

On uniform approximation by some classical Bernstein-type operators [☆]

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Abstract

We investigate the functions for which certain classical families of operators of probabilistic type over noncompact intervals provide uniform approximation on the whole interval. The discussed examples include the Szász operators, the Szász–Durrmeyer operators, the gamma operators, the Baskakov operators, and the Meyer–König and Zeller operators. We show that some results of Totik remain valid for unbounded functions, at the same time that we give simple rates of convergence in terms of the usual modulus of continuity. We also show by a counterexample that the result for Meyer–König and Zeller operators does not extend to Cheney and Sharma operators.

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1. Introduction

The classical Szász–Mirakyan operator S_t ($t > 0$) over the interval $[0, \infty)$ is defined by

$$S_t f(x) := \sum_{k=0}^{\infty} f\left(\frac{k}{t}\right) \pi_k(tx), \quad x \geq 0, \quad f \in \mathcal{S}, \quad (1)$$

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where

$$\pi_k(u) := e^{-u} \frac{u^k}{k!}, \quad u \geq 0, \quad k = 0, 1, \dots, \quad (2)$$

and \mathcal{S} is the set of all real functions on $[0, \infty)$ such that the right-hand side in (1) makes sense for all $t > 0$ and $x \geq 0$. In particular, \mathcal{S} contains all the real functions on $[0, \infty)$ which are bounded or uniformly continuous.

The approximation properties of S_t have been extensively investigated in the literature. It is well known that

$$|S_t f(x) - f(x)| \leq (1 + \sqrt{x}) \omega\left(f; \frac{1}{\sqrt{t}}\right), \quad t > 0, \quad x \geq 0, \quad f \in \mathcal{S},$$

where $\omega(f; \cdot)$ stands for the usual modulus of continuity of f . This implies that, when f is uniformly continuous, $S_t f$ converges to f uniformly on each subinterval $[0, a]$ as $t \rightarrow \infty$. An interesting problem is to determine the class of all continuous functions $f \in \mathcal{S}$ such that $S_t f$ converges to f uniformly on the whole interval $[0, \infty)$ as $t \rightarrow \infty$. A major step in this direction is the following result of Totik, which is actually a part of [7, Theorem 1] (see also [9]).

Theorem A. *Let f be a real continuous and bounded function on $[0, \infty)$. Then, $S_t f$ converges to f uniformly on $[0, \infty)$, as $t \rightarrow \infty$, if and only if the function*

$$f^*(x) := f(x^2), \quad x \geq 0, \quad (3)$$

is uniformly continuous.

It is natural to ask what happens if the boundedness assumption on the functions is dropped. Obviously, part *only if* is no longer true, because we have $S_t e_1 = e_1$ ($t > 0$), where e_1 is the monomial $e_1(x) := x$. In the present paper, we however show that part *if* continues to hold, at the same time that we give rates of convergence in terms of the usual modulus of continuity. More precisely, we show the following.

Theorem 1. *Let $f \in \mathcal{S}$ and let f^* be defined as in (3). We have, for all $t > 0$ and $x \geq 0$,*

$$|S_t f(x) - f(x)| \leq 2\omega\left(f^*; \frac{1}{\sqrt{t}}\right).$$

Therefore, $S_t f$ converges to f uniformly on $[0, \infty)$ as $t \rightarrow \infty$, whenever f^ is uniformly continuous.*

The proof of Theorem 1 is given in Section 3. In Sections 4–7, we deal with the same problem, and establish analogous results, for other classical operators over noncompact intervals, namely, the Szász–Durrmeyer, the gamma, the Baskakov, and the Meyer–König and Zeller operators. Finally, in Section 8, we show by a counterexample that the convergence result in Section 7 for Meyer–König and Zeller operators does not extend to Cheney and Sharma operators.

All the mentioned operators are well-known examples of operators of probabilistic type (also called Bernstein-type operators), and throughout the paper we use probabilistic

tools, in particular, the representations of the operators in terms of appropriate stochastic processes, as they are given in [1–3]. Theorem 1 and the results in Sections 4–7 can actually be viewed as particular versions of the general theorem stated in Section 2.

