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# Eventually norm continuous semigroups on Hilbert space and perturbations<sup>☆</sup>

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## Abstract

The eventually norm continuous semigroups on Hilbert space  $\mathcal{H}$  and perturbation are studied in this paper. By resolvent of infinitesimal generator, the sufficient and necessary conditions for eventually norm continuous semigroups are given. Using the result obtained, it is proved that if  $\mathcal{A}$  is infinitesimal generator of an eventually norm continuous semigroup  $T(t)$ , then there is a subspace  $\mathcal{E}_A$  of  $\mathcal{L}(\mathcal{H})$  such that, for any  $\mathcal{B} \in \mathcal{E}_A$ , the semigroup  $S(t)$  generated by  $\mathcal{A} + \mathcal{B}$  preserves the property of  $T(t)$ .

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## 1. Introduction

A  $C_0$  semigroup  $T(t)$  on a Banach space  $X$  is said to be eventually norm continuous if there exists  $t_0 \geq 0$  such that the map  $t \rightarrow T(t)$  is norm continuous for  $t > t_0$ . If  $t_0 = 0$ ,  $T(t)$  is said to be norm continuous. Let  $\mathcal{A}$  be the infinitesimal generator of  $T(t)$ . Pazy in [1] pointed out that “so far there are no known necessary and sufficient conditions, in terms of  $\mathcal{A}$  or the resolvent  $R(\lambda; \mathcal{A})$ , which assure the continuous for  $t > 0$  of  $T(t)$  in the uniform operator topology.” If  $X$  is a Hilbert space, denote by  $\mathcal{H}$ , Puhong in [2] answered the problem proposed by Pazy. He obtained the following result.

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Let  $T(t)$  be a  $C_0$  semigroup on a Hilbert space with generator  $\mathcal{A}$ . Then the following assertions are equivalent:

- (1)  $T(t)$  is norm continuous for  $t > 0$ ;
- (2) For all  $\omega > \omega(\mathcal{A})$  one has  $\lim_{\tau \rightarrow \pm\infty} \|R(\omega + i\tau; \mathcal{A})\| = 0$ ;
- (3) There exist  $\omega \in \mathbb{R}$  and  $r \geq 0$  such that  $\lim_{|\tau| \geq r, \tau \rightarrow \pm\infty} \|R(\omega + i\tau; \mathcal{A})\| = 0$ .

El-Mennaoui and Engel in [3] gave a simple proof for this result. Naturally, an interesting problem is: what conditions are the complete characterization of the eventually norm continuous semigroup in terms of its infinitesimal generator? Many authors have wrought out the problem (e.g., [4,5] and references therein). Here we refer to the result worked by Blasco and Martinez [4]. They discussed the property of norm continuous semigroup on Banach space and proved the following assertion (see [4, Theorem 5]) in Hilbert space.

Let  $\mathcal{A}$  be the infinitesimal generator of a  $C_0$  semigroup  $T(t)$  with  $\|T(t)\| \leq Me^{-t}$  on a Hilbert space  $\mathcal{H}$ , then the following assertions are equivalent:

- (1)  $T(t)$  is norm continuous for  $t > t_0$ ;
- (2) There exists  $C > 0$  such that

$$\limsup_{|s| \rightarrow \infty} \|n! R^n(is; \mathcal{A})\|^{1/n} \leq C, \quad \forall n \in \mathbb{N};$$

- (3) There exists  $t_0 > 0$  such that

$$\lim_{n \rightarrow \infty} \frac{\rho_n}{t_0^n} = 0,$$

where

$$\rho_n = \lim_{k \rightarrow \infty} \sup_{\|x\| \leq 1, \|y\| \leq 1} \int_{|s| > k} (n+1)! |(R^{n+2}(is; \mathcal{A})x, y)| ds.$$

The above result gives complete characterization of eventually norm continuous semigroup in term of resolvent involving integral of the resolvent.

It is well known that if  $T(t)$  is a  $C_0$  semigroup on Banach space  $X$  with infinitesimal generator  $\mathcal{A}$ , and  $\mathcal{B}$  is a bounded linear operator, then  $\mathcal{A} + \mathcal{B}$  is the infinitesimal generator of a  $C_0$  semigroup  $S(t)$  on  $X$ . In particular, the semigroup  $S(t)$  is determined via the integral equation

$$S(t)x = T(t)x + \int_0^t T(t-r)\mathcal{B}S(r)x dr, \quad \forall x \in X.$$

Set  $S_0(t) = T(t)$  and define

$$S_{n+1}(t)x = \int_0^t T(t-r)\mathcal{B}S_n(r)x dr.$$

When  $T(t)$  satisfies  $\|T(t)\| \leq M e^{\omega t}$ , we have

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \quad \|S_n(t)\| \leq M e^{\omega t} \frac{M^n \|B\|^n t^n}{n!},$$

and for any  $\operatorname{Re} \lambda \geq \omega + M\|B\|$ ,  $x \in \mathcal{H}$ ,

$$\int_0^{\infty} e^{-\lambda t} S_n(t)x \, dt = [R(\lambda; \mathcal{A})B]^n R(\lambda; \mathcal{A})x = R(\lambda; \mathcal{A})[BR(\lambda; \mathcal{A})]^n x.$$

Therefore,

$$R(\lambda; \mathcal{A} + B) = \sum_{n=0}^{\infty} R(\lambda; \mathcal{A})[BR(\lambda; \mathcal{A})]^n = \sum_{n=0}^{\infty} [R(\lambda; \mathcal{A})B]^n R(\lambda; \mathcal{A}).$$

From the above equality and the result in [2], we can easily see that if  $T(t)$  is norm continuous (compact) for  $t > 0$ , then  $S(t)$  is norm continuous (compact, respectively) for  $t > 0$ .

As pointed out in [1, p. 79], not all the properties of the semigroup  $T(t)$  are preserved by a bounded perturbation of its infinitesimal generator. For example, if  $\mathcal{A}$  is the infinitesimal generator of a semigroup  $T(t)$  which is norm continuous for  $t > t_0 \geq 0$ , or is differentiable for  $t > t_0 \geq 0$ , or is compact for  $t > t_0 \geq 0$ , then  $S(t)$  need not possess the corresponding property. The key of the problem is that the asymptotic behavior of resolvent of the infinitesimal generator is not clear when the parameter tends to infinity along the line parallel to imaginary axis. The asymptotic behavior of resolvent of infinitesimal generator along the vertical line has an essential meaning in the study of property of semigroup.

The purