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Multigrid methods for space fractional partial differential equations ☆

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ABSTRACT

We propose some multigrid methods for solving the algebraic systems resulting from finite element approximations of space fractional partial differential equations (SFPDEs). It is shown that our multigrid methods are optimal, which means the convergence rates of the methods are independent of the mesh size and mesh level. Moreover, our theoretical analysis and convergence results do not require regularity assumptions of the model problems. Numerical results are given to support our theoretical findings.

1. Introduction

Fractional partial differential equations (FPDEs) have found many impressive applications in lots of fields, such as finance, phase transitions, stratified materials, anomalous diffusions (see [25] and references therein). To solve them, both analytical and numerical methods are used in the literature. The analytical methods like the Fourier transform method, the Laplace transform method and the Mellin transform method have been developed to seek closed-form analytical solutions [29]. Since such closed-form analytical solutions are unavailable in most cases, extensive researches have already been carried out on the development of numerical methods for fractional partial differential equations like finite difference methods (see e.g., [4,8,14,22,23,34,36]), finite element methods (see e.g., [9,10,19]), and spectral methods [15,17].

Let Ω be a polyhedral domain in $\mathbb{R}^d$, we consider the space fractional partial differential equations (SFPDEs): find $u(x)$ such that (see [11])

$$-\int_{S^{d-1}} D_z^{2\alpha} u(x)\, \tilde{M}(z) dz + cu(x) = f(x), \quad x \in \Omega, \tag{1.1}$$

$$u|_{R^d \setminus \Omega} = 0, \tag{1.2}$$

where $1/2 < \alpha \le 1$, $c \ge 0$, f is a source term, $S^{d-1} = \{z \in R^d; ||z||_2 = 1\}$, $\tilde{M}(z)$ is a probability density function on S^{d-1}, $||\cdot||_2$ denotes the standard Euclidean norm, and $D_z^{2\alpha}$, which will be given later, denotes the directional derivative of order

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2α in the direction of the unit vector z. Here we assume $\tilde{M}$ is symmetric about origin, i.e., $\tilde{M}(z) = \tilde{M}(z')$ if $z, z' \in S^{d-1}$ satisfy $z + z' = 0$, which means that the considered problem is a symmetric one.

One special case of (1.1) is

$$-\sum_{i=1}^{d}(p_{i\ -\infty}D_{x_i}^{2\alpha} + q_{i\ x_i}D_{\infty}^{2\alpha})u + cu = f \tag{1.3}$$

and $p_i, q_i \geq 0$ satisfying $p_i = q_i$ and $\sum_{i=1}^{d}(p_i + q_i) = 1$, where ${}_{-\infty}D_{x_i}^{2\alpha}$, ${}_{x_i}D_{\infty}^{2\alpha}$ denote Riemann–Liouville fractional derivatives. Actually, (1.3) can be obtained from (1.1) by taking $\tilde{M} = \sum_{i=1}^{d} p_i\delta(z - e_i) + q_i\delta(z + e_i)$, where e_i is the ith column of identity matrix in $\mathbb{R}^{d\times d}$ and δ the Dirac function on S^{d-1}. The corresponding time-dependent equation of (1.1) can be used to describe a general super-diffusion process (see [20]), which is an appropriate extension from one dimensional problem

$$\frac{\partial u}{\partial t} - (p_{-\infty}D_x^{2\alpha} + q_x D_{\infty}^{2\alpha})u + cu = f. \tag{1.4}$$

As to the super-diffusion, please refer to [24] for details.

One of the greatest challenges for numerically solving SFPDEs is how to reduce the computation costs. Due to the non-local properties of fractional differential operators, numerical methods for linear SFPDEs tend to yield the linear equations $Ax = b$ with the following characteristics: (1) the coefficient matrix A is dense or full; (2) the condition number of A increases fast, as the mesh becomes fine. Reducing the computation costs for SFPDEs is harder than doing it for the integer order PDEs. Some methods have already been designed to overcome this difficulty, such as alternating-direction implicit methods (ADI) [23,39,40], and iterative methods [16,27,28,30,40–43].

Iterative methods seem to be efficient tools for solving SFPDEs. Actually two issues in this situation need to be concerned for efficiency: one is to do the matrix–vector multiplications efficiently, and the other is to find good preconditioners. As to the first issue, some literatures are contributed: in [38], with the notice of Toeplitz-like structure of the coefficient matrix, the matrix–vector multiplications are done with $O(N\log N)$ complexity by using a fast Fourier transform (FFT) [5,6]. This technique of "matrix–vector multiplication" has been widely used to improve the efficiency of iterative methods for the SFPDEs [16,28,40–43]. As regards the second issue, some literatures should be listed as follows: the first relevant paper may be [2] in which a multilevel preconditioner of fractional power was put forward; in [16], the authors propose preconditioners constructed by some banded matrices of fixed bandwidth; in [42], the authors present a preconditioner by some symmetric positive Toeplitz matrices; moreover a new preconditioner is designed in [13] through some circulant matrices.

It is known that multigrid methods are optimal iterative procedures, which have been widely used for integer order PDEs (see e.g., [3,35]). In recent years, some researchers begin to investigate multigrid methods for solving SFPDEs. For instance, in [46], Zhou and Wu apply the multigrid method to solve one dimensional steady SFPDEs, and in [28], the authors consider the V-cycle multigrid method for solving corresponding time-dependent problems. But till now, no satisfactory convergence results have been obtained for the multigrid methods for solving SFPDEs. Actually, in [28], the authors only conduct the theoretical analysis for the two-level multi-grid method, and Zhou and Wu in [46] get the convergence results only under the assumption that the adjoint problem has sufficiently smooth solution.

In this paper, we introduce a V-cycle multigrid method with one smoothing step on each level to solve linear algebraic systems resulting from the finite element approximations of the SFPDEs (1.1). It is shown that our V-cycle multigrid methods are optimal, which means the convergence rates are independent of the mesh size and mesh level. Moreover, our theoretical analysis and the convergence results in this paper do not require any regularity assumptions of the model problems. To the best of our knowledge, this paper is a first attempt to give a rigorous theoretical analysis for the V-cycle multigrid methods for the finite element approximations of SFPDEs in any dimensions.

This paper is also the first work to design the fast solver for the SFPDE (1.1) with M being a continuous function. Among the current numerical methods for SFPDEs, most of them are for one dimensional problems and for some special high dimensional problems like (1.3), and only a few are for more general problems like (1.1). Actually, only [11,31] study the numerical methods for (1.1): in [11], the authors consider the finite element approximation for (1.1) and in [31], the author studies the corresponding time-dependent case.

In the rest of the paper, without loss of generality, we restrict ourselves to the case $d = 2$, namely, we consider the problem (1.1) in $\mathbb{R}^2$. For $\Lambda \subset \mathbb{R}^2$, denote by $L^2(\Lambda)$ the space of all measurable functions v on Λ satisfying $\int_\Lambda (v(x))^2 dx < \infty$, and by $C_0^\infty(\Lambda)$ the space of infinitely differentiable functions with compact support in Λ. Set

$$(v, w)_\Lambda = \int\limits_\Lambda vw\,dxdy, \quad ||v||_\Lambda = (v, v)_\Lambda^{1/2},$$

and they are abbreviated as (v, w) and $||v||$ respectively if $\Lambda = \mathbb{R}^2$.

To simplify our statement, we make a convention here: function v defined on a domain $\Lambda \subset \mathbb{R}^2$ also denotes its extension on $\mathbb{R}^2$ which extends v by zero outside Λ. The constant C with or without subscript will denote a generic positive constant which may take on different values in different places. These constants will always be independent of the mesh sizes and levels in the multigrid methods. Following [44], we also use symbols $\lesssim, \gtrsim$ and $\approx$ in this paper. That $a_1 \lesssim b_1$, $a_2 \gtrsim b_2$ and $a_3 \approx b_3$ means that $a_1 \le C_1 b_1$, $a_2 \ge C_2 b_2$ and $C_3 b_3 \le a_3 \le C_3' b_3$ for some positives C_1, C_2, C_3 and C_3'.

The rest of the paper is organized as follows: for the sake of completeness, in Section 2.1, we give our model problem and the corresponding finite element discretization. In Section 3, we present our V-cycle multigrid methods and introduce some basic theoretical results. In Section 4, we shall prove the convergence of the multigrid methods. In Section 5, the numerical results are given to verify our theoretical findings.

2. The model problem and its discretization

In this section, we shall present the SFPDE in $\mathbb{R}^2$, and then introduce its variational formulation and corresponding finite element discretization.

2.1. The model problem

We first introduce the concepts of directional integrals and derivatives [11].

Definition 2.1. (See [11].) Let $\mu > 0$, $\theta \in \mathbb{R}$. The μth order fractional integral in the direction $z = (\cos\theta, \sin\theta)$ is defined by

$$D_z^{-\mu} v(x, y) := D_\theta^{-\mu} v(x, y) = \int_0^\infty \frac{\tau^{\mu-1}}{\Gamma(\mu)} v(x - \tau\cos\theta, y - \tau\sin\theta) d\tau,$$

where Γ is the Gamma function.

Definition 2.2. (See [11].) Let n be a positive integer, and $\theta \in \mathbb{R}$. The nth order derivative in the direction of $z = (\cos\theta, \sin\theta)$ is given by

$$D_\theta^n v(x, y) := \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right)^n v(x, y).$$

Definition 2.3. (See [11].) Let $\mu > 0$, $\theta \in \mathbb{R}$. Let n be the integer such that $n - 1 \le \mu < n$, and define $\sigma = n - \mu$. Then the μth order directional derivative in the direction of $z = (\cos\theta, \sin\theta)$ is defined by

$$D_z^\mu v(x, y) := D_\theta^\mu v(x, y) = D_\theta^n D_\theta^{-\sigma} v(x, y).$$

If v is viewed as a function in x, D_0^μ, D_π^μ are just the left and the right Riemann–Liouville derivatives (see e.g., [29,33]). The fractional derivative operators in problem (1.1) are related to the following fractional derivative:

Definition 2.4. (See [11].) Assume that $v : \mathbb{R}^2 \to \mathbb{R}$, $\mu > 0$. The μth order fractional derivative with respect to the measure $\tilde{M}$ is defined as

$$D_{\tilde{M}}^\mu v(x, y) := \int_{S^1} D_\theta^\mu v(x, y) \tilde{M}(\theta) d\theta,$$

where $S^1 = [0 + \nu, 2\pi + \nu)$ with a suitable scalar ν, and $\tilde{M}(\theta)$, which satisfies $\int_\nu^{2\pi+\nu} \tilde{M}(\theta) d\theta = 1$, is a periodic function with period 2π. Without loss of generality, we take $\nu = 0$.