2. A general result

Let I be an interval of the real line and let $L := \{L_t: t > 0\}$ be a family of positive linear operators over I having the form

$$L_t f(x) = E f(Z_t(x)), \quad t > 0, x \in I, f \in \mathcal{L}, \quad (4)$$

where (here and hereafter) E denotes mathematical expectation, $\{Z_t(x): t > 0, x \in I\}$ is a double-indexed integrable stochastic process taking values in I , and \mathcal{L} stands for the domain of L , i.e., the set of all real functions defined on I for which the right-hand side in (4) makes sense, for all $t > 0$ and $x \in I$, i.e., such that

$$E|f(Z_t(x))| < \infty, \quad t > 0, x \in I.$$

Remark. The integrability assumption

$$E|Z_t(x)| < \infty, \quad t > 0, x \in I,$$

is equivalent to saying that \mathcal{L} contains the monomial $e_1(x) := x$, and it also guarantees that \mathcal{L} contains all the real (measurable) functions f on I such that $\omega(f; 1) < \infty$. To see this, let $a \in I$ be a fixed point. Then, for all $t > 0$ and $x \in I$, we have

$$\begin{aligned} |f(Z_t(x))| &\leq |f(a)| + |f(Z_t(x)) - f(a)| \leq |f(a)| + \omega(f; |Z_t(x) - a|) \\ &\leq |f(a)| + (1 + |a| + |Z_t(x)|)\omega(f; 1), \end{aligned}$$

and we conclude that

$$E|f(Z_t(x))| \leq |f(a)| + (1 + |a| + E|Z_t(x)|)\omega(f; 1) < \infty.$$

Note also that the condition $\omega(f; 1) < \infty$ is fulfilled, if f is bounded or uniformly continuous.

For this family of operators, we assert the following.

Theorem 2. Let $f \in \mathcal{L}$, let φ be a one-to-one monotone function from I onto the interval J , and let f^* be the function defined by

$$f^*(x) := f(\varphi^{-1}(x)), \quad x \in J. \quad (5)$$

We have, for all $t > 0$ and $x \in I$,

$$|L_t f(x) - f(x)| \leq 2\omega(f^*; EY_t(x)),$$

where $Y_t(x)$ is the random variable given by

$$Y_t(x) := |\varphi(Z_t(x)) - \varphi(x)|. \quad (6)$$

Therefore, we have $L_t f - f = o(1)$ ($t \rightarrow \infty$) uniformly in I , whenever f^* is uniformly continuous and $\sup_{x \in I} EY_t(x) = o(1)$ ($t \rightarrow \infty$).

Proof. Let $t > 0$, $x \in I$, and $f \in \mathcal{L}$ be fixed. We have from (5)

$$f(z) = f^*(\varphi(z)), \quad z \in I,$$

and, therefore, by (4)

$$L_t f(x) = E f^*(\varphi(Z_t(x))).$$

Thus, we can write

$$|L_t f(x) - f(x)| \leq E |f^*(\varphi(Z_t(x))) - f^*(\varphi(x))| \leq E \omega(f^*; Y_t(x)),$$

where $Y_t(x)$ is the same as in (6). Finally, from the inequality

$$\omega(f^*; \alpha\delta) \leq (1 + \alpha)\omega(f^*; \delta), \quad \alpha, \delta \geq 0,$$

we obtain

$$E \omega(f^*; Y_t(x)) \leq 2 \omega(f^*; E Y_t(x)),$$

completing the proof of Theorem 2. \square

3. Proof of Theorem 1

The Szász operators in (1) can be represented in the following way

$$S_t f(x) = E f\left(\frac{N(tx)}{t}\right), \quad t > 0, \quad x \geq 0, \quad f \in \mathcal{S},$$

