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Rescaling of the Roe scheme in low Mach-number flow regions

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ABSTRACT

A rescaled matrix-valued dissipation is reformulated for the Roe scheme in low Mach-number flow regions from a well known family of local low-speed preconditioners popularized by Turkel. The rescaling is obtained explicitly by suppressing the pre-multiplication of the preconditioner with the time derivative and by deriving the full set of eigenspaces of the Roe–Turkel matrix dissipation. This formulation preserves the time consistency and does not require to reformulate the boundary conditions based on the characteristic theory. The dissipation matrix achieves by construction the proper scaling in low-speed flow regions and returns the original Roe scheme at the sonic line. We find that all eigenvalues are nonnegative in the subsonic regime. However, it becomes necessary to formulate a stringent stability condition to the explicit scheme in the low-speed flow regions based on the spectral radius of the rescaled matrix dissipation. With the large disparity of the eigenvalues in the dissipation matrix, this formulation raises a two-timescale problem for the acoustic waves, which is circumvented for a steady-state iterative procedure by the development of a robust implicit characteristic matrix time-stepping scheme. The behaviour of the modified eigenvalues in the incompressible limit and at the sonic line also suggests applying the entropy correction carefully, especially for complex non-linear flows.

1. Introduction

Low-speed preconditioning based on Chorin artificial compressibility has become widely used for computing low-speed flow configurations with numerical schemes developed for compressible flows. This local preconditioning technique was designed to achieve an optimal conditioning of the iterative procedure and to guaranty the proper scaling of the artificial dissipation when the Mach number approaches zero. The low-speed preconditioning approach has proved to be very efficient to overcome the accuracy issue of compressible flow solvers in the incompressible limit. Actually, the low-speed preconditioning should always be used since many industrial applications are characterized by mixed compressible and incompressible flows, over a wide range of Reynolds numbers.

However, this approach suffers from the complexity of its practical implementation. Since the local preconditioner modifies the characteristic relations, all boundary conditions based on characteristic variables or Riemann invariants must be reformulated accordingly. For large aerodynamics codes in which a large number of boundary conditions may be implemented, it is then necessary to reformulate most of the boundary conditions. Furthermore, the extension to unsteady flows is not trivial and without a special treatment, time-accuracy may be lost.

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Thus, a practical point of view has essentially motivated this contribution. For the compressible Euler equations, it was found interesting to investigate the effect of removing the pre-multiplication of the preconditioning matrix with the time-derivative of the independent flow variables. This formulation sometimes called improperly "preconditioning of the stabilization terms only" has been investigated particularly by Guillard and Viozat in [1,2] and by Birken and Meister in [3]. This formulation actually doesn't improve the conditioning and yields a large disparity in the eigenvalues of the matrix dissipation. However, this issue can be circumvented with some augmented Jacobi preconditioning, as mentioned in [4]. In addition, with this simplification, the explicit scheme recovers a basic structure with the centred scheme and stabilization terms. Then it becomes no longer necessary to reformulate the characteristic curves for the preconditioned Jacobian matrix and the time accuracy is preserved. Some recent attempts of improving the accuracy of conservative schemes in the low speed limit may have been also motivated by a drastic simplification of the implementation of the low-speed preconditioning. This is especially the case of the Rieper low Mach-number fix proposed in [5], and the Thornber et al. "Low Mach" LMRoe scheme [6] modifying jumps of the discrete velocities, which was further developed by Oßwald et al. with the "Low dissipation Low Mach" L^2Roe scheme [7]. This may also explain the success of the AUSM-family schemes and their modification for low Mach-number flows [8,9]. An alternative to modifying the matrix dissipation is to introduce a modified speed of sound, as considered by Rossow [10], and further extended in [11]. This approach was also investigated by Li and Gu in [12].

The problem of the accuracy in the asymptotic limit of the incompressible flow for the discretization of the normalized Euler equations was first addressed by Guillard and Viozat [1,2]. For their "preconditioning of the stabilization terms only" formulated with the Roe scheme as defined in [2], it was shown that the checkerboard modes for the leading and second-order pressure fields are cancelled out by the rescaled matrix dissipation and that the pressure field should be constant in space up to a fluctuation in space of order two. Furthermore, the authors have clearly pointed out a lack of dissipation of the standard conservative schemes in the incompressible limit.

A number of authors have further considered the discrete analysis based on the normalized equations for the asymptotic behaviour of the pressure and velocity fields. An "all-speed Roe scheme" has been developed by Li and Gu [13] in order to recover at the discrete level the divergence constraint of the leading order velocity and the Poisson equation for the second-order pressure, which are not satisfied by the preconditioned Roe scheme formulated in [1,2]. However, checkerboards modes are not automatically suppressed by their numerical flux and a low Mach number fix proposed by Rieper for the Roe's approximate Riemann solver [5] seems attractive, as combining the advantages in the incompressible limit of both approaches investigated in [1] and [13].

Over the last years, many modified Roe-type [1,5–7,13,14,12], AUSM-type [8,9], flux-splitting [15] or Godunov-type schemes [16,18] have been formulated to apply conservative finite-difference schemes to low-speed flows. It has been found necessary to propose a unified theoretical framework to analyse their respective discrete properties and to understand why they fail to be accurate in the incompressible limit without specific corrections. It is worth mentioning the work of Li and Gu for the analysis of Roe-type schemes [12], based on the flux splitting of the dissipation vector introduced in [17] and the contribution of Dellacherie for Godunov-type schemes [18] using the Hodge decomposition for solutions derived from the one-dimensional barotropic Euler equations.

On the other hand, few contributions have addressed the numerical stability of shock-capturing schemes adapted for low-speed flows. A Fourier Analysis is carried out by Dellacherie for the one-dimensional wave equations using the low Mach Godunov scheme and an explicit CFL condition is formulated for both the explicit and the implicit scheme [18]. For a formulation of the compressible Euler equations with "preconditioning of the stabilization terms only", the issue of the stability for a matrix-valued dissipation formulated from the Lax–Friedrich scheme is pointed out for the first time in [3] on the basis of the asymptotic behaviour of the largest eigenvalue in the incompressible limit. Formulating a stability criteria is also an essential feature when designing numerical schemes for complex flows, especially when an "all-speed scheme" is being developed. Results obtained by Birken and Meister clearly show for the Euler equations that the standard CFL condition used for the computation of compressible flows is no longer valid in the incompressible limit and that a stringent stability condition for the time step with $\Delta t \simeq \mathcal{O}(M^2)$ when the Mach number $M \to 0$ must be accounted for when the "preconditioning of the stabilization terms only" is considered. Nevertheless, the eigenspaces of the matrix-valued dissipation are not derived and a practical CFL condition for the local time step is not formulated explicitly for the fastest acoustic speed.

The main concern of this contribution is to reformulate a consistent matrix-valued dissipation with the low-speed limit and the transonic regime, and the corresponding stability condition, in the multidimensional case. We have considered the Roe scheme [19] as baseline formulation for the matrix dissipation. The necessary rescaling of the Roe scheme in the incompressible limit is formulated from a family of preconditioners popularized by Turkel [20–26]. The scheme is also sometimes termed as the Roe–Turkel scheme [2,18]. This reformulation corresponds to a drastic change of the stabilization terms and therefore the necessary Von Neumann criteria for the linear stability must be reconsidered completely. This can be achieved only by deriving the eigenvalues and the full set of the right and left eigenvectors of the matrix-valued dissipation, which surprisingly has never been done so far. The diagonalization of the rescaled matrix-valued dissipation must be achieved for the computation of complex flows, also because somehow an entropy fix may be used to prevent eigenvalues from approaching zero and to select the relevant physical solution satisfying the entropy condition across shocks.

With this reformulation, we lose the optimal conditioning but we save the essential feature of the accuracy in the incompressible limit. In particular, we see that by forcing the proper scaling of the dissipation matrix, we cannot avoid the two-timescale issue, since the isentropic solution breaks up into a fast component that should disappear when the Mach number goes to zero, and a slow component solution of the incompressible equations [16]. The system then becomes very stiff and numerically difficult to solve with the explicit scheme and a time step defined by the stability condition for the fastest acoustic speed. For steady-flow problems, a robust implicit scheme must be developed to improve both the stability and the damping properties of the iterative procedure. An augmented Runge–Kutta/implicit scheme developed by Rossow in [10] especially addresses this analytical stiffness in the incompressible limit. However, we must stress that our final objective is not to develop an optimal approach specialized to handle almost incompressible flows, but only a more accurate compressible flow solver in low speed flow regions, without reformulating the boundary conditions and keeping the consistency in time for unsteady flows.

In the following, we shall reformulate the rescaled Roe scheme following Turkel's analysis, and the corresponding stability condition for the explicit scheme. For steady-state problems, a robust implicit scheme is developed in order to circumvent the severe stability bound occurring for low Mach number flows. The implicit scheme is characterized by a characteristic time-step matrix, enforcing the damping properties of the numerical procedure. For transonic flows, it is shown how that the entropy fix and the characteristic time-step matrix must be carefully formulated in order to transition smoothly to the Roe scheme. Preliminary results for low-speed and transonic flows are presented. As expected, it is experienced that a consistent approximation is achieved with the rescaled matrix dissipation in the incompressible limit. It is also shown in the transonic case that the known spurious entropy produced by the Roe scheme at the stagnation point is strongly reduced by the rescaling of the Roe matrix dissipation.

2. Formulation of the rescaled Roe scheme

2.1. General framework

The rescaled Roe scheme is based on the generalization of low-speed preconditioners due to Turkel [21–23], formulated for the independent variables $\boldsymbol{W_0} = [p, u, v, S]^T$ in the two-dimensional case, where p is the pressure u, v are the velocity components and S is the entropy. In all the following, ρ is the density, a denotes the speed of sound and M the Mach number.

The family of preconditioners we shall consider contain a free parameter β as formulated by Choi–Merkle and is defined for the $\boldsymbol{W_0}$ variables with

$$\boldsymbol{P_0}^{-1} = \begin{pmatrix} \dfrac{a^2}{\beta^2} & 0 & 0 & \delta \\ \dfrac{\alpha u}{\rho\beta^2} & 1 & 0 & 0 \\ \dfrac{\alpha v}{\rho\beta^2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \boldsymbol{P_0} = \begin{pmatrix} \dfrac{\beta^2}{a^2} & 0 & 0 & -\dfrac{\beta^2}{a^2}\delta \\ -\dfrac{\alpha u}{\rho a^2} & 1 & 0 & \dfrac{\alpha u}{\rho a^2}\delta \\ -\dfrac{\alpha v}{\rho a^2} & 0 & 1 & \dfrac{\alpha v}{\rho a^2}\delta \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

With $\delta = 0$ and $\alpha = 0$ we get the family of preconditioners including the Weiss–Smith formulation [17], while setting $\delta = 1$ and $\alpha = 0$ we recover the preconditioner introduced by Choi and Merkle [27]. The case $\alpha = 0$ corresponds to the original artificial compressibility method formulated for the primitive variables while $\alpha = -1$ corresponds to the artificial compressibility method formulated in conservation form [20]. The value $\alpha = 1$ may be also considered for viscous flow [29]. To transition smoothly to transonic flow, we demand that $\frac{\beta^2}{a^2} \to 1$ with $\alpha = 0$ and $\delta = 0$ as $M \to 1$. In that case the preconditioning matrix returns the identity matrix.

In our implementation, the Choi–Merkle preconditioning parameter will assume the following formulation

$$\frac{\beta^2}{a^2} = min(max(M^2, \epsilon^2), 1), \tag{2}$$

where ϵ^2 is a cut-off value, which may depend on the flow physics. It may become necessary to enforce the accuracy and especially the robustness of the preconditioned scheme in stagnation point regions and in the boundary layer with high-aspect ratio grids. It has been especially demonstrated in [28] that a better control of the free parameter $\frac{\beta^2}{a^2}$ has to be achieved in the low Reynolds number flow regions. The authors suggest the cut-off value ϵ^2 to be related to the isentropic Mach number inside the boundary layer. In [29,31], the “inviscid” preconditioning parameter is modified for high lift configurations and a cell Reynolds number is considered as a viscous correction.

It is worth mentioning that for subcritical or transonic inviscid flows, it is essential to set the small parameter ϵ^2 to a much smaller value than the reference Mach number M_∞^2 to preserve the higher accuracy of the rescaled Roe scheme in the low-speed flow regions. In our low-speed flow computations, the small parameter ϵ^2 is set to a cut-off value for all inflow Mach numbers considered. In [1], the use of the small parameter ϵ^2 was not mentioned. This doesn't seem to create any numerical difficulties. On the other hand, no spurious dissipation effects due to corrected eigenvalues are introduced. For low Mach-number flow computations, the Van Leer formulation was also considered

$$\frac{\beta^2}{a^2} = min(\frac{2M^2}{1-2M^2}, 1), \quad \text{so} \quad \frac{\beta^2}{a^2} = 1 \quad \text{as} \quad M \geq 0.5. \tag{3}$$

The rescaled Roe scheme is intended to be used in the compressible regime, and is formulated in a final analysis for the conservative variables $\boldsymbol{W}$ by removing the pre-multiplication of the time derivative with the preconditioning matrix

$$\Delta \boldsymbol{W} = -\Delta t \left[\left(\frac{\partial \boldsymbol{F}}{\partial x} + \frac{\partial \boldsymbol{G}}{\partial y} \right) - \frac{\partial}{\partial x} \left(\frac{h_x}{2} \boldsymbol{P}^{-1} |\boldsymbol{P A}| \frac{\partial \boldsymbol{W}}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{h_y}{2} \boldsymbol{P}^{-1} |\boldsymbol{P B}| \frac{\partial \boldsymbol{W}}{\partial y} \right) \right], \tag{4}$$

where h_x and h_y are some space increments, $\boldsymbol{A}$ and $\boldsymbol{B}$ are the flux Jacobian matrices in the (x, y) space directions and $\boldsymbol{P}$ is the preconditioner (1) expressed for the conservative variables. We can see that this semi-discrete scheme is a time-consistent discretization of the Euler equations based on a backward Euler approximation of the time derivative. In the framework of a steady state problem, the time step Δt is replaced by a local time step. Additionally the formulation (4) alleviates its implementation in a CFD code since once removed the pre-multiplication of the preconditioner with the time derivative, there is no need to reformulate the boundary conditions using the characteristic theory or the Riemann invariants.

Thus, this reformulation corresponds to a different approach, aiming at only modifying the Roe matrix dissipation as investigated over the last decade by many authors (Li & Gu, Rossow, Rieper followed by Thornber and Oßwald, and some others). The scheme must be interpreted as a consistent rescaling of the dissipation for low-speed flows, but not as a preconditioning method, as it doesn't improve the condition number of the iterative procedure. The stability bound is drastically modified in the low-speed flow regions. As demonstrated in [3], the stability must be reconsidered from matrices $\boldsymbol{P}^{-1}|\boldsymbol{PA}|$ and $\boldsymbol{P}^{-1}|\boldsymbol{PB}|$. It becomes then essential to reformulate the Von Neumann condition by deriving algebraically the corresponding eigenspaces and eigenvalues. This can be achieved step by step as indicated in the two next sections, starting from the stream-aligned formulation of the Euler equations for the symmetrizing entropic variables.

