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Demiclosedness Principle and Convergence Theorems for Strictly Pseudocontractive Mappings of Browder–Petryshyn Type

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Let E be a real q-uniformly smooth Banach space which is also uniformly convex (for example, L_p or l_p spaces, $1 < p < \infty$) and K a nonempty closed convex subset of E. Let $T\colon K \to K$ be a strictly pseudocontractive mapping in the sense of F. E. Browder and W. V. Petryshyn (1967, *J. Math. Anal. Appl.* **20,** 197–228). It is proved that $(I - T)$ is demiclosed at zero. If $F(T) = \{x \in K : Tx = x\} \neq \emptyset$, weak and strong convergence of the Mann and Ishikawa iteration methods to a fixed point of T is proved.

Key Words: fixed points; strictly pseudocontractive maps; Mann iteration; Ishikawa iteration.

1. INTRODUCTION

Let E be an arbitrary real Banach space and let J_q $(q > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by

$$J_q(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1} \right\},$$

where E^* denotes the dual space of E and $\langle \cdot , \cdot \rangle$ denotes the generalized duality pairing. In particular, J_2 is called the normalized duality mapping and it is usually denoted by J. It is well known (see, for example, [11]) that $J_q(x) = \|x\|^{q-2} J(x)$ if $x \neq 0$, and that if E^* is strictly convex then J_q is

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single-valued. In the sequel we shall denote the single-valued generalized duality mapping by j_q.

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strictly pseudocontractive* in the terminology of Browder and Petryshyn [1] if for all $x, y \in D(T)$ there exist $\lambda > 0$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 - \lambda\|x - y - (Tx - Ty)\|^2. \tag{1}$$

Without loss of generality we may assume $\lambda \in (0,1)$. If I denotes the identity operator, then (1) can be written in the form

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \ge \lambda\|(I-T)x - (I-T)y\|^2. \tag{2}$$

In Hilbert spaces, (1) (and hence (2)) is equivalent to the inequality

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2,\ k = (1-\lambda) < 1.$$

T is said to be *demiclosed at a point p* if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx = p$. Furthermore, T is said to be *demicompact* if whenever $\{x_n\}$ is a bounded sequence in $D(T)$ such that $\{x_n - Tx_n\}$ converges strongly, then $\{x_n\}$ has a subsequence which converges strongly.

In [1] Browder and Petryshyn proved the following:

THEOREM BP. *Let H be a real Hilbert space and K a nonempty closed convex and bounded subset of H. Let $T: K \to K$ be a strictly pseudocontractive map. Then for any fixed $\gamma \in (1-k, 1)$, the sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in K$ by*

$$x_{n+1} := \gamma x_n + (1-\gamma)Tx_n = [\gamma I + (1-\gamma)T]^n(x_1), \qquad n \ge 1$$

converges weakly to a fixed point of T. If additionally T is demicompact, then $\{x_n\}$ converges strongly to a fixed point of T.

In [9] Rhoades also proved the following convergence theorem using the *Mann iteration method* [5]:

THEOREM R. *Let H be a real Hilbert space and K a nonempty compact convex subset of H. Let $T: K \to K$ be a strictly pseudocontractive map and let $\{\alpha_n\}$ be a real sequence satisfying the conditions: (i) $\alpha_0 = 1$, (ii) $0 < \alpha_n < 1$, $n \ge 1$, (iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and (iv) $\lim_{n\to\infty}\alpha_n = \alpha < 1 - k$. Then the sequence of the Mann iteration method generated from an arbitrary $x_0 \in K$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \qquad n \ge 0$$

converges strongly to a fixed point of T.

Let E be a q-uniformly smooth Banach space which is also uniformly convex, K a nonempty closed convex (not necessarily bounded) subset of E, and $T\colon K \to K$ a strictly pseudocontractive map.

It is our purpose in this paper to first prove that $(I - T)$ is demiclosed at zero. If $F(T) := \{x \in K : Tx = x\} \neq \emptyset$, we then prove weak and strong convergence theorems for the iterative approximation of fixed points of T using the Mann and Ishikawa iteration methods. Theorems BP and R will be special cases of our theorem. Our class of Banach spaces includes the Lp, l_p spaces and the Sobolev spaces W_m^P, $1 < p < \infty$.