where $N := \{N(u) : u \geq 0\}$ is a standard Poisson process, i.e., a stochastic process starting at 0, having independent stationary increments, and such that, for each $u \geq 0$, $N(u)$ has the Poisson distribution given by

$$P(N(u) = k) = \pi_k(u), \quad k = 0, 1, \dots,$$

where $\pi_k(u)$ is the same as in (2). Thus, according to Theorem 2, we only need to show that we have, for all $t > 0$ and $x > 0$,

$$E Y_t(x) \leq \frac{1}{\sqrt{t}}, \tag{7}$$

where

$$Y_t(x) := \left| \sqrt{\frac{N(tx)}{t}} - \sqrt{x} \right| = \frac{|N(tx)/t - x|}{\sqrt{N(tx)/t} + \sqrt{x}} \leq \frac{1}{\sqrt{x}} \left| \frac{N(tx)}{t} - x \right|. \tag{8}$$

Since the expectation and the variance of $N(tx)$ are, respectively, given by

$$E N(tx) = tx, \quad \text{var}(N(tx)) = tx,$$

we obtain from (8) and the Cauchy–Schwarz inequality

$$E Y_t(x) \leq \frac{1}{\sqrt{x}} E \left| \frac{N(tx)}{t} - x \right| \leq \frac{1}{\sqrt{x}} \sqrt{\text{var}\left(\frac{N(tx)}{t}\right)} = \frac{1}{\sqrt{t}},$$

showing (7), and completing the proof. \square

4. Szász–Durrmeyer operators

For $t > 0$, Mazhar and Totik [6] introduced the following two Durrmeyer-type modifications of the operator S_t

$$L_t f(x) := t \sum_{k=0}^{\infty} \pi_k(tx) \int_0^{\infty} \pi_k(tu) f(u) du, \quad f \in \mathcal{L}, \quad (9)$$

and

$$L_t^{\circ} f(x) := f(0)\pi_0(tx) + t \sum_{k=1}^{\infty} \pi_k(tx) \int_0^{\infty} \pi_{k-1}(tu) f(u) du, \quad f \in \mathcal{L}^{\circ}, \quad (10)$$

where $x \geq 0$, $\pi_k(\cdot)$ is the same as in (2), and $f \in \mathcal{L}$ (respectively, \mathcal{L}°) := the set of all real functions on $[0, \infty)$ such that the right-hand side of (9) (respectively, (10)) makes sense for all $t > 0$ and $x \geq 0$.

These operators can be represented in the following way (cf. [1])

$$L_t f(x) = Ef\left(\frac{U_{N(tx)+1}}{t}\right), \quad L_t^{\circ} f(x) = Ef\left(\frac{U_{N(tx)}}{t}\right),$$

where $U := \{U_r: r \geq 0\}$ is a standard gamma process, i.e., a stochastic process starting at 0, having independent stationary increments, and such that, for each $r > 0$, U_r has the gamma distribution with density

$$g_r(u) := \frac{u^{r-1} e^{-u}}{\Gamma(r)} 1_{(0, \infty)}(u),$$

and $N := \{N(u): u \geq 0\}$ is a standard Poisson process independent of U and defined on the same probability space.

Mazhar and Totik [6, Theorems 4 and 7] showed that Theorem A also holds when S and S_t are, respectively, replaced by \mathcal{L} and L_t , or by \mathcal{L}° and L_t° . Here, we establish the following.

Theorem 3. *Let $f \in \mathcal{L}$ (respectively, $f \in \mathcal{L}^{\circ}$) and let f^* be defined as in (3). We have, for all $t > 0$ and $x \geq 0$,*

$$|L_t f(x) - f(x)| \leq 2\omega\left(f^*; \frac{3}{\sqrt{t}}\right), \quad (11)$$

respectively,

$$|L_t^{\circ} f(x) - f(x)| \leq 2\omega\left(f^*; \frac{1}{\sqrt{t}}\right). \quad (12)$$

Therefore, $L_t f$ (respectively, $L_t^{\circ} f$) converges to f uniformly on $[0, \infty)$ as $t \rightarrow \infty$, whenever f^* is uniformly continuous.