Remark 2.5. It is easy to check that

$$D_{\tilde{M}}^2 v(x, y) = a_{11} \frac{\partial^2 v}{\partial x^2} + a_{22} \frac{\partial^2 v}{\partial y^2} + 2a_{12} \frac{\partial^2 v}{\partial x \partial y},$$

where $a_{11} = \int_0^{2\pi} \cos^2\theta \tilde{M}(\theta) d\theta$, $a_{22} = \int_0^{2\pi} \sin^2\theta \tilde{M}(\theta) d\theta$ and $a_{12} = 2\int_0^{2\pi} \cos\theta \sin\theta \tilde{M}(\theta) d\theta$ (see also [21]). Denote by L a positive integer, let $\theta_k \in [0, 2\pi)$ and $p_k \ge 0$, $k = 1, 2, \ldots, L$, satisfy $\sum_{k=1}^L p_k = 1$. Assume that $D_\theta^\mu v$ is continuous in θ, and then

$$D_{\tilde{M}}^\mu v = \sum_{k=1}^L p_k D_{\theta_k}^\mu v(x, y), \tag{2.1}$$

if

$$\tilde{M} = \sum_{k=1}^{L} p_k \delta(\theta - \theta_k), \tag{2.2}$$

where δ denotes Dirac delta function.

For $u : \mathbb{R}^2 \to \mathbb{R}$, define differential operator L_α in $\mathbb{R}^2$ as

$$L_\alpha u = -D_{\tilde{M}}^{2\alpha} u + cu.$$

Denote by Ω a polygonal domain in $\mathbb{R}^2$, set $1/2 < \alpha \le 1$, and then the model problem of this paper is to find $u : \bar{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} L_\alpha u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where f is a source term and we assume that $\tilde{M}(\theta)$ satisfies $\tilde{M}(\theta) = \tilde{M}(\theta + \pi)$ for $\theta \in \mathbb{R}$, i.e., (2.3) is a symmetric problem. Here, we recall the convection made in Section 1, i.e., u also denotes its extension by zero outside Ω.

2.2. The variational formulation

Definition 2.6. (See [37].) Let $\mu \ge 0$, $\mathcal{F}v(\xi_1, \xi_2)$ be the Fourier transform of $v(x, y)$, $|\xi| = \sqrt{\xi_1^2 + \xi_2^2}$. Define norm

$$||v||_{H^\mu(\mathbb{R}^2)} := \left\| (1 + |\xi|^2)^{\mu/2} |\mathcal{F}v| \right\|.$$

Let $H^\mu(\mathbb{R}^2) := \{v \in L^2(\mathbb{R}^2); ||v||_{H^\mu(\mathbb{R}^2)} < \infty\}$.

For $v \in H_0^\mu(\Omega)$, we also denote $||v||_{H^\mu(\mathbb{R}^2)}$ by $||v||_{H^\mu(\Omega)}$. It is known that $H^\mu(\mathbb{R}^2)$ is a Hilbert space equipped with the inner product $(v, w)_{H^\mu(\mathbb{R}^2)} = ((1+|\xi|^2)^\mu \mathcal{F}v, \overline{\mathcal{F}w})$ and $C_0^\infty(\mathbb{R}^2)$ is dense in $H^\mu(\mathbb{R}^2)$ (see [37]). Now, we introduce and prove some useful results for the fractional directional derivatives of functions in $C_0^\infty(\mathbb{R}^2)$.

Lemma 2.7. *(See [11].) For $\mu \in \mathbb{R}$, $v \in C_0^\infty(\mathbb{R}^2)$, the Fourier transform of $D_\theta^\mu v$ is*

$$\mathcal{F}(D_\theta^\mu v(x, y)) = (2\pi i(\xi_1 \cos\theta + \xi_2 \sin\theta))^\mu \, \mathcal{F}v(\xi_1, \xi_2).$$

Lemma 2.8. *For $\mu, s > 0$, $v, w \in C_0^\infty(\mathbb{R}^2)$,*

$$(D_\theta^\mu v, w) = (D_\theta^{\mu-s} v, D_{\theta+\pi}^s w),$$

where $D_\theta^0 v = v$.

Proof. By Lemma 2.7 and (A.1), we know $\mathcal{F}D_\theta^\mu v = (2\pi i(\xi_1 \cos\theta + \xi_2 \sin\theta))^\mu \, \mathcal{F}v$, $\mathcal{F}D_\theta^{\mu-s} v = (2\pi i(\xi_1 \cos\theta + \xi_2 \sin\theta))^{\mu-s} \times \mathcal{F}v$, $\mathcal{F}D_{\theta+\pi}^s w = \overline{(2\pi i(\xi_1 \cos\theta + \xi_2 \sin\theta))^s} \mathcal{F}w$. Then the lemma follows by Parseval's formula. □

We define the weak fractional directional derivative according to the relation $(D_\theta^\mu v, w) = (v, D_{\theta+\pi}^\mu w)$ which is a special case of Lemma 2.8 (see also Lemma 5.7 in [11]). Let $L_{loc}^1(\mathbb{R}^2)$ denote the set of locally integrable functions on $\mathbb{R}^2$.

Definition 2.9. Given $\mu > 0$, $\theta \in \mathbb{R}$, let $v \in L^2(\mathbb{R}^2)$. If there is a function $v_\mu \in L_{loc}^1(\mathbb{R}^2)$ such that

$$(v, D_{\theta+\pi}^\mu w) = (v_\mu, w), \quad \forall w \in C_0^\infty(\mathbb{R}^2),$$

then v_μ is called the weak μth order derivative in the direction of θ for v, denoted by $D_\theta^\mu v$, i.e., $v_\mu = D_\theta^\mu v$.

It is not hard to see that the weak derivative $D_\theta^\mu v$ is unique if it exists and that the weak derivative coincides with the correspondent derivative defined in Definition 2.3 if $v \in C_0^\infty(\mathbb{R}^2)$. In the following, we use $D_\theta^\mu v$ to denote the weak derivative.

Lemma 2.10. *Let $\mu > 0$. For any $v \in H^\mu(\mathbb{R}^2)$, $0 < s \le \mu$ and $\theta \in \mathbb{R}$, the weak derivative $D_\theta^s v$ exists and satisfies*

$$\mathcal{F}D_\theta^s v(\xi_1,\xi_2) = (2\pi i\xi_1\cos\theta + 2\pi i\xi_2\sin\theta)^s \mathcal{F}v(\xi_1,\xi_2), \tag{2.4}$$

$$||D_\theta^s v|| \le C||v||_{H^\mu(\mathbb{R}^2)}. \tag{2.5}$$

Proof. Since $C_0^\infty(\mathbb{R}^2)$ is dense in $H^\mu(\mathbb{R}^2)$, there is a Cauchy sequence $\{v_n\} \subset C_0^\infty(\mathbb{R}^2)$ such that $||v_n - v||_{H^\mu(\mathbb{R}^2)} \to 0$ as $n \to 0$. By Lemma 2.7, $\mathcal{F}D_\theta^s w = (2\pi i(\xi_1\cos\theta + \xi_2\sin\theta))^s\mathcal{F}w$ for $w \in C_0^\infty(\mathbb{R}^2)$. By Parseval's formula and $0 < s \le \mu$, it is not hard to see that $||D_\theta^s w|| = ||\mathcal{F}D_\theta^s w|| \le C||w||_{H^\mu(\mathbb{R}^2)}$. So we have $||D_\theta^s v_n - D_\theta^s v_m|| \le C||v_n - v_m||_{H^\mu(\mathbb{R}^2)}$ and $\{D_\theta^s v_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^2)$. Denote by $v_s \in L^2(\mathbb{R}^2)$ the function to which $\{D_\theta^s v_n\}$ converges to. By Lemma 2.8, for any $w \in C_0^\infty(\mathbb{R}^2)$,

$$(v_n, D_{\theta+\pi}^s w) = (D_\theta^s v_n, w).$$

Taking the limits of both sides of the above equation, we obtain $(v, D_{\theta+\pi}^s w) = (v_s, w)$ for any $w \in C_0^\infty(\mathbb{R}^2)$. So $D_\theta^s v$ exists and is equal to v_s by Definition 2.9. By the definition of Fourier transform for the function in $L^2(\mathbb{R}^2)$,

$$((2\pi i(\xi_1\cos\theta + \xi_2\sin\theta))^s\mathcal{F}v_n, v) = (D_\theta^s v_n, \mathcal{F}v), \quad \forall v \in C_0^\infty(\mathbb{R}^2). \tag{2.6}$$

Because

$$||v_n - v||_{H^\mu(\mathbb{R}^2)} = \left\| (1+|\xi|^2)^{\mu/2}|\mathcal{F}(v_n - v)| \right\| \to 0,$$

it is not hard to see that $(2\pi i(\xi_1\cos\theta + \xi_2\sin\theta))^s\mathcal{F}v_n$ converges to $(2\pi i(\xi_1\cos\theta + \xi_2\sin\theta))^s\mathcal{F}v$ in $L^2(\mathbb{R}^2)$. Taking the limits of both sides of (2.6), we obtain (2.4) by the definition of Fourier transform. (2.5) can be proved directly by (2.4) and Parseval's formula. □

Lemma 2.11. *Let $\mu, s > 0$ with $\mu - s > 0$. For $v, w \in H^{\mu+s}(\mathbb{R}^2)$,*

$$(D_\theta^\mu v, D_{\theta+\pi}^\mu w) = (D_\theta^{\mu+s} v, D_{\theta+\pi}^{\mu-s} w). \tag{2.7}$$

Proof. For any $g \in H^{\mu+s}(\mathbb{R}^2)$, $||D_\theta^\mu g||$, $||D_{\theta+\pi}^\mu g||$, $||D_\theta^{\mu+s} g||$ and $||D_{\theta+\pi}^{\mu-s} g||$ are all bounded by $C||g||_{H^\mu(\mathbb{R}^2)}$ by Lemma 2.10. Then the lemma follows from that $C_0^\infty(\mathbb{R}^2)$ is dense in $H^{\mu+s}(\mathbb{R}^2)$ and Lemma 2.8. □