2. PRELIMINARIES

From (2) we have

$$\|x - y\| \geq \lambda\|x - y - (Tx - Ty)\| \geq \lambda\|Tx - Ty\| - \lambda\|x - y\|,$$

so that

$$\|Tx - Ty\| \leq L\|x - y\|, \qquad \forall x, y \in K \text{ where } L = \frac{(1 + \lambda)}{\lambda}.$$

Since $\|x - y\| \geq \lambda\|x - y - (Tx - Ty)\|$, we have

$$\begin{aligned} &\left\langle x - Tx - (y - Ty), j_q(x - y)\right\rangle \\ &\quad = \|x - y\|^{q-2}\langle x - Tx - (y - Ty), j(x - y)\rangle \\ &\quad \geq \lambda\|x - y\|^{q-2}\|x - Tx - (y - Ty)\|^2 \\ &\quad \geq \lambda^{q-1}\|x - Tx - (y - Ty)\|^q. \end{aligned} \tag{3}$$

Let E be a real Banach space. The *modulus of smoothness* of E is the function

$$\rho_E\colon [0, \infty) \to [0, \infty)$$

defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1,\ \|y\| \leq \tau\right\}.$$

E is *uniformly smooth* if and only if $\lim_{\tau \to 0}(\rho_E(\tau)/\tau) = 0$.

Let $q > 1$. E is said to be *q-uniformly smooth* (or to have a modulus of smoothness of power type $q > 1$) if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. Hilbert spaces, L_p(or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces, W_m^p, $1 < p < \infty$, are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$L_p(or\, l_p) \text{ or } W_m^p \text{ is } \begin{cases} p - \text{uniformly smooth if } 1 < p \leq 2 \\ 2 - \text{uniformly smooth if } p \geq 2. \end{cases}$$

THEOREM HKX [11, p. 1130]. *Let $q > 1$ and let E be a real Banach space. Then the following are equivalent*:

(1) *E is q-uniformly smooth.*

(2) *There exists a constant $c_q > 0$ such that for all $x, y \in E$*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q. \tag{4}$$

(3) *There exists a constant d_q such that for all $x, y \in E$, and $t \in [0, 1]$*

$$\|(1 - t)x + ty\|^q \geq (1 - t)\|x\|^q + t\|y\|^q - \omega_q(t)d_q\|x - y\|^q, \tag{5}$$

where $\omega_q(t) = t^q(1 - t) + t(1 - t)^q$.

Furthermore, it is proved in [12, Remark 5, p. 208] that if E is q-uniformly smooth ($q > 1$), then for all $x, y \in E$ there exists a constant $L_* > 0$ such that

$$\|j_q(x) - j_q(y)\| \leq L_*\|x - y\|^{q-1}. \tag{6}$$

E is said to have a *Fréchet differentiable norm* if for all $x \in U = \{x \in E : \|x\| = 1\}$

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly in $y \in U$. In this case there exists an increasing function b: $[0, \infty) \to [0, \infty)$ with $\lim_{t \to 0^+} b(t) = 0$ such that

$$\frac{1}{2}\|x\|^2 + \langle h, j(x)\rangle \leq \frac{1}{2}\|x + h\|^2 \leq \frac{1}{2}\|x\|^2 + \langle h, j(x)\rangle + b(\|h\|), \quad \forall x, h \in E. \tag{7}$$

In the sequel we shall need the following results:

LEMMA TX [10, p. 303]. *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=0}^\infty$ be sequences of nonnegative real numbers such that $\sum_{n=1}^\infty b_n < \infty$ and*

$$a_{n+1} \leq a_n + b_n, \qquad n \geq 1.$$

Then $\lim_{n\to\infty} a_n$ exists.

THEOREM GK [3, p. 109]. *Let E be a uniformly convex Banach space, K a closed convex subset of E, and $T\colon K \to E$ a nonexpansive mapping. Then $(I - T)$ is demiclosed at zero.*

3. MAIN RESULTS

For the rest of this paper, λ is the constant appearing in (1), L is the Lipschitz constant of T, and c_q, d_q, $\omega_q(t)$, and L_* are the constants appearing in inequalities (4)–(6). We now prove the following:

LEMMA 1. *Let E be a real q-uniformly smooth Banach space and K a nonempty convex subset of E. Let $T\colon K \to K$ be a strictly pseudocontractive map and let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ be real sequences in $[0,1]$. Define $T_n\colon K \to K$ by*

$$T_n x := (1-\alpha_n)x + \alpha_n T\big((1-\beta_n)x + \beta_n Tx\big), \qquad x \in K.$$

Then for all $x, y \in K$

$$\|T_n x - T_n y\|^q \le [1+\delta_n]\|x-y\|^q - \alpha_n\big[\lambda^{q-1}q(1-\beta_n) - c_q\alpha_n^{q-1}\big] \times \big\|x - T(g_n(x)) - (y - T(g_n(y)))\big\|^q, \tag{8}$$

where $\delta_n := 2q\alpha_n\beta_n\lambda^{q-1}d_q(1+L)^q + q\alpha_n L_*(1+L)^{q+1}\beta_n^{q-1}$, $g_n(x) := (1-\beta_n)x + \beta_n Tx$ *and* $g_n(y) := (1-\beta_n)y + \beta_n Ty$.

Proof.

$$\begin{aligned}
&\|T_n x - T_n y\|^q \\
&\quad = \big\|x - y - \alpha_n[x - T(g_n(x)) - (y - T(g_n(y)))]\big\|^q \\
&\quad \le \|x-y\|^q - q\alpha_n\big\langle x - T(g_n(x)) - (y - T(g_n(y))), j_q(x-y)\big\rangle \\
&\qquad + \alpha_n^q c_q\big\|x - T(g_n(x)) - (y - T(g_n(y)))\big\|^q \qquad (\text{using } (4)).
\end{aligned} \tag{9}$$

Observe that

$$\begin{aligned}
&\big\langle x - T(g_n(x)) - (y - T(g_n(y))), j_q(x-y)\big\rangle \\
&\quad = \big\langle x - g_n(x) - (y - g_n(y)), j_q(x-y)\big\rangle \\
&\qquad + \big\langle g_n(x) - T(g_n(x)) - (g_n(y) - T(g_n(y))), j_q(x-y)\big\rangle \\
&\quad \ge \beta_n\lambda^{q-1}\|x - Tx - (y - Ty)\|^q + M_n(x,y) \qquad (\text{using } (3)),
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
M_n(x,y) &= \left\langle g_n(x) - T(g_n(x)) - \left(g_n(y) - T(g_n(y))\right), j_q(x-y)\right\rangle \\
&= \Big\langle g_n(x) - T(g_n(x)) - \left(g_n(y) - T(g_n(y))\right), \\
&\qquad\qquad j_q(x-y) - j_q\left(g_n(x) - g_n(y)\right)\Big\rangle \\
&\quad + \Big\langle g_n(x) - T(g_n(x)) - \left(g_n(y) - T(g_n(y))\right), \\
&\qquad\qquad j_q\left(g_n(x) - g_n(y)\right)\Big\rangle \\
&\geq \lambda^{q-1}\left\| g_n(x) - T(g_n(x)) - \left(g_n(y) - T(g_n(y))\right)\right\|^q \\
&\quad + \Big\langle g_n(x) - T(g_n(x)) - \left(g_n(y) - T(g_n(y))\right), \\
&\qquad\qquad j_q(x-y) - j_q\left(g_n(x) - g_n(y)\right)\Big\rangle. \qquad (11)
\end{aligned}$$

Furthermore, if we set $Z_n(x, y) := \|g_n(x) - T(g_n(x)) - (g_n(y) - T(g_n(y)))\|^q$, then

$$\begin{aligned}
Z_n(x,y) &= \Big\|(1-\beta_n)\left(x - T(g_n(x))\right) - \left(y - T(g_n(y))\right) \\
&\qquad + \beta_n\left[Tx - T(g_n(x)) - \left(Ty - T(g_n(y))\right)\right]\Big\|^q \\
&\geq (1-\beta_n)\left\| x - T(g_n(x)) - \left(y - T(g_n(y))\right)\right\|^q \\
&\quad + \beta_n\left\| Tx - T(g_n(x)) - \left(Ty - T(g_n(y))\right)\right\|^q \\
&\quad - \omega_q(\beta_n) d_q \| x - Tx - (y - Ty)\|^q \qquad (\text{using } (5)). \quad (12)
\end{aligned}$$

From (9)–(12) we obtain

$$\begin{aligned}
&\|T_n x - T_n y\|^q \\
&\quad \leq \|x-y\|^q - q\alpha_n\Big\{\beta_n \lambda^{q-1}\| x - Tx - (y - Ty)\|^q \\
&\qquad\qquad + \lambda^{q-1}(1-\beta_n)\left\| x - T(g_n(x)) - \left(y - T(g_n(y))\right)\right\|^q \\
&\qquad\qquad + \lambda^{q-1}\beta_n\left\| Tx - T(g_n(x)) - \left(Ty - T(g_n(y))\right)\right\|^q \\
&\qquad\qquad - \lambda^{q-1}\omega_q(\beta_n) d_q\| x - Tx - (y - Ty)\|^q \\
&\qquad\qquad + \Big\langle g_n(x) - T(g_n(x)) - \left(g_n(y) - T(g_n(y))\right), \\
&\qquad\qquad\quad j_q(x-y) - j_q\left(g_n(x) - g_n(y)\right)\Big\rangle\Big\} \\
&\qquad + \alpha_n^q c_q \left\| x - T(g_n(x)) - \left(y - T(g_n(y))\right)\right\|^q \\
&\quad \leq \|x-y\|^q - \alpha_n\left[q\lambda^{q-1}(1-\beta_n) - \alpha_n^{q-1}c_q\right]
\end{aligned}$$

$$\times \left\| x - T\left(g_n(x)\right) - \left(y - T\left(g_n(y)\right)\right) \right\|^q$$
$$+ q d_q \lambda^{q-1} \alpha_n \omega_q(\beta_n) \| x - Tx - (y - Ty) \|^q$$
$$+ q\alpha_n \left\| g_n(x) - T\left(g_n(x)\right) - \left(g_n(y) - T\left(g_n(y)\right)\right) \right\|$$
$$\times \left\| j_q(x-y) - j_q\left(g_n(x) - g_n(y)\right) \right\|. \tag{13}$$

Observe that $\omega_q(\beta_n) = \beta_n(1-\beta_n)^q + \beta_n^q(1-\beta_n) \le 2\beta_n$; $\|x - Tx - (y - Ty)\| \le (1+L)\|x-y\|$;

$$\|j_q(x-y) - j_q\left(g_n(x) - g_n(y)\right)\| \le L_* \beta_n^{q-1} \|x - Tx - (y - Ty)\|^{q-1} \quad (\text{using } (6))$$
$$\le L_*(1+L)^{q-1} \beta_n^{q-1} \|x-y\|^{q-1};$$

and

$$\left\| g_n(x) - T\left(g_n(x)\right) - \left(g_n(y) - T\left(g_n(y)\right)\right) \right\|$$
$$\le (1+L)\|g_n(x) - g_n(y)\|$$
$$\le (1+L)\left[(1-\beta_n)\|x-y\| + \beta_n L\|x-y\|\right]$$
$$\le (1+L)^2 \|x-y\|.$$

Hence

$$\|T_n x - T_n y\|^q$$
$$\le \left[1 + 2\alpha_n \beta_n \lambda^{q-1} q d_q (1+L)^q + \alpha_n \beta_n^{q-1} q L_* (1+L)^{q+1}\right] \|x-y\|^q$$
$$- \alpha_n \left[q\lambda^{q-1}(1-\beta_n) - \alpha_n^{q-1} c_q\right]$$
$$\times \left\| x - T\left(g_n(x)\right) - \left(y - T\left(g_n(y)\right)\right) \right\|^q,$$

completing the proof of Lemma 1. ∎

Remark 1. Let $\gamma = \min\{1, \lambda(q/c_q)^{1/(q-1)}\}$, choose any $\alpha \in (0, \gamma]$, and set $\alpha_n = \alpha$, $\beta_n = 0$, $\forall n \ge 1$ in Lemma 1. Then we obtain $T_\alpha \colon K \to K$ defined for all $x \in K$ by

$$T_\alpha x = (1-\alpha)x + \alpha T x.$$

Furthermore,

$$\|T_\alpha x - T_\alpha y\|^q \le \|x-y\|^q - \alpha\left[q\lambda^{q-1} - \alpha^{q-1} c_q\right] \|x - Tx - (y - Ty)\|^q, \quad \forall x, y \in K. \tag{14}$$

By the choice of α, we have $[q\lambda^{q-1} - \alpha^{q-1}c_q] \geq 0$ so that it follows from (14) that $\|T_\alpha x - T_\alpha y\| \leq \|x - y\|, \forall x, y \in K$. Thus T_α is nonexpansive and $F(T) = F(T_\alpha)$.

LEMMA 2. *Let E be a real q-uniformly smooth Banach space and let K be a nonempty convex subset of E. Let $T\colon K \to K$ be a strictly pseudocontractive mapping with a nonempty fixed-point set $F(T)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the conditions*

(i) $0 \leq \alpha_n, \beta_n \leq 1$, $n \geq 1$

(ii) $0 < a \leq \alpha_n^{q-1} \leq b < (q\lambda^{q-1}/c_q)(1 - \beta_n)$, $\forall n \geq 1$ *and for some constants* $a, b \in (0,1)$

(iii) $\sum_{n=1}^{\infty} \beta_n^{\tau} < \infty$, *where* $\tau = \min\{1, (q-1)\}$.

Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \qquad n \geq 1$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \qquad n \geq 1.$$

Then

(a) $\lim_{n\to\infty}\|x_n - x^*\|$ *exists for every* $x^* \in F(T)$.

(b) $\lim_{n\to\infty}\|y_n - Ty_n\| = 0$.