2.2. One-dimensional formulation

In order to investigate the limit to incompressible equations as the Mach number goes to zero, we follow Turkel [25] and explicitly describe the scheme for the one-dimensional equations, including a second-order artificial viscosity term. For the equations termed in entropic variables, we shall consider first a streamwise two-dimensional coordinate system with the x-axis aligned to the flow. Then setting $\frac{\partial v}{\partial y} = 0$, $\frac{\partial p}{\partial y} = 0$ in the Euler equations and taking as independent variables for the pressure $d\Phi = \frac{dp}{\rho a}$ and $dS = \frac{dp - a^2 d\rho}{\rho a}$ proportional to the entropy for the symmetrizing variables, the equations for v and S decouple and so we shall consider the two acoustic equations. Let introduce some artificial dissipation terms with constant coefficients. We then get the following scheme in differential $\boldsymbol{d\tilde{W}_0} = [d\Phi, du]^T$ variables

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} + u\dfrac{\partial \Phi}{\partial x} + a\dfrac{\partial u}{\partial x} = Q_{1,1}\dfrac{\partial^2 \Phi}{\partial x^2} + Q_{1,2}\dfrac{\partial^2 u}{\partial x^2}, \\ \dfrac{\partial u}{\partial t} + a\dfrac{\partial \Phi}{\partial x} + u\dfrac{\partial u}{\partial x} = Q_{2,1}\dfrac{\partial^2 \Phi^2}{\partial x^2} + Q_{2,2}\dfrac{\partial^2 u}{\partial x^2}. \end{cases} \tag{5}$$

In matrix form, the previous system reads

$$\frac{\partial \boldsymbol{\tilde{W}_0}}{\partial t} + \boldsymbol{\tilde{A}_0}\frac{\partial \boldsymbol{\tilde{W}_0}}{\partial x} = \boldsymbol{Q}\,\frac{\partial^2 \boldsymbol{\tilde{W}_0}}{\partial x^2}. \tag{6}$$

The limit to incompressible equations for the scheme (5) has been investigated in [25,26]. Within a formal asymptotic analysis, the mesh size being kept constant, it was found that in the incompressible limit, the proper scaling of the matrix-valued dissipation requires

$$Q_{1,1} = \mathcal{O}(\frac{1}{M^2}), \quad Q_{1,2} = \mathcal{O}(\frac{1}{M}), \quad Q_{2,1} = \mathcal{O}(\frac{1}{M}), \quad Q_{2,2} = \mathcal{O}(1). \tag{7}$$

This is not satisfied when the dissipation is some function of the matrix $|\boldsymbol{\tilde{A}_0}|$. For the system in variables $\boldsymbol{d\tilde{W}_0}$, the preconditioner becomes

$$\boldsymbol{\tilde{P}_0}^{-1} = \begin{pmatrix} \dfrac{a^2}{\beta^2} & 0 \\ \dfrac{\alpha u a}{\beta^2} & 1 \end{pmatrix} \quad , \quad \boldsymbol{\tilde{P}_0} = \begin{pmatrix} \dfrac{\beta^2}{a^2} & 0 \\ -\dfrac{\alpha u}{a} & 1 \end{pmatrix}. \tag{8}$$

The preconditioned Jacobian matrix

$$\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{A}}_{\mathbf{0}} = \begin{pmatrix} \dfrac{\beta^2}{a^2}u & \dfrac{\beta^2}{a} \\ a\left(1-\alpha\dfrac{u^2}{a^2}\right) & (1-\alpha)u \end{pmatrix}, \tag{9}$$

has the two acoustic eigenvalues $\lambda_\pm$

$$\begin{cases} \lambda_0 = u, \\ \lambda_\pm = zu \, \pm \sqrt{(zu)^2 + \dfrac{\beta^2}{a^2}(a^2-u^2)}, \qquad z = 0.5\left(1-\alpha+\dfrac{\beta^2}{a^2}\right). \end{cases} \tag{10}$$

We have the following diagonalization $|\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{A}}_{\mathbf{0}}| = \tilde{\boldsymbol{R}}|\boldsymbol{\Lambda}|\tilde{\boldsymbol{R}}^{-1}$, where $|\boldsymbol{\Lambda}| = \mathrm{diag}(|\lambda_+|, |\lambda_-|)$, $\boldsymbol{R}$ and $\tilde{\boldsymbol{R}}^{-1}$ are respectively the matrices of the right and left eigenvectors

$$\tilde{\boldsymbol{R}} = \begin{pmatrix} \dfrac{\beta^2}{a} & \dfrac{\beta^2}{a} \\ \lambda_+ - \dfrac{\beta^2}{a^2}u & \lambda_- - \dfrac{\beta^2}{a^2}u \end{pmatrix}, \quad \tilde{\boldsymbol{R}}^{-1} = \frac{1}{(\lambda_+-\lambda_-)}\begin{pmatrix} \dfrac{a}{\beta^2}[\lambda_+ - (1-\alpha)u] & 1 \\ -\dfrac{a}{\beta^2}[\lambda_- - (1-\alpha)u] & -1 \end{pmatrix}. \tag{11}$$

In the $\boldsymbol{d\tilde{W}_0}$ variables, the rescaled Roe scheme is characterized by the matrix-valued dissipation

$$\tilde{\boldsymbol{Q}}_{\mathbf{0}} = \tilde{\boldsymbol{P}}_{\mathbf{0}}^{-1}|\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{A}}_{\mathbf{0}}| = \tilde{\boldsymbol{P}}_{\mathbf{0}}^{-1}\tilde{\boldsymbol{R}}|\boldsymbol{\Lambda}|\tilde{\boldsymbol{R}}^{-1}.$$

In the following, we will assume the flow subsonic. Let $\chi = (zu)^2 + \frac{\beta^2}{a^2}(a^2-u^2)$. We always have $\chi > 0$ and

$$\chi \geq (zu)^2 \Rightarrow \left(zu - \sqrt{\chi}\right)\left(zu + \sqrt{\chi}\right) \leq 0.$$

Hence, eigenvalues λ_+ and λ_- (10) have opposite signs for a subsonic flow with

$$\text{if } u \leq 0 \;\rightarrow\; \lambda_- \leq 0 \Rightarrow \lambda_+ \geq 0,$$
$$\text{if } u \geq 0 \;\rightarrow\; \lambda_+ \geq 0 \Rightarrow \lambda_- \leq 0.$$

Thus $|\lambda_+| = \lambda_+$ and $|\lambda_-| = -\lambda_-$ and we get an explicit expression of the dissipation matrix

$$\tilde{\boldsymbol{P}}_{\mathbf{0}}^{-1}|\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{A}}_{\mathbf{0}}| = \frac{1}{\sqrt{\chi}}\begin{pmatrix} \left[zM^2 + \left(1-M^2\right)\right]a^2 & z\,u\,a \\ \left[z+\alpha\left(1-M^2\right)\right]u\,a & \left[zM^2 + \dfrac{\beta^2}{a^2}\left(1-M^2\right)\right]a^2 \end{pmatrix}, \tag{12}$$

where $M = \dfrac{|u|}{a}$. Note that for the two acoustic equations

$$\tilde{\boldsymbol{P}}_{\mathbf{0}}^{-1}|\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{A}}_{\mathbf{0}}| = \frac{1}{\sqrt{\chi}}\left[zu\tilde{\boldsymbol{A}}_{\mathbf{0}} + \frac{\beta^2}{a^2}(a^2-u^2)\tilde{\boldsymbol{P}}_{\mathbf{0}}^{-1}\right] \tag{13}$$

as a consequence of the Cayley–Hamilton theorem (see Appendix A). Thus, the dissipation matrix can be interpreted as a weighted average between the Jacobian matrix and the preconditioner.

At a sonic point the second coefficient vanishes and it can be easily verified that for both possible values $u = \pm a$ the rescaled matrix dissipation returns $|\tilde{\boldsymbol{A}}_{\mathbf{0}}|$ for $\alpha = 0$. Let be Q_{ij} the coefficients of matrix (12). In the limit to the incompressible equations, assuming the flow isentropic, with $\beta^2 \simeq u^2 = \mathcal{O}(1)$, we successively have

$$\frac{\beta^2}{a^2} \simeq \frac{u^2}{a^2} = \mathcal{O}(M^2), \qquad \chi = z^2u^2 + \beta^2\left(1-M^2\right) \simeq u^2 = \mathcal{O}(1),$$

and

$$Q_{11} = \frac{1}{\sqrt{\chi}}\left[zM^2 + \left(1-M^2\right)\right]a^2 \simeq \frac{a^2}{|u|} = \mathcal{O}(\frac{1}{M^2}), \qquad Q_{12} = \frac{1}{\sqrt{\chi}}\,z\,u\,a \simeq a = \mathcal{O}(\frac{1}{M})$$
$$Q_{21} = \frac{1}{\sqrt{\chi}}\left[z+\alpha\left(1-M^2\right)\right]\,u\,a \simeq a = \mathcal{O}(\frac{1}{M}), \qquad Q_{22} = \frac{1}{\sqrt{\chi}}\left[zM^2 + \frac{\beta^2}{a^2}\left(1-M^2\right)\right]a^2 \simeq |u| = \mathcal{O}(1).$$

Then, as $M \to 0$

$$\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}| \simeq \begin{pmatrix} \mathcal{O}(\frac{1}{M^2}) & \mathcal{O}(\frac{1}{M}) \\ \mathcal{O}(\frac{1}{M}) & \mathcal{O}(1) \end{pmatrix}, \tag{14}$$

and the scheme gives by construction the correct asymptotic order for the artificial dissipation terms.

We also see that matrix $\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|$ has two real eigenvalues given by

$$\mu_{\pm} = \frac{a}{2\sqrt{\tilde{\chi}}}\left[2z + \alpha\left(1 - M^2\right) \pm \sqrt{\Delta}\right], \tag{15}$$

with

$$\Delta = \left(1 - \frac{\beta^2}{a^2}\right)^2 \left(1 - M^2\right)^2 + 4z\left[z + \alpha\left(1 - M^2\right)\right]M^2, \qquad \tilde{\chi} = z^2 M^2 + \frac{\beta^2}{a^2}(1 - M^2).$$

These acoustic eigenvalues are also given in [3] for the one-dimensional system in conservative variables, with a different formalism. We also find that both eigenvalues $\mu_{\pm}$ are positive as long as the flow is subsonic with

$$\mu_+\mu_- = Q_{11}Q_{22} - Q_{12}Q_{21} = \det\left(\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|\right),$$

and

$$\det\left(\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|\right) = \frac{\left[\left(z - \frac{\beta^2}{a^2}\right)^2 - \alpha\frac{\beta^2}{a^2}\right]u^2 + \beta^2}{\left[z^2 - \frac{\beta^2}{a^2}\right]u^2 + \beta^2}\left(a^2 - u^2\right) = a^2 - u^2 = \det\left(|\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|\right) > 0.$$

Note that rescaled matrix dissipation is symmetric for $\alpha = 0$ and therefore is positive definite with the two positive eigenvalues for in case of subsonic flow. This property is lost by any change of variables that does not preserve the symmetry of $\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|$. In the incompressible limit, the eigenvalues behave as

$$\mu_{\pm} \simeq \frac{a^2}{2|\beta|}\left[1 + \frac{\beta^2}{a^2} \pm |1 - \frac{\beta^2}{a^2}|\right].$$

Since $\frac{\beta^2}{a^2} \leq 1$

$$\mu_+ \simeq \frac{a^2}{|\beta|} = \mathcal{O}(\frac{1}{M^2}), \qquad \mu_- \simeq |\beta| = \mathcal{O}(1).$$

Thus, in the asymptotic limit acoustics waves associated to μ_+ will travel at infinite speed while acoustic waves associated to μ_- are slowed down to the flow velocity. This is a general behaviour of the eigenvalues $\mu_{\pm}$ in the low speed limit, because (7) holds necessarily with the coefficients Q_{ij}, for a proper scaling of the artificial dissipation, whatever the definition of the preconditioner $\tilde{\boldsymbol{P}}_{\boldsymbol{0}}$. Note that this asymptotic behaviour is also independent of the parameter α. This raises the well known singularity of the two time-scale problem. The optimal conditioning is lost and is even worse that the original system. The behaviour of eigenvalues $\mu_{\pm}$ in the low Mach number limit will become critical for the stability of the rescaled Roe scheme and a CFL condition must be formulated accordingly. For the other extreme case, at a sonic line with $\alpha = 0$, we see that we recover the absolute values of the unpreconditioned system

$$\mu_+ = 2a, \qquad \mu_- = 0.$$

When the preconditioning is not activated, setting $\alpha = 0$ and $\frac{\beta^2}{a^2} = 1$ in expressions (15) gives $\mu_{\pm} = a(1 \pm M) = a \pm |u|$, the eigenvalues of $|\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|$ recast for the velocity module.

We can also easily derive the following diagonalization of the rescaled matrix dissipation

$$\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}| = \tilde{\boldsymbol{R}}_{\boldsymbol{0}}\tilde{\boldsymbol{\Lambda}}\tilde{\boldsymbol{R}}_{\boldsymbol{0}}^{-1}, \tag{16}$$

where $\tilde{\boldsymbol{\Lambda}} = \mathrm{diag}(\mu_+, \mu_-)$, $\tilde{\boldsymbol{R}}_{\boldsymbol{0}}$ is the matrix of the right eigenvectors and $\tilde{\boldsymbol{R}}_{\boldsymbol{0}}^{-1}$ is obtained by direct inversion of $\tilde{\boldsymbol{R}}_{\boldsymbol{0}}$

$$\tilde{\boldsymbol{R}}_{\boldsymbol{0}} = \begin{pmatrix} [\mu_+ - Q_{22}] & Q_{12} \\ Q_{21} & [\mu_- - Q_{11}] \end{pmatrix}, \quad \tilde{\boldsymbol{R}}_{\boldsymbol{0}}^{-1} = \frac{1}{\det\left(\tilde{\boldsymbol{R}}_{\boldsymbol{0}}\right)}\begin{pmatrix} [\mu_- - Q_{11}] & -Q_{12} \\ -Q_{21} & [\mu_+ - Q_{22}] \end{pmatrix}, \tag{17}$$

with the determinant of matrix $\tilde{\boldsymbol{R}}_{\mathbf{0}}$ given by

$$\begin{aligned}\det\left(\tilde{\boldsymbol{R}}_{\mathbf{0}}\right) &= [\mu_- - Q_{11}][\mu_+ - Q_{22}] - Q_{12}Q_{21} = \mu_+[\mu_- - Q_{11}] + \mu_-[\mu_+ - Q_{22}] \\ &= [\mu_+ - Q_{22}][\mu_- - \mu_+] = [Q_{11} - \mu_-][\mu_- - \mu_+].\end{aligned} \tag{18}$$

At a stagnation point, $Q_{12} = 0$, $Q_{21} = 0$, and

$$\mu_+ = Q_{11} = \frac{a^2}{|\beta|}, \quad \mu_- = Q_{22} = |\beta|, \quad \mu_+ - \mu_- = |\beta|\left(\frac{a^2}{\beta^2} - 1\right).$$

Thus with $\mu_+ - \mu_- \neq 0$, we always have $\mu_- - Q_{11} \neq 0$, $\mu_+ - Q_{22} \neq 0$, and the decomposition (16) can never become singular in the range $|u| < a$. Additionally, with $\alpha = 0$, since by construction $\mu_+ + \mu_- = Q_{11} + Q_{22}$, we have

$$(\mu_+ - Q_{22})Q_{12} + (\mu_- - Q_{11})Q_{21} = 0 \quad \text{with} \quad Q_{12} = Q_{21}, \tag{19}$$

and we can see that the right and left eigenvectors are orthogonal. In this special case

$$\det\left(\tilde{\boldsymbol{R}}_{\mathbf{0}}\right) = -\left[(\mu_+ - Q_{22})^2 + Q_{12}^2\right].$$

Then defining

$$\tilde{\boldsymbol{M}}_{\mathbf{0}} = \frac{1}{\sqrt{(\mu_+ - Q_{22})^2 + Q_{12}^2}}\tilde{\boldsymbol{R}}_{\mathbf{0}} \tag{20}$$

the symmetric matrix $\tilde{\boldsymbol{P}}_{\mathbf{0}}^{-1}|\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{A}}_{\mathbf{0}}|$ is diagonalized by the following unitary congruence

$$\tilde{\boldsymbol{P}}_{\mathbf{0}}^{-1}|\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{A}}_{\mathbf{0}}| = \tilde{\boldsymbol{M}}_{\mathbf{0}}\tilde{\Lambda}\tilde{\boldsymbol{M}}_{\mathbf{0}}^T. \tag{21}$$

2.3. Multidimensional extension

We derive the multidimensional extension of the previous analysis, assuming the flow subsonic. Let be $\boldsymbol{d\tilde{W}_0} = [d\Phi, du, dv, dS]^T$ the set of entropy variables for the two-dimensional Euler equations. In matrix form, the two-dimensional rescaled Roe scheme is formulated as follows

$$\frac{\partial \tilde{\boldsymbol{W}}_{\mathbf{0}}}{\partial t} + \tilde{\boldsymbol{A}}_{\mathbf{0}}\frac{\partial \tilde{\boldsymbol{W}}_{\mathbf{0}}}{\partial x} + \tilde{\boldsymbol{B}}_{\mathbf{0}}\frac{\partial \tilde{\boldsymbol{W}}_{\mathbf{0}}}{\partial y} = \frac{h_x}{2}\tilde{\boldsymbol{P}}_{\mathbf{0}}^{-1}|\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{A}}_{\mathbf{0}}|\frac{\partial^2 \tilde{\boldsymbol{W}}_{\mathbf{0}}}{\partial x^2} + \frac{h_y}{2}\tilde{\boldsymbol{P}}_{\mathbf{0}}^{-1}|\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{B}}_{\mathbf{0}}|\frac{\partial^2 \tilde{\boldsymbol{W}}_{\mathbf{0}}}{\partial y^2}. \tag{22}$$

We know from Turkel that the eigenvalues of the preconditioned Jacobian matrix $\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{A}}_{\mathbf{0}}$ are independent of the parameter δ in the preconditioner [23]. This is also true for the rescaled matrix dissipation $\tilde{\boldsymbol{P}}_{\mathbf{0}}^{-1}|\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{A}}_{\mathbf{0}}|$. Thus, in the following, we will set $\delta = 0$ in the family of preconditioners, as we are basically interested in finding out a stability criteria for the rescaled Roe scheme.

For the multidimensional extension, we consider a general direction $\boldsymbol{n} = [n_x, n_y]^T$ and we shall express the Jacobian matrix in the general form $\tilde{\boldsymbol{A}}_{\mathbf{0}}(\boldsymbol{n}) = n_x\tilde{\boldsymbol{A}}_{\mathbf{0}} + n_y\tilde{\boldsymbol{B}}_{\mathbf{0}}$. It is indeed important in the following analysis to differentiate the local Mach number $M = \dfrac{\sqrt{u^2+v^2}}{a}$ defining the preconditioning parameter $\frac{\beta^2}{a^2}$ and the directional Mach number $M_n = \dfrac{|q|}{a|\boldsymbol{n}|}$ arising explicitly in the multidimensional formulation of the rescaled matrix dissipation.

The preconditioner $\tilde{\boldsymbol{P}}_{\mathbf{0}}$ is formulated from (1) with $\delta = 0$ in $\boldsymbol{d\tilde{W}_0}$ variables. The preconditioned Jacobian matrix then reads

$$\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{A}}_{\mathbf{0}}(\boldsymbol{n}) = \begin{pmatrix} \dfrac{\beta^2}{a^2}q & \dfrac{\beta^2}{a}n_x & \dfrac{\beta^2}{a}n_y & 0 \\ a\left(n_x - \dfrac{\alpha u q}{a^2}\right) & q - \alpha u n_x & -\alpha u n_y & 0 \\ a\left(n_y - \dfrac{\alpha v q}{a^2}\right) & -\alpha v n_x & q - \alpha v n_y & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

where in all the following $q = un_x + vn_y$ is the normal velocity. In the multidimensional case, we have the following diagonalization $|\tilde{\boldsymbol{P}}_{\mathbf{0}}\tilde{\boldsymbol{A}}_{\mathbf{0}}(\boldsymbol{n})| = \tilde{\boldsymbol{R}}|\Lambda|\tilde{\boldsymbol{R}}^{-1}$ given in [23] for eigenvalues

$$\begin{cases} \lambda_0 = q, \\ \lambda_\pm = zq \pm \sqrt{(zq)^2 + \dfrac{\beta^2}{a^2}\left(a^2|\boldsymbol{n}|^2 - q^2\right)} \quad \text{with} \quad z = 0.5\left(1 - \alpha + \dfrac{\beta^2}{a^2}\right). \end{cases} \tag{23}$$

In case of subsonic flow, we know the sign of the two acoustic eigenvalues, with $\lambda_+ \geq 0$ and $\lambda_- \leq 0$. Therefore

$$|\Lambda| = \mathrm{diag}(\lambda_+, -\lambda_-, |q|, |q|).$$

We proceed as for the one-dimensional case to find an explicit expression of the matrix valued dissipation

$$\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}(\boldsymbol{n})| = \tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{R}}|\Lambda|\tilde{\boldsymbol{R}}^{-1}.$$

Let first introduce the following coefficients, functions of the directional Mach number

$$Q_{11} = \frac{1}{\sqrt{\chi}}\,|\boldsymbol{n}|^2\left[zM_n^2 + \left(1 - M_n^2\right)\right]a^2, \qquad Q_{12} = \frac{1}{\sqrt{\chi}}\,z\,q\,a,$$
$$Q_{21} = \frac{1}{\sqrt{\chi}}\,|\boldsymbol{n}|^2\left[z + \alpha\left(1 - M_n^2\right)\right]q\,a, \qquad Q_{22} = \frac{1}{\sqrt{\chi}}\,|\boldsymbol{n}|^2\left[zM_n^2 + \frac{\beta^2}{a^2}\left(1 - M_n^2\right)\right]a^2,$$

and

$$\chi = (zq)^2 + \frac{\beta^2}{a^2}\left(a^2|\boldsymbol{n}|^2 - q^2\right) = a^2|\boldsymbol{n}|^2\left[z^2M_n^2 + \frac{\beta^2}{a^2}(1 - M_n^2)\right].$$

With $\lambda_\pm$ given by (23), and using the identity $(\lambda_+ - q)(\lambda_- - q) = \alpha q^2 - \beta^2|\boldsymbol{n}|^2$, we also define

$$\xi_x = \frac{\alpha u q - \beta^2 n_x}{\alpha q^2 - \beta^2|\boldsymbol{n}|^2}, \quad \xi_y = \frac{\alpha v q - \beta^2 n_y}{\alpha q^2 - \beta^2|\boldsymbol{n}|^2},$$

and

$$\zeta_x = \frac{n_y(un_y - vn_x)}{\alpha q^2 - \beta^2|\boldsymbol{n}|^2}, \quad \zeta_y = -\frac{n_x(un_y - vn_x)}{\alpha q^2 - \beta^2|\boldsymbol{n}|^2}.$$

We see that

$$\xi_x n_x + \xi_y n_y = 1 \quad \text{and} \quad \zeta_x n_x + \zeta_y n_y = 0. \tag{24}$$

The rescaled matrix dissipation can be expressed as follows

$$\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}(\boldsymbol{n})| = \begin{pmatrix} Q_{11} & n_x Q_{12} & n_y Q_{12} & 0 \\ \xi_x Q_{21} - \alpha\zeta_x a|q| & n_x\xi_x(Q_{22} - |q|) + |q| & n_y\xi_x(Q_{22} - |q|) & 0 \\ \xi_y Q_{21} - \alpha\zeta_y a|q| & n_x\xi_y(Q_{22} - |q|) & n_y\xi_y(Q_{22} - |q|) + |q| & 0 \\ 0 & 0 & 0 & |q| \end{pmatrix}. \tag{25}$$

In the incompressible limit, as $M \longrightarrow 0$, we find that

$$\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}(\boldsymbol{n})| \simeq \begin{pmatrix} \mathcal{O}(\frac{1}{M^2}) & \mathcal{O}(\frac{1}{M}) & \mathcal{O}(\frac{1}{M}) & 0 \\ \mathcal{O}(\frac{1}{M}) & \mathcal{O}(1) & \mathcal{O}(1) & 0 \\ \mathcal{O}(\frac{1}{M}) & \mathcal{O}(1) & \mathcal{O}(1) & 0 \\ 0 & 0 & 0 & \mathcal{O}(1) \end{pmatrix} \quad \text{as} \quad M \longrightarrow 0,$$

which gives by construction the proper scaling in the incompressible limit [25,26].

From expression (25), we can easily find the eigenvalues μ of the matrix $\tilde{\boldsymbol{Q}}_{\boldsymbol{0}}$. Using the identities (24), we find for the linear waves

$$\mu_0 = |q| \qquad \text{(double)},$$

and for the acoustic waves

$$\mu_\pm = \frac{1}{2}\left[Q_{11} + Q_{22} \pm \sqrt{(Q_{11} - Q_{22})^2 + 4\,Q_{12}\,Q_{21}}\right].$$

Upon substitution with the expressions of the Q_{ij}

$$\mu_\pm = \frac{a|\boldsymbol{n}|}{2\sqrt{\tilde{\chi}}}\left[2z + \alpha(1-M_n^2) \pm \sqrt{\Delta}\right]$$
$$\text{with} \quad \Delta = \left(1-\frac{\beta^2}{a^2}\right)^2\left(1-M_n^2\right)^2 + 4z\left[z+\alpha(1-M_n^2)\right]M_n^2, \qquad \tilde{\chi} = z^2M_n^2 + \frac{\beta^2}{a^2}(1-M_n^2). \tag{26}$$

We find as expected from the one-dimensional analysis that as long as the flow remains subsonic, the eigenvalues $\mu_\pm$ are positive with the following identity

$$\mu_+\mu_- = a^2|\boldsymbol{n}|^2 - q^2. \tag{27}$$

Thus, with $\alpha = 0$ and $\mu_0 \geq 0$, the rescaled matrix dissipation formulated for the set of independent variables $\boldsymbol{d\tilde{W}_0}$ is positive semidefinite in the subsonic range. For the two extreme cases

$$\text{as } M_n \longrightarrow 0 \Rightarrow \mu_+ \simeq \frac{a^2}{|\beta|}|\boldsymbol{n}| = \mathcal{O}(\frac{1}{M^2}), \;\; \mu_- \simeq |\beta||\boldsymbol{n}| = \mathcal{O}(1), \;\; \mu_0 \simeq \mathcal{O}(1)$$
$$\text{as } M_n \longrightarrow 1 \Rightarrow \mu_+ \simeq 2a|\boldsymbol{n}|, \;\; \mu_- \simeq 0, \;\; \mu_0 \simeq a|\boldsymbol{n}|.$$

As anticipated in the one-dimensional case, we can see that severe stability constraints will occur in the incompressible limit, as it is discussed later.

In the multidimensional case, we find the following decomposition as a consequence of the Cayley–Hamilton theorem for 4×4 matrices

$$\boldsymbol{\tilde{P}_0}^{-1}|\boldsymbol{\tilde{P}_0\tilde{A}_0(n)}| = \frac{1}{\sqrt{\chi}}\left[zq\boldsymbol{\tilde{A}_0(n)} + \frac{\beta^2}{a^2}(a^2|\boldsymbol{n}|^2 - q^2)\boldsymbol{\tilde{P}_0}^{-1}\right] + (|q| - Q_{22})\boldsymbol{\tilde{L}_0(n)} \tag{28}$$

with

$$\boldsymbol{\tilde{L}_0(n)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\alpha\zeta_x a & n_y\xi_y & 0 & 0 \\ -\alpha\zeta_y a & 0 & n_x\xi_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix $(|q| - Q_{22})\boldsymbol{\tilde{L}_0(n)}$ has essentially $|q| - Q_{22}$ as non-zero eigenvalue with $|q| \simeq Q_{22}$ in the whole subsonic range except in the low speed limit. It doesn't affect the pressure equation. On the other hand, the resulting matrix of the two first contributions has the two acoustic eigenvalues $\mu_\pm > 0$ and double eigenvalue $Q_{22} > 0$. For the two acoustic equations, it was found that the contribution $(|q| - Q_{22})\boldsymbol{\tilde{L}_0(n)}$ does not exist. In addition, this "acoustic" contribution has the same behaviour as the rescaled matrix dissipation in the low speed limit. So, the resulting matrix of the two first contributions is close to the rescaled matrix dissipation (25). At a sonic point, i.e. $|q| = a|\boldsymbol{n}|$, if $\alpha = 0$ we also get $Q_{22} = a|\boldsymbol{n}|$. Therefore the two last contributions vanish and again for both possible values $q = \pm a|\boldsymbol{n}|$ we check that $\boldsymbol{\tilde{P}_0}^{-1}|\boldsymbol{\tilde{P}_0\tilde{A}_0(n)}| = |\boldsymbol{\tilde{A}_0(n)}|$.

In the multidimensional case, we derive the following diagonalization of the rescaled matrix dissipation

$$\boldsymbol{\tilde{P}_0}^{-1}|\boldsymbol{\tilde{P}_0\tilde{A}_0(n)}| = \boldsymbol{\tilde{R}_0\tilde{\Lambda}\tilde{R}_0}^{-1},$$

with $\boldsymbol{\tilde{\Lambda}} = \text{diag}(\mu_+, \mu_-, |q|, |q|)$, $\boldsymbol{\tilde{R}_0}$ is the matrix of the right eigenvectors

$$\boldsymbol{\tilde{R}_0} = \begin{pmatrix} (\mu_+ - Q_{22}) & Q_{12} & 0 & 0 \\ [\xi_x Q_{21} - \alpha\zeta_x a|q|R_1] & [\xi_x(\mu_- - Q_{11}) - \alpha\zeta_x a|q|R_2] & -n_y & 0 \\ \left[\xi_y Q_{21} - \alpha\zeta_y a|q|R_1\right] & \left[\xi_y(\mu_- - Q_{11}) - \alpha\zeta_y a|q|R_2\right] & n_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{29}$$

where

$$R_1 = \frac{\mu_+ - Q_{22}}{\mu_+ - |q|}, \qquad R_2 = \frac{Q_{12}}{\mu_- - |q|},$$

and $\boldsymbol{\tilde{R}_0}^{-1}$ is the matrix of the left eigenvectors

$$\tilde{\boldsymbol{R}}_{\boldsymbol{0}}^{-1} = \begin{pmatrix} \dfrac{(\mu_- - Q_{11})}{D} & -\dfrac{Q_{12}}{D} n_x & -\dfrac{Q_{12}}{D} n_y & 0 \\ -\dfrac{Q_{21}}{D} & \dfrac{(\mu_+ - Q_{22})}{D} n_x & \dfrac{(\mu_+ - Q_{22})}{D} n_y & 0 \\ -\alpha\zeta a|q|L_1 & -\xi_y - \alpha\zeta_y a|q|L_2 & \xi_x + \alpha\zeta_x a|q|L_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{30}$$

with

$$\zeta = \frac{\zeta_x}{n_y} = -\frac{\zeta_y}{n_x} = \frac{(un_y - vn_x)}{\alpha q^2 - \beta^2|\boldsymbol{n}|^2},$$

and

$$L_1 = \frac{Q_{22} - |q|}{(\mu_+ - |q|)(\mu_- - |q|)}, \qquad L_2 = \frac{Q_{12}}{(\mu_+ - |q|)(\mu_- - |q|)},$$

where $D = \det\left(\tilde{\boldsymbol{R}}_{\boldsymbol{0}}\right)$ given by (18).

The above scaling of the eigenvectors corresponds to the most straightforward extension of the one-dimensional formulation (17). It is worth noticing that the apparent singularity $\mu_- - |q| = 0$ which may occur in the definition of coefficients R_1, L_1 and L_2 if $\alpha \neq 0$ for a specific directional Mach number $M_n < 1$, root of the non-linear equation

$$M_n^2 + \frac{\mu_+}{a|\boldsymbol{n}|} M_n - 1 = 0,$$

doesn't create any numerical difficulty. We actually find by a deep inspection of the eigenspaces corresponding the second right eigenvector and to the third left eigenvector that if $\mu_- = |q|$, then necessarily the velocity vector is aligned with the actual direction $\boldsymbol{n}$ and therefore $\zeta_x = \zeta_y = \zeta = 0$. Thus, for this specific state, the second right eigenvector and the third left eigenvector recover non-singular expressions. It will be also shown in the stability analysis that $\mu_+ > |q|$ for all possible values of the preconditioning parameter α. So the above diagonalization can never become singular in practice in the subsonic range.

We also remind that by turning off the preconditioning matrix with $\frac{\beta^2}{a^2} = 1$ with $\alpha = 0$, expressions (29)–(30) do not return the right and left eigenvector matrix of $|\tilde{\boldsymbol{A}}_{\boldsymbol{0}}(\boldsymbol{n})|$, associated to eigenvalues $|\lambda_+| = |a|\boldsymbol{n}| + q|$ and $|\lambda_-| = |a|\boldsymbol{n}| - q|$. They actually correspond to the diagonalization of matrix $|\tilde{\boldsymbol{A}}_{\boldsymbol{0}}(\boldsymbol{n})|$ with the acoustic eigenvalues recast for the module of the normal velocity $a|\boldsymbol{n}| + |q|$ and $a|\boldsymbol{n}| - |q|$. In the case $q \longrightarrow -a|\boldsymbol{n}|$, we especially see that $|\lambda_+| \longrightarrow 0$ and $|\lambda_-| \longrightarrow 2a|\boldsymbol{n}|$ while $\mu_+ \longrightarrow 2a|\boldsymbol{n}|$ and $\mu_- \longrightarrow 0$ respectively. This justifies to formulate carefully a possible entropy fix at the sonic line as described later.

In the special case $\alpha = 0$, we also see that the "shear" and "entropic" eigenvectors are both orthogonal to the acoustic eigenvectors. Taking $|\boldsymbol{n}| = 1$, with (19) the acoustic eigenvectors are also orthogonal to each others and the diagonalization with unitary congruence (20)–(21) can be applied. So the symmetry properties of the original unpreconditioned system are preserved for the rescaled matrix dissipation. However, this is no longer true when the rescaled matrix dissipation is formulated from the primitive or conservative variables since the eigenvectors are modified by any change of variables. We know from [30] that the non-normality of the preconditioned system leads to significant robustness problems at stagnation points.

3. Stability analysis for the explicit scheme

We now consider the explicit scheme with rescaling of the dissipation matrix derived in the previous section, expressed for the conservative variables $\boldsymbol{W}$. In the two-dimensional case, the explicit discretization at time $t^n = n\Delta t$ for $\boldsymbol{x}_{ij} = (i\delta x, j\delta y)$, where δx and δy are the constant mesh spacing in the (x, y) directions, can be formulated with the finite-difference scheme in conservation form

$$\frac{\Delta \boldsymbol{W}_{ij}}{\Delta t} + \frac{\boldsymbol{h}_{i+\frac{1}{2}j} - \boldsymbol{h}_{i-\frac{1}{2}j}}{\delta x} + \frac{\boldsymbol{g}_{ij+\frac{1}{2}} - \boldsymbol{g}_{ij-\frac{1}{2}}}{\delta y} = 0, \tag{31}$$

with the time increment $\Delta \boldsymbol{W}_{ij} = \boldsymbol{W}_{ij}^{n+1} - \boldsymbol{W}_{ij}^n$. For the first-order Roe scheme, the numerical flux writes in the x-direction

$$\boldsymbol{h}_{i+\frac{1}{2}j} = \frac{1}{2}\left[\boldsymbol{F}(\boldsymbol{W}_{ij}^n) + \boldsymbol{F}(\boldsymbol{W}_{i+1j}^n)\right] - \frac{1}{2}|\boldsymbol{A}_{roe}|_{i+\frac{1}{2}j}\left(\boldsymbol{W}_{i+1j}^n - \boldsymbol{W}_{ij}^n\right). \tag{32}$$

For the rescaled Roe scheme, we substitute (32) with

$$\boldsymbol{h}_{i+\frac{1}{2}j} = \frac{1}{2}\left[\boldsymbol{F}(\boldsymbol{W}_{ij}^n) + \boldsymbol{F}(\boldsymbol{W}_{i+1j}^n)\right] - \frac{1}{2}\boldsymbol{P}^{-1}|\boldsymbol{P}\boldsymbol{A}|_{i+\frac{1}{2}j}\left(\boldsymbol{W}_{i+1j}^n - \boldsymbol{W}_{ij}^n\right), \tag{33}$$

and matrix $\boldsymbol{P}^{-1}|\boldsymbol{PA}|$ formulated for the Roe average. There is no need to explicitly formulate the matrix dissipation for the conservative variables. In practice, the dissipation vector can always be computed using similarity transformations following the algebraic algorithm described in [23]

$$\boldsymbol{P}^{-1}|\boldsymbol{PA}|_{i+\frac{1}{2}j}\left(\boldsymbol{W}^n_{i+1j}-\boldsymbol{W}^n_{ij}\right)=\left[\frac{\partial \boldsymbol{W}}{\partial \tilde{\boldsymbol{W}}_\mathbf{0}}\tilde{\boldsymbol{R}}_\mathbf{0}\tilde{\boldsymbol{\Lambda}}\tilde{\boldsymbol{R}}_\mathbf{0}^{-1}\frac{\partial \tilde{\boldsymbol{W}}_\mathbf{0}}{\partial \boldsymbol{W}}\right]_{i+\frac{1}{2}j}\left(\boldsymbol{W}_{i+1j}-\boldsymbol{W}_{ij}\right).$$

The stability constraint is strongly concerned with the spectral radius $\rho(\boldsymbol{P}^{-1}|\boldsymbol{PA}|)$ of the matrix-valued dissipation. It can be demonstrated in the multi-dimensional framework that (see Appendix B)

$$\rho(\boldsymbol{P}^{-1}|\boldsymbol{PA}|)=\mu_+ \quad \forall M_n\le 1, \tag{34}$$

for all possible values of the parameter α. Furthermore, if $\alpha=0$, we also have $\mu_+\ge\rho(\boldsymbol{A})=|q|+a|\boldsymbol{n}|$ for all $M_n\le 1$ which is not always the case with $\alpha=1$ or $\alpha=-1$. At the sonic line, with $M_n=1$ in expressions (26), μ_+ returns $\rho(\boldsymbol{A})$ for all α.

Following the linear Von-Neumann analysis carried out by Birken and Meister for the Lax–Friedrich scheme in [3], we finally conclude that the stability condition for the explicit rescaled Roe scheme is formulated for the fasted wave speed as follows

$$\Delta t\le\frac{h}{\mu_+}\simeq\mathcal{O}(M^2)\quad\text{as}\quad M_n\to 0,\quad\forall\alpha \tag{35}$$

and

$$\Delta t\le\frac{h}{|q|+a|\boldsymbol{n}|}\quad\text{as}\quad M_n\to 1\quad\text{and}\quad M_n\ge 1,\quad\text{with}\quad\alpha=0 \tag{36}$$

where h represents some characteristic cell distance. So for steady-state problems, the local time step will be very small $\simeq\mathcal{O}(hM^2)$ in the low-speed flow regions only. In supersonic flow regions, the "standard" stability condition (36) must be applied. Therefore, with the strong limitation on the time step, an efficient implicit stage must be added to the baseline explicit rescaled Roe scheme for the steady-state problem.

It is interesting to compare the acoustic eigenvalues $\mu_\pm$ of the rescaled matrix dissipation with the original eigenvalues $|\lambda_\pm|=|a|\boldsymbol{n}|\pm q|$ in the whole subsonic range $M_n\le 1$. In the next figures, all eigenvalues are normalized by the local speed of sound and therefore can be expressed only as functions of the dimensionless normal velocity $\frac{q}{a|\boldsymbol{n}|}\in[-1,1]$ ($=\frac{u}{a}$ in the one-dimensional case). The eigenvalues $\mu_\pm$ strongly depend on the local Mach number M within the parameter $\frac{\beta^2}{a^2}$ with a noticeable effect of the second parameter α for low Mach numbers. In Fig. 1–3 we have compared the effect of $\alpha=-1$, $\alpha=0$ and $\alpha=1$, considering different values of the local Mach number. As demonstrated above, we see that μ_+ is the largest eigenvalue in the entire subsonic range with $\mu_+>|q|$, $\forall\alpha$, $\mu_+\ge\rho(\boldsymbol{A})$ for $\alpha=0$ and that μ_- takes small values especially when $M\le 0.1$. The value of α has little impact on μ_- but has a large effect on μ_+ in the incompressible limit, as illustrated in Fig. 1 for $M=0.1$. As it can be clearly seen, the stiffness of the rescaled matrix dissipation increases dramatically as the local Mach number is becoming smaller and smaller. The worst situation arises for $\alpha=1$, which could not be tested in our computations of low Mach-number flows. Globally $\alpha=-1$ gives the better conditioning except when $\frac{q}{a|\boldsymbol{n}|}\to 0$ where the asymptotic value of μ_+ is reached. This asymptotic value given by $\frac{a^2}{|\beta|}|\boldsymbol{n}|$ is independent of α but strongly depends on the local Mach number. When $M\to 1$, the effect of α is almost negligible and the stiffness of the rescaled Roe matrix becomes identical to the original Roe matrix (see Fig. 3 for the case $M=1$ with $\mu_+=\rho(\boldsymbol{A})$ if $\alpha=0$).

4. Implicit matrix time-stepping scheme for the steady-state problem

4.1. Formulation

The Jacobi preconditioning or preconditioning-squared is known to smooth out high frequency errors and is especially efficient with the multigrid technique to accelerate the convergence. It basically doesn't help to improve the stiffness of the rescaled Roe scheme in the low speed limit. On the other hand the low-speed preconditioning is not designed to treat high frequency errors. Turkel suggested to combine both approaches [4,31]. However, here we apply a block Jacobi preconditioner to the rescaled Roe scheme in order to formulate an implicit matrix time-stepping scheme for the steady-state problem in the sense that each characteristic variables is updated with its own $\Delta t^{(k)}$ [32]. Actually the matrix time-stepping approach provides a mechanism enforcing the damping properties of the scheme, as it is described next within a linear stability analysis. In the case of the rescaled Roe scheme, the formulation of a time-step matrix is not obvious, since the dissipation matrix is not an explicit function of the original Jacobian matrix. For the fully discrete scheme in a two-dimensional Cartesian mesh (31) with the numerical flux corresponding to the rescaled Roe scheme (33), the Jacobi preconditioning may be formulated as follows

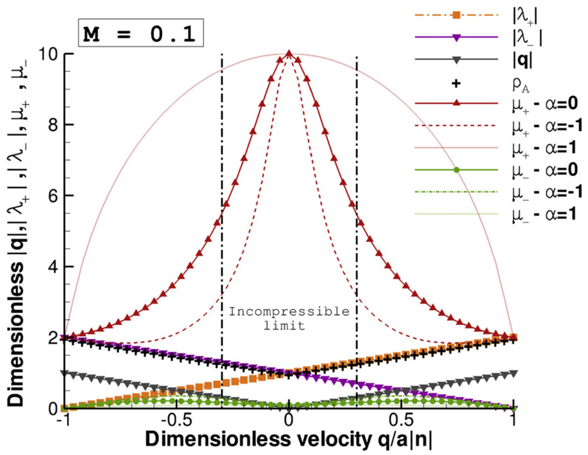

Fig. 1. Dimensionless eigenvalues for the original matrix-valued dissipation (Roe) and for the rescaled Roe scheme when $M = 0.1$.

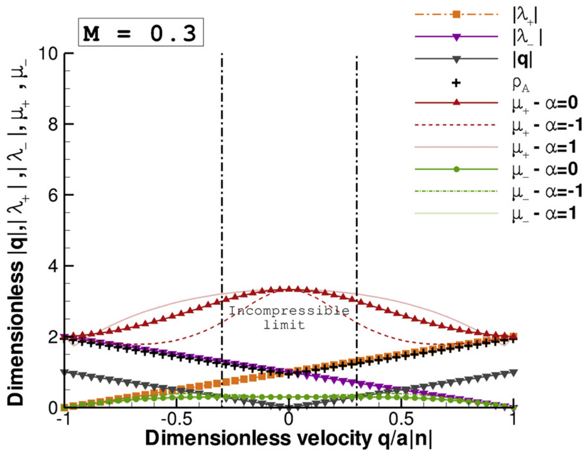

Fig. 2. Dimensionless eigenvalues for the original matrix-valued dissipation (Roe) and for the rescaled Roe scheme when $M = 0.3$.

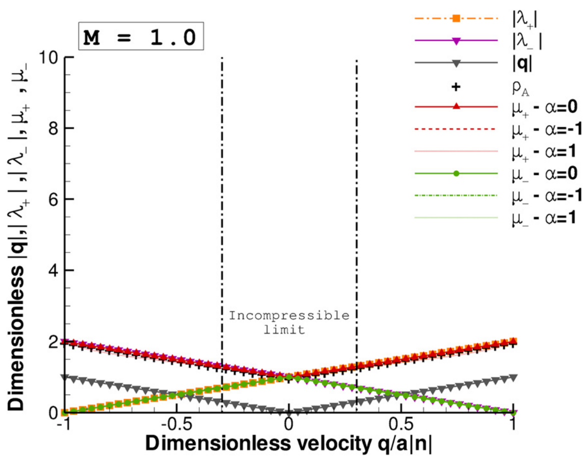

Fig. 3. Dimensionless eigenvalues for the original matrix-valued dissipation (Roe) and for the rescaled Roe scheme when $M = 1$.

$$\left(\frac{\boldsymbol{A_p}+\boldsymbol{B_p}}{\rho(\boldsymbol{P}^{-1}|\boldsymbol{PA}|)+\rho(\boldsymbol{P}^{-1}|\boldsymbol{PB}|)}\Delta \boldsymbol{W}\right)_{ij} = -\Delta t\left(\frac{\boldsymbol{h}_{i+\frac{1}{2}j}-\boldsymbol{h}_{i-\frac{1}{2}j}}{\delta x}+\frac{\boldsymbol{g}_{ij+\frac{1}{2}}-\boldsymbol{g}_{ij-\frac{1}{2}}}{\delta y}\right), \tag{37}$$

where the matrix coefficients $\boldsymbol{A_p}$ and $\boldsymbol{B_p}$ must fulfil the following conditions:

1. The eigenvectors must be the same as the original Jacobian matrix in order to preserve the formulation of the characteristics variables;
2. The eigenvalues must be nonnegative in order to let unchanged the signs of the characteristic speeds and bounded away from zero (thus positive) for the formulation of a characteristic time-step matrix;
3. The stability condition (35)–(36) must be satisfied for the characteristic variable corresponding to the fastest wave speed;
4. The matrix coefficients $\boldsymbol{A_p}$ and $\boldsymbol{B_p}$ should achieve a smooth transition with $|\boldsymbol{A}|$ and $|\boldsymbol{B}|$ at the sonic line.

Matrix coefficients $\boldsymbol{A_p}=\boldsymbol{P}^{-1}|\boldsymbol{PA}|$ and $\boldsymbol{B_p}=\boldsymbol{P}^{-1}|\boldsymbol{PB}|$ perfectly match conditions 2., 3. and 4. but must be rejected as not satisfying the first condition. Setting $\boldsymbol{A_p}=|\boldsymbol{A}|=\boldsymbol{R_A}|\boldsymbol{\Lambda_A}|\boldsymbol{R_A}^{-1}$ and $\boldsymbol{B_p}=|\boldsymbol{B}|=\boldsymbol{R_B}|\boldsymbol{\Lambda_B}|\boldsymbol{R_B}^{-1}$ satisfies conditions 1., 2. and 4., provided that the vanishing eigenvalues $|\lambda^{(k)}|$ are corrected by some entropy fix. This latter choice reduces the spread of the characteristic speeds from a factor $1/M$ with respect to the formulation with the rescaled matrix dissipation. However, requirement 3. is lost with $\rho(\boldsymbol{A}) \le \mu_+$ in general and in practice, this formulation is unstable and can be used only with small or moderate values of the CFL number. An intermediate choice is to build matrix coefficients from the eigenvectors of the original Jacobian matrix and the eigenvalues $\mu^{(k)}$ of the rescaled matrix dissipation supposed to be all bounded away from zero. In that case, the eigenvalues $\mu^{(k)}$ must be corrected to ensure the transition at the sonic line with $|\lambda^{(k)}|$ the eigenvalues of the original Roe matrix dissipation. This intermediate formulation provides optimal stability properties and yields a robust scheme but let the spread of the characteristic speeds unchanged with respect to the rescaled Roe scheme.

Hence, let consider $\boldsymbol{A_p}=\boldsymbol{R_A}\tilde{\boldsymbol{\Lambda}}_{\boldsymbol{A}}\boldsymbol{R_A}^{-1}$ and $\boldsymbol{B_p}=\boldsymbol{R_B}\tilde{\boldsymbol{\Lambda}}_{\boldsymbol{B}}\boldsymbol{R_B}^{-1}$, with $\tilde{\boldsymbol{\Lambda}}=diag(\mu^{(k)})$, $\boldsymbol{R_A}$ and $\boldsymbol{R_B}$ being the right eigenvector matrices of the original Jacobian matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. In a Cartesian mesh, the CFL condition can be expressed with a more restrictive formulation of the local time-step

$$\Delta t = CFL\frac{h}{\rho(\boldsymbol{P}^{-1}|\boldsymbol{PA}|)+\rho(\boldsymbol{P}^{-1}|\boldsymbol{PB}|)}.$$

We introduce a characteristic time-step matrix

$$\boldsymbol{\Delta t_c} = CFL\,h\left(\boldsymbol{A_p}+\boldsymbol{B_p}\right)^{-1} = \left[(\boldsymbol{R_A}\boldsymbol{\Delta t_A}^{-1}\boldsymbol{R_A}^{-1})+(\boldsymbol{R_B}\boldsymbol{\Delta t_B}^{-1}\boldsymbol{R_B}^{-1})\right]^{-1}, \tag{38}$$

where $\boldsymbol{\Delta t}$ is the diagonal matrix

$$\boldsymbol{\Delta t} = diag(\Delta t^+, \Delta t^-, \Delta t^0, \Delta t^0) \tag{39}$$

with

$$\Delta t^+ = CFL\frac{h}{\mu_*^+}, \quad \Delta t^- = CFL\frac{h}{\mu_*^-}, \quad \Delta t^0 = CFL\frac{h}{\mu_*^0}. \tag{40}$$

The star symbol indicates that the eigenvalues may be bounded away from zero and especially that $\mu_\pm$ must be modified to ensure that the above condition 4 is satisfied. This point is discussed later, in the supersonic flow extension of the rescaled Roe scheme.

Thus, the Jacobi preconditioner amounts to reformulate the original conservative scheme (31) with a time-step matrix

$$\Delta \boldsymbol{W}_{ij} = -(\boldsymbol{\Delta t_c})_{ij}\left(\frac{\boldsymbol{h}_{i+\frac{1}{2}j}-\boldsymbol{h}_{i-\frac{1}{2}j}}{\delta x}+\frac{\boldsymbol{g}_{ij+\frac{1}{2}}-\boldsymbol{g}_{ij-\frac{1}{2}}}{\delta y}\right).$$

This scheme has optimal properties in damping high frequency errors, and is well adapted to transonic flows. By setting $\boldsymbol{P}$ to the identity matrix $\boldsymbol{Id}$ in (37), we can also formulate a matrix time-stepping scheme for the original Roe scheme with

$$\Delta t^+ = CFL\frac{h}{|\lambda_+|}, \quad \Delta t^- = CFL\frac{h}{|\lambda_-|}, \quad \Delta t^0 = CFL\frac{h}{|\lambda_0|},$$

and $\lambda_\pm = q \pm a|\boldsymbol{n}|$, $\lambda_0 = q$.

In order to overcome the issue of using very small time steps in the incompressible limit, we see that the matrix time-stepping scheme must be combined with a robust implicit scheme allowing the use of large CFL numbers, as recommended in [3]. A usual way to define an implicit scheme for the Roe scheme, is to consider the following backward Euler scheme as formulated in [33] for an approximated differentiation of the Roe flux

$$
\begin{aligned}
&- \sigma_x \boldsymbol{A^+}_{i-\frac{1}{2}j}\, \Delta \boldsymbol{W}_{i-1j} + \sigma_x \boldsymbol{A^-}_{i+\frac{1}{2}j}\, \Delta \boldsymbol{W}_{i+1j} \\
&\quad + \left(\boldsymbol{Id} - \sigma_x \boldsymbol{A^-}_{i+\frac{1}{2}j} + \sigma_x \boldsymbol{A^+}_{i-\frac{1}{2}j} - \sigma_y \boldsymbol{B^-}_{ij+\frac{1}{2}} + \sigma_y \boldsymbol{B^+}_{ij-\frac{1}{2}} \right) \Delta \boldsymbol{W}_{ij} \\
&\quad - \sigma_y \boldsymbol{B^+}_{ij-\frac{1}{2}}\, \Delta \boldsymbol{W}_{ij-1} + \sigma_y \boldsymbol{B^-}_{ij+\frac{1}{2}}\, \Delta \boldsymbol{W}_{ij+1} = \Delta \boldsymbol{W}^{exp}_{ij}
\end{aligned}
\tag{41}
$$

where $\Delta \boldsymbol{W}^{exp}_{ij}$ is the time increment for the explicit scheme, $\sigma_x = \frac{\Delta t}{\delta x}$, $\sigma_y = \frac{\Delta t}{\delta y}$ and $\boldsymbol{A^+}$, $\boldsymbol{A^-}$, $\boldsymbol{B^+}$, $\boldsymbol{B^-}$ are respectively the positive and negative parts of the flux Jacobian matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in the (x, y) space directions. For the rescaled Roe scheme, a baseline implicit scheme is then formulated by replacing in (41) respectively

$$
\boldsymbol{A}^{\pm} \longmapsto \frac{\boldsymbol{A} \pm \boldsymbol{P}^{-1}|\boldsymbol{PA}|}{2} = \boldsymbol{P}^{-1}(\boldsymbol{PA})^{\pm} \quad \text{and} \quad \boldsymbol{B}^{\pm} \longmapsto \frac{\boldsymbol{B} \pm \boldsymbol{P}^{-1}|\boldsymbol{PB}|}{2} = \boldsymbol{P}^{-1}(\boldsymbol{PB})^{\pm}. \tag{42}
$$

The implicit scheme (41) can be also formulated for the characteristic matrix time-stepping scheme, upon the following substitutions

$$
\sigma_x \boldsymbol{Id} \longmapsto \frac{1}{\delta x} \boldsymbol{\Delta t_c}, \quad \sigma_y \boldsymbol{Id} \longmapsto \frac{1}{\delta y} \boldsymbol{\Delta t_c}. \tag{43}
$$

4.2. Stability of the implicit scheme

In the following, the stability of the implicit rescaled Roe scheme with scalar and matrix time-steps is considered. Both versions correspond actually to small perturbations of the original implicit Roe scheme (41) in the transonic regime. Both are especially unconditionally stable for the linear problem as shown below. But as the free-stream Mach number is decreasing, their dissipative properties substantially differ from the original Roe scheme. This can be pointed out within a Fourier analysis. Let (ϕ, θ) be the normalized wave numbers in the two space directions. Then, for the linear problem the amplification matrix takes the general form

$$
\boldsymbol{G}(\phi, \theta) = \boldsymbol{Id} - \boldsymbol{H}^{-1}(\phi, \theta)\boldsymbol{K}(\phi, \theta),
$$

with for the implicit scheme $\boldsymbol{H}(\phi, \theta) = \boldsymbol{Id} + \boldsymbol{K}(\phi, \theta)$. We get successively

- For the original Roe scheme

$$
\boldsymbol{K}(\phi, \theta) = (1 - cos(\phi))\sigma_x |\boldsymbol{A}| + (1 - cos(\theta))\sigma_y |\boldsymbol{B}| + i\, sin(\phi)\sigma_x \boldsymbol{A} + i\, sin(\theta)\sigma_y \boldsymbol{B}
$$

- For the rescaled Roe scheme

$$
\boldsymbol{K}(\phi, \theta) = (1 - cos(\phi))\sigma_x \boldsymbol{P}^{-1}|\boldsymbol{PA}| + (1 - cos(\theta))\sigma_y \boldsymbol{P}^{-1}|\boldsymbol{PB}| + i\, sin(\phi)\sigma_x \boldsymbol{A} + i\, sin(\theta)\sigma_y \boldsymbol{B}
$$

- For the matrix time-stepping schemes, we replace σ_x and σ_y according to (43).

Stability and damping properties of the four schemes can be characterized by the following estimates, assuming all matrix coefficients simultaneously diagonalizable according to the Lerat's general framework [34,35]. Let introduce $\lambda^{(k)}_{\boldsymbol{G}}$ the eigenvalues of the amplification matrix $\boldsymbol{G}(\phi, \theta)$ and respectively $\lambda^{(k)}_{\boldsymbol{A}}$, $\lambda^{(k)}_{\boldsymbol{B}}$ and $\mu^{(k)}_{\boldsymbol{A}}$, $\mu^{(k)}_{\boldsymbol{B}}$ the eigenvalues of the matrix coefficients $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{A_p}$, $\boldsymbol{B_p}$, $\boldsymbol{P}^{-1}|\boldsymbol{PA}|$, $\boldsymbol{P}^{-1}|\boldsymbol{PB}|$. For the matrix time-stepping schemes, the time-step matrix has eigenvalues

$$
\Delta t^{(k)} = CFL \frac{h}{|\lambda^{(k)}_{\boldsymbol{A}}| + |\lambda^{(k)}_{\boldsymbol{B}}|} \quad \text{(Roe scheme)}, \ \Delta t^{(k)} = CFL \frac{h}{\mu^{(k)}_{\boldsymbol{A}} + \mu^{(k)}_{\boldsymbol{B}}} \quad \text{(rescaled Roe scheme)}.
$$

Important for differentiating the different formulations is to have a close upper estimate of the spectral radius of the amplification matrix. Without loss of generality, we have for the Roe scheme in one space-dimension

$$
\rho(\boldsymbol{G}) = \frac{1}{min_k(|1 + \lambda^{(k)}_{\boldsymbol{K}}|)},
$$

with

$$
|1 + \lambda_{\boldsymbol{K}}| = \sqrt{(1 + 2(1 - cos(\phi))\sigma|\lambda|(1 + \sigma|\lambda|))} \geq 1, \quad \forall \sigma.
$$

Denoting $\nu = \sigma|\lambda|$, we see that $\nu \longmapsto \rho(\boldsymbol{G})(\phi, \nu)$ is a strictly diminishing function and therefore

$$
\rho(\boldsymbol{G}) = \frac{1}{\sqrt{(1 + 2(1 - cos(\phi))\nu_{min}(1 + \nu_{min}))}},
$$

where $\nu_{min} = \sigma min_k(|\lambda^{(k)}|)$ if no matrix time-stepping scheme is formulated, $\nu_{min} = min_k(\sigma^{(k)}|\lambda^{(k)}|)$, $\sigma^{(k)} = \frac{\Delta t^{(k)}}{\delta x}$ with the matrix time-stepping scheme. The optimal rate of convergence to a steady state is then obtained for large CFL numbers with $\lim_{CFL\to\infty} \rho(\boldsymbol{G}) = 0,\ \forall\phi \in]0, 2\pi[,\ \forall\lambda \neq 0$. The same results hold for the two-dimensional problem with the upper-bound estimate

$$\rho(\boldsymbol{G}) \leq \frac{1}{\sqrt{(1+2(1-cos(\phi))(\nu_{min})_x + 2(1-cos(\theta))(\nu_{min})_y}} \leq 1, \quad \forall\sigma_x, \sigma_y.$$

So the higher are $(\nu_{min})_x$ or $(\nu_{min})_y$, the lower is $\rho(\boldsymbol{G})$ and the better is the convergence rate to a steady state.

In the case of the rescaled Roe scheme, using similar arguments, we get in the one-dimensional case

$$\rho(\boldsymbol{G}) = \frac{1}{\sqrt{(1+(1-cos(\phi))[2+(1-cos(\phi))(\nu_\mu)_{min}](\nu_\mu)_{min} + sin^2(\phi)(\nu_\lambda)_{min}}},$$

where $(\nu_\mu)_{min} = \sigma min_k(\mu^{(k)})$, $(\nu_\lambda)_{min} = \sigma min_k(|\lambda^{(k)}|)$ if no matrix time-stepping scheme is formulated, $(\nu_\mu)_{min} = min_k(\sigma^{(k)}\mu^{(k)})$, $(\nu_\lambda)_{min} = min_k(\sigma^{(k)}|\lambda^{(k)}|)$ with the matrix time-stepping scheme.

Now comparing the effect of the matrix time-stepping scheme on the non-dimensional quantities $(\nu_{min})_x$ and $(\nu_{min})_y$ for both the Roe scheme and the rescaled Roe scheme, we get

– For the Roe scheme:

$$\sigma_x|\lambda_{\boldsymbol{A}}^{(k)}| = \frac{\Delta t}{\delta x}|\lambda_{\boldsymbol{A}}^{(k)}| = CFL\frac{h}{\delta x}\frac{|\lambda_{\boldsymbol{A}}^{(k)}|}{\rho(\boldsymbol{A})+\rho(\boldsymbol{B})} \leq CFL\frac{h}{\delta x}\frac{|\lambda_{\boldsymbol{A}}^{(k)}|}{|\lambda_{\boldsymbol{A}}^{(k)}|+|\lambda_{\boldsymbol{B}}^{(k)}|} = \frac{\Delta t^{(k)}}{\delta x}|\lambda_{\boldsymbol{A}}^{(k)}| = \sigma_x^{(k)}|\lambda_{\boldsymbol{A}}^{(k)}| \quad \forall k,$$

and similarly $\sigma_y|\lambda_{\boldsymbol{B}}^{(k)}| \leq \sigma_y^{(k)}|\lambda_{\boldsymbol{B}}^{(k)}|,\ \forall k$. Assuming that $(\nu_{min})_x$ is reached for $k = k_0$, we see that

$$(\nu_{min})_x = \sigma_x|\lambda_{\boldsymbol{A}}^{(k_0)}| \leq \sigma_x|\lambda_{\boldsymbol{A}}^{(k)}| \leq \sigma_x^{(k)}|\lambda_{\boldsymbol{A}}^{(k)}|, \quad \forall k \quad \Rightarrow \quad (\nu_{min})_x \leq min_k(\sigma_x^{(k)}|\lambda_{\boldsymbol{A}}^{(k)}|).$$

– For the rescaled Roe scheme, the previous result writes

$$[(\nu_\mu)_{min}]_x = \sigma_x\mu_{\boldsymbol{A}}^{(k_0)} \leq min_k(\sigma_x^{(k)}\mu_{\boldsymbol{A}}^{(k)}) \quad \text{and} \quad [(\nu_\lambda)_{min}]_x = \sigma_x|\lambda_{\boldsymbol{A}}^{(k_0)}| \leq min_k(\sigma_x^{(k)}|\lambda_{\boldsymbol{A}}^{(k)})|.$$

Therefore the matrix time-stepping scheme always yields higher $(\nu_{min})_x$ and $(\nu_{min})_y$ and thus may improve the damping properties of the iterative procedure. It is also interesting to look at the situation in the incompressible limit. Since for the Roe scheme in a fixed mesh

$$|\lambda_+| \simeq |\lambda_-| \simeq \mathcal{O}(\frac{1}{M}) \quad \text{and} \quad |\lambda_0| \simeq \mathcal{O}(1) \quad \text{as} \quad M \to 0,$$

with find that

$$(\nu_{min})_x \simeq \mathcal{O}(M) < min_k(\sigma_x^{(k)}|\lambda^{(k)}|) \simeq \mathcal{O}(1) \quad \text{as} \quad M \to 0.$$

For the rescaled Roe scheme in the same conditions

$$\mu_+ \simeq \mathcal{O}(\frac{1}{M^2}) \quad \text{and} \quad \mu_- \simeq \mu_0 \simeq \mathcal{O}(1) \quad \text{as} \quad M \to 0,$$

thus

$$[(\nu_\mu)_{min}]_x \simeq \mathcal{O}(M^2) << min_k(\sigma_x^{(k)}\mu^{(k)}) \simeq \mathcal{O}(1) \quad \text{as} \quad M \to 0.$$

So the optimal damping properties of the matrix time-stepping scheme should be enforced in the incompressible limit.

It can also be demonstrated that the rescaled Roe scheme is also second-order dissipative in the sense of Kreiss [35] and can be used for the computation of non-linear flows.

5. Extension to transonic flows

The behaviour of the modified eigenvalues $\mu_\pm$ and $\mu_0 = |q|$ suggests to apply selectively the so-called entropy fix. With $\mu_+ > 0$, a correction has to be found only for μ_0 at the stagnation point and for μ_- at the sonic line, since the formulation of the rescaled matrix prevents μ_- from being too small near a stagnation point. With $\mu_- \simeq |q|$ near the stagnation point, this suggests to equalize μ_0 and μ_- when $M \to 0$. We suppose the Mach number to be small enough so that the preconditioning parameter recovers its cut-off value $\frac{\beta^2}{a^2} = \epsilon^2$. Then, a possible correction will assume the following form

$$\mu_0 = max(|q|, \epsilon a|\boldsymbol{n}|) \quad \text{with} \quad \epsilon a|\boldsymbol{n}| = \lim_{M\to 0}\mu_- \simeq \mathcal{O}(1). \tag{44}$$

This correction might be interpreted formally as similar to the first entropy correction proposed by Harten and Hyman [36]. However, notice that the quantity $\epsilon a|\boldsymbol{n}|$ is actually a bit smaller than the threshold employed for the entropy fix, which is usually based on some fraction of the spectral radius $|q| + a|\boldsymbol{n}|$ (even for the linear waves) and not on the local speed of sound only. For viscous or turbulent flows, as mentioned previously, the formulation of the cut-off value ϵ^2 must be reconsidered, and the correction (44) may assume a different formulation.

Regarding the acoustic eigenvalue μ_- at the sonic line when $M_n = 1$, it seems reasonable to be consistent with the formulation of the original matrix dissipation, by considering the same formulation of the entropy fix used for the Roe scheme. This choice is motivated by the extension to a supersonic flow described in the following. Considering again the symmetrizing entropic variables, in the case of subsonic flow $|u| \leq a$ we have the following diagonalization

$$|\tilde{\boldsymbol{A}}_{\boldsymbol{0}}| = \begin{pmatrix} a & a \\ a & -a \end{pmatrix} \begin{pmatrix} |u+a| & 0 \\ 0 & |u-a| \end{pmatrix} \begin{pmatrix} \frac{1}{2a} & \frac{1}{2a} \\ \frac{1}{2a} & -\frac{1}{2a} \end{pmatrix}.$$

Although in case of transonic flow the entropic variables are not appropriate, suppose for the analysis that $u \to a$. Then the acoustic eigenvalue $|\lambda_-| = |u - a|$ may be corrected by some entropy fix and therefore $|\lambda_-| \to 0$ is modified with $|\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|$ formulated for $diag(|u + a|, \Psi(|u - a|))$, where Ψ is some given cut-off function preventing $|\lambda_-|$ from reaching zero. On the other hand, assuming $\frac{\beta^2}{a^2} \to 1$ with $\alpha = 0$ in the expression of the rescaled matrix dissipation as $|u| \to a$, we get

$$\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}| = \begin{pmatrix} |u| & u \\ u & -|u| \end{pmatrix} \begin{pmatrix} a+|u| & 0 \\ 0 & a-|u| \end{pmatrix} \begin{pmatrix} \frac{1}{2|u|} & \frac{1}{2u} \\ \frac{1}{2u} & -\frac{1}{2|u|} \end{pmatrix}.$$

Thus at the sonic line, $\mu_- = a - |u| \to 0$ and $\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|$ may be also reformulated with $diag(a + |u|, \Psi_p(a - |u|))$. The cut-off function $\Psi_p(\mu_-)$ must be defined in order to assure $\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}| \to |\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|$ when $|u| \to a$. In the limit $|u| \to a$, when applying the entropy fix, we can see that we get the identity $\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}| = |\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|$ provided that

$$\begin{cases} \Psi_p(\mu_-) = \Psi(|\lambda_+|) \quad \text{and} \quad \mu_+ = |\lambda_-| \quad \text{if} \quad u \to -a \\ \Psi_p(\mu_-) = \Psi(|\lambda_-|) \quad \text{and} \quad \mu_+ = |\lambda_+| \quad \text{if} \quad u \to a \end{cases} \tag{45}$$

If no entropy fix is applied

$$|\tilde{\boldsymbol{A}}_{\boldsymbol{0}}| = \tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}| = \begin{pmatrix} a & \pm a \\ \pm a & a \end{pmatrix} \quad \text{when} \quad u = \pm a.$$

The formulation (45) of the entropy fix for the rescaled Roe scheme is unchanged by similarity transformations and can be also extended to the Euler equations in conservation form. The definition of Ψ_p is also formally unchanged in the multidimensional case, as indicated below. For complex non-linear flows, we may consider the classical Harten's entropy fix formulated in [37]

$$\begin{cases} \Psi(|\lambda|) = \dfrac{\lambda^2 + \delta_h^2}{2\delta_h} & \text{if } |\lambda| < \delta_h \\ \Psi(|\lambda|) = |\lambda| & \text{if } |\lambda| \geq \delta_h \end{cases} \tag{46}$$

with δ_h being the threshold for the correction of the eigenvalues approaching zero which is usually a fraction of $\rho(\boldsymbol{A})$.

If the flow becomes locally supersonic, the formulation of the rescaled Roe scheme is no longer valid and the original Roe scheme must be selected. For the explicit scheme, it becomes then necessary to introduce a "switch" between the subsonic and supersonic flow conditions. The matrix valued dissipation $\boldsymbol{Q}$ must be formulated as follows

$$\text{if } a|\boldsymbol{n}| - |q| \geq 0 \text{ then } \boldsymbol{Q} = \boldsymbol{P}^{-1}|\boldsymbol{PA}| \text{ else } \boldsymbol{Q} = |\boldsymbol{A}_{roe}|,$$

with $\boldsymbol{P}^{-1}|\boldsymbol{PA}| = |\boldsymbol{A}_{roe}|$ if $|q| = a|\boldsymbol{n}|$ and $\alpha = 0$ provided that the rescaled matrix dissipation is formulated for the Roe's average. In the supersonic flow regions, the CFL condition must be selected accordingly with (36). As aforementioned, the entropy fix must be applied selectively in order to assure a smooth transition with the Roe scheme near the sonic line. For the rescaled matrix dissipation, the entropy fix must be applied according to (45) with

$$\Psi_p(\mu_-) = \Psi(|\lambda_\pm|) \quad \text{and} \quad \mu_+ = |\lambda_\mp| \quad \text{if} \quad |q \pm a|\boldsymbol{n}|| \leq \delta_h,$$

whenever the Harten's entropy fix (46) is considered.

Since $\mu_+ \to \rho(\boldsymbol{A})$ as $M \to 1$, no special treatment has to be considered for the formulation of the scalar time-step. However, this is not true for the eigenvalues as illustrated in Fig. 1–3. So for the implicit scheme, in the formulation of the time-step matrix, a smooth transition $\boldsymbol{A_p} = \boldsymbol{R_A}\tilde{\boldsymbol{\Lambda}}_{\boldsymbol{A}}\boldsymbol{R_A}^{-1} \to |\boldsymbol{A}_{roe}|$ can only be achieved by modifying the eigenvalues $\mu^{(k)}$. A possible reformulation of the acoustic speeds $\mu_\pm^*$ in the time-step matrix (38)–(40) has been found by considering

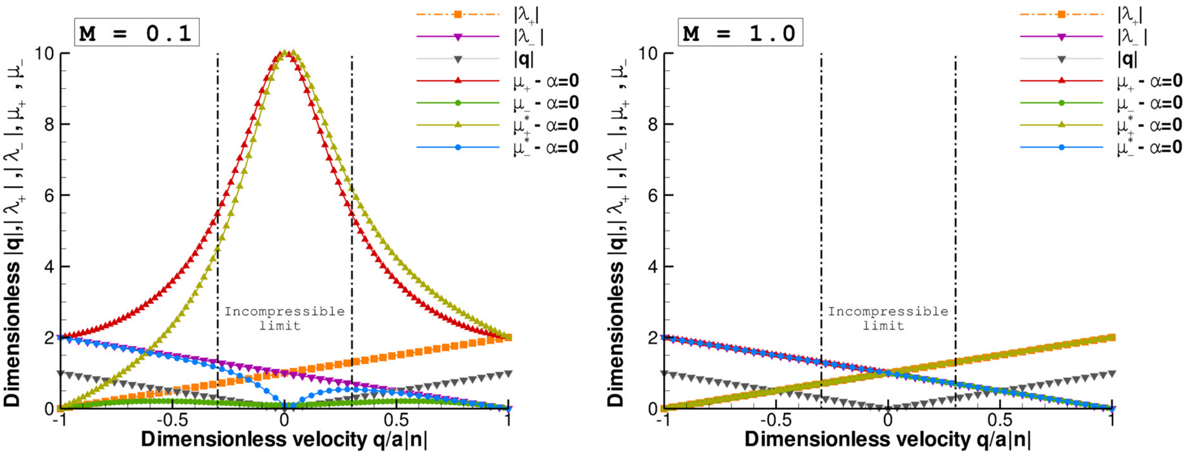

Fig. 4. Dimensionless corrected eigenvalues $\mu^*_\pm$ in the definition of time-step matrix ($\alpha = 0$). Left: $M = 0.1$ with $\mu^*_- = |Q_{22} - a|\boldsymbol{n}|\frac{zq}{\sqrt{\chi}}|$, Right: $M = 1.0$ with $\mu^*_- = |q - a|\boldsymbol{n}|| \Rightarrow \mu^*_\pm = |\lambda_\pm|$.

the "acoustic portion" of decomposition (28). Upon substitution of matrices $\boldsymbol{A}$ and $\boldsymbol{P}^{-1}$ by their respective eigenvalues, we get

$$\mu^*_+ = Q_{11} + a|\boldsymbol{n}|\frac{zq}{\sqrt{\chi}}, \quad \mu^*_- = |Q_{22} - a|\boldsymbol{n}|\frac{zq}{\sqrt{\chi}}|, \tag{47}$$

where the absolute value on μ^*_- ensures positivity. The behaviour of the normalized corrected eigenvalues (47) as function of $\dfrac{q}{a|\boldsymbol{n}|}$ is plotted in Fig. 4 left for $M = 0.1$ and $\alpha = 0$. It can be especially seen that $\mu^*_\pm \to |\lambda_\pm|$ the eigenvalues of $|\boldsymbol{A}_{roe}|$ as $q \to \pm a|\boldsymbol{n}|$ and that the asymptotic behaviour of $\mu^*_+ \simeq \mu_+$ as $|q| \to 0$ is reproduced by the correction. Note that $\mu^*_- \simeq |\lambda_-|$ except when $|q| \to 0$ where μ^*_- may become singular. So the final expression of μ^*_- is further simplified in order to avoid any singularity, and we simply take $\mu^*_- = |\lambda_-| = |q - a|\boldsymbol{n}||$ in the whole subsonic range $|q| \le a|\boldsymbol{n}|$. So for the rescaled Roe scheme, the time-step matrix will assume the following expression for all inflow Mach numbers

$$\Delta t^+ = CFL\frac{h}{Q_{11} + a|\boldsymbol{n}|\dfrac{zq}{\sqrt{\chi}}}, \quad \Delta t^- = CFL\frac{h}{|q - a|\boldsymbol{n}||}, \quad \Delta t^0 = CFL\frac{h}{|q|}. \tag{48}$$

Thus, when $M \to 1$ we have the smooth transition $\mu^*_+ \to |\lambda_+| = |q + a|\boldsymbol{n}||$ with the matrix time-step formulated for the Roe scheme (see Fig. 4 right). In practice, $|q - a|\boldsymbol{n}||$ and $|q|$ may locally vanish or be very small and must be bounded away from zero with the entropy fix.

6. Computational results

We were basically interested in testing numerically the stability and the accuracy of the rescaled matrix dissipation for the Roe scheme in the two extreme cases $M \to 0$ and $M \to 1$. Flow configurations were then selected assuming steady conditions. In order to frame the assumption of inviscid flow considered previously and to demonstrate numerically the properties of the rescaled Roe scheme, we have considered a NACA0012 airfoil configuration widely used for the assessment of CFD solvers. The meshes used are the Vasseberg–Jameson meshes, a family of O-meshes generated from a conformal transformation, yielding quadrilateral cells with an aspect ratio of one, with gridlines essentially orthogonal at each vertex within the mesh [38]. This mesh topology allows to avoid introducing poor grid quality or highly grid stretching effects in the numerical solutions. Note that the original formulation in the CFD solver of the boundary conditions using the characteristic theory has been left unchanged when the rescaled Roe scheme was selected. In our implementation of the rescaled Roe scheme, we have considered the finite-volume method in structured grids. The implicit scheme (41) is solved iteratively using a LU-SGS method, with successive forward and backward diagonal sweeps within the mesh.

6.1. Low Mach number flows about the NACA0012 airfoil

The experiment of reference [1] was reproduced for the reformulation of the rescaled Roe matrix dissipation. In [1] the rescaled matrix dissipation was computed from the diagonalization of preconditioned Jacobian matrix $|\boldsymbol{PA}|$, while in our implementation, the diagonalization was formulated from the full matrix $\boldsymbol{P}^{-1}|\boldsymbol{PA}|$ and the implicit scheme (41) with matrix coefficients (42) was used. The primary interest of this test case is to demonstrate that the proper scaling of the pressure

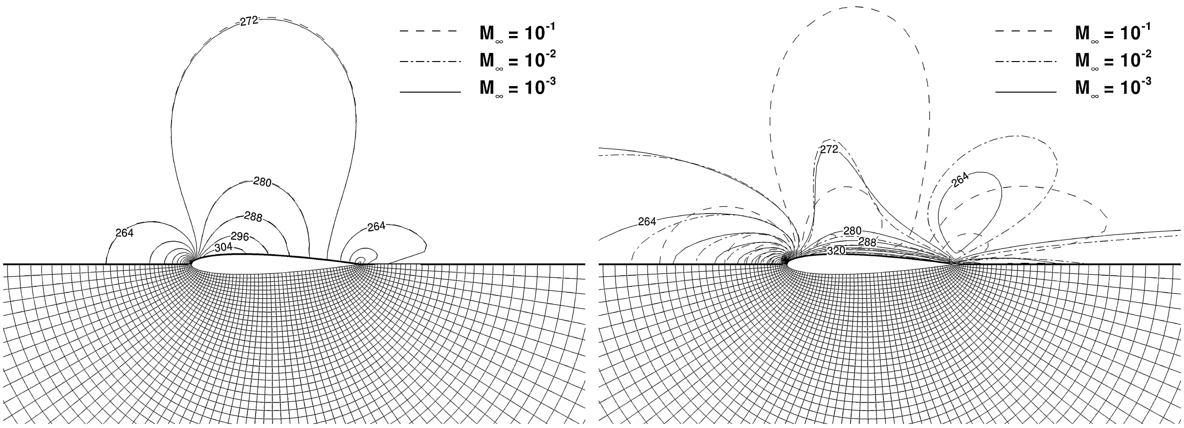

Fig. 5. Velocity contours with the inflow Mach number in the Vasseberg–Jameson mesh. Left: rescaled Roe scheme. Right: Roe scheme ($V_\infty = 270$ m/s).

field is recovered at the discrete level in the incompressible limit by the rescaled Roe scheme. The test-case corresponds to a nonlifting solution at zero angle-of-attack. The grid considered for this preliminary assessment has $N_c = 128$ cells in both directions.

For the normalized equations, we know that the discrete solutions of the first-order Roe scheme support pressure fluctuations in space of order M_∞, with $p(\boldsymbol{x}, t) = p_0(t) + p_1(\boldsymbol{x}, t)M_\infty$, while in the continuous case, the pressure fluctuations scale as M_∞^2. It was also demonstrated that the rescaled Roe scheme (with $\alpha = 0$) support pressure fluctuations in space of second-order $p(\boldsymbol{x}, t) = p_0(t) + p_2(\boldsymbol{x}, t)M_\infty^2$. Thus, without preconditioning or rescaling of the dissipation operator, a compressible flow solver usually fails to compute low Mach number flows.

A rigorous way to prove this result numerically consists in considering a sequence of computations with decreasing inflow Mach number using the same mesh and investigating whether the pressure fluctuations undergo a linear or quadratic behaviour with the Mach number. Looking for steady solutions, the pressure field will then assume the form $p(\boldsymbol{x}) = p_0 + \delta p(\boldsymbol{x})$, with $\delta p(\boldsymbol{x}) = p_f(\boldsymbol{x})M_\infty^n$, where p_0 is the leading order pressure corresponding to the thermodynamic static pressure, p_f is the magnitude of the pressure fluctuations and $n = 1$ or 2 is the order of the fluctuations. Following [1], the accuracy of the computations in the incompressible limit can be characterized by the evolution of a normalized pressure field $\tilde{p} \in [0, 1]$ and an indicator characterizing the order of the pressure fluctuations as the Mach number goes to zero. The normalized pressure used for the comparison of the pressure fields at different inflow Mach numbers is defined as

$$\tilde{p}(\boldsymbol{x}) = \frac{p(\boldsymbol{x}) - p_{min}}{p_{max} - p_{min}} = \frac{p_f(\boldsymbol{x}) - (p_f)_{max}}{(p_f)_{max} - (p_f)_{min}} = \tilde{p}_f(\boldsymbol{x}),$$

and then also corresponds to a normalized field for pressure fluctuations. Note that the normalized pressure $\tilde{p}$ is independent from the inflow Mach number and therefore as $M_\infty \to 0$, numerical solutions should also converge to a consistent approximation of the incompressible solution. A normalized pressure fluctuation $\delta\tilde{p}$ is also introduced with

$$\delta\tilde{p} = \frac{p_{max} - p_{min}}{p_{max}} \simeq \left[\frac{(p_f)_{max} - (p_f)_{min}}{p_0}\right] M_\infty^n \quad \text{for} \quad M_\infty << 1.$$

Since in the normalization process all flow variables are of the same order of magnitude (around unity), the above definition of $\delta\tilde{p}$ gives the variation of the pressure fluctuations with the Mach number.

The behaviour $p = \mathcal{O}(1/M^2)$ for the pressure field was reproduced for a fixed mesh within a sequence of computations by increasing the inflow pressure while keeping constant the inflow velocity and the density, for decreasing Mach numbers M_∞ in the range 10^{-1} to 10^{-3}. The entropy correction was deactivated for all inflow Mach numbers considered. Numerical experiments are represented in the next figures for the Van Leer formulation of the preconditioning parameter (3) and $\alpha = -1$. As expected, solutions obtained with the rescaled matrix dissipation converge to a unique isentropic solution with constant density, corresponding to a decoupling of the velocity and pressure fields, which cannot be achieved with the Roe scheme, as illustrated in Fig. 5–6. The behaviour of the pressure fluctuations with the inflow Mach number is represented in Fig. 7. With the rescaled Roe scheme, the pressure fluctuations exactly scale with M_∞^2. As also indicated, the same experience was conducted for the Choi–Merkle formulation of the preconditioning parameter (2) and $\alpha = 0$, with exactly the same quadratic behaviour reproduced for the pressure fluctuations. As shown by many authors, we see that the Roe scheme supports fluctuations of order M_∞ in the incompressible limit. Additionally, for inflow Mach numbers lower than 0.1, a grid-convergence study carried out in [2] rigorously demonstrates that the effort in grid density required to the Roe scheme to consistently converge to the correct pressure field is inversely proportional to the Mach number. This is a consequence of the behaviour of the leading order term of the truncation error in the incompressible limit.

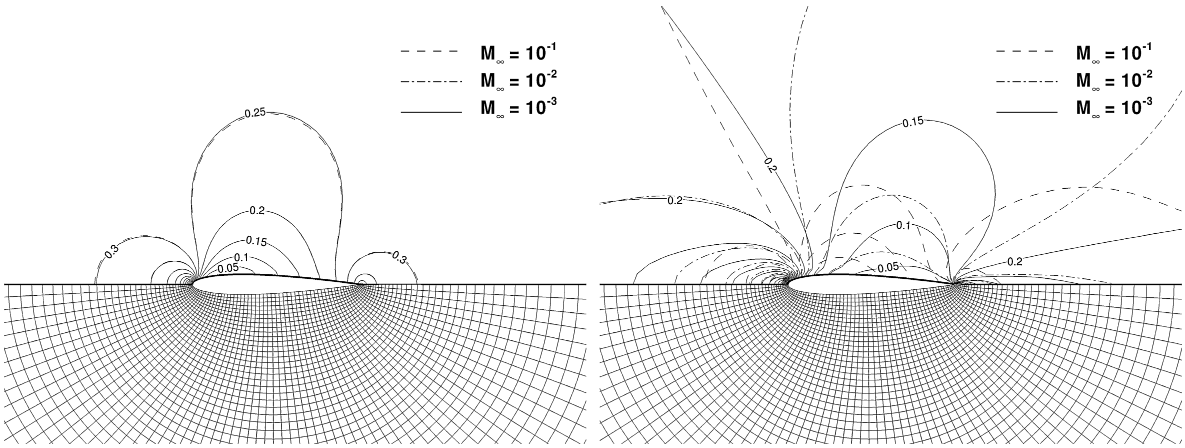

Fig. 6. Contours of the normalized pressure with the inflow Mach number in the Vasseberg–Jameson mesh. Left: rescaled Roe scheme. Right: Roe scheme.

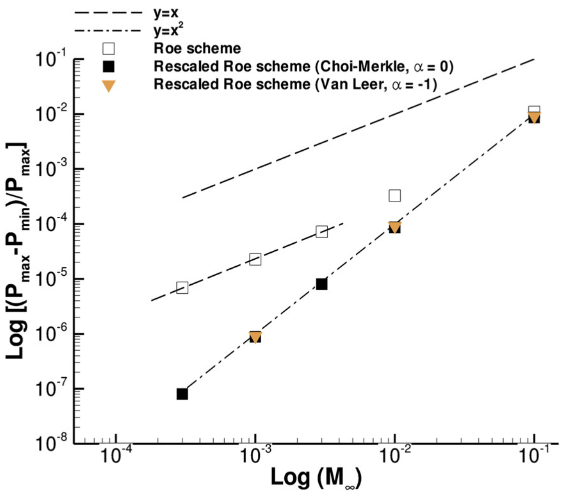

Fig. 7. Pressure fluctuations with the inflow Mach number for the Roe scheme and the rescaled Roe scheme.

The effects of the matrix time-stepping scheme are illustrated in Fig. 8 for the rescaled Roe scheme. As demonstrated in the stability analysis, we experienced that the convergence rate is drastically increased when large CFL numbers are used, and the time-step matrix proved to be more efficient than the standard scalar time for any tested CFL number and any inflow Mach number. Largest CFL numbers used correspond to $\Delta t_+ \simeq \mathcal{O}(h)$, so typically $\mathrm{CFL} \simeq \mathcal{O}(1/M_\infty^2)$, with $\mathrm{CFL} = 10^2$ for $M_\infty = 10^{-1}$, $\mathrm{CFL} = 10^4$ for $M_\infty = 10^{-2}$ and $\mathrm{CFL} = 10^6$ for $M_\infty = 10^{-3}$. The convergence rate is almost independent of the Mach number with the time-step matrix. In the Fig. 8, the limit cycle is defined with the numerical procedure unable to further damp out round-off errors. On the other hand, the zero level machine corresponds to the machine precision. Practically, the limit cycle is reached when the machine accuracy cannot further represent pressure disturbances of the order of round-off errors, corresponding to pressure fluctuations of order $\mathcal{O}(M^2)$. This is a well known behaviour, as the rescaled Roe scheme does not damp out second-order pressure fluctuations occurring in the incompressible limit, without specific corrections not investigated in the present work. Therefore, as represented in the figures with the horizontal dotted lines, the limit cycle is shifted according to the inflow Mach number, from 13 orders of convergence when $M_\infty = 10^{-1}$ for pressure fluctuations of order $\mathcal{O}(10^{-2})$, to 9 orders of convergence when $M_\infty = 10^{-3}$, with pressure fluctuations of order $\mathcal{O}(10^{-6})$. A similar behaviour for the convergence histories is reported by Rossow [10]. However this is not a major issue, as the consistent solutions are obtained in the limit cycle.

The effect of a usual grid stretching was also investigated for the rescaled Roe scheme. The unique isentropic solution for the velocity field and the convergence history are plotted in Fig. 9. The same CFL numbers could be used in this conventional mesh, with a similar effect observed on the convergence rate with matrix time-stepping scheme. However, a much larger number of iterations is required to reach the limit cycle with the standard scalar time-step. The effect of the characteristic matrix time-step on the damping properties of the numerical procedure is much more pronounced in this highly stretched grid. It can be noted that the convergence rate to the steady state may be up to fifteen times faster with the matrix time-step.

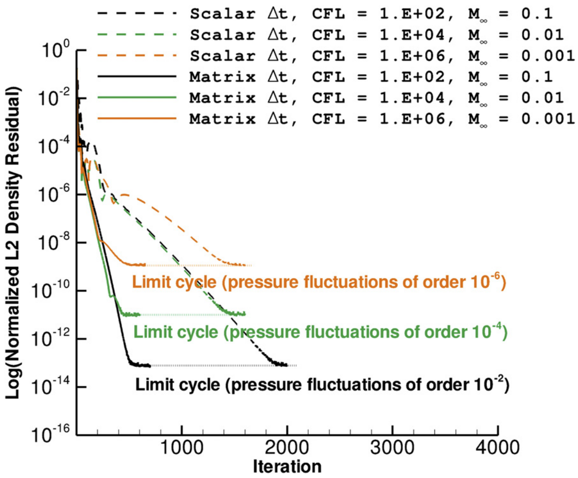

Fig. 8. Convergence history for the rescaled Roe scheme with the inflow Mach number 10^{-1}, 10^{-2} and 10^{-3}. Solid lines: matrix time-step. Dashed lines: scalar time step. Normalized residuals by the maximum residual computed during the iterations.

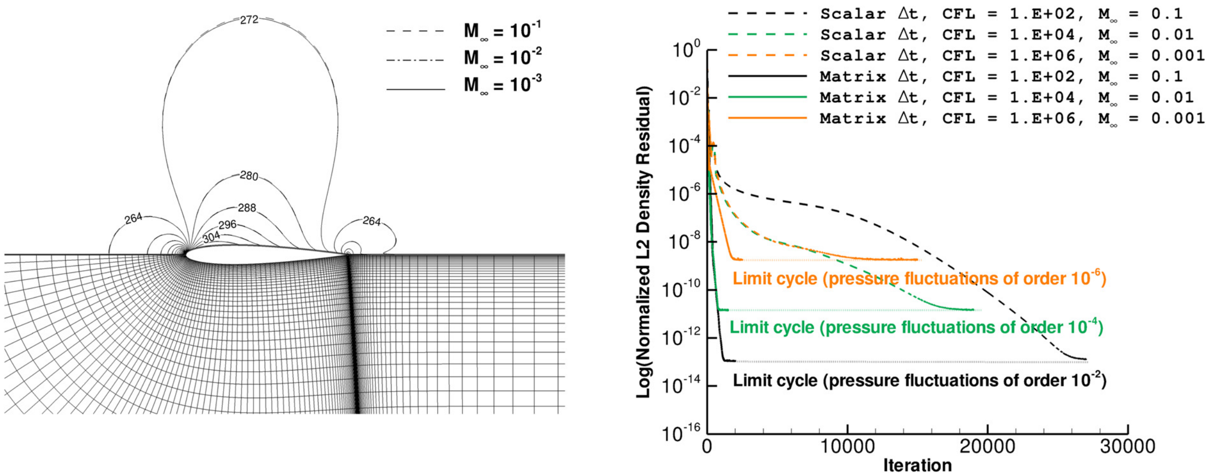

Fig. 9. Effect of grid stretching on the rescaled Roe scheme. Left: Velocity contours with the inflow Mach number. Right: Convergence history with the inflow Mach number 10^{-1}, 10^{-2} and 10^{-3} for the scalar (dashed lines) and the matrix (solid lines) time steps.

