

Necessary and Sufficient Conditions for Oscillation of Second Order Neutral Differential Equations

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Consider the second order nonlinear neutral differential equation with delays: (E) $d^2/dt^2[y(t) - py(t - \tau)] + q(t)f(y(t - \sigma)) = 0$, for $t \in [0, \infty)$, where $q(t), f(x)$ are continuous functions, $q(t) \geq 0$, $yf(y) > 0$ if $y \neq 0$, and $0 < p < 1$, $\tau > 0$, $\sigma > 0$. When $f(y)$ satisfies either the superlinear or sublinear conditions which include the special case $f(y) = y|y|^{\gamma-1}$ of $\gamma > 1$ and $0 < \gamma < 1$, respectively, we give necessary and sufficient conditions for the oscillation of all continuable solutions of (E). When $p = \tau = \sigma = 0$ in (E), these results reduce to the well known theorems of Atkinson and Belohorec in the special case when $f(y) = y|y|^{\gamma-1}$, $\gamma \neq 1$. © 2000 Academic Press

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1. We are here concerned with the second order neutral differential equation with constant delays

$$\frac{d^2}{dt^2}(y(t) - py(t - \tau)) + q(t)f(y(t - \sigma)) = 0, \quad (1)$$

on $[0, \infty)$, where $q(t) \in C[0, \infty)$, $f(y) \in C^1(-\infty, \infty)$, $q(t) \geq 0$, $yf(y) > 0$ whenever $y \neq 0$, $0 < p < 1$, $\tau > 0$, $\sigma > 0$. For any continuous function $\phi(t)$ defined on $[-\rho, 0]$, $\rho = \max(\tau, \sigma)$, Eq. (1) has a solution $y(t)$ extendable on $[0, \infty)$ satisfying the initial condition $y(t) \equiv \phi(t)$ for $t \in [-\rho, 0]$; see, e.g., Hale [11]. A solution $y(t)$ of (1) is oscillatory if it has arbitrarily large zeros; i.e., for any $t_0 \in [0, \infty)$ there exists $t_1 \geq t_0$ such that $y(t_1) = 0$. Equation (1) is said to be oscillatory if all continuable solutions are oscillatory.

We shall consider a class of nonlinear functions $f(y)$ satisfying $f'(y) \geq 0$ and certain nonlinear conditions typified by the Emden–Fowler equation

$$y''(t) + q(t)y|y|^{\gamma-1} = 0, \quad (2)$$

where $\gamma > 0$. We say that $f(y)$ satisfied the superlinear condition if

$$0 < \int_{\varepsilon}^{\infty} \frac{dy}{f(y)}, \int_{-\infty}^{-\varepsilon} \frac{dy}{f(y)} < \infty, \quad \text{for all } \varepsilon > 0; \quad (3)$$

and $f(y)$ satisfies the sublinear condition if

$$0 < \int_0^{\varepsilon} \frac{dy}{f(y)}, \int_{-\varepsilon}^0 \frac{dy}{f(y)} < \infty, \quad \text{for all } \varepsilon > 0. \quad (4)$$

Conditions (3) and (4) correspond to $\gamma > 1$ and $0 < \gamma < 1$ in Eq. (2), respectively.

For Eq. (2), there are necessary and sufficient conditions for the oscillation of all its solutions due to Atkinson and Belohorec, respectively:

THEOREM A (Atkinson [1]). *Let $q(t) \in C[0, \infty)$ and $q(t) \geq 0$. Then, if $\gamma > 1$, Eq. (2) is oscillatory if and only if*

$$\int^{\infty} tq(t)dt = \infty. \quad (5)$$

THEOREM B (Belohorec [2]). *Let $q(t) \in [0, \infty)$ and $q(t) \geq 0$. Then, if $0 < \gamma < 1$, Eq. (2) is oscillatory if and only if*

$$\int^{\infty} t^{\gamma}q(t)dt = \infty. \quad (6)$$

The purpose of this paper is to prove analogous results of Theorems A and B for the neutral differential equation with delays in the form of Eq. (1).