Proof. Let $t > 0$ and $x \geq 0$. As in the proof of Theorem 2, to show (11), we only need to check that

$$E Y_t(x) \leq \frac{3}{\sqrt{t}},$$

where

$$Y_t(x) := \left| \sqrt{\frac{U_{N(tx)+1}}{t}} - \sqrt{x} \right| \leq \left| \sqrt{\frac{U_{N(tx)+1}}{t}} - \sqrt{\frac{tx+1}{t}} \right| + \frac{1}{\sqrt{t}}.$$

Since

$$E U_{N(tx)+1} = tx + 1 \quad \text{and} \quad \text{var}(U_{N(tx)+1}) = 2tx + 1,$$

we obtain

$$E Y_t(x) \leq \frac{1}{\sqrt{t}} + \sqrt{\frac{t}{tx+1}} \sqrt{\text{var}\left(\frac{U_{N(tx)+1}}{t}\right)} \leq \frac{3}{\sqrt{t}},$$

showing the claim. The proof of (12) is analogous, and we omit the details. \square

The following result actually shows that both L_t and L_t° provide uniform convergence for the same functions.

Theorem 4. Let f be a real uniformly continuous function on $[0, \infty)$. For $t > 0$ and $x \geq 0$, we have

$$|L_t f(x) - L_t^\circ f(x)| \leq 2\omega\left(f; \frac{1}{t}\right).$$

Proof. Let $t > 0$ and $x \geq 0$. An analogous argument to that used in the proof of Theorem 2 yields

$$|L_t f(x) - L_t^\circ f(x)| \leq 2\omega(f; E Z_t(x)),$$

where

$$Z_t(x) := \frac{|U_{N(tx)+1} - U_{N(tx)}|}{t}.$$

Since $U_{N(tx)+1} - U_{N(tx)}$ has the same probability distribution as U_1 , we have $E Z_t(x) = 1/t$, and the conclusion follows. \square

5. Gamma operators

For $t > 0$, the gamma operator G_t over the interval $(0, \infty)$ is the integral operator given by

$$G_t f(x) := \frac{1}{\Gamma(t)} \int_0^\infty f\left(\frac{x\theta}{t}\right) \theta^{t-1} e^{-\theta} d\theta = E f\left(\frac{x U_t}{t}\right), \quad (13)$$

where $x > 0$, $U := \{U_t: t \geq 0\}$ is a standard gamma process, and $f \in \mathcal{G} :=$ the set of all real functions on $(0, \infty)$ such that the right-hand side of (13) makes sense, for all $t, x > 0$.

For these operators, we assert the following

Theorem 5. Let $f \in \mathcal{G}$, and let f^* be the function on $(-\infty, \infty)$ defined by

$$f^*(x) := f(e^x), \quad -\infty < x < \infty. \quad (14)$$

We have, for all $t > 2$ and $x > 0$,

$$|G_t f(x) - f(x)| \leq 2\omega\left(f^*; \frac{2}{\sqrt{t-2}}\right).$$

Therefore, we have $G_t f - f = o(1)$ ($t \rightarrow \infty$) uniformly in $(0, \infty)$, whenever f^* is uniformly continuous.

Proof. According to Theorem 2, it suffices to show that we have, for all $t > 2$,

$$E Y_t \leq \frac{2}{\sqrt{t-2}},$$

where

$$Y_t := \left| \log \frac{x U_t}{t} - \log x \right| = \left| \log \frac{U_t}{t} \right|.$$