Proof. Set $x = x_n$, $y = x^*$ in Lemma 1. Then

$$\|x_{n+1} - x^*\|^q \leq [1 + \delta_n]\|x_n - x^*\|^q - \alpha_n\left[q\lambda^{q-1}(1 - \beta_n) - \alpha_n^{q-1}c_q\right]\|x_n - Ty_n\|^q. \quad (15)$$

Condition (ii) implies that

$$q\lambda^{q-1}(1 - \beta_n) - c_q\alpha_n^{q-1} \geq \left[q\lambda^{q-1}(1 - \beta_n) - c_q b\right] > 0, \qquad \forall n \geq 1. \quad (16)$$

Hence (15) reduces to

$$\|x_{n+1} - x^*\|^q \leq [1 + \delta_n]\|x_n - x^*\|^q, \qquad n \geq 1. \quad (17)$$

It follows from condition (iii) that $\sum_{n=1}^{\infty} \delta_n < \infty$, and hence (17) implies that $\{\|x_n - x^*\|\}$ is bounded. Let $\|x_n - x^*\| \leq M$, $n \geq 1$. Then (17) implies that

$$\|x_{n+1} - x^*\|^q \leq \|x_n - x^*\|^q + M^q\delta_n,$$

and it follows from Lemma TX that $\lim_{n\to\infty}\|x_n - x^*\|$ exists, completing the proof of (a). Using (16) in (15) we obtain

$$\|x_{n+1} - x^*\|^q \le \|x_n - x^*\|^q - \alpha_n\left[q\lambda^{q-1}(1-\beta_n) - c_q b\right]\|x_n - Ty_n\|^q + M^q\delta_n. \quad (18)$$

Since $\lim_{n\to\infty}[q\lambda^{q-1}(1-\beta_n) - c_q b] = q\lambda^{q-1} - c_q b > 0$, then there exists a positive integer N_0 such that $q\lambda^{q-1}(1-\beta_n) - c_q b \ge \frac{1}{2}[q\lambda^{q-1} - c_q b] > 0$, $\forall n \ge N_0$. Hence it follows from (18) that

$$\frac{a^{\frac{1}{q-1}}}{2}\left[q\lambda^{q-1} - c_q b\right]\|x_n - Ty_n\|^q \le \|x_n - x^*\|^q - \|x_{n+1} - x^*\|^q + M^q\delta_n, \quad \forall n \ge N_0,$$

so that

$$\frac{a^{\frac{1}{q-1}}}{2}\left[q\lambda^{q-1} - c_q b\right]\sum_{j=N_0}^{n}\|x_j - Ty_j\|^q \le \|x_{N_0} - x^*\|^q + M^q\sum_{j=N_0}^{n}\delta_n \le \|x_{N_0} - x^*\|^q + M^q\sum_{j=0}^{\infty}\delta_j < \infty.$$

Hence $\sum_{n=0}^{\infty}\|x_n - Ty_n\|^q < \infty$, and this implies $\lim\|x_n - Ty_n\| = 0$. Since $\|x_n - Tx_n\| \le (1+L)\|x_n - x^*\| \le M(1+L)$, $\forall n \ge 1$ and

$$\begin{aligned} 0 \le \|y_n - Ty_n\| &\le \|y_n - x_n\| + \|x_n - Ty_n\| \\ &\le \beta_n(1+L)M + \|x_n - Ty_n\| \to 0 \qquad \text{as } n\to\infty, \end{aligned}$$

we have $\lim\|y_n - Ty_n\| = 0$, completing the proof of Lemma 2. ∎

COROLLARY 1. *Let E be a real q-uniformly smooth Banach space and K a nonempty closed convex subset of E. Let T, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{x_n\}$ be as in Lemma 2. If $\{x_n\}$ clusters strongly at some point p, then $p \in F(T)$ and $\{x_n\}$ converges strongly to p.*

Proof. $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ which converges strongly to $p \in K$. It follows from Lemma 2 that $\{x_n\}$ is bounded. Since $\|Tx_n - x^*\| \le L\|x_n - x^*\|$, $\forall x^* \in F(T)$, then $\{Tx_n\}$ is bounded. Consequently, $\|Tx_n - p\| \le D$, $\forall n \ge 1$ and for some $D > 0$. Observe that

$$\begin{aligned} 0 \le \|y_{n_j} - p\| &\le \left(1 - \beta_{n_j}\right)\|x_{n_j} - p\| + \beta_{n_j}\|Tx_{n_j} - p\| \\ &\le \|x_{n_j} - p\| + \beta_{n_j}\|Tx_{n_j} - p\| \\ &\le \|x_{n_j} - p\| + \beta_{n_j}D \to 0 \qquad \text{as } j\to\infty, \end{aligned}$$

so that $y_{n_j} \to p$ as $j \to \infty$. Since T is continuous at p, then $Ty_{n_j} \to Tp$ as $j \to \infty$. Hence $\lim_j \|y_{n_j} - Ty_{n_j}\| = \|p - Tp\| = 0$, so that $p \in F(T)$. From Lemma 2, $\lim \|x_n - p\|$ exists, and since $\lim_j \|x_{n_j} - p\| = 0$, we have $\lim \|x_n - p\| = 0$, completing the proof of Corollary 1. ∎

Remark 2. In view of Corollary 1, we can conclude that if K is also closed in Lemma 2, then either $\{x_n\}$ converges strongly to a fixed point of T or $\{x_n\}$ has no subsequence which converges strongly. In particular, if K is compact, then $\{x_n\}$ converges strongly to a fixed point of T.