6.2. Transonic flow about the NACA0012 airfoil

A transonic flow condition for $M_\infty = 0.8$, $\alpha = 0^0$ was also considered in order to assess the smooth transition of the rescaled Roe scheme with the original Roe scheme at the sonic line, and the effect of the rescaled matrix dissipation at the stagnation point. This was achieved within grid convergence, considering 5 out of the 8 coarsest-to-finest meshes defined in [38], with dimensions 128×128 to 2048×2048 cells. So the finest grid used is composed of over 4 million cells. For the rescaled Roe scheme, the Choi–Merkle formulation of the preconditioning parameter (2) was selected, with $\epsilon^2 = 0.01 << M_\infty^2$ and $\alpha = 0$. For both schemes, the entropy fix was activated, with the special treatment described previously for the rescaled Roe scheme, and a second-order MUSCL extrapolation was considered. For a transonic flow, both schemes have a similar conditioning and therefore similar convergence rates were obtained, with zero-level machine convergence obtained for each mesh. Note that with the matrix time-stepping scheme, the Roe scheme also achieves a faster convergence to the steady state compared to the standard scalar time step.

The comparison for the pressure coefficient obtained in the finest mesh is plotted in Fig. 10 left. The respective solutions have a very similar discrete shock structure, with the same computed shock location. The effect of the proper scaling of the matrix dissipation has to be found near the stagnation point, where a slightly higher peak pressure is predicted with the rescaled Roe scheme. Then, the suction effect on the leading edge is slightly reduced and a larger pressure drag coefficient is found with the rescaled Roe scheme, for all meshes used in the grid-convergence study. A consequence of a better control of the numerical dissipation can be readily pointed out with the entropy distribution on the airfoil. The effect of the grid refinement on the entropy distribution is illustrated in Fig. 10 right for both schemes. A significant effect of the rescaled matrix dissipation can be especially observed on the spurious entropy generation at the stagnation point. Since for

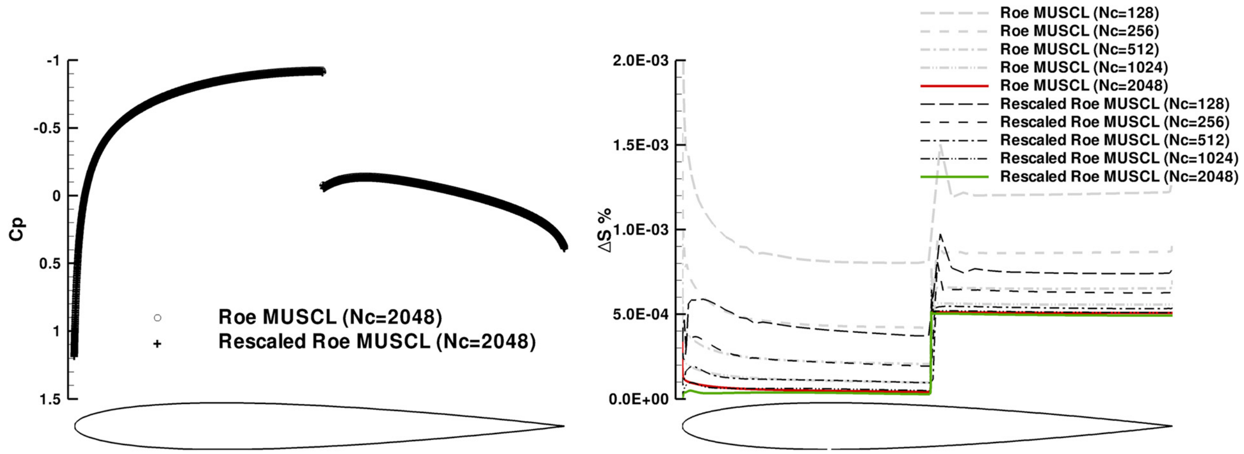

Fig. 10. Comparison of the flow solution for the Roe scheme and the rescaled Roe scheme at $M_\infty = 0.8$, $\alpha = 0^0$. Left: pressure coefficient. Right: entropy distribution.

both schemes, the same definition of the entropy fix was considered, the strong reduction of the spurious entropy is the consequence of the rescaling of the Roe scheme in the low speed flow regions where high gradients of the solution are present. Similar effects were observed in sub-critical and in low-speed flow conditions.

7. Conclusions

A consistent matrix-valued dissipation with the low-speed limit and the transonic regime was derived from the Roe scheme, considering a family of preconditioners popularized by Turkel. The numerical method is characterized by the suppression of the pre-multiplication of the preconditioner with the time derivative and the explicit scheme recovers a basic structure with the centred scheme and stabilization terms. This formulation of the matrix dissipation is interpreted as a rescaling of the Roe scheme for low Mach number flows. When the entropic variables are considered as independent variables, it is demonstrated that the rescaled matrix dissipation is positive semidefinite and can be diagonalized by unitary congruence. So, robustness problems could be significantly alleviated especially at stagnation points. However, our final objective is to developed a more accurate compressible flow solver in low speed flow regions, and the optimal normality properties of the rescaled matrix dissipation are lost by any change of variables. By construction, the rescaled Roe scheme gives the correct asymptotic order for the dissipation terms, and in particular returns the proper quadratic behaviour of the pressure fluctuations in the incompressible limit. However, as the Mach number goes to zero, the system becomes very stiff and very small time steps must be used, independently of the definition of the local preconditioner used for the formulation of the Roe–Turkel scheme. With the rescaling of the Roe scheme, it is then necessary to completely reformulate the stability condition. A CFL condition was formulated explicitly, based on the spectral radius of the rescaled matrix dissipation. For steady-state problems, it is shown that a robust implicit scheme can be formulated in order to circumvent the severe stability bound occurring for low Mach number flows. The implicit scheme is characterized by the introduction of a characteristic time-step matrix, enforcing the damping properties of the scheme. Very large CFL numbers should be used in the incompressible limit, with tipically CFL $\simeq \mathcal{O}(1/M^2)$. However, as indicated in the introduction, the rescaled Roe scheme is intended to improve locally the prediction of complex high Reynolds number flows characterized by the coexistence of high gradients of the solution in mixed incompressible and compressible regions, arising typically in industrial configurations. For such complex flows, it is anticipated that the overall stability of the numerical procedure will not be dominated by the limited number of computational cells where the flowfield can be retained as incompressible. In transonic flow conditions, the rescaled Roe scheme ensures a smooth transition with the original Roe scheme, provided that the entropy fix is carefully formulated. A grid-convergence study demonstrates that the discrete shock structure is preserved with the TVD property enforced by the MUSCL extrapolation. As a consequence of the proper scaling of the matrix dissipation, a significant effect can be especially observed for the inviscid flow, with a drastic reduction of the spurious entropy generation at the stagnation point.

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Appendix A

The Cayley–Hamilton theorem states that substituting any square matrix $\boldsymbol{M}$ for its eigenvalues in the characteristic polynomial p results in the zero matrix $p(\boldsymbol{M}) = \boldsymbol{0}$. Considering the special case of the 2×2 matrix $|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|$ with positive eigenvalues, then $|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|^2 = (\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}})^2$ and the second-order characteristic polynomial reads

$$(\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}})^2 + c_1|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}| + c_0\boldsymbol{Id} = \boldsymbol{0},$$

where c_1 is the sum of the diagonal elements and c_0 the determinant. With the preconditioner being invertible, we then get by multiplying the characteristic polynomial with $\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}$

$$\tilde{\boldsymbol{A}}_{\boldsymbol{0}}(\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}) + c_1\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}| + c_0\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1} = \boldsymbol{0}. \tag{A.1}$$

Similarly, for matrix $\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}$, the Cayley–Hamilton theorem amounts to the identity

$$\tilde{\boldsymbol{A}}_{\boldsymbol{0}}(\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}) + d_1\tilde{\boldsymbol{A}}_{\boldsymbol{0}} + d_0\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1} = \boldsymbol{0}. \tag{A.2}$$

By subtracting the polynomial identities (A.1) and (A.2), it can be easily observed that the dissipation matrix $\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}|$ can be interpreted as a weighted average between the Jacobian matrix $\tilde{\boldsymbol{A}}_{\boldsymbol{0}}$ and the preconditioner $\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}$, with

$$\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}|\tilde{\boldsymbol{P}}_{\boldsymbol{0}}\tilde{\boldsymbol{A}}_{\boldsymbol{0}}| = \frac{1}{c_1}\left[d_1\tilde{\boldsymbol{A}}_{\boldsymbol{0}} + (d_0 - c_0)\tilde{\boldsymbol{P}}_{\boldsymbol{0}}^{-1}\right]. \tag{A.3}$$

This is explicitly formulated with (13). For the multidimensional 2D problem with 4×4 matrices, the decomposition (28) can be also demonstrated by routine calculations, considering a similar formal proof with 4^{th} order characteristic polynomials. The existence of the $\tilde{\boldsymbol{L}}_0$ matrix can be easily established, as function of the preconditioning matrix $\boldsymbol{P}$ and the Jacobian matrix $\boldsymbol{A}$. However the formulation of decomposition (28) and especially the formulation of the resulting sparse matrix $\tilde{\boldsymbol{L}}_0$ would require too much algebra if one tries to apply the Cayley–Hamilton theorem. Result (28) was actually obtained directly from matrix (25), following the decomposition derived for the 2×2 acoustic matrix.

Appendix B

In this appendix, it is demonstrated in a multi-dimensional framework that the spectral radius of the rescaled Roe matrix is given by the fastest acoustic speed μ_+ for all possible values of the free parameter α.

We have $\rho(\boldsymbol{P}^{-1}|\boldsymbol{P}\boldsymbol{A}|) = max(\mu_+, |q|)$ since $\mu_+ > \mu_-$. We found that $\mu_+ > |q|$ when $M_n \to 0$ and $M_n \to 1$. This is actually true $\forall M_n \leq 1$ and for $\alpha = 0$, $\alpha = 1$ and $\alpha = -1$ as demonstrated in the following.

Using identity (27) and taking advantage of having both $\mu_\pm \geq 0$, we can derive the following expression

$$(\mu_+ - |q|)(\mu_- + |q|) = a^2|\boldsymbol{n}|^2\left[1 + M_n\sqrt{\frac{\Delta}{\tilde{\chi}}} - 2M_n^2\right]. \tag{B.1}$$

From (B.1), we have the sufficient condition that as long as $M_n \leq \frac{\sqrt{2}}{2}$, then $\mu_+ - |q| \geq 0$ and $\rho(\boldsymbol{P}^{-1}|\boldsymbol{P}\boldsymbol{A}|) = \mu_+$, without condition on the free parameter α. We also find that if $\Delta - \tilde{\chi} \geq 0$, then $\sqrt{\frac{\Delta}{\tilde{\chi}}} \geq 1$ and $\mu_+ - |q| \geq 0$ with

$$1 + M_n\sqrt{\frac{\Delta}{\tilde{\chi}}} - 2M_n^2 \geq 1 + M_n - 2M_n^2 \geq 0 \quad \text{for all} \quad 0 \leq M_n \leq 1.$$

We can formulate the quantity $\Delta - \tilde{\chi}$ as follows

$$\Delta - \tilde{\chi} = P_0(1 - M_n^2) + P_1 M_n^2,$$

with

$$P_0 = (1 - \frac{\beta^2}{a^2})^2(1 - M_n^2) + 2\alpha(1 - \alpha + \frac{\beta^2}{a^2})M_n^2 - \frac{\beta^2}{a^2} \quad \text{and} \quad P_1 = \frac{3}{4}(1 - \alpha + \frac{\beta^2}{a^2})^2.$$

Since $P_1 > 0$, a sufficient condition for $\Delta - \tilde{\chi} \geq 0$ is $P_0 \geq 0$. With $\alpha = 1$, we see that as long as M_n lies in the range $\frac{\sqrt{2}}{2} \leq M_n \leq 1$, $P_0 \geq 0$. Thus setting $\alpha = 1$ ensures $\mu_+ - |q| > 0$ and then $\rho(\boldsymbol{P}^{-1}|\boldsymbol{P}\boldsymbol{A}|) = \mu_+$ for all $M_n \leq 1$. On the other hand, considering now the cases $\alpha = 0$ and $\alpha = -1$, we cannot conclude on the sign of $\mu_+ - |q|$ by using the expression $\Delta - \tilde{\chi}$. However, using identity (B.1), we also see that

$$(\mu_+ - |q|)(\mu_- + |q|)\left(1 + M_n\sqrt{\frac{\Delta}{\tilde{\chi}}} + 2M_n^2\right) = a^2|\boldsymbol{n}|^2\left[1 + 2M_n\sqrt{\frac{\Delta}{\tilde{\chi}}} + M_n^2\left(\frac{\Delta - 4M_n^2\tilde{\chi}}{\tilde{\chi}}\right)\right], \tag{B.2}$$

with $\tilde{\chi} > 0$. Following the same idea, we also have

$$(\mu_+ - \rho(\boldsymbol{A}))(\mu_- + \rho(\boldsymbol{A}))\left(\sqrt{\frac{\Delta}{\tilde{\chi}}} + 2M_n\right) = a^2|\boldsymbol{n}|^2\,(1+M_n)\left(\frac{\Delta - 4M_n^2\tilde{\chi}}{\tilde{\chi}}\right), \tag{B.3}$$

where $\rho(\boldsymbol{A}) = |q| + a|\boldsymbol{n}|$ is the spectral radius of the original Jacobian matrix.

With $\alpha = 0$

$$\Delta - 4M_n^2\tilde{\chi} = (1 - \frac{\beta^2}{a^2})^2(1 - M_n^4) \geq 0 \quad \text{for all} \quad 0 \leq M_n \leq 1.$$

With $\alpha = -1$

$$\Delta - 4M_n^2\tilde{\chi} = \left((1 - \frac{\beta^2}{a^2})^2 - M_n^2\right)(1 - M_n^2) \quad => \quad 1 + M_n^2\left(\frac{\Delta - 4M_n^2\tilde{\chi}}{\tilde{\chi}}\right) > 0 \quad \text{for all} \quad 0 \leq M_n \leq 1.$$

Then with identity (B.2), we conclude that $\rho(\boldsymbol{P^{-1}}|\boldsymbol{PA}|) = \mu_+\ \forall M_n \leq 1$, and for all possible values of the parameter α. With $\alpha = 0$ in identity (B.3), we also have $\mu_+ \geq \rho(\boldsymbol{A})$ for all $M_n \leq 1$. However, this is not always the case with $\alpha = 1$ or $\alpha = -1$.

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