This claim is a consequence of the following two lemmas. \square

Lemma 1. If V is a positive random variable, then

$$E|\log V| \leq [E(V-1)^2 + E(V^{-1}-1)^2]^{1/2}.$$

Proof. The inequality

$$\log x \leq x - 1, \quad x \geq 1, \quad (15)$$

leads to

$$|\log V|^2 \leq (V-1)^2 + (V^{-1}-1)^2,$$

and the conclusion follows from this and the fact that, by the Cauchy–Schwarz inequality, we have

$$E|\log V| \leq (E|\log V|^2)^{1/2}. \quad \square$$

Lemma 2. We have

$$E\left[\frac{U_t}{t} - 1\right]^2 = \frac{1}{t}, \quad t > 0,$$

and

$$E\left[\frac{t}{U_t} - 1\right]^2 = \frac{t+2}{(t-1)(t-2)}, \quad t > 2.$$

Proof. The conclusions readily follow from the facts that, for $t > 0$,

$$EU_t = t = \text{var}(U_t), \quad t > 0,$$

and, for $t > 2$,

$$E(U_t^{-1}) = \frac{1}{t-1}, \quad \text{var}(U_t^{-1}) = \frac{1}{(t-1)^2(t-2)}. \quad \square$$

Another family $(G_t^*)_{t>0}$ of gamma-type operators over the interval $(0, \infty)$, which is substantially different from $(G_t)_{t>0}$, is the one defined by

$$G_t^* f(x) := \frac{1}{\Gamma(t+1)} \int_0^\infty f\left(\frac{xt}{\theta}\right) \theta^t e^{-\theta} d\theta = Ef\left(\frac{xt}{U_{t+1}}\right), \quad (16)$$

where x and U_t are the same as above, and $f \in \mathcal{G}^* :=$ the set of all real functions on $(0, \infty)$ such that the right-hand side of (16) makes sense for all $t, x > 0$.

These operators have been considered in [5,9]. Here, we state the following theorem which extends a result of Totik in [9, p. 178]. The proof mimics that of Theorem 5 and is therefore omitted.

Theorem 6. Let $f \in \mathcal{G}^*$ and let f^* be the same as in (14). We have, for all $t > 1$ and $x > 0$,

$$|G_t^* f(x) - f(x)| \leq 2\omega\left(f^*; \sqrt{\frac{3}{t-1}}\right).$$

Therefore, we have $G_t^* f - f = o(1)$ ($t \rightarrow \infty$) uniformly in $(0, \infty)$, whenever f^* is uniformly continuous.

6. Baskakov operators

For $t > 0$, the Baskakov operator B_t over the interval $[0, \infty)$ is given by

$$B_t f(x) := \sum_{k=0}^{\infty} f\left(\frac{k}{t}\right) \binom{t+k-1}{k} \frac{x^k}{(1+x)^{t+k}} = Ef\left(\frac{N(xU_t)}{t}\right), \quad (17)$$

where $x \geq 0$, $N := \{N(u) : u \geq 0\}$ and $U := \{U_t : t \geq 0\}$ are the same as in Section 4, and $f \in \mathcal{B} :=$ the set of all real functions on $[0, \infty)$ such that the right-hand side of (17) makes sense for all $t > 0$ and $x \geq 0$.

The following result is related to [8, Theorem 1].

Theorem 7. Let $f \in \mathcal{B}$ and let f^* be defined by

$$f^*(x) := f(e^x - 1), \quad x \geq 0. \quad (18)$$

We have, for all $t > 2$ and $x \geq 0$,

$$|B_t f(x) - f(x)| \leq 2\omega\left(f^*; \sqrt{\frac{6}{t-2} \frac{x}{1+x}}\right).$$

Therefore, we have $B_t f - f = o(1)$ ($t \rightarrow \infty$) uniformly in $[0, \infty)$, whenever f^* is uniformly continuous.