As a general reference on oscillatory theory for neutral differential equations, we refer to Gyóri and Ladas [10]. Oscillation theorems for second order neutral equations were discussed by Ladas et al. [14, 15] and Graef et al. [8, 9]. In the delay differential case, i.e., Eq. (1) when $p = 0$, reference should also be made to Ladde et al. [16]. Extensions of Atkinson and Belohorec oscillation theorems to more general nonlinear differential equations were given in earlier papers by this author [18, 19] and Coffman and Wong [5]. They were also extended to include delay equations in [20] and to forced equations in [21]. These results were further extended by more recent papers of Nasr [17] and Das and Misra [7].

2. In this section, we shall prove that condition (5) is necessary and sufficient for the oscillation of (1) if the nonlinear function $f(y)$ satisfies the superlinear condition (3).

THEOREM 1. *Let $q(t) \in C[0, \infty)$, $q(t) \geq 0$ and $f(y) \in C^1(-\infty, \infty)$, satisfying $yf(y) > 0$ whenever $y \neq 0$, $f'(y) \geq 0$ for all y and also condition (3). Then Eq. (1) is oscillatory if and only if (5) holds.*

Proof. To prove sufficiency, let $y(t)$ be a nonoscillatory solution of (1). Since $yf(y) > 0$ whenever $y \neq 0$, we may without loss of generality assume that $y(t) > 0$ for all $t \geq t_0 \geq 0$, where t_0 depends on the solution $y(t)$. Denote $z(t) = y(t) - py(t - \tau)$. Since $q(t) \geq 0$, Eq. (1) implies $z''(t) \leq 0$ and $z'(t)$ is nonincreasing. Hence $\lim_{t \rightarrow \infty} z'(t) = l$. Suppose that $l < 0$; then $\lim_{t \rightarrow \infty} z(t) = -\infty$. We claim that $z(t)$ cannot be eventually negative on $[t_0, \infty)$. Suppose it is the case consider two mutually exclusive cases: (i) there exists a sequence $\{t_k\}$, $t_k \rightarrow \infty$ as $k \rightarrow \infty$ and $y(t_k) = \sup_{t \leq t_k} y(t)$ or otherwise (ii) there exists a sequence $\{\tau_k\}$, $\tau_k \rightarrow \infty$ and $y(\tau_k) = \inf_{t \leq \tau_k} y(t)$. In the first case (i), we find

$$z(t_k) = y(t_k) - py(t_k - c) \geq y(t_k)(1 - p) > 0,$$

which shows that $z(t)$ cannot be eventually negative. In the second case (ii), we have $z(\tau_k - \tau) = y(\tau_k - \tau) - py(\tau_k) \geq y(\tau_k)(1 - p) > 0$, which again shows that $z(t)$ cannot be eventually negative. In particular, this rules out $l < 0$. Thus we must have $l \geq 0$, which implies that $z(t)$ must be eventually positive; i.e., there exists $t_* \geq t_0 \geq 0$ such that $z(t) > 0$ for all $t \geq t_*$. Otherwise, since $\lim_{t \rightarrow \infty} z'(t) = l \geq 0$ and $z'(t)$ is nonincreasing, we must have $z'(t) < 0$ for all sufficiently large t . As $z(t)$ cannot be eventually negative, there exists $\bar{t} \geq t_*$ such that $z(\bar{t}) > 0$, so $z(t) \geq z(\bar{t}) > 0$.