Assume that the solution u of (2.3) is sufficiently smooth (indeed, that $u \in C^2(\Omega)$ with $u|_{\partial\Omega} = 0$ is sufficient). Multiplying both sides of the first equation in (2.3) with $v \in C_0^\infty(\Omega)$ and integrating over Ω give

$$-\int_0^{2\pi} (D_\theta^{2\alpha} u, v)\tilde{M}(\theta)d\theta + c(u,v) = (f,v), \quad v \in C_0^\infty(\Omega). \tag{2.8}$$

Then employing the relation $(D_\theta^1 w, v) = (w, D_{\theta+\pi}^1 v)$ (it can be obtained by integration by parts), we obtain

$$-\int_0^{2\pi} (D_\theta^{2\alpha-1} u, D_{\theta+\pi}^1 v)\tilde{M}(\theta)d\theta + c(u,v) = (f,v), \quad v \in C_0^\infty(\Omega). \tag{2.9}$$

Then by Lemma 2.11, (2.9) can be rewritten as

$$-\int_0^{2\pi} (D_\theta^{\alpha} u, D_{\theta+\pi}^\alpha v)\tilde{M}(\theta)d\theta + c(u,v) = (f,v), \quad v \in C_0^\infty(\Omega). \tag{2.10}$$

Define the bilinear form $\tilde{B} : H_0^\alpha(\Omega) \times H_0^\alpha(\Omega) \to \mathbb{R}$ as

$$\tilde{B}(u,v) := -\int_0^{2\pi} (D_\theta^{\alpha} u, D_{\theta+\pi}^\alpha v)\tilde{M}(\theta)d\theta + c(u,v).$$

By $\tilde{M}(\theta) = \tilde{M}(\theta+\pi)$ for $\theta \in \mathbb{R}$, it is easy to check that $\tilde{B}(v,w)$ is a symmetric bilinear form, i.e., $\tilde{B}(v,w) = \tilde{B}(w,v)$ for $v, w \in H_0^\alpha(\Omega)$. The variational formulation of (2.3) is (see also [11]) to find $u \in H_0^\alpha(\Omega)$ such that

$$\tilde{B}(u,v) = (f,v), \quad \forall v \in H_0^\alpha(\Omega). \tag{2.11}$$

Now we restate some results in [11] about the solvability of (2.11). To guarantee the existence of the solution of (2.11), we assume that $\tilde{M}(\theta)$ satisfies

$$\int_0^{2\pi} |(\xi_1\cos\theta+\xi_2\sin\theta)|^{2\alpha}\tilde{M}(\theta)d\theta \geq C_0|\xi|^{2\alpha} \tag{2.12}$$

for some positive C_0. Denote $\kappa = 2\pi(\xi_1\cos\theta+\xi_2\sin\theta)$, $E_1 = \{(\xi_1,\xi_2)\in\mathbb{R}^2 : \xi_1\cos\theta+\xi_2\sin\theta > 0\}$, $E_2 = \{(\xi_1,\xi_2)\in\mathbb{R}^2 : \xi_1\cos\theta+\xi_2\sin\theta < 0\}$, and then by Parseval's formula and Lemma 2.10,

$$\begin{aligned}(D_\theta^\alpha v, D_{\theta+\pi}^\alpha v) &= ((i\kappa)^{2\alpha}\mathcal{F}v, \overline{\mathcal{F}v}) \\ &= (|\kappa|^{2\alpha}\exp(i\alpha\mathrm{sign}(\kappa)\pi)\mathcal{F}v, \overline{\mathcal{F}v}) \\ &= (|\kappa|^{2\alpha}\exp(i\alpha\pi)\mathcal{F}v, \overline{\mathcal{F}v})_{E_1} + (|\kappa|^{2\alpha}\exp(-i\alpha\pi)\mathcal{F}v, \overline{\mathcal{F}v})_{E_2} \\ &= \cos(\alpha\pi)(|\kappa|^{2\alpha}\mathcal{F}v, \overline{\mathcal{F}v}) + i\sin(\alpha\pi)\left((|\kappa|^{2\alpha}\mathcal{F}v, \overline{\mathcal{F}v})_{E_1} - (|\kappa|^{2\alpha}\mathcal{F}v, \overline{\mathcal{F}v})_{E_2}\right) \\ &= \cos(\alpha\pi)(|\kappa|^{2\alpha}\mathcal{F}v, \overline{\mathcal{F}v}),\end{aligned} \tag{2.13}$$

where for the computation of complex please refer to Appendix A, in the fourth equality, the Euler formula $\exp(i\kappa) = \cos(\kappa) + i\sin(\kappa)$ is used, the last equality is because the value of $(D_\theta^\alpha v, D_{\theta+\pi}^\alpha v)$ is real and the imaginary part must be zero (for another proof for this equality please refer to [11]). Furthermore, by (2.12) and $\cos(\alpha\pi) < 0$

$$\begin{aligned}-\int_0^{2\pi}(D_\theta^\alpha v, D_{\theta+\pi}^\alpha v)\tilde{M}(\theta)d\theta &= -\cos(\alpha\pi)\iint_{\mathbb{R}^2}|\mathcal{F}v|^2\int_0^{2\pi}|2\pi(\xi_1\cos\theta+\xi_2\sin\theta)|^{2\alpha}\tilde{M}(\theta)d\theta\, d\xi_1 d\xi_2 \\ &\gtrsim \iint_{\mathbb{R}^2}|\xi|^{2\alpha}|\mathcal{F}v|^2 d\xi_1 d\xi_2.\end{aligned} \tag{2.14}$$

For $v \in H_0^\alpha(\Omega)$, we have

$$\begin{aligned}||v||^2 \leq C_1||D_\theta^\alpha v||^2 &= C_1\iint_{\mathbb{R}^2}|2\pi(\xi_1\cos\theta+\xi_2\sin\theta)|^{2\alpha}|\mathcal{F}v|^2 d\xi_1 d\xi_2 \\ &\leq C_2\iint_{\mathbb{R}^2}|\xi|^{2\alpha}|\mathcal{F}v|^2 d\xi_1 d\xi_2,\end{aligned} \tag{2.15}$$

where the inequality is by (5.15) in [11] and the equality is by Parseval's formula. With the combination of (2.14) and (2.15), we conclude under condition (2.12),

$$\tilde{B}(v,v) \gtrsim ||v||^2_{H^\alpha(\Omega)}, \quad v \in H_0^\alpha(\Omega). \tag{2.16}$$

By Lemma 2.10, it is easy to verify that

$$\tilde{B}(v,w) \lesssim ||v||_{H^\alpha(\Omega)}||w||_{H^\alpha(\Omega)}, \quad v, w \in H_0^\alpha(\Omega). \tag{2.17}$$

By (2.16) and (2.17), using Lax–Milgram theorem, we know that the variational formulation (2.11) admits a unique solution in $H_0^\alpha(\Omega)$.

Remark 2.12. Condition (2.12) is easily satisfied. For example, it holds if $\tilde{M}(\theta)$ is non-zero over a connected set of positive measure in $[0, 2\pi)$ (see [11]), and it holds when $\tilde{M}(\theta) = \sum_{k=1}^4 p_k\delta(\theta - k\pi/2)d\theta$, with $p_k \geq 0$ and $p_1 + p_3 = 1$, $p_2 + p_4 = 1$.

2.3. *The finite element discretization*

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω such that $\bar{\Omega} = \cup_{K\in\mathcal{T}_h}K$, h_K be the maximal length of the sides of the triangle K and $h = \max_{K\in\mathcal{T}_h}h_K$. Denote by $P_1(K)$ the space of polynomials of degree less than or equal to 1 on $K \in \mathcal{T}_h$. Define the finite dimensional subspace V associated with $\mathcal{T}_h$ as

$$V := \{v \in C^0(\bar{\Omega}) : v|_{\partial\Omega} = 0, v|_K \in P_1(K), \forall K \in \mathcal{T}_h\}.$$

It is known that $V \subset H_0^1(\Omega) \subset H_0^\alpha(\Omega)$. Thus the finite element approximation for (2.11) is to find $\tilde{u}_h \in V$ such that

$$\tilde{B}(\tilde{u}_h, v) = (f, v), \quad \forall v \in V. \tag{2.18}$$

The error estimates for the finite element solution $\tilde{u}_h$ are given in [11].

Although our V-cycle methods and the relevant theories are valid for (2.18), the finite element discretization (2.18) can not be implemented when $\tilde{M}$ is a continuous function. So in practical applications, we use the finite element discretization (2.18) only when the probability density function $\tilde{M}$ has the discrete form as that in (2.2). For the case that $\tilde{M}(\theta)$ is the continuous function, we propose an alternative finite element discretization instead of (2.18). Here we focus on the case $\tilde{M}(\theta) \in C^1[0, 2\pi]$ is a periodic function with period 2π to present our alternative finite element problem: find $\bar{u}_h \in V$ such that

$$\bar{B}(\bar{u}_h, v) = (f, v), \quad \forall v \in V, \tag{2.19}$$

where $\bar{B}(\cdot,\cdot)$ is an approximation of $\tilde{B}(\cdot,\cdot)$. Exactly in this paper, set a positive integer N_θ such that N_θ is a multiple of 4. Letting $\theta_i = 2i\pi/N_\theta$, $i = 0, \ldots, N_\theta - 1$ and denoting $\Delta\theta = 2\pi/N_\theta$, we use the compound trapezoid formula to get $\bar{B}(\cdot,\cdot)$, i.e., for $v, w \in V$,

$$\begin{aligned}\tilde{B}(v, w) &= -\int_0^{2\pi} (D_\theta^\alpha v, D_{\theta+\pi}^\alpha w)\tilde{M}(\theta)d\theta + c(v, w)\\ &\approx -\Delta\theta \sum_{i=0}^{N_\theta-1} (D_{\theta_i}^\alpha v, D_{\theta_i+\pi}^\alpha w)\tilde{M}(\theta_i) + c(v, w) := \bar{B}(v, w).\end{aligned}$$