Proof. By Theorem 2, we only need to show that, for $t > 2$ and $x \geq 0$, we have

$$E Y_t(x) \leq \sqrt{\frac{6}{t-2} \frac{x}{1+x}},$$

where

$$Y_t(x) := \left| \log \left(1 + \frac{N(xU_t)}{t} \right) - \log(1+x) \right| = \left| \log \left[\frac{1}{1+x} \left(1 + \frac{N(xU_t)}{t} \right) \right] \right|.$$

This fact directly follows from Lemma 1 and Lemma 4 below. The proof of Lemma 4 is based upon the following auxiliary result which can be found in [3]. \square

Lemma 3. Let $t > 0$ and let Z be a nonnegative integer-valued random variable. Then, for $k = 1, 2, \dots$,

$$E \left(\frac{t}{t+Z} \right)^k = \int_0^1 g(s) \frac{t^k}{(k-1)!} s^{t-1} \log^{k-1} \left(\frac{1}{s} \right) ds,$$

where $g(\cdot)$ is the probability generating function of Z , i.e.,

$$g(s) := E s^Z, \quad 0 < s < 1.$$

Lemma 4. Let $x \geq 0$ and let $N(xU_t)$ be the same as in (17). We have, for $t > 0$,

$$E \left[\frac{1}{1+x} \frac{t + N(xU_t)}{t} - 1 \right]^2 = \frac{1}{t} \frac{x}{1+x} \quad (19)$$

and, for $t > 2$,

$$E \left[(1+x) \frac{t}{t + N(xU_t)} - 1 \right]^2 \leq \frac{5}{t-2} \frac{x}{1+x}. \quad (20)$$

Proof. Let $x \geq 0$ be fixed. For $t > 0$, we have

$$E N(xU_t) = tx \quad \text{and} \quad \text{var}(N(xU_t)) = tx(1+x),$$

and, therefore,

$$E \left[\frac{1}{1+x} \frac{t + N(xU_t)}{t} - 1 \right]^2 = \text{var} \left[\frac{t + N(xU_t)}{(1+x)t} \right] = \frac{tx(1+x)}{(1+x)^2 t^2} = \frac{1}{t} \frac{x}{1+x},$$

showing (19). To show (20), fix $t > 2$. Since the probability generating function $g(\cdot)$ of $N(xU_t)$ is given by

$$g(s) := \left(\frac{1-y}{1-ys} \right)^t, \quad 0 < s < 1,$$

where $y := x/(1+x)$, we have from Lemma 3

$$E\left[\frac{1}{1-y} \frac{t}{t+N(xU_t)}\right] = \frac{1}{1-y} \int_0^1 t s^{t-1} \left(\frac{1-y}{1-ys}\right)^t ds$$

and

$$E\left[\frac{1}{1-y} \frac{t}{t+N(xU_t)}\right]^2 = \frac{1}{(1-y)^2} \int_0^1 t^2 s^{t-1} \log\left(\frac{1}{s}\right) \left(\frac{1-y}{1-ys}\right)^t ds.$$

The change of variable

$$\frac{1-s}{1-ys} = u$$

and several routine integrations by parts lead to

$$E\left[\frac{1}{1-y} \frac{t}{t+N(xU_t)}\right] = 1 + y \int_0^1 \frac{(1-u)^t}{(1-yu)^2} du$$

and

$$\begin{aligned} E\left[\frac{1}{1-y} \frac{t}{t+N(xU_t)}\right]^2 &= 1 + y \left(2 + \frac{t}{t+1}\right) \int_0^1 \frac{(1-u)^t}{(1-yu)^3} du \\ &\quad + \frac{2y^2}{1-y} \frac{t}{t+1} \int_0^1 \frac{(1-u)^{t+1}}{(1-yu)^3} \log \frac{1-yu}{1-u} du. \end{aligned}$$

From these equalities, we obtain

$$\begin{aligned} E\left[\frac{1}{1-y} \frac{t}{t+N(xU_t)} - 1\right]^2 &= \frac{yt}{t+1} \int_0^1 \frac{(1-u)^t}{(1-yu)^3} du + 2y^2 \int_0^1 \frac{u(1-u)^t}{(1-yu)^3} du \\ &\quad + \frac{2y^2}{1-y} \frac{t}{t+1} \int_0^1 \frac{(1-u)^{t+1}}{(1-yu)^3} \log \frac{1-yu}{1-u} du, \end{aligned}$$