We therefore have $z(t) > 0$, $z'(t) > 0$ and $z''(t) \leq 0$ on an open interval $[t_*, \infty)$ for some t_* sufficiently large. Since $f'(y) \geq 0$, we have $f(y(t - \sigma)) \geq f(z(t - \sigma))$ for all $t \geq t_* + \sigma$ and we find Eq. (1) implies the second order differential inequality for $z(t)$,

$$0 = z''(t) + q(t)f(y(t - \sigma)) \geq z''(t) + q(t)f(z(t - \sigma)) \quad (7)$$

on $t \in [t_*, \infty)$. Define $w(t) = tz'(t)/f(z(t - \sigma))$ which satisfies, on account of (7), the Riccati differential inequality:

$$w'(t) + tq(t) \leq \frac{z'(t)}{f(z(t - \sigma))} - \frac{tf'(z(t - \sigma))z'(t)}{[f(z(t - \sigma))]^2} z'(t - \sigma). \quad (8)$$

Now integrate (8) from t_* to t to obtain

$$w(t) + \int_{t_*}^t sq(s)ds \leq w(t_2) + \int_{z(t_* - \sigma)}^{\infty} \frac{dy}{f(y)} - \int_{z(t - \sigma)}^{\infty} \frac{dy}{f(y)}, \quad (9)$$

where we had dropped the last integral in (8) which is non-negative. Now, the last term in (9) above is positive by (3) whilst the first term $w(t)$ is also positive since $z'(t) > 0$. Hence,

$$\int_{t_2}^t sq(s)ds \leq w(t_2) + \int_{z(t_2-\sigma)}^{\infty} \frac{dy}{f(y)} = M_0, \quad (10)$$

where M_0 depends only on the solution $y(t)$. Letting $t \rightarrow \infty$ in (10), one easily sees that it is incompatible with (5). This proves the sufficiency part of Theorem 1.

To prove the necessity of condition (5) for the oscillation of Eq. (1), we shall apply the contraction mapping principle. Consider the Banach space $C^2[t_0, \infty)$ with the supnorm, $\|y\| = \sup_{t \geq t_0} |y(t)|$ for $y(t) \in C^2[t_0, \infty)$, where $t_0 > 0$ is to be chosen later. Assume that condition (5) fails, i.e., $tq(t) \in L^1[t_0, \infty)$; then there is a nonoscillatory solution $y(t)$. In fact, we shall show the existence of a solution $y(t)$ of (1) such that $\lim_{t \rightarrow \infty} y(t) = \frac{1}{1-p}$.

Let Y be a subset of $C^2[t_0, \infty)$ consisting of functions $y(t)$ satisfying the estimate

$$\frac{1}{2} \leq |y(t)| \leq \frac{1}{1-p} \quad \text{for } t \geq t_0. \quad (11)$$

Define an operator $T: Y \rightarrow Y$ by

$$Ty(t) = 1 + py(t - \tau) - \int_t^{\infty} (s - t)q(s)f(y(s - \sigma))ds. \quad (12)$$

Since $f(y) \in C^1(-\infty, \infty)$, f is Lipschitzian with Lipschitzian constant $L = \sup_{1/2 \leq y \leq 1/(1-p)} f'(y)$; i.e.,

$$|f(y_1) - f(y_2)| \leq L|y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R}. \quad (13)$$

Choose t_0 sufficiently large so that $L \int_{t_0}^{\infty} tq(t) \leq \frac{1-p}{2}$. Let $y \in Y$; then from (11), we have $Ty(t) \geq 1 + \frac{p}{2} - \frac{L}{1-p} \int_t^{\infty} sq(s)ds = 1 + \frac{p}{2} - \frac{1}{2} \geq \frac{1}{2}$ and $Ty(t) \leq 1 + \frac{p}{1-p} = \frac{1}{1-p}$, so $T(Y) \subseteq Y$. On the other hand, using (13) in (12) we find

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| &\leq p|y_1(t - \tau) - y_2(t - \tau)| \\ &\quad + L \left\{ \int_t^{\infty} sq(s)ds \right\} \|y_1 - y_2\| \\ &\leq p\|y_1 - y_2\| + \frac{1+p}{2}\|y_1 - y_2\| = \frac{1+p}{2}\|y_1 - y_2\|. \end{aligned}$$

Hence T is a contraction since $0 < p < 1$, so T has a fixed point in Y . For $Ty = y$ in (12), $y(t)$ is a solution of (1), and that $\lim_{t \rightarrow \infty} y(t) = \frac{1}{1-p}$. This completes the proof of Theorem 1.