The fact that $\tilde{M}(\theta) = \tilde{M}(\theta + \pi)$ and N_θ is a multiple of 4 guarantees that $\bar{B}(v, w)$ is a symmetric bilinear form as well, i.e., $\bar{B}(v, w) = \bar{B}(w, v)$. By Parseval's formula, we have

$$\begin{aligned}(D_\theta^\alpha v, D_{\theta+\pi}^\alpha w)_\Omega &= ((2\pi i\xi_1 \cos\theta + 2\pi i\xi_2 \sin\theta)^{2\alpha}\mathcal{F}v, \overline{\mathcal{F}w})\\ &\le C||v||_{H^\alpha(\Omega)}||w||_{H^\alpha(\Omega)}\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\frac{d}{d\theta}(D_\theta^\alpha v, D_{\theta+\pi}^\alpha w)_\Omega &= 2\alpha\left((-2\pi i\xi_1 \sin\theta + 2\pi i\xi_2 \cos\theta)(2\pi i\xi_1 \cos\theta + 2\pi i\xi_2 \sin\theta)^{2\alpha-1}\mathcal{F}v, \overline{\mathcal{F}w}\right)\\ &\le C||v||_{H^\alpha(\Omega)}||w||_{H^\alpha(\Omega)}.\end{aligned} \tag{2.21}$$

By the error formula for the compound trapezoid formula, it is easy to verify that

$$|\tilde{B}(v, w) - \bar{B}(v, w)| \le C\Delta\theta||v||_{H^\alpha(\Omega)}||w||_{H^\alpha(\Omega)}, \tag{2.22}$$

where C is a positive constant independent of θ, v and w. Combining (2.22) with (2.16) and (2.17), we know for sufficiently small $\Delta\theta$,

$$\begin{aligned}&\bar{B}(v, v) \gtrsim ||v||^2_{H^\alpha(\Omega)}, \quad v \in H_0^\alpha(\Omega),\\ &\bar{B}(v, w) \lesssim ||v||_{H^\alpha(\Omega)}||w||_{H^\alpha(\Omega)}, \quad v, w \in H_0^\alpha(\Omega).\end{aligned} \tag{2.23}$$

By Lax–Milgram theorem, (2.19) has a unique solution. The first Strang lemma (see [7]) holds here, i.e.,

$$\begin{aligned}||u - \bar{u}_h||_{H^\alpha(\Omega)} &\lesssim C \inf_{v\in V}\left\{||u - v||_{H^\alpha(\Omega)} + \sup_{w\in V}\frac{|B(v, w) - \bar{B}(v, w)|}{||w||_{H^\alpha(\Omega)}}\right\}\\ &\lesssim C \inf_{v\in V}\left\{||u - v||_{H^\alpha(\Omega)} + \Delta\theta||v||_{H^\alpha(\Omega)}\right\}.\end{aligned}$$

Finally, the finite element approximation of (2.3) is unitedly presented as: find $u_h \in V$ such that

$$B(u_h, v) = (f, v), \quad \forall v \in V, \tag{2.24}$$

where $B(v, w) = -\int_0^{2\pi}(D_\theta^\alpha v, D_{\theta+\pi}^\alpha w)M(\theta)d\theta + c(v, w)$, $M(\theta)$ is equal to a discrete form $\sum_{k=1}^L p_k\delta(\theta - \theta_k)$ such that $B(\cdot,\cdot)$ is a symmetric bilinear form,

$$B(v, v) \gtrsim ||v||^2_{H^\alpha(\Omega)}, B(v, w) \lesssim ||v||_{H^\alpha(\Omega)}||w||_{H^\alpha(\Omega)}, \quad v, w \in H_0^\alpha(\Omega), \tag{2.25}$$

and $\int_0^{2\pi} M(\theta)d\theta \lesssim 1$. Specially for the cases mentioned above, the finite element problem (2.24) represents problem (2.18) if $M(\theta) = \tilde{M}(\theta) = \sum_{k=1}^L p_k\delta(\theta - \theta_k)$ and problem (2.19) if $M(\theta) = \Delta\theta\sum_{i=0}^{N_\theta-1}\delta(\theta - \theta_i)\tilde{M}(\theta_i)$.

3. Multigrid algorithm

In this section, for (2.24), we shall present our V-cycle multigrid algorithm and a general framework for our convergence analysis.

Take $f_h \in V$ such that $(f_h, v) = (f, v)$, $\forall v \in V$ and define a linear operator $A : V \to V$ as follows:

$$(Av, w) = B(v, w), \quad \forall v, w \in V. \tag{3.1}$$

The finite element approximation of system (2.24) can be restated as to find $u_h \in V$ such that

$$Au_h = f_h. \tag{3.2}$$

In the following, we shall use the operator equation (3.2) to construct our multigrid algorithm. Since $B(v, w)$ is a symmetric bilinear form, we know, by (2.25), that $A : V \to V$ is symmetric positive definite with respect to $(\cdot,\cdot)$, i.e.,

$$(Av, w) = (v, Aw), \quad v, w \in V; \quad (Av, v) > 0, \quad 0 \neq v \in V.$$

Then bilinear form

$$(v, w)_A := (Av, w), \quad v, w \in V,$$

also induces an inner product on V. Set norm

$$||v||_A = (Av, v)^{1/2}, \quad v \in V.$$

By (2.25), we have

$$||v||_A \approx ||v||_{H^\alpha(\Omega)}, \quad \forall v \in V. \tag{3.3}$$

3.1. Algorithm

Assume that the triangulation $\mathcal{T}_h$ of Ω is constructed by a successive refinement process. To be precise, let $\mathcal{T}_J = \mathcal{T}_h$ for some $J > 1$, and $\mathcal{T}_k$ for $k \geq 0$ be a nested sequence of quasi-uniform triangulations, i.e., $\mathcal{T}_k = \{\tau_k^i\}$ consists of simplexes τ_k^i of size h_k such that $\Omega = \cup_i \tau_k^i$; τ_{k-1}^l is a union of simplexes of τ_k^i. We further assume that there is a positive constant $\gamma < 1$, independent of k, such that h_k is proportional to γ^k and the simplexes in $\mathcal{T}_1$ are of diameter ≈ 1.

For each partition $\mathcal{T}_k$, we may define finite element spaces V_k by

$$V_k = \{v \in C^0(\bar{\Omega}) : v|_{\partial\Omega} = 0, v|_\tau \in P_1(\tau), \forall \tau \in \mathcal{T}_k\}. \tag{3.4}$$

Obviously, the following inclusion relation holds: $V_1 \subset V_2 \subset \cdots \subset V_J = V$. Our V-cycle multigrid methods are based on the subspace decomposition $V = V_1 + V_2 + \cdots + V_J$.

For each $k \in \{1, 2, \ldots, J\}$, define projectors $Q_k, P_k : V \to V_k$ by

$$(Q_k v, w) = (v, w), \quad (P_k v, w)_A = (v, w)_A, \quad v \in V, w \in V_k,$$

specially, set $Q_0 : V \to V$ as $Q_0 v = 0$, and define the linear operator $A_k : V_k \to V_k$

$$(A_k v, w) = (Av, w), \quad v, w \in V_k.$$

It is easy to verify that

$$A_k P_k = Q_k A, \quad k = 1, 2, \ldots, J. \tag{3.5}$$

It is obvious that A_k is symmetric and positive definite with respect to $(\cdot,\cdot)$. Denote by $\lambda_k \in \mathbb{R}$, $k = 1, 2, \ldots, J$, the maximal eigenvalue of A_k.

Let $u_k = P_k u_h$ and $f_k = Q_k f_h$, we may get the operator equation in subspace

$$A_k u_k = f_k. \tag{3.6}$$

Our multigrid algorithm is essentially an iterative procedure in which the subspace equation (3.6) is approximately solved successively to get new approximations to (3.2) from old approximations. More precisely, denote by $R_k : V_k \to V_k$ the approximate inverse of A_k, and by u^{old} the old approximation to u. Correcting the residual of u^{old} in V_k gives

$$u^{new} = u^{old} + R_k Q_k (f_h - Au^{old}).$$

We take R_k to be symmetric with respect to $(\cdot,\cdot)$ such that

$$(R_k v, v) \approx \frac{1}{\lambda_k}(v, v), \quad \forall v \in V_k, k = 1, 2, \ldots, J. \tag{3.7}$$

Remark 3.1. In this paper, we have $h_1 = O(1)$ and take $R_1 = A_1^{-1}$. By Lemma 4.3, (3.3) and the definition of norm $||\cdot||_{H^\mu(\Omega)}$, we know that $(v, v) \lesssim (A_1 v, v) \lesssim h_1^{-2\alpha}(v, v)$, and $\lambda_1 = O(1)$. Then we have $(R_1^{-1} v, v) \approx \frac{1}{\lambda_1}(v, v)$.

Next we give our V-cycle multigrid algorithm.

V-cycle multigrid algorithm. Let $u^0 = 0 \in V$, assume that $u^k \in V$ has been obtained. Then u^{k+1} is generated by

$$u^{k+1} = u^k + B_J(f_h - Au^k), \tag{3.8}$$

where B_J is defined inductively: Let $B_1 = A_1^{-1}$, and assume that $B_{k-1} : V_{k-1} \to V_{k-1}$ has been defined; then for $g \in V_k$, $B_k : V_k \to V_k$ is defined as follows:

Step 1. $v^1 = R_k g$;
Step 2. $v^2 = v^1 + B_{k-1} Q_{k-1}(g - A_k v^1)$;
Step 3. $B_k g = v^2 + R_k(g - A_k v^2)$.

3.2. A general framework

For the V-cycle multigrid method, we have

$$u_h - u^{k+1} = (I - B_J A)(u_h - u^k).$$

Denote

$$E_J = (I - T_J)(I - T_{J-1}) \cdots (I - T_1), \quad E_J^* = (I - T_1) \cdots (I - T_{J-1})(I - T_J) \tag{3.9}$$

with $T_1 = P_1$, $T_k = R_k A_k P_k$, $k = 2, 3, \dots, J$. Then we have $(I - B_J A) = E_J E_J^*$. Define the operator norm as

$$||E_J||_A = \sup_{v \in V} \frac{||E_J v||_A}{||v||_A}.$$

It is easy to see that E_J^* is the $(\cdot,\cdot)_A$-adjoint of E_J, i.e.,

$$(E_J v, w)_A = (v, E_J^* w)_A, \quad v, w \in V$$

and that

$$||E_J||_A = ||E_J^*||_A, \quad ||E_J E_J^*||_A \le ||E_J||_A^2.$$

The main work in this paper is to establish the contraction property: there is a constant $0 < \delta < 1$ independent of the mesh size and mesh level such that

$$||E_J||_A \le \sqrt{\delta}. \tag{3.10}$$

By (3.10), we may obtain $||u_h - u^k||_A \le \delta^k ||u_h - u^0||$.

