \left[v'(\xi)(\xi - x)^{1-s} \right] d\xi + \frac{1}{1-s} v'(\xi)(\xi - x)^{1-s}|_{\xi = x} \right\} \\
&= \frac{v(1)}{\Gamma(1-s)(1-x)^s} + \frac{-1}{\Gamma(1-s)} \int_x^1 \frac{v'(\xi)}{(\xi - x)^s}\, d\xi \\
&= \text{RHS}.
\end{aligned}$$

On the other hand, for $w \in H^s(\Lambda)$, we have, by Lemma 2.2 in [11],

$$\lim_{x \to -1^+} \int_{-1}^x \frac{w(\xi)}{(x - \xi)^s}\, d\xi = 0.$$

Then, by employing again integration by parts and using the above two equalities, we obtain

$$\begin{aligned}
\left(D^s_x w(x),\ v(x)\right)_\Lambda &= \frac{1}{\Gamma(1-s)} \int_{-1}^1 \frac{d}{dx} \int_{-1}^x \frac{w(\xi)}{(x - \xi)^s}\, d\xi\ v(x)\, dx \\
&= \frac{v(x)}{\Gamma(1-s)} \int_{-1}^x \frac{w(\xi)}{(x - \xi)^s}\, d\xi \Bigg|_{-1}^1 - \frac{1}{\Gamma(1-s)} \int_{-1}^1 \int_{-1}^x \frac{w(\xi)}{(x - \xi)^s}\, d\xi\ v'(x)\, dx \\
&= \frac{v(1)}{\Gamma(1-s)} \int_{-1}^1 \frac{w(\xi)}{(1 - \xi)^s}\, d\xi - \frac{1}{\Gamma(1-s)} \int_{-1}^1 \int_\xi^1 \frac{v'(x)}{(x - \xi)^s}\, dx\ w(\xi)\, d\xi
\end{aligned}$$

$$= \int_{-1}^{1} w(\xi)\Big[\frac{v(1)}{\Gamma(1-s)(1-\xi)^s} + \frac{-1}{\Gamma(1-s)}\int_{\xi}^{1}\frac{v'(x)}{(x-\xi)^s}\,dx\Big]\,d\xi$$
$$= \big(w(\xi),\ {}_{\xi}D^s v(\xi)\big)_\Lambda. \qquad \square$$

For general positive real $s > 0$, we have the following result.

Lemma A.2. *For all positive real $s > 0$, if $w \in {}^lH^s(\Lambda)$, $v \in C_0^\infty(\Lambda)$, then*

$$\big(D_x^s w(x),\ v(x)\big)_\Lambda = \big(w(x),\ {}_xD^s v(x)\big)_\Lambda. \tag{A.8}$$

Proof. Let n be the integer such that $n-1 \le s < n$. By repeating integration by parts n times, we get

$${}_xD^s v(x) = {}_x^C D^s v(x) + \sum_{j=0}^{n-1}(-1)^j \frac{v^{(j)}(1)(1-x)^{j-s}}{\Gamma(1+j-s)} = {}_x^C D^s v(x). \tag{A.9}$$

In virtue of the definition of $D_x^s w$, we have, for the left-hand side of (A.8),

$$\big(D_x^s w(x),\ v(x)\big)_\Lambda = \frac{1}{\Gamma(n-s)}\int_{-1}^{1}\frac{d^n}{dx^n}\int_{-1}^{x}\frac{w(\xi)}{(x-\xi)^{s-n+1}}\,d\xi\ v(x)\,dx$$
$$= \frac{(-1)^n}{\Gamma(n-s)}\int_{-1}^{1}\int_{-1}^{x}\frac{w(\xi)}{(x-\xi)^{s-n+1}}\,d\xi\ v^{(n)}(x)dx$$
$$= \frac{(-1)^n}{\Gamma(n-s)}\int_{-1}^{1}\int_{\xi}^{1}\frac{v^{(n)}(x)}{(x-\xi)^{s-n+1}}dx\ w(\xi)d\xi$$
$$= \Big(w(\xi),\ {}_\xi^C D^s v(\xi)\Big)_\Lambda.$$