and (20) readily follows by using the estimates

$$\begin{aligned} \int_0^1 \frac{(1-u)^t}{(1-yu)^3} du &\leq \int_0^1 (1-u)^{t-3} du = \frac{1}{t-2}, \\ \int_0^1 \frac{u(1-u)^t}{(1-yu)^3} du &\leq \int_0^1 u(1-u)^{t-3} du = \frac{1}{(t-1)(t-2)}, \end{aligned}$$

and

$$\int_0^1 \frac{(1-u)^{t+1}}{(1-yu)^3} \log \frac{1-yu}{1-u} du \leq \int_0^1 \frac{(1-u)^{t+1}}{(1-yu)^3} \frac{(1-y)u}{1-u} du \leq \frac{1-y}{(t-1)(t-2)},$$

where we have used (15). The proof of Lemma 4 is complete. \square

7. Meyer–König and Zeller operators

For $t > 0$, the Meyer–König and Zeller operator M_t over $[0, 1)$ (as modified by Cheney and Sharma) is defined by

$$M_t f(x) := (1-x)^{t+1} \sum_{k=0}^{\infty} f\left(\frac{k}{t+k}\right) \binom{t+k}{k} x^k = Ef\left(\frac{N(q(x)U_{t+1})}{t+N(q(x)U_{t+1})}\right), \quad (21)$$

where $x \in [0, 1)$, $q(x)$ is given by

$$q(x) := \frac{x}{1-x}, \quad (22)$$

$N := \{N(u) : u \geq 0\}$ and $U := \{U_t : t \geq 0\}$ are the same as in Section 4, and $f \in \mathcal{M} :=$ the set of all real functions on $[0, 1)$ such that the right-hand side of (21) makes sense for all $t > 0$ and $x \in [0, 1)$.

Our main result in this section is the following theorem which relates to [8, Theorem 3].

Theorem 8. Let $f \in \mathcal{M}$ and let f^* be the function on $[0, \infty)$ defined by

$$f^*(x) := f(1 - e^{-x}), \quad x \geq 0.$$

We have, for all $t > 1$ and $x \in [0, 1)$,

$$|M_t f(x) - f(x)| \leq 2\omega\left(f^*; \sqrt{\frac{3x}{t-1}}\right).$$

Therefore, we have $M_t f - f = o(1)$ ($t \rightarrow \infty$) uniformly in $[0, 1)$, whenever f^* is uniformly continuous.

Proof. By the same arguments as in the preceding proofs, we only need to show that we have, for all $t > 1$ and $x \in [0, 1)$,

$$EY_t(x) \leq \sqrt{\frac{3x}{t-1}},$$

where

$$Y_t(x) := \left| \log \frac{t + N(q(x)U_{t+1})}{t} - \log \frac{1}{1-x} \right| = \left| \log \left\{ (1-x) \frac{t + N(q(x)U_{t+1})}{t} \right\} \right|.$$

Such a fact follows from Lemma 1 and the following result, the proof of which is omitted, since it is completely analogous to that of Lemma 4. \square

Lemma 5. Let $x \in [0, 1)$. We have, for $t > 0$,

$$E \left[(1-x) \frac{t + N(q(x)U_{t+1})}{t} - 1 \right]^2 = \frac{(t+1)x + x^2}{t^2}$$

and, for $t > 1$,

$$E \left[\frac{1}{1-x} \frac{t}{t + N(q(x)U_{t+1})} - 1 \right]^2 = x \int_0^1 \frac{(1-u)^t}{(1-xu)^2} du \leq \frac{x}{t-1}.$$