3. In this section, we shall prove an analogous result for the oscillation of Eq. (1) in the sublinear case.

THEOREM 2. *Let $q(t) \in C[0, \infty)$, $q(t) \geq 0$ and $f(y) \in C^1(-\infty, \infty)$, satisfying $yf(y) > 0$ whenever $y \neq 0$, $f'(y) \geq 0$ for all y , the sublinear condition (3), and also*

$$f(uv) \geq f(u)f(v) \quad \text{if } uv \geq 0 \quad \text{and} \quad |v| \geq M, \quad (14)$$

for some large $M > 0$. Then Eq. (1) is oscillatory if and only if

$$\int^{\infty} f(t)q(t)dt = \infty. \quad (15)$$

Proof. Let $y(t)$ be a nonoscillatory solution of (1), which can be assumed to be positive on $[t_0, \infty)$ for some $t_0 \geq 0$, and proceed as in the proof of Theorem 1. Denote $z(t) = y(t) - py(t - \tau)$. Then $z(t) < y(t)$, and $z''(t) \leq 0$, $z'(t) > 0$, $z(t) > 0$ for $t \geq t_0$ as before. Returning to (1), we obtain the second order differential inequality (7); namely,

$$z''(t) + q(t)f(z(t - \sigma)) \leq 0, \quad t \geq t_0 + \sigma. \quad (16)$$

Observe that $z(t) = z(t_0) + \int_{t_0}^t z'(s)ds \geq z'(t)(t - t_0)$ so $f(z(t - \sigma)) \geq f(z'(t - \sigma)(t - \sigma - t_0))$. For any λ , $0 < \lambda < 1$, if t_0 is sufficiently large, then $t - \sigma - t_0 \geq \lambda t$, for $t \geq t_1 > t_0$. Thus by (14), we have

$$f(z'(t - \sigma)(t - \sigma - t_0)) \geq f(\lambda z'(t - \sigma))f(t), \quad t \geq t_1$$

from which (16) can be rewritten as follows:

$$\frac{z''(t)}{f(\lambda z'(t - \sigma))} + q(t)f(t) \leq 0, \quad t \geq t_1. \quad (17)$$

Integrating (17) and using the sublinear condition (4), we find

$$\int_{t_0}^t f(s)q(s)ds \leq \int_0^{\lambda z'(t_1 - \sigma)} \frac{dy}{f(y)} - \int_0^{\lambda z'(t - \sigma)} \frac{dy}{f(y)} \leq K_0, \quad (18)$$

where $K_0 = \int_0^{\lambda z'(t_1 - \sigma)} \frac{dy}{f(y)}$. Clearly (18) is incompatible with (15). This proves the sufficiency part of Theorem 2.

To prove that condition (15) is also necessary for the oscillation of Eq. (1), we assume that condition (5) fails and proceed to establish the

existence of a nonoscillatory solution. In this case, we choose t_0 sufficiently large such that

$$\int_{t_0}^{\infty} q(t)f(t)dt < \lambda \left(\frac{1-p}{4} \right), \quad (19)$$

where $0 < \lambda < 1$. Let $\rho = \max\{\tau, \sigma\} > 0$ and $I_0^\rho = [t_0 - \rho, t_0]$. Consider the linear function $\phi \in C(I_0^\rho)$ defined by $\phi(s) = \lambda(s - t_0 + \rho)$ for $s \in I_0^\rho$. Here $\phi(s) \geq 0$, $\phi'(s) = \lambda$ for all $s \in I_0^\rho$, $\phi(t_0) = \lambda\rho > 0$, and $\phi(t_0 - \tau) = \lambda(\rho - \tau) \geq 0$. For such a given initial function $\phi(t)$, the neutral differential equation (1) has a solution $y_\phi(t)$ which we shall denote by $y(t)$ for short and $y(t) = \phi(t)$ for all $t \in I_0^\rho$. We shall prove that this solution is nonoscillatory. In fact, $y'(t_0) = \phi'(t_0) = \lambda$ and we shall show that $y'(t) \geq \frac{\lambda}{2}$ for all $t \geq t_0$.