Remark 3.2. For the V-cycle multigrid method, the spectral radius of the iterative matrix $\rho = \rho(I - B_J A) \le \delta$. It is known that the condition number $\kappa(B_J A) \le \frac{1+\rho}{1-\rho} \le \frac{1+\delta}{1-\delta}$ and $B_J A$ is self-adjoint and positive with respect to inner product $(\cdot,\cdot)_A$. The δ's independence of the mesh size implies that B_J is a good preconditioner for A which can be used to design efficient preconditioned conjugate gradient methods.

Define K_0 and K_1 as two smallest positive constants satisfying the following conditions:

1. For any $v \in V$, there exists a decomposition $v = \sum_{i=1}^J v_i$ for $v_i \in V_i$ such that

$$\sum_{i=1}^J (R_i^{-1} v_i, v_i) \le K_0 (Av, v). \tag{3.11}$$

2. For any $S \subset \{1, 2, \dots, J\} \times \{1, 2, \dots, J\}$ and $v_i, w_i \in V$ for $i = 1, 2, \dots, J$,

$$\sum_{(i,j)\in S} (T_i v_i, T_j w_j)_A \le K_1 \left(\sum_{i=1}^J (T_i v_i, v_i)_A \right)^{\frac{1}{2}} \left(\sum_{j=1}^J (T_j w_j, w_j)_A \right)^{\frac{1}{2}}. \tag{3.12}$$

The estimate of the upper bound of $||E_J||_A$ relies on the following lemma:

Lemma 3.3. *(See [2,44].) Let E_J be defined by (3.9). We have*

$$||E_J||_A \le 1 - \frac{2-\omega_1}{K_0(1+K_1)^2},$$

where $\omega_1 = \max_k \rho(R_k A_k)$, $\rho(R_k A_k)$ denotes the spectral radius of $R_k A_k$.

The estimate of the parameter ω_1 is straightforward. Since $R_1 = A_1^{-1}$, $\rho(R_1 A_1) = 1$. From (3.7), for $v \in V_k$ $(k = 2, \dots, J)$

$$\frac{C_1}{\lambda_k}(v, v) \le (R_k v, v) \le \frac{C_2}{\lambda_k}(v, v),$$

and furthermore

$$(R_k A_k v, v)_A = (R_k A_k v, A_k v) \le \frac{C_2}{\lambda_k}(A_k v, A_k v) \le C_2(v, A_k v) = (v, v)_A, \tag{3.13}$$

where the last inequality is obtained from that A_k is symmetric positive matrix and λ_k is the maximal eigenvalue of A_k. Combining (3.13) with the fact that $R_k A_k$ is symmetric with respect to inner product $(\cdot,\cdot)_A$, we have $\rho(R_k A_k) \le C_2$. Taking R_k such that C_2 is suitably small can guarantee the $\omega_1 < 2$.

Next, we shall estimate the parameters K_1, K_2. The following lemma is helpful for the analysis.

Lemma 3.4. *(See [2,44].) Let $\epsilon = (\epsilon_{ij}) \in R^{J\times J}$ be a nonnegative symmetric matrix, with components ϵ_{ij} being the smallest constant satisfying*

$$(T_i v, T_j w)_A \le \epsilon_{ij}(T_i v, v)_A^{\frac{1}{2}}(T_j w, w)_A^{\frac{1}{2}}, \quad \forall v, w \in V. \tag{3.14}$$

Then we have

$$K_1 \le \rho(\epsilon),$$

where $\rho(\epsilon)$ denotes the spectral radius of matrix ϵ. Furthermore, if $\epsilon_{ij} \lesssim \gamma^{|i-j|}$ for some $\gamma \in (0, 1)$, then $\rho(\epsilon) \lesssim (1-\gamma)^{-1}$.

4. Convergence analysis

We here first introduce two interpolation norms and relevant Sobolev spaces (see e.g., [37]). Let Λ be a domain in $\mathbb{R}^2$. For integer m, denote by $||\cdot||_{\tilde{H}^m(\Lambda)}$ the Sobolev norm of integer order m, i.e.,

$$||v||_{\tilde{H}^m(\Lambda)} := \left(\sum_{|l|\le m} ||D^l v||^2_{L^2(\Lambda)}\right)^{1/2},$$

with $l = (l_1, l_2)$, $|l| = l_1 + l_2$ and $D^l = (\frac{\partial}{\partial x})^{l_1}(\frac{\partial}{\partial y})^{l_2}$. Let $\mu > 0$ be a non-integer and $0 < s < 1$, n is a non-negative integer such that $n < \mu < n+1$. We introduce the interpolation norms

$$||v||_{\tilde{H}^\mu(\Lambda)} := \left(\int_0^\infty \tilde{K}(v,t)t^{-2\mu-1}dt\right)^{1/2}, \; ||v||_{\hat{H}^s(\Lambda)} := \left(\int_0^\infty \hat{K}(v,t)t^{-2s-1}dt\right)^{1/2} \tag{4.1}$$

where

$$\tilde{K}(v,t) := \inf_{w\in\tilde{H}^{n+1}(\Lambda)} \left(||v-w||^2_{\tilde{H}^n(\Lambda)} + t^2||w||^2_{\tilde{H}^{n+1}(\Lambda)}\right),$$

$$\hat{K}(v,t) := \inf_{w\in\tilde{H}^1_0(\Lambda)} \left(||v-w||^2_{L^2(\Lambda)} + t^2||w||^2_{\tilde{H}^1(\Lambda)}\right).$$

Relevant Sobolev spaces are

$$\tilde{H}^\mu(\Lambda) := \{v \in L^2(\Lambda); ||v||_{\tilde{H}^\mu(\Lambda)} < \infty\}, \quad \hat{H}^s(\Lambda) := \{v \in L^2(\Lambda); ||v||_{\hat{H}^s(\Lambda)} < \infty\}. \tag{4.2}$$

Let Λ_1, Λ_2 be two domains in $\mathbb{R}^2$ with $\Lambda_1 \subset \Lambda_2$, and then

$$\begin{aligned}
&\left(\int_0^\infty \inf_{w\in\tilde{H}^{n+1}(\Lambda_1)} \left(||v-w||^2_{\tilde{H}^n(\Lambda_1)} + t^2||w||^2_{\tilde{H}^{n+1}(\Lambda_1)}\right) t^{-2\mu-1}dt\right)^{1/2} \\
\le &\left(\int_0^\infty \inf_{w\in\tilde{H}^{n+1}(\Lambda_2)} \left(||(v-w)|_{\Lambda_1}||^2_{\tilde{H}^n(\Lambda_1)} + t^2|| w|_{\Lambda_1}||^2_{\tilde{H}^{n+1}(\Lambda_1)}\right) t^{-2\mu-1}dt\right)^{1/2} \\
\le &\left(\int_0^\infty \inf_{w\in\tilde{H}^{n+1}(\Lambda_2)} \left(||v-w||^2_{\tilde{H}^n(\Lambda_2)} + t^2||w||^2_{\tilde{H}^{n+1}(\Lambda_2)}\right) t^{-2\mu-1}dt\right)^{1/2}.
\end{aligned} \tag{4.3}$$

So we have, for $v \in \tilde{H}^\mu(\Lambda_2)$,

$$||v||_{\tilde{H}^\mu(\Lambda_1)} \le ||v||_{\tilde{H}^\mu(\Lambda_2)}. \tag{4.4}$$

Remark 4.1. The following space relations can be found in literature: (1) $\mu > 0$, $\tilde{H}^\mu(\mathbb{R}^2)$ and $\tilde{H}^1_0(\Omega)$ coincide with $H^\mu(\mathbb{R}^2)$ and $H^1_0(\Omega)$ respectively; (2) for $1/2 < \mu < 1$, $\tilde{H}^\mu_0(\Omega)$ coincides with $\hat{H}^\mu(\Omega)$ (see [18,37]); for $1/2 < \mu < 1$, $\tilde{H}^\mu_0(\Omega)$ coincides with $H^\mu_0(\Omega)$ (this can been shown by (1), (2) and the definitions of the interpolation spaces).

Combining with Remark 4.1 and the well known interpolation property (see e.g., Lemma 22.3 in [37]), we know, for $1/2 < \mu \le 1$,

$$||(I - Q_k)v|| \lesssim h^\mu_k ||v||_{H^\mu(\Omega)}, \quad v \in H^\mu_0(\Omega). \tag{4.5}$$

Now, we develop some results for the finite element spaces $V_k, k \ge 1$. Let $\Omega' \subset \mathbb{R}^2$ be a suitable polygonal domain such that $\Omega \subset \Omega'$ and $\mathrm{dist}(\partial\Omega', \Omega) > C$ for a positive C. $\mathcal{T}'_k$, $k \ge 1$, are the quasi-uniform triangulations obtained by extending $\mathcal{T}_k$ from Ω to Ω', that is, $\mathcal{T}'_k$ in Ω coincides with $\mathcal{T}_k$. Furthermore we still make sure that $\mathcal{T}'_k = \{\tau^i_k\}$ consists of simplexes τ^i_k of size h_k. Let $V'_k = \{v \in C^0(\bar{\Omega}') : v|_{\partial\Omega'} = 0, v|_\tau \in P_l(\tau), \forall \tau \in \mathcal{T}'_k\}$. In the following, for $v \in V_k$, v always denotes its extension (on Ω' and on $\mathbb{R}^2$), which is extended by zero outside Ω, and so we also have $v \in V'_k$.