COROLLARY 2. *Let E be a real q-uniformly smooth Banach space and K a nonempty closed convex subset of E. Let $T: K \to K$ be a demicompact strictly pseudocontractive map with a nonempty fixed point-set. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{x_n\}$ be as in Lemma 2. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Since $\{y_n\}$ is bounded and $\{y_n - Ty_n\}$ converges strongly, then there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ which converges strongly to some point $p \in K$. The continuity of T and $\lim \|y_n - Ty_n\| = 0$ implies $p \in F(T)$. Furthermore, $y_{n_j} - p = x_{n_j} - p + \beta_{n_j}(Tx_{n_j} - x_{n_j})$, so that

$$\begin{aligned} \|x_{n_j} - p\| &\leq \|y_{n_j} - p\| + \beta_{n_j}\|Tx_{n_j} - x_{n_j}\| \\ &\leq \|y_{n_j} - p\| + \beta_{n_j} M(1 + L) \to 0 \qquad \text{as } j \to \infty. \end{aligned}$$

Hence $\{x_{n_j}\}$ converges strongly to $p \in F(T)$, and it now follows from Lemma 2 (or Remark 2) that $\{x_n\}$ converges strongly to $p \in F(T)$, completing the proof of Corollary 2. ∎

LEMMA 3. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty convex subset of E and let $T: K \to K$ be a strictly pseudocontractive map with a nonempty fixed-point set $F(T)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{x_n\}$ be as in Lemma 2. Then for all $p_1, p_2 \in F(T)$, the limit*

$$\lim_{n \to \infty} \|tx_n + (1 - t)p_1 - p_2\|$$

exists for all $t \in [0, 1]$.

Proof. Let $a_n(t) := \|tx_n + (1 - t)p_1 - p_2\|$. Then $\lim_{n \to \infty} a_n(0) = \|p_1 - p_2\|$, and from Lemma 2, $\lim_{n \to \infty} a_n(1) = \lim_{n \to \infty} \|x_n - p_2\|$ exists. It now remains to prove the lemma for $t \in (0, 1)$. Let T_n be as in Lemma 1. Then

$$\|T_n x - T_n y\| \leq [1 + \delta_n]\|x - y\| = k_n\|x - y\|, \qquad \forall x, y \in K,$$

where $k_n = 1 + \delta_n$. Since $\sum_{n=1}^{\infty} \delta_n < \infty$, then $\prod_{n=1}^{\infty} k_n < \infty$. Set

$$S_{n,m} := T_{n+m-1} T_{n+m-2} \ldots T_n, \qquad m \geq 1.$$

Then

$$\|S_{n,m}x - S_{n,m}y\| \le \left(\prod_{j=n}^{n+m-1} k_j\right)\|x-y\|,\ \forall x, y \in K; \qquad S_{n,m}x_n = x_{n+m};$$

$$\text{and} \qquad S_{n,m}p = p,\ \forall p \in F(T).$$

Set $b_{n,m} := \|S_{n,m}(tx_n + (1-t)p_1) - tS_{n,m}x_n - (1-t)S_{n,m}p_1\|$; $D := (\prod_{J=1}^{\infty} k_j)^2\|x_1 - p_1\|$. Let δ denote the *modulus of convexity* of E. We prove that

$$\frac{D}{2}\delta\left(\frac{4}{D}b_{n,m}\right) \le \left(\prod_{j=n}^{n+m-1} k_j\right)\|x_n - p_1\| - \|x_{n+m} - p_1\|. \tag{19}$$

If $\|x_n - p_1\| = 0$ for some n_0, then $x_n = p_1$, $\forall n \ge n_0$ so that clearly (19) holds and in fact $\{x_n\}$ converges strongly to $p_1 \in F(T)$. Thus we may assume $\|x_n - p_1\| > 0$, $\forall n \ge 1$. It is well known (see, for example, 2, p. 108]) that

$$\begin{aligned}\|tx + (1-t)y\| &\le 1 - 2\min\{t, (1-t)\}\,\delta(\|x-y\|)\\ &\le 1 - 2t(1-t)\,\delta(\|x-y\|)\end{aligned} \tag{20}$$

for all $t \in [0,1]$ and for all $x, y \in E$ such that $\|x\| \le 1$, $\|y\| \le 1$. Set

$$w_{n,m} := \frac{S_{n,m}p_1 - S_{n,m}(tx_n + (1-t)p_1)}{t\left(\prod_{j=n}^{n+m-1}k_j\right)\|x_n - p_1\|},$$

$$z_{n,m} := \frac{S_{n,m}(tx_n + (1-t)p_1) - S_{n,m}x_n}{(1-t)\left(\prod_{j=n}^{n+m-1}k_j\right)\|x_n - p_1\|}.$$