Since $y'(t_0) = \phi'(t_0) = \lambda > 0$ and $y(t_0) = \phi(t_0) = \lambda\rho > 0$, observe that $y''(t_0 + \varepsilon) + q(t_0 + \varepsilon)f(y(t_0 + \varepsilon)) = 0$ for $\varepsilon > 0$ small. So $f(y(t_0 + \varepsilon)) > 0$ implies $y''(t_0 + \varepsilon) \leq 0$ and $y'(t)$ is nonincreasing in the right neighborhood of t_0 . Let $\tau_1 > t_0$ be the largest t to the right of t_0 such that $m_1 = y'(\tau_1) = \inf\{y'(t) : t_0 \leq t \leq \tau_1\}$. We first claim that $m_1 > 0$. Consider the definite integral

$$\begin{aligned} I_0 &= \int_{t_\sigma}^{t_0 + \sigma} q(s)f(y(s - \sigma))ds \\ &= \int_{t_0}^{t_0 + \sigma} q(s)f(\lambda(s - \sigma - t_0 + \rho))ds. \end{aligned} \quad (20)$$

Since $s - \sigma - t_0 + \rho \geq 0$ for all $s \geq t_0$ so $I_0 \geq 0$. Furthermore, since $f'(y) \geq 0$ for all y , we have $f(\lambda(s - \sigma - t_0 + \rho)) \leq f(\lambda(s - t_0)) \leq f(\lambda s) \leq f(s)$, so by (19) we can estimate (20) as follows:

$$I_0 \leq \int_{t_0}^{t_0 + \sigma} q(s)f(\lambda(s - t_0))ds \leq \int_{t_0}^{\infty} q(s)f(s)ds \leq \frac{\lambda(1-p)}{4}. \quad (21)$$

Suppose that $m_1 \leq 0$; then there exists τ_0 , $t_0 \leq \tau_0 < \tau_1$, such that $y'(\tau_0) = 0$ and $y'(s) > 0$ if $s < \tau_0$. Integrating (1) from t_0 to τ_0 , we find

$$\begin{aligned} y'(\tau_0) - y'(\tau_0 - \tau) &= y'(t_0) - py'(t_0 - \tau) \\ &\quad - \int_{t_0}^{\tau_0} q(s)f(y(s - \sigma))ds. \end{aligned} \quad (22)$$

We note from (22) that

$$0 > -y'(\tau_0 - \tau) = \lambda(1 - p) - I_0 - \int_{t_0 + \sigma}^{\tau_0} q(s)f(y(s - \sigma))ds. \quad (23)$$

Since $y'(t) \leq y'(t_0) = \lambda$ for all $t_0 \leq t \leq \tau_0$, we have

$$\begin{aligned} y(s - \sigma) &= y(t_0) + \int_{t_0}^{s - \sigma} y'(\zeta)d\zeta \leq \lambda\rho + \lambda(s - \sigma - t_0) \\ &= \lambda(\rho - \sigma) + \lambda(s - t_0) \leq \lambda(\rho - \sigma) + \lambda s. \end{aligned}$$

As $0 < \lambda < 1$, we can choose t_0 sufficiently large so that $\lambda(\rho - \sigma) + \lambda s \leq s$ if $s \geq t_0$. Thus, $f(y(s - \sigma)) \leq f(s)$ and the last integral in (23) satisfies

$$\int_{t_0 + \sigma}^{\tau_0} q(s)f(y(s - \sigma))ds \leq \int_{t_0 + \sigma}^{\infty} q(s)f(s)ds \leq \frac{\lambda(1 - p)}{4}. \quad (24)$$

Using (21), (24) in (23), we find $0 > -y'(\tau_0 - \tau) > \frac{\lambda}{2}(1 - p) > 0$, which is impossible. Hence $m_1 > 0$.