Lemma 4.2. *Let $\mu > 0$, $v \in \tilde{H}^\mu(\Omega')$ with $supp(v) \subset \Omega$ (v also denotes its extension on $\mathbb{R}^2$ which is extended by zero outside Ω'). Then we have $||v||_{\tilde{H}^\mu(\Omega')} \approx ||v||_{H^\mu(\mathbb{R}^2)}$.*

Proof. For μ being an integer, the conclusion is direct. For the case that μ is not an integer, denote n as a non-negative integer such that $n < \mu < n+1$. From (4.3), $||v||_{\tilde{H}^\mu(\Omega')} \le ||v||_{\tilde{H}^\mu(\mathbb{R}^2)} \approx ||v||_{H^\mu(\mathbb{R}^2)}$. Now we prove the converse relation. Let Λ be a domain in $\mathbb{R}^2$ with C^{n+1}-smooth boundary such that $\Omega \subset\subset \Lambda \subset \Omega'$. Then by (4.4), $v \in \tilde{H}^\mu(\Lambda)$. Following the proof for the strong extension of Sobolev space (see e.g., Theorem 4.26 in [1]), we can show that there is a linear operator E continuous from $\tilde{H}^j(\Lambda)$ into $\tilde{H}^j(\mathbb{R}^2)$ for integers $0 \le j \le n+1$, such that $E(v|_\Lambda) = v$. Then we have

$$\begin{aligned}
||v||_{\tilde{H}^\mu(\mathbb{R}^2)} &= \left(\int_0^\infty \inf_{w\in\tilde{H}^{n+1}(\mathbb{R}^2)} \left(||v-w||^2_{\tilde{H}^n(\mathbb{R}^2)} + t^2||w||^2_{\tilde{H}^{n+1}(\mathbb{R}^2)}\right) t^{-2\mu-1}dt\right)^{1/2} \\
&\le \left(\int_0^\infty \inf_{w\in\tilde{H}^{n+1}(\Lambda)} \left(||E(v|_\Lambda - w)||^2_{\tilde{H}^n(\mathbb{R}^2)} + t^2||Ew||^2_{\tilde{H}^{n+1}(\mathbb{R}^2)}\right) t^{-2\mu-1}dt\right)^{1/2} \\
&\lesssim \left(\int_0^\infty \inf_{w\in\tilde{H}^{n+1}(\Lambda)} \left(||v-w||^2_{\tilde{H}^n(\Lambda)} + t^2||w||^2_{\tilde{H}^{n+1}(\Lambda)}\right) t^{-2\mu-1}dt\right)^{1/2} \\
&= ||v||_{\tilde{H}^\mu(\Lambda)},
\end{aligned} \tag{4.6}$$

where the last inequality is by the continuity of E. Combining with (4.4), we obtain $||v||_{H^\mu(\mathbb{R}^2)} \approx ||v||_{\tilde{H}^\mu(\mathbb{R}^2)} \lesssim ||v||_{\tilde{H}^\mu(\Omega')}$. $\square$

Lemma 4.3. *For $0 < \mu < 3/2$, $v \in V_k$, we have*

$$||v||_{H^\mu(\mathbb{R}^2)} \lesssim h_k^{-\mu} ||v||, \tag{4.7}$$

and then $V_k \subset H^\mu(\mathbb{R}^2)$.

Proof. By $v \in V_k$, we known $v \in V_k'$, $||v||_{\tilde{H}^\mu(\Omega')} \lesssim h_k^{-\mu}||v||$ from [3,44,45] and further $||v||_{H^\mu(\mathbb{R}^2)} \lesssim h_k^{-\mu}||v||$ by Lemma 4.2. □

Let β be a positive with $\alpha+\beta<3/2$ and $\alpha-\beta \geq 0$ in the rest of this paper. We have the following results:

Lemma 4.4. *It holds that*

$$(v,w)_A \lesssim ||v||_{H^{\alpha+\beta}(\mathbb{R}^2)}||w||_{H^{\alpha-\beta}(\mathbb{R}^2)}, \quad v,w \in V.$$

Proof. Since $v,w \in V$, by Lemma 4.3, we know that $v,w \in H^{\alpha+\beta}(\mathbb{R}^2)$. Then

$$\begin{aligned}
(v,w)_A &= (Av,w) = B(v,w)\\
&= -\int_0^{2\pi} \left(D_\theta^\alpha v, D_{\theta+\pi}^\alpha w\right) M(\theta)d\theta + c(v,w)\\
&= -\int_0^{2\pi} \left(D_\theta^{\alpha+\beta} v, D_{\theta+\pi}^{\alpha-\beta} w\right) M(\theta)d\theta + c(v,w)\\
&\leq \int_0^{2\pi} ||D_\theta^{\alpha+\beta} v||_{L^2(\Omega)}||D_{\theta+\pi}^{\alpha-\beta} w||_{L^2(\Omega)} M(\theta)d\theta + c||v||_{L^2(\Omega)}\,||w||_{L^2(\Omega)}\\
&\lesssim ||v||_{H^{\alpha+\beta}(\mathbb{R}^2)}||w||_{H^{\alpha-\beta}(\mathbb{R}^2)} + c||v||_{L^2(\Omega)}\,||w||_{L^2(\Omega)}\\
&\lesssim ||v||_{H^{\alpha+\beta}(\mathbb{R}^2)}||w||_{H^{\alpha-\beta}(\mathbb{R}^2)},
\end{aligned}$$

where the third equality is by Lemma 2.11, and the second inequality is by Lemma 2.10 and $\int_0^{2\pi} M(\theta)d\theta \lesssim 1$. □

Lemma 4.5. *Let $i \leq j$, then*

$$(v,w)_A \lesssim \gamma^{(j-i)\beta} h_i^{-\alpha} h_j^{-\alpha} ||v||\,||w||, \quad v \in V_i, w \in V_j. \tag{4.8}$$

Here we recall that $\gamma \in (0,1)$ is a constant such that $h_k = O(\gamma^k)$.

Proof. For $v \in V_i$, $w \in V_j$, we know that

$$\begin{aligned}
(v,w)_A &\lesssim ||v||_{H^{\alpha+\beta}(\mathbb{R}^2)}||w||_{H^{\alpha-\beta}(\mathbb{R}^2)} \lesssim h_i^{-(\alpha+\beta)}||v||\,h_j^{-(\alpha-\beta)}||w||\\
&= (h_j/h_i)^\beta h_i^{-\alpha} h_j^{-\alpha}||v||\,||w|| \lesssim \gamma^{(j-i)\beta} h_i^{-\alpha} h_j^{-\alpha}||v||\,||w||,
\end{aligned}$$

where the first inequality is by Lemma 4.4, the second inequality is by Lemma 4.3, and the last inequality is by the relation $h_k \approx O(\gamma^k)$. □

Lemma 4.6. *Let $W_i = (Q_i - Q_{i-1})V$, then*

$$(v,w)_A \lesssim \gamma^{|j-i|\beta}||v||_A||w||_A, \quad \forall u \in W_i, v \in W_j. \tag{4.9}$$

Proof. By (4.5) and (3.3), we have

$$||v|| \lesssim h_k^\alpha ||v||_A, \quad \forall v \in W_k.$$

Combining the above inequality with Lemma 4.5 gives the lemma.

Lemma 4.7. *It holds that*

$$(T_i v, T_j w)_A \lesssim \gamma^{|i-j|\beta}(T_i v, v)_A^{\frac{1}{2}}(T_j w, w)_A^{\frac{1}{2}}, \quad \forall v,w \in V. \tag{4.10}$$

Proof. It suffices to prove (4.10) holds for $i \leq j$. Assume that $i \leq j$, and then for $v,w \in V$,

$$\begin{aligned}
(T_i v, T_j w)_A &= (R_i A_i P_i v, R_j A_j P_j w)_A\\
&\lesssim \gamma^{(j-i)\beta} h_i^{-\alpha} h_j^{-\alpha}||R_i A_i P_i v||\,||R_j A_j P_j w||,
\end{aligned} \tag{4.11}$$

where the inequality is by Lemma 4.5.

$$||R_i A_i P_i v||^2 = (R_i A_i P_i v, R_i A_i P_i v) \approx \frac{1}{\lambda_i}(R_i A_i P_i v, A_i P_i v) = \frac{1}{\lambda_i}(T_i v, v)_A,$$

where the second equality is by (3.7) and the symmetry of R_k. Then we obtain

$$||R_i A_i P_i v|| \lesssim \lambda_i^{-1/2}(T_i v, v)_A^{1/2}, \tag{4.12}$$

and similarly

$$||R_j A_j P_j w|| \lesssim \lambda_j^{-1/2}(T_j w, w)_A^{1/2}. \tag{4.13}$$

For $v \in V_k$, we have

$$(Av, v) = ||v||_A^2 \approx ||v||^2_{H^\alpha(\Omega)} \lesssim h_k^{-2\alpha}||v||^2, \tag{4.14}$$

where the second equality is by (3.3) and the last inequality is by Lemma 4.3. For $w \in V$, $v = (Q_k - Q_{k-1})w \in V_k$, by (4.5), we have

$$h_k^{-2\alpha}||v||^2 \lesssim ||v||^2_{H^\alpha(\Omega)} \approx (Av, v). \tag{4.15}$$

By (4.14) and (4.15), it is not hard to see that

$$\lambda_k \approx h_k^{-2\alpha}, \quad k = 1, 2, \ldots, J. \tag{4.16}$$

Combining (4.11) with (4.12), (4.13) and (4.16) gives

$$(T_i v, T_j w)_A \lesssim \gamma^{(j-i)\beta}(T_i v, v)_A^{\frac{1}{2}}(T_j w, w)_A^{\frac{1}{2}}, \quad \forall v, w \in V.$$

The lemma is proved. □

Lemma 4.8. *Let*

$$||v||_M^2 := \sum_{k=1}^{J} ||(Q_k - Q_{k-1})v||_A^2, \tag{4.17}$$

and then for $v \in V$, we have

$$||v||_M \approx ||v||_A.$$

Proof. It is not hard to see that the space $H_0^\alpha(\Omega)$ coincides with $\tilde{H}^\alpha(\Omega)$ in [26]. Combining with Theorem 1 of [26], we know that $||w||^2_{H^\alpha(\Omega)} \approx \sum_{k=1}^{\infty} h_k^{-2\alpha}||(Q_k - Q_{k-1})w||^2$ holds for $w \in H_0^\alpha(\Omega)$. For $v \in V$, $||(Q_k - Q_{k-1})v||^2 \approx h_k^{2\alpha}||(Q_k - Q_{k-1})v||^2_{H^\alpha(\Omega)}$ by (4.14) and (4.15). Combining with (3.3) gives the lemma. □

Theorem 4.9. *We have*

$$K_0 \lesssim 1, \quad K_1 \lesssim 1.$$

That is to say, our V-cycle multigrid method is optimal, which means that the convergence rate is independent of the mesh size and mesh level.

Proof. For $v \in V$, decompose v as $v = \sum_{i=1}^{J} v_i$ with $v_i = (Q_i - Q_{i-1})v$. By (4.5) and (3.3) we have $||v_i|| \lesssim h_i^\alpha ||v_i||_A$. Furthermore combining (3.7) with (4.16), we have $(R_i^{-1}v_i, v_i) \lesssim ||v_i||_A^2$. Using Lemma 4.8 gives $K_0 \lesssim 1$. Finally combining Lemma 4.7 with Lemma 3.4 gives that $K_1 \lesssim 1$. □

5. Implementation

Let ϕ_k^i, $i = 1, \ldots, N_k$, be the nodal basis of the finite element space V_k. The implementation is a classical procedure in literature (see e.g., [2]), and we here only illustrate how to generate the stiff matrices of the finite element systems and how to choose $R_k : V_k \to V_k$, $k = 2, \ldots, J$, the approximations of A_k.