Then $\|w_{n,m}\| \le 1$ and $\|z_{n,m}\| \le 1$ so that it follows from (20) that

$$2t(1-t)\,\delta(\|w_{n,m} - z_{n,m}\|) \le 1 - \|tw_{n,m} + (1-t)z_{n,m}\|. \tag{21}$$

Observe that

$$\|w_{n,m} - z_{n,m}\| = \frac{b_{n,m}}{t(1-t)\left(\prod_{j=n}^{n+m-1}k_j\right)\|x_n - p_1\|} \qquad \text{and}$$

$$\|tw_{n,m} + (1-t)z_{n,m}\| = \frac{\|S_{n,m}x_n - S_{n,m}p_1\|}{\left(\prod_{j=n}^{n+m-1}k_j\right)\|x_n - p_1\|},$$

so that it follows from (21) that

$$2t(1-t)\left(\prod_{j=n}^{n+m-1} k_j\right)\|x_n - p_1\|\delta\left(\frac{b_{n,m}}{t(1-t)\left(\prod_{j=n}^{n+m-1}k_j\right)\|x_n - p_1\|}\right)$$
$$\le \left(\prod_{j=n}^{n+m-1} k_j\right)\|x_n - p_1\| - \|S_{n,m}x_n - S_{n,m}p_1\|. \tag{22}$$

Observe that

$$t(1-t)\left(\prod_{j=n}^{n+m-1} k_j\right)\|x_n - p_1\| \le \frac{1}{4}\left(\prod_{j=1}^{\infty} k_j\right)^2 \|x_1 - p_1\| = \frac{D}{4}$$
$$\left(\text{since } t(1-t) \le \frac{1}{4}, \forall t \in [0,1]\right).$$

Since E is uniformly convex, then $\frac{\delta(s)}{s}$ is nondecreasing and hence it follows from (22) that

$$\frac{D}{2}\delta\left(\frac{4}{D}b_{n,m}\right) \le \left(\prod_{j=n}^{n+m-1} k_j\right)\|x_n - p_1\| - \|S_{n,m}x_n - S_{n,m}p_1\|$$
$$= \left(\prod_{j=n}^{n+m-1} k_j\right)\|x_n - p_1\| - \|x_{n+m} - p_1\|,$$

establishing (19). From Lemma 2, $\lim_{n\to\infty}\|x_n - p_1\|$ exists and hence $\lim_{n\to\infty}\|x_n - p_1\| = \lim_{n\to\infty}\|x_{n+m} - p_1\|$. Since $\delta(0) = 0$ and $\lim_{n\to\infty}\prod_{j=n}^{\infty}k_j = 1$, then the continuity of δ yields $\lim_{n\to\infty}b_{n,m} = 0$ uniformly for all m. Observe that

$$\begin{aligned}
a_{n+m}(t) &\le \big\| tx_{n+m} + (1-t)p_1 - p_2 \\
&\qquad + \left(S_{n,m}(tx_n + (1-t)p_1) - tS_{n,m}x_n - (1-t)S_{n,m}p_1\right)\big\| \\
&\qquad + \big\| -\left(S_{n,m}(tx_n + (1-t)p_1) - tS_{n,m}x_n - (1-t)S_{n,m}p_1\right)\big\| \\
&= \|S_{n,m}(tx_n + (1-t)p_1) - p_2\| + b_{n,m} \\
&= \|S_{n,m}(tx_n + (1-t)p_1) - S_{n,m}p_2\| + b_{n,m} \\
&\le \left(\prod_{j=n}^{n+m-1} k_j\right)\|tx_n + (1-t)p_1 - p_2\| + b_{n,m} \\
&= \left(\prod_{j=n}^{n+m-1} k_j\right)a_n(t) + b_{n,m}.
\end{aligned}$$

Hence $\limsup_{n\to\infty} a_n(t) \le \liminf_{n\to\infty} a_n(t)$, completing the proof of Lemma 3. ∎

LEMMA 4. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty convex subset of E and let $T\colon K \to K$ be a strictly pseudocontractive map with a nonempty fixed-point set $F(T)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{x_n\}$ be as in Lemma 2. Then for all $p_1, p_2 \in F(T)$, $\lim_{n\to\infty}\langle x_n, j(p_1 - p_2)\rangle$ exists. Furthermore if $\omega_w(x_n)$ denotes the set of weak subsequential limits of $\{x_n\}$, then $\langle p - q, j(p_1 - p_2)\rangle = 0$, $\forall p_1, p_2 \in F(T)$, and $\forall p, q \in \omega_w(x_n)$.*