Integrating (1) as in the case of (22) with τ_1 , replacing τ_0 , we find

$$\begin{aligned} y'(\tau_1)(1 - p) &\geq y'(\tau_1) - py'(\tau_1 - \tau) \\ &= \lambda(1 - p) - I_0 - \int_{t_0 + \sigma}^{\tau_1} q(s)f(y(s - \sigma))ds \end{aligned} \quad (25)$$

By (21) and (24) again, we obtain from (25) that

$$y'(\tau_1)(1 - p) \geq \frac{\lambda}{2}(1 - p) \quad \text{or} \quad y'(\tau_1) \geq \frac{\lambda}{2}. \quad (26)$$

Recall that τ_1 is chosen to be the largest t to the right of t_0 ; i.e., $y'(\tau_1) = \inf\{y'(t) : t_0 \leq t \leq \tau_1\}$. If such τ_1 does not exist then $y'(t)$ must be nondecreasing, and (26) shows that $y'(t) \geq \frac{\lambda}{2}$ for all $t \geq t_0$ (where the arguments in (25) and (26) with τ_1 substituted by t remain valid). Thus, the assertion is proved. So, we can assume that such τ_1 does exist.

Next, we define t_1 to be the largest t to the right of τ_1 so that $y'(t_1) = \sup\{y'(t) : t_0 \leq t \leq t_1\}$. If such t_1 does not exist then $y'(t)$ is nondecreasing for $t \geq \tau_1$. Hence $y'(t) \geq y'(\tau_1) \geq \frac{\lambda}{2}$ for all $t \geq t_0$. Again, the assertion is proved. In this manner, we can define inductively sequences of $\{\tau_k\}$ and $\{t_k\}$, $k = 1, 2, \dots$ so that

$$\begin{aligned} m_k &= y'(\tau_k) = \inf\{y'(t) : t_0 \leq t \leq \tau_k\}, \\ M_k &= y'(t_k) = \sup\{y'(t) : t_0 \leq t \leq t_k\}. \end{aligned}$$

We first assume that $m_k \geq \frac{\lambda}{2}$ and show that $M_k \leq \lambda$. Once again integrate (1) from t_0 to t_k to obtain

$$\begin{aligned} M_k(1-p) &\leq y'(t_k) - py'(t_k - \tau) \\ &= y'(t_0) - py'(t_0 - \tau) - \int_{t_0}^{t_k} q(s)f(y(s - \sigma))ds \\ &= \lambda(1-p) - I_0 - \int_{t_0+\sigma}^{t_k} q(s)f(y(s - \sigma))ds. \end{aligned} \quad (27)$$

Since $m_k \geq \frac{\lambda}{2}$ so $y'(t) \geq m_k \geq \frac{\lambda}{2}$ for all $t \in [t_0 + \sigma, t_k]$. Recall $y(t_0) = \phi(t_0) = \lambda p > 0$. Hence the last integral in (27) is positive. This together with the fact that $I_0 \geq 0$ implies $M_k \leq \lambda$.

Now we show by induction that if $m_k \geq \frac{\lambda}{2}$, hence $M_k \leq \lambda$, then $m_{k+1} \geq \frac{\lambda}{2}$. Integrate (1) from t_0 to τ_{k+1} and find similar to (27) that

$$\begin{aligned} m_{k+1}(1-p) &= y'(\tau_{k+1})(1-p) \geq y'(\tau_{k+1}) - py'(\tau_{k+1} - \tau) \\ &= \lambda(1-p) - I_0 - \int_{t_0+\sigma}^{\tau_{k+1}} q(s)f(y(s - \sigma))d\sigma. \end{aligned} \quad (28)$$

Note that $M_k \leq \lambda$ implies $y'(t) \leq \lambda$ for $t \in [t_0 + \sigma, \tau_{k+1}]$, so we have for $t_0 \geq \rho$

$$y(s) = \phi(t_0) + \int_{t_0}^s y'(\xi)d\xi \leq \lambda\rho + \lambda(s - t_0) \leq \lambda s. \quad (29)$$

Using (29) we can estimate the last integral in (28) as follows:

$$\int_{t_0+\sigma}^{\tau_{k+1}} q(s)f(y(s - \sigma))ds \leq \int_{t_0+\sigma}^{\tau_{k+1}} \lambda q(s)(s - \sigma)ds \leq \lambda \int_{t_0}^{\infty} q(s)s ds. \quad (30)$$

Thus by (21), (30), and (19), we conclude from (28) that $m_{k+1} \geq \frac{\lambda}{2}$.