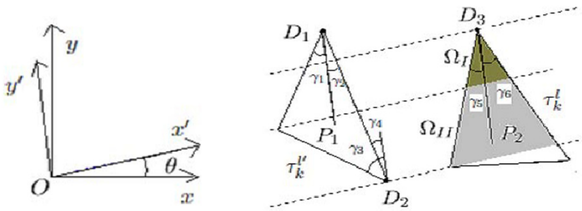

Fig. 1. Illustration for computing I_θ.

5.1. The stiffness matrices and R_k

For A_k, denote its corresponding stiffness matrix by $\tilde{A}_k \in \mathbb{R}^{N_k \times N_k}$ with entries

$$(\tilde{A}_k)_{ij} = B(\phi_k^i, \phi_k^j). \tag{5.1}$$

Since M has the discrete form $M(\theta) = \sum_{l=1}^{L} p_l \delta(\theta - \theta_l)$,

$$(\tilde{A}_k)_{ij} = -\sum_{l=1}^{L} p_l (D_{\theta_l}^{2\alpha-1}\phi_k^i, D_{\theta_l+\pi}\phi_k^j) + c(\phi_k^i, \phi_k^j).$$

We need only discuss how to numerically compute

$$\begin{aligned} I_\theta &= (D_\theta^{2\alpha-1}\phi_k^i, D_{\theta+\pi}\phi_k^j) = (D_\theta^{-\nu} D_\theta \phi_k^i, D_{\theta+\pi}\phi_k^j) \\ &= \int\limits_{\mathrm{ssup}(\phi_k^j)} D_\theta^{-\nu} D_\theta \phi_k^i \times D_{\theta+\pi}\phi_k^j dxdy \end{aligned} \tag{5.2}$$

for a fixed θ, where $\nu = (2 - 2\alpha)$, and then the entries of the stiff matrices can be numerically computed. If $\alpha = 1$, the computation of the stiffness matrices is easy, since the original problem is an integer order one. Now we focus on the case of $1/2 < \alpha < 1$. Define the index set K_i as

$$K_i = \{l;\, \tau_k^l \in \mathcal{T}_k,\, \tau_k^l \subset \mathrm{supp}(\phi_k^i)\}.$$

Then

$$\begin{aligned} I_\theta &= \sum_{l \in K_j} \int\limits_{\tau_k^l} D_\theta^{-\nu} D_\theta \phi_k^i \times D_{\theta+\pi}\phi_k^j dxdy \\ &= \sum_{l \in K_j} \sum_{l' \in K_i} \int\limits_{\tau_k^l} D_\theta^{-\nu} (\chi_{\tau_k^{l'}} D_\theta \phi_k^i) \times D_{\theta+\pi}\phi_k^j dxdy, \end{aligned}$$

where for a set S in $\mathbb{R}^2$,

$$\chi_S(x, y) = \begin{cases} 1, & \text{if } (x, y) \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Noting that $D_\theta(\phi_k^i)|_{\tau_k^{l'}}$, $D_{\theta+\pi}(\phi_k^j)|_{\tau_k^l}$ are both constants, we numerically compute

$$\int\limits_{\tau_k^l} D_\theta^{-\nu} \chi_{\tau_k^{l'}}(x, y) \times \chi_{\tau_k^l}(x, y) dxdy, \tag{5.3}$$

and then I_θ can be computed. Next we illustrate how to compute the integral in (5.3) by an example. On the left of Fig. 1 is Cartesian coordinate systems xOy and $x'Oy'$, and the angle between axes Ox and Ox' is θ. On the right of Fig. 1, the two triangles are $\tau_k^{l'}$ and τ_k^l; D_1, D_2, D_3 denote the corresponding vertices of the triangles; Ω_I, Ω_{II} denote the corresponding shadow areas respectively; lines D_1P_1 and D_3P_2 are both parallel to axis Oy'; $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$ are correspondent angles. Denote the coordinates of D_1, D_2 and D_3 under coordinate system $x'Oy'$ by (x'_{11}, y'_{12}), (x'_{21}, y'_{22}) and (x'_{31}, y'_{32}) respectively. Then we have

$$\begin{aligned}
&\int_{\tau_k^l} D_\theta^{-\nu} \chi_{\tau_k^{l'}}(x,y) \times \chi_{\tau_k^l}(x,y)dxdy \\
&= \int_{\Omega_I} D_\theta^{-\nu} \chi_{\tau_k^{l'}}(x,y)dxdy + \int_{\Omega_{II}} D_\theta^{-\nu} \chi_{\tau_k^{l'}}(x,y)dxdy \\
&= \int_{\Omega_I} {}_{-\infty}D_{x'}^{-\nu} \chi_{\tau_k^{l'}}(x',y')dx'dy' + \int_{\Omega_{II}} {}_{-\infty}D_{x'}^{-\nu} \chi_{\tau_k^{l'}}(x',y')dx'dy' \\
&= \int_{\Omega_I} \frac{1}{\Gamma(\nu+1)} \left(x' - x_1' + (y_1' - y')\tan\gamma_1\right)^\nu dx'dy' \\
&\quad - \int_{\Omega_I} \frac{1}{\Gamma(\nu+1)} \left(x' - x_1' - (y_1' - y')\tan\gamma_2\right)^\nu dx'dy' \\
&\int_{\Omega_{II}} \frac{1}{\Gamma(\nu+1)} \left(x' - x_2' + (y' - y_2')\tan\gamma_3\right)^\nu dx'dy' \\
&\quad - \int_{\Omega_{II}} \frac{1}{\Gamma(\nu+1)} \left(x' - x_2' + (y' - y_2')\tan\gamma_4\right)^\nu dx'dy'.
\end{aligned}$$

The last four integrals above can be computed directly. Finally we know that the entries of the stiffness matrices can be numerically computed.

We choose R_k as

$$R_k v = \frac{1}{\tilde{\lambda}_k} \sum_{i=1}^{N_k} (v, \phi_k^i)\phi_k^i, \quad v \in V_k, \tag{5.4}$$

with $\tilde{\lambda}_k \approx \lambda_k h_k^2$. Define mass matrix $M_k \in \mathbb{R}^{N_k \times N_k}$ with entries

$$(M_k)_{ij} = (\phi_k^i, \phi_k^j).$$

For $v \in V_k$, denote by $\tilde{v} \in \mathbb{R}^{N_k}$ the vector of coefficients of v in the basis $\{\phi_k^i\}_{i=1}^{N_k}$. It is known that $\tilde{v}^T M_k \tilde{v} \approx h_k^2 \tilde{v}^T \tilde{v}$ and $\tilde{v}^T M_k^2 \tilde{v} \approx h_k^2 \tilde{v}^T M_k \tilde{v}$. Hence we have

$$(R_k v, v) \approx \frac{1}{\tilde{\lambda}_k} \tilde{v}^T M_k^2 \tilde{v} \approx \frac{1}{\lambda_k} \tilde{v}^T M_k \tilde{v} \approx \frac{1}{\lambda_k}(v, v), \tag{5.5}$$

which means (3.7) holds. In the numerical tests, we take $\tilde{\lambda}_k = \frac{3}{2}(\tilde{A}_k)_{ii}$, $k = 2, \ldots, J$. It is not hard to verify that $(\tilde{A}_k)_{ii} \approx h_k^{2-2\alpha} \approx h_k^2 \lambda_k$.

5.2. Computation complexity

For the numerical approximation of SFPDEs, one of the key issues is how to reduce the computation complexity. We confine ourself to the case that Ω is a square domain. Of course the technique here is also helpful for effectively designing schemes for the case that Ω is a general domain (for example, the domain decomposition method can be used with interior sub-domains being chosen as square domains (see [12])).

The triangulations $\mathcal{T}_k$, $k = 1, 2, \ldots, J$ are those in Fig. 2, where dashed curve denotes the ellipsis, $n_k = n_0 2^k - 1$, $l_k = l_0 2^k - 1$ with positive integers n_0, l_0, and p_k^m, $m = 1, \ldots, n_k l_k$ are the interior points. The finite element space $V_k = \{v \in H_0^1(\Omega) : v|_\tau \in P_1(\tau), \forall \tau \in \mathcal{T}_k\}$. Let $\phi_k^m = \phi_k^m(x, y)$, $m = 1, \ldots, n_k l_k$, be the nodal basis functions, i.e., ϕ_k^m is a piecewise linear polynomial whose values are 1 at p_k^m and zeros at other nodes (including interior and exterior nodes).

Denote $U = (U_1, U_2 \ldots, U_{n_k}, \ldots, U_{2n_k}, \ldots, U_{l_k n_k})^T$. Next we discuss how to effectively conduct the multiplication of matrix $\tilde{A}_k$ and vector U. Let $v = (v_1, v_2, \ldots, v_{(2n_k-1)l_k - n_k + 1})^T \in \mathbb{R}^{(2n_k-1)l_k - n_k + 1}$ with

$$v_{(2n_k-1)j+i} = B(\phi_k^1, \phi_k^{jn_k+i}),\ i = 1, \ldots, n_k,\ j = 0, \ldots, l_k - 1,$$
$$v_{(2n_k-1)j-i+2} = B(\phi_k^i, \phi_k^{jn_k+1}),\ i = 2, \ldots, n_k,\ j = 1, \ldots, l_k - 1.$$

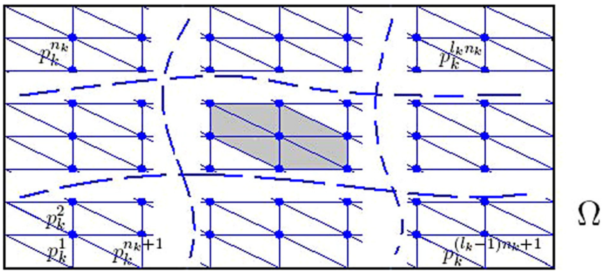

Fig. 2. Uniform triangulation.