Proof. Since E is both uniformly convex and uniformly smooth, it has a Fréchet differentiable norm. Set $x = p_1 - p_2$ and $h = t(x_n - p_1)$ in (7) to obtain

$$\begin{aligned}
&\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle \\
&\qquad \le \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \\
&\qquad \le \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle + b(t\|x_n - p_1\|).
\end{aligned}$$

Since b is increasing and $\|x_n - p_1\| \le M$, $\forall n \ge 1$ and for some $M > 0$, then

$$\begin{aligned}
&\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle \\
&\qquad \le \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \\
&\qquad \le \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle + b(tM).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{2}\|p_1 - p_2\|^2 + t\limsup_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle \\
&\qquad \le \frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|^2 \\
&\qquad \le \frac{1}{2}\|p_1 - p_2\|^2 + t\liminf_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle + b(tM).
\end{aligned}$$

Hence $\limsup_{n\to\infty}\langle x_n, j(p_1 - p_2)\rangle \le \liminf_{n\to\infty}\langle x_n, j(p_1 - p_2)\rangle + \frac{b(tM)}{t}$. Since $\lim_{t\to 0^+}\frac{b(t)}{t} = 0$, then $\lim_{n\to\infty}\langle x_n, j(p_1 - p_2)\rangle$ exists. Since $\lim_{n\to\infty}\langle x_n, j(p_1 - p_2)\rangle = \langle p, j(p_1 - p_2)\rangle$, $\forall p \in \omega_w(x_n)$, we have $\langle p - q, j(p_1 - p_2)\rangle = 0$, $\forall p_1, p_2 \in F(T)$ and $\forall p, q \in \omega_w(x_n)$. ▮

THEOREM 1. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and T: $K \to K$ a strictly pseudocontractive map. Then $(I - T)$ is demiclosed at zero.*

Proof. Let $\{x_n\}$ be a sequence in K such that $\{x_n\}$ converges weakly to p and $\{(I - T)(x_n)\}$ converges strongly to 0. Let α and T_α be as in Remark 1. Then $(I - T_\alpha)(x_n) = \alpha(I - T)(x_n)$, so that $\{(I - T_\alpha)(x_n)\}$ converges strongly to 0. Since T_α is nonexpansive, it follows from Theorem GK that $(I - T\alpha)$ is demiclosed at 0, so that $(I - T_\alpha)(p) = 0$. Consequently, $(I - T)(p) = 0$, completing the proof of Theorem 1.

THEOREM 2. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and let T: $K \to K$ be a strictly pseudocontractive map with a nonempty fixed-point set $F(T)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{x_n\}$ be as in Lemma 2. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Since $\{x_n\}$ is bounded, it has a weakly convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$. Suppose $\{x_{n_j}\}$ converges weakly to p. Then $p \in K$ because K is weakly closed.

Since $\|Tx_n - x^*\| \le L\|x_n - x^*\| \le LM$, $\forall x^* \in F(T)$, $n \ge 1$, and for some $M > 0$, it follows that $\{Tx_n\}$ is bounded. Let $f \in E^*$ be arbitrary. Then

$$\begin{aligned}
0 &\le \left| f(y_{n_j}) - f(p) \right| \\
&= \left| \left(1 - \beta_{n_j}\right)\left[f(x_{n_j}) - f(p)\right] + \beta_{n_j}\left[f\left(Tx_{n_j}\right) - f(p)\right] \right| \\
&\le \left| f(x_{n_j}) - f(p) \right| + \beta_{n_j}\|f\|\,\|Tx_{n_j} - p\| \to 0 \qquad \text{as } n \to \infty,
\end{aligned}$$

so that $\{y_{n_j}\}$ converges weakly to p. Since $\lim_{n\to\infty}\|y_n - Ty_n\| = 0$ and $(I - T)$ is demiclosed at zero, we must have $p - Tp = 0$, so that $p \in F(T)$. If $\{x_{m_k}\}$ is another subsequence of $\{x_n\}$ which converges weakly to some q. Then as for p, we must have $q \in K$ and $q \in F(T)$, and it follows from Lemma 4 that $p = q$. Hence $\omega_w(x_n)$ is singleton, so that $\{x_n\}$ converges weakly to a fixed point of T. ▮

Remark 3. If we set $\beta_n = 0$, $\forall n \ge 1$ in our lemmas, corollaries, and theorems, we obtain the corresponding results for the Mann iteration method.

Remark 4. Hilbert spaces are 2-uniformly smooth and satisfy (4) with $c_q = 1$. If we set $q = 2$, $c_q = 1$, and $\beta_n = 0$, $\forall n \geq 1$ in Lemma 1, then Theorem R follows from Remark 2. Furthermore, Theorem BP follows from Corollary 2 and Theorem 2 by setting $q = 2$, $c_q = 1$, $\beta_n = 0$, $\alpha_n = \alpha \in (0, 1 - k)$, $\forall n \geq 1$.

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