Finally, we note that regardless whether the sequence $\{\tau_k\}$, hence $\{t_k\}$, becomes finite one always obtains $y'(t) \geq \frac{\lambda}{2}$ for $t \geq t_0$, so $y(t)$ is nonoscillatory. Indeed, $\lim_{t \rightarrow \infty} y(t) = \infty$ and the proof of Theorem 2 is complete.

4. In this final section, we close with some remarks concerning the results proved in the previous sections and pose some open problems for further research.

(a) The proofs for Theorems 1 and 2 used explicitly the assumption that p , τ , and σ are positive constants. It is easy to check that these proofs remain valid if any or all of p , τ , and σ become zero. Thus, Theorems 1 and 2 are true extensions of Alkinson and Belohorec theorems for the Emden–Fowler differential equation (2).

(b) It should be noted that neither the superlinear condition (3) nor the sublinear condition (4) was required in the necessity part of the proofs for Theorems 1 and 2. Moreover, the superhomogeneous condition (14) was also not used in the necessity part for the sublinear Theorem 2.

(c) In Theorem 2 of the sublinear equation, conditions (14) and (15) were first introduced by Wong [18, 22] and Coles [6]. The proofs of sufficiency parts of these theorems were based upon improved versions of original proofs by Atkinson and Belohorec as given in our earlier paper on second order delay differential equations [20].

(d) Condition (5) for the superlinear equation (2) in case $\gamma > 1$ and condition (15) for the sublinear equation (2) with $0 < \gamma < 1$ are known to be sufficient for oscillation of (2) without the assumption that $q(t) \geq 0$; see Kiguradze [13] and Belohorec [3]. Their results were further extended to the more general function $f(y)$ satisfying (3) and (4) by Kamenev [12]. It will be of great interest to know whether the same results remain valid for the neutral differential equation (1), even in the Emden–Fowler case; i.e., $f(y) = y|y|^{\gamma-1}$, $\gamma \neq 1$.

(e) A neutral equation with variable coefficient, i.e., Eq. (1) when p is replaced by a continuous function $p(t)$, is of great interest in applications; see Györi and Ladas [10]. Our proofs are valid in most parts when $\lim_{t \rightarrow \infty} p(t) = p_0$, $0 < p_0 < 1$, but weaker assumptions such as $\limsup_{t \rightarrow \infty} p(t) < 1$ will be a significant improvement upon results given here.

(f) For our proofs to work it is important that $0 \leq p < 1$, $\tau \geq 0$, and $\sigma \geq 0$. If any one of p , τ , σ is negative, then neutral equation (1) is of mixed type. It would be of considerable interest to develop oscillation results for these equations. For some partial results in this direction, we refer to Grammatikopoulos et al. [4, 9].

(g) Finally, it is easy to give examples of nonlinear functions $f(y)$ other than $f(y) = y|y|^{\gamma-1}$, $\gamma > 0$, which satisfy the superlinear or the sublinear conditions (3) and (4). To exhibit an example in the sublinear case (4) subject also to the supermultiplicative condition (14), we consider

an odd function $f(y) = \frac{\sqrt{y}}{1+y}$, $y > 0$ with $f(y) = -f(-y)$. Clearly $f(y)$ satisfies (4). For $u, v > 0$, observe

$$f(uv) = \frac{\sqrt{uv}}{1+uv} \geq \frac{\sqrt{u}\sqrt{v}}{(1+u)(1+v)} = f(u)f(v),$$

so $f(y)$ satisfies (14) and Theorem 2 is applicable.