8. Cheney and Sharma operators

An interesting generalization of the operator M_t is the double-indexed operator $P_{t,r}$ ($t > 0, r \geq 0$) introduced by Cheney and Sharma [4], and defined by

$$P_{t,r}f(x) := (1-x)^{t+1} \exp\left(\frac{-rx}{1-x}\right) \sum_{k=0}^{\infty} f\left(\frac{k}{t+k}\right) L_k^{(t)}(-r)x^k,$$

where $x \in [0, 1)$, and $L_k^{(t)}$ denotes the generalized Laguerre polynomial of degree k , i.e.,

$$L_k^{(t)}(r) := \sum_{i=0}^k \binom{t+k}{k-i} \frac{(-r)^i}{i!}.$$

We actually have $P_{t,0} = M_t$. It is shown in [3] that $P_{t,r}f$ converges to f uniformly on $[0, 1)$ as $t \rightarrow \infty$ and $r/t \rightarrow 0$, whenever f is uniformly continuous on $[0, 1)$, so extending the corresponding property for the Meyer–König and Zeller operators (rates of convergence are also given in the same paper).

In this section, we however show that the last part of Theorem 8 does not extend to the operators $P_{t,r}$. To do this, let f be the function given by

$$f(x) := \log \log \left(\frac{e}{1-x} \right), \quad x \in [0, 1).$$

This function fulfills the condition that the function

$$f^{**}(x) := f(1 - \exp(1 - e^x)) = x, \quad x \geq 0,$$

is uniformly continuous on $[0, \infty)$. This condition is much more restrictive than the one required in the last part of Theorem 8. Therefore, we have

$$\sup_{0 \leq x < 1} |M_t f(x) - f(x)| \rightarrow 0 \quad (t \rightarrow \infty).$$

We however claim that, for all $t > 0$ and $r > 0$,

$$\sup_{0 \leq x < 1} |P_{t,r} f(x) - M_t f(x)| \geq \log 2,$$

so that $P_{t,r}f$ cannot converge to f , uniformly on $[0, 1)$ as $t \rightarrow \infty$ and $r/t \rightarrow 0$.

To show the claim, we use the probabilistic representation for the Cheney and Sharma operators introduced in [2] (see also [3]). Let $\{N(t): t \geq 0\}$, $\{N'(t): t \geq 0\}$, $\{N''(t): t \geq 0\}$, $\{U_t: t \geq 0\}$, and $\{V_t: t \geq 0\}$ be five mutually independent stochastic processes defined on the same probability space, $\{N(t): t \geq 0\}$, $\{N'(t): t \geq 0\}$, and $\{N''(t): t \geq 0\}$ being standard Poisson processes, and $\{U_t: t \geq 0\}$ and $\{V_t: t \geq 0\}$ being standard gamma processes. Set, for $t > 0$, $r \geq 0$, and $x \in [0, 1)$,

$$Z_{t,r}(x) := N(q(x)U_{t+1}) + N'(rq(x)) + N''(q(x)V_{N'(rq(x))}),$$

where $q(x)$ is the same as in (22). Then, we have

$$P_{t,r}f(x) = Ef\left(\frac{Z_{t,r}(x)}{t + Z_{t,r}(x)}\right) = E\left[\log \log \frac{e(t + Z_{t,r}(x))}{t}\right]$$

and, therefore,

$$|P_{t,r}f(x) - M_t f(x)| = |P_{t,r}f(x) - P_{t,0}f(x)| = EY_{t,r}(x),$$

where

$$Y_{t,r}(x) := \log \frac{\log(e(t + Z_{t,r}(x))/t)}{\log(e(t + Z_{t,0}(x))/t)}.$$

Let $t > 0$ and $r > 0$ be fixed. Using the strong laws of large numbers for standard Poisson processes and for standard gamma processes, it is easy to see that, with probability 1,

$$\lim_{x \uparrow 1} \frac{Z_{t,r}(x)}{r[q(x)]^2} = 1$$

and

$$\lim_{x \uparrow 1} \frac{Z_{t,0}(x)}{q(x)U_{t+1}} = 1.$$

From these facts, it readily follows that

$$\lim_{x \uparrow 1} Y_{t,r}(x) = \log 2,$$

with probability 1, and, from Fatou's lemma, we conclude that

$$\liminf_{x \uparrow 1} |P_{t,r}f(x) - M_t f(x)| \geq E \lim_{x \uparrow 1} Y_{t,r}(x) = \log 2,$$

showing the claim.

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