Finally, using (A.9) gives (A.8). $\square$

References

[1] C. Bernardi, Y. Maday, Approximations spectrales de problemès aux limites elliptiques, Springer-Verlag, 1992.
[2] L.Z. Chen, J. Shen, C.J. Xu, Spectral direction splitting scheme for incompressible Navier–Stokes equations, East Asian J. Appl. Math. 1 (3) (2011) 215–234.
[3] M. Chen, W. Deng, A second-order numerical method for two-dimensional two-sided space fractional convection diffusion equation, Appl. Math. Model. 38 (13) (2014) 3244–3259.
[4] W. Deng, Finite element method for the space and time fractional Fokker–Planck equation, SIAM J. Numer. Anal. 47 (2008) 204–226.
[5] J. Douglas Jr., S. Kim, On accuracy of alternating direction implicit methods for parabolic equations, Preprint, 1999.
[6] V.J. Ervin, J.P. Roop, Variational formulation for the stationary fractional advection dispersion equation, Numer. Methods Partial Differ. Equ. 22 (3) (2006) 558–576.
[7] J.L. Guermond, P.D. Minev, Start-up flow in a three-dimensional lid-driven cavity by means of a massively parallel direction splitting algorithm, Int. J. Numer. Methods Fluids 68 (7) (2012) 856–871.
[8] X. Ji, H. Tang, High-order accurate Runge–Kutta (local) discontinuous Galerkin methods for one- and two-dimensional fractional diffusion equations, Numer. Math. Theor. Meth. 5 (2012) 333–358.
[9] B. Jin, R. Lazarov, Z. Zhou, Error estimates for a semidiscrete finite element method for fractional order parabolic equations, SIAM J. Numer. Anal. 51 (1) (2013) 445–466.
[10] X.J. Li, C.J. Xu, A space–time spectral method for the time fractional diffusion equation, SIAM J. Numer. Anal. 47 (3) (2009) 2108–2131.
[11] X.J. Li, C.J. Xu, Existence and uniqueness of the weak solution of the space–time fractional diffusion equation and a spectral method approximation, Commun. Comput. Phys. 8 (5) (2010) 1016–1051.
[12] Y.M. Lin, X.J. Li, C.J. Xu, Finite difference/spectral approximations for the fractional cable equation, Math. Comput. 80 (275) (2010) 1369–1396.
[13] Y.M. Lin, C.J. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, J. Comput. Phys. 225 (2) (2007) 1533–1552.
[14] F. Liu, P. Zhuang, V. Anh, I. Turner, K. Burrage, Stability and convergence of the difference methods for the space–time fractional advection–diffusion equation, Appl. Math. Comput. 191 (1) (2007) 12–20.
[15] F. Mainardi, Fractional diffusive waves in viscoelastic solids, in: Nonlinear Waves in Solids, 1995, pp. 93–97.
[16] M.M. Meerschaert, H.P. Scheffler, C. Tadjeran, Finite difference methods for two dimensional fractional dispersion equation, J. Comput. Phys. 211 (2006) 249–261.
[17] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for two-sided space-dimensional fractional partial differential equations, Appl. Numer. Math. 56 (2006) 80–90.
[18] H.P. Müller, R. Kimmich, J. Weis, NMR flow velocity mapping in random percolation model objects: evidence for a power-law dependence of the volume-averaged velocity on the probe-volume radius, Phys. Rev. E 54 (1996) 5278–5285.
[19] H. Scher, M. Lax, Stochastic transport in a disordered solid, Phys. Rev. B 7 (1973) 4491–4502.
[20] H. Scher, E. Montroll, Anomalous transit-time dispersion in amorphous solids, Phys. Rev. B 12 (1975) 2455–2477.

[21] E. Sousa, Numerical approximations for fractional diffusion equations via splines, Comput. Math. Appl. 62 (2011) 938–944.

[22] H. Sun, W. Chen, C. Li, Y. Chen, Finite difference schemes for variable-order time fractional diffusion equation, Int. J. Bifurc. Chaos 22 (4) (2012) 1250085.

[23] C. Tadjeran, M.M. Meerschaert, A second-order accurate numerical method for the two-dimensional diffusion equation, J. Comput. Phys. 220 (2007) 813–823.

[24] H. Wang, T.S. Basu, A fast finite difference method for two-dimensional space-fractional diffusion equations, SIAM J. Sci. Comput. 34 (2012) A2444–A2458.

[25] H. Wang, N. Du, Fast alternating-direction finite difference methods for three-dimensional space-fractional diffusion equations, J. Comput. Phys. 258 (2013) 305–318.

[26] H. Wang, K. Wang, An $O(N\log^2 N)$ alternating-direction finite difference method for two-dimensional fractional diffusion equations, J. Comput. Phys. 230 (2011) 7830–7839.

[27] H. Wang, K. Wang, T. Sircar, A direct $O(N\log^2 N)$ finite difference method for fractional diffusion equations, J. Comput. Phys. 229 (2010) 8095–8104.

[28] H. Wang, D. Yang, Well-posedness of variable-coefficient conservative fractional elliptic differential equations, SIAM J. Numer. Anal. 51 (2013) 1088–1107.

[29] H. Wang, D. Yang, S. Zhu, Inhomogeneous Dirichlet boundary value problems of space-fractional diffusion equations and their finite element approximations, SIAM J. Numer. Anal. 52 (2014) 1292–1310.

[30] B.D. Welfert, Unconditional stability of second-order ADI schemes applied to multi-dimensional diffusion equations with mixed derivative terms, Appl. Numer. Math. 59 (3) (2009) 677–692.

[31] Q. Yang, F. Liu, I. Turner, Numerical methods for fractional partial differential equations with Riesz space fractional derivatives, Appl. Math. Model. 34 (1) (2010) 200–218.

[32] M. Zayernouri, G.E. Karniadakis, Exponentially accurate spectral and spectral element methods for fractional ODEs, J. Comput. Phys. 257 (2014) 460–480.

[33] N. Zhang, W. Deng, Y. Wu, Finite difference/element method for a two-dimensional modified fractional diffusion equation, Adv. Appl. Math. Mech. 4 (2012) 496–518.

[34] Y. Zhang, M.M. Meerschaert, B. Baeumer, Particle tracking for time-fractional diffusion, Phys. Rev. E 78 (2008) 036705.

[35] Y. Zhang, Z. Sun, Alternating direction implicit schemes for the two-dimensional fractional sub-diffusion equation, J. Comput. Phys. 230 (2011) 8713–8728.