REFERENCES

1. F. V. Atkinson, On second order nonlinear oscillation, *Pacific J. Math.* **5** (1955), 643–647.
2. S. Belohorec, Oscillatory solutions of certain nonlinear differential equations of second order, *Mat. Fyz. Casopis Sloven Akad. Vied.* **11** (1961), 250–255.
3. S. Belohorec, Two remarks on the properties of solutions of a nonlinear differential equation, *Acta Tac. Rerum Natur. Univ. Comen. Math.* **22** (1969), 19–26.
4. S. J. Bilchev, M. K. Grammatikopoulos, and L. P. Stavioulakis, Oscillations of second order neutral differential equations with deviating arguments, in “Oscillation and Dynamics in Delay Equations,” AMS Contemporary Math., Vol. 129, pp. 1–21, Am. Math. Soc., Providence, 1992.
5. C. V. Coffman and J. S. W. Wong, Oscillation and nonoscillation theorems for second order ordinary differential equations, *Funkcial. Ekvac.* **15** (1972), 119–130.
6. W. J. Coles, A nonlinear oscillation theorem, in “International Conference on Differential Equations” (H. A. Antosiewicz, Ed.), pp. 193–202, Academic Press, New York, 1975.
7. P. Das and N. Misra, A necessary and sufficient condition for the solutions of a functional differential equation to be oscillatory or tend to zero, *J. Math. Anal. Appl.* **204** (1997), 78–87.
8. J. R. Graef, M. K. Grammatikopoulos, and P. W. Spikes, Asymptotic properties of solutions of nonlinear neutral delay differential equations of the second order, *Rad. Mat.* **4** (1988), 133–149.
9. J. R. Graef, M. K. Grammatikopoulos, and P. W. Spikes, Some results on the asymptotic behavior of solutions of a second order nonlinear neutral delay equation, in “Oscillation and Dynamics in Delay Equations,” AMS Contemporary Math., Vol. 129, pp. 105–114, Am. Math. Soc., Providence, 1992.
10. I. Györi and G. Ladas, “Oscillation Theory of Delay Differential Equations,” Clarendon, Oxford, 1991.
11. J. K. Hale, “Theory of Functional Differential Equations,” Springer-Verlag, New York, 1997.
12. I. V. Kamenev, Oscillation of solutions of second order nonlinear equations, *Trans. Moscow Electrical Machine Design Inst.* **5** (1969/70), 125–136.
13. I. T. Kiguradze, On condition for oscillation of solution of the equation $u'' + a(t)|u|^n \operatorname{sgn} u = 0$, *Casopis Pest Mat.* **87** (1962), 492–495 (in Russian).
14. G. Ladas, E. C. Partheniadis, and Y. G. Sficas, Oscillations of second order neutral equations, *Canad. J. math.* **41** (1988), 1301–1314.
15. G. Ladas, E. C. Partheniadis, and Y. G. Sficas, Necessary and sufficient conditions for oscillations of second order neutral equations, *J. Math. Anal. Appl.* **138** (1989), 214–231.
16. G. S. Ladde, V. Lakshmikantham, and B. G. Zhang, “Oscillation Theory of Differential Equations with Deviating Arguments,” Dekker, New York, 1987.

17. A. H. Nasr, Necessary and sufficient conditions for the oscillation of forced nonlinear second order differential equations with delayed arguments. *J. Math. Anal. Appl.* **212** (1997), 51–59.
18. J. S. W. Wong, Some properties of solutions of $u''(t) = a(t)f(u)g(u') = 0$ III, *Siam J. Appl. Math.* **14** (1966), 209–214.
19. J. S. W. Wong, A note on second order nonlinear oscillation, *SIAM Rev.* **10** (1968), 88–91.
20. J. S. W. Wong, Second order oscillation with retarded arguments, in “Ordinary Differential Equations, 1971 NRL-MRC Conference” (L. Weiss, Ed.), pp. 581–596, Academic Press, New York, 1972.
21. J. S. W. Wong, Second order nonlinear forced oscillation, *SIAM J. Math. Anal.* **19** (1988), 667–675.
22. J. S. W. Wong, A second order sublinear oscillation theorem, *J. Math. Anal. Appl.* **1** (1989), 408–412.