Define a symmetric Toeplitz matrix

$$\tilde{\tilde{A}} = \begin{pmatrix} \nu_1 & \nu_2 & \cdots & \nu_{(2n_k-1)l_k-n_k} & \nu_{(2n_k-1)l_k-n_k+1} \\ \nu_2 & \nu_1 & \cdots & \nu_{(2n_k-1)l_k-n_k-1} & \nu_{(2n_k-1)l_k-n_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \nu_{(2n_k-1)l_k-n_k} & \nu_{(2n_k-1)l_k-n_k-1} & \cdots & \nu_1 & \nu_2 \\ \nu_{(2n_k-1)l_k-n_k+1} & \nu_{(2n_k-1)l_k-n_k} & \cdots & \nu_2 & \nu_1 \end{pmatrix}.$$

Toeplitz matrix, also called diagonal-constant matrix, is a matrix in which each descending diagonal from left to right is a constant.

For any i, j with $1 \le i \le j \le n_k l_k$, let d_i, r_i, d_j, r_j be nonnegative integers satisfying $i = n_k d_i + r_i$, $j = n_k d_j + r_j$, $1 \le r_i, r_j \le n_k$. Let $j' = (d_j - d_i)$, $i' = |r_j - r_i|$, and then by the property of the operator $B(\cdot,\cdot)$, it is easy to see that

$$B(\phi_k^i, \phi_k^j) = \begin{cases} B(\phi_k^1, \phi_k^{j'n_k+i'+1}) = \nu_{j'(2n_k-1)+i'+1}, & \text{if } r_j \ge r_i; \\ B(\phi_k^{i'+1}, \phi_k^{j'n_k+1}) = \nu_{j'(2n_k-1)-i'+1}, & \text{if } r_j < r_i. \end{cases}$$

And thereby any component of matrix $\tilde{\tilde{A}}$ is also one of vector ν. Define sets

$$\mathcal{I}_m = \{m(2n_k-1)+1, m(2n_k-1)+2, \ldots, m(2n_k-1)+n_k\}, \quad m = 0, 1, \ldots, l_k - 1$$

and $\mathcal{I} = \bigcup_{0 \le m \le l_k - 1} \mathcal{I}_m$. We have the relation

$$\tilde{A}_k = \tilde{\tilde{A}}_{\mathcal{I},\mathcal{I}}, \tag{5.6}$$

where $\tilde{\tilde{A}}_{\mathcal{I},\mathcal{I}}$ denotes the sub-matrix of $\tilde{\tilde{A}}$ which consists of entries $\tilde{\tilde{A}}_{ij}$ of $\tilde{\tilde{A}}$ indexed by $i, j \in \mathcal{I}$. Denote $U' \in \mathbb{R}^{(2n_k-1)l_k-n_k+1}$ as

$$U' = (U_1, \ldots, U_{n_k}, \overbrace{0, \ldots, 0}^{n_k-1}, U_{n_k+1}, \ldots, U_{2n_k}, \overbrace{0, \ldots, 0}^{n_k-1}, U_{2n_k+1}, \ldots, U_{l_k n_k}).$$

It is not hard to see that

$$\tilde{A}_k U = (\tilde{\tilde{A}} U')_{\mathcal{I}},$$

where for a given vector v, $v_{\mathcal{I}}$ denotes the vector which consists of entries v_i indexed by $i \in \mathcal{I}$. So the multiplication of the matrix $\tilde{A}_k$ and any vector $U \in R^{n_k l_k}$ can be obtained by conducting the multiplication of the Toeplitz matrix $\tilde{\tilde{A}}$ and $U' \in R^{(2n_k-1)l_k-n_k+1}$. The multiplication of a Toeplitz matrix in $\mathbb{R}^{n\times n}$ and a vector in $\mathbb{R}^n$ can be done with computation complexity $O(n \log n)$. Recall that $N_J = n_J l_J$ denotes the number of the unknowns in the finite element problem (3.2), and then by the above analysis, we conclude that for the V-cycle multigrid methods developed in Section 4, each iteration needs computation complexity $O(N_J \log N_J)$.

5.3. Numerical results

In this section, we shall present some numerical results to confirm our theoretical findings. In our numerical test, we take $n_0 = l_0 = 4$, and take $N_\theta = 4(n_J + 1)$ if M is a continuous function.

We shall check our V-cycle multigrid method and the preconditioned conjugate gradient algorithm (PCG) with B_J as the preconditioner. Meanwhile, the numerical result for the conjugate gradient algorithm (CG) is also presented for comparison. Our tests are carried out using Matlab software. The stopping criterion of the algorithm is

$$||u^k - u^{k-1}||_\infty \le 10^{-6}.$$

Table 1
Numerical results for Example 5.1.

Level J	DOFs	V-cycle Iter	PCG Iter	CG Iter
4	4096	11	7	49
5	16 384	11	7	73
6	65 536	11	7	118
7	262 144	11	7	197
8	1 048 576	11	7	313

Table 2
Numerical results for Example 5.2.

Level	DOFs	V-cycle Iter	PCG Iter	CG Iter
4	4096	10	6	47
5	16 384	10	6	69
6	65 536	10	6	114
7	262 144	10	6	185
8	1 048 576	10	6	308

Table 3
Numerical results of the V-cycle method for Example 5.3.

Level	DOFs	$\tilde{M}(\theta) = M_1$ Iter	$\tilde{M}(\theta) = M_2$ Iter
4	4096	19	11
5	16 384	18	11
6	65 536	17	11
7	262 144	15	11
8	1 048 576	14	11

We first present two examples: one is with the probability measure $\tilde{M}$ having a discrete form and the other with $\tilde{M}$ being a continuous function. Table 1 and Table 2 list the numerical results for Example 5.1 and Example 5.2 respectively, where "DOFs" denotes the degree of freedoms and "Iter" denotes the iterative steps on each level. It is seen that the numbers of iterations of our V-cycle multigrid and PCG per level are bounded independent of the mesh size and mesh level, which confirms our theoretical results.

Example 5.1. Let $\Omega = [0, 2] \times [0, 2]$, the equation to be solved is

$$-\frac{1}{4}({}_{-\infty}D_x^{1.5} + {}_xD_{\infty}^{1.5} + {}_{-\infty}D_y^{1.5} + {}_yD_{\infty}^{1.5})u = 1. \tag{5.7}$$

Example 5.2. Let $\Omega = [0, 2] \times [0, 2]$ and $\tilde{M}(\theta) = 1$. The equation to be solved is

$$-D_{\tilde{M}}^{1.5}u = 1. \tag{5.8}$$

We choose smooth $f(x, y)$ in the examples such that the solutions have singularity near the boundaries. The computation complexity of our multigrid methods is shown in Fig. 3, where "Time" denotes the CPU time (in seconds) spent by one iteration. As can be seen from the Fig. 3, the CPU time of each iteration is almost linear with respect to the degree of freedoms. So the computation complexity of our multigrid method is also optimal.

Finally, we end this section with a numerical test for our V-cycle multigrid method solving the finite element discretizations of nonsymmetric SFPDEs. In the future, we will try to derive the relevant theoretical analysis for the nonsymmetric case.

Example 5.3. Let $\Omega = [0, 2] \times [0, 2]$, and the equation to be solved is

$$-D_{\tilde{M}}^{1.5}u = 1. \tag{5.9}$$

Here we test two cases: 1. $\tilde{M}(\theta) = M_1 = 0.5\delta(\theta) + 0.5\delta(\theta - \pi/2)$; 2. $\tilde{M}(\theta) = M_2 = \sin^2(\theta/2)$. The numerical results are listed in Table 3.

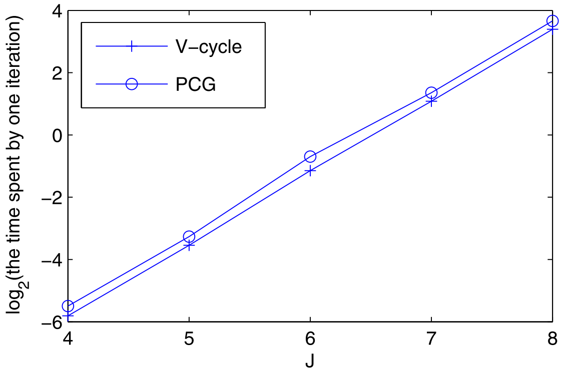

Fig. 3. The CPU time per iteration.

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Appendix A

The Fourier analysis plays critical roles in this paper: for $g \in L^1(\mathbb{R}^2)$, the Fourier transform of g is the function $\mathcal{F}g$ defined on (the dual of) $\mathbb{R}^2$ by

$$\mathcal{F}g(\xi_1, \xi_2) = \int_{\mathbb{R}^2} e^{-2i\pi(x\xi_1 + y\xi_2)} g(x, y)dxdy,$$

where i denotes the imaginary unit; for $g \in L^2(\mathbb{R}^2)$, the Fourier transform $\mathcal{F}g$ of g is defined in the following distribution sense (see [37]):

$$(\mathcal{F}g, v) = (g, \mathcal{F}v), \quad \forall v \in C_0^\infty(\mathbb{R}^2),$$

and more precisely, $\mathcal{F}$ is an isometry from $L^2(\mathbb{R}^2)$ into itself, which satisfies Parseval's formula (see [32])

$$||\mathcal{F}g|| = ||g||$$

and

$$(v, \overline{w}) = (\mathcal{F}v, \overline{\mathcal{F}w}),$$

where $\bar{z}$ denotes the complex conjugate of the complex number z. The Fourier transform of the μth order fractional derivative consists of the complex in the form $(i\kappa)^\mu$ with $\mu > 0, \kappa \in \mathbb{R}$ (see [29]). So it may be a multi-valued function. To guarantee the Fourier transform to be univalent, we express complex variable $z = |z|\exp(i\theta)$, $-\pi \le \theta < \pi$, where $\exp(i\theta) = \cos\theta + i\sin\theta$, $|z|$ and θ respectively denote the modulus and the argument of z. Then

$$(i\kappa)^\mu = (\mathrm{sign}(\kappa)i|\kappa|)^\mu = (|\kappa|\exp(i\mathrm{sign}(\kappa)\pi/2))^\mu = |\kappa|^\mu \exp(i\mu\mathrm{sign}(\kappa)\pi/2),$$
$$(-i\kappa)^\mu = |\kappa|^\mu \exp(-i\mu\mathrm{sign}(\kappa)\pi/2).$$

It is easy to see that, for $\mu > 0$,

$$\overline{(-i\kappa)^\mu} = (i\kappa)^\mu, \quad \forall \kappa \in \mathbb{R}. \tag{A.1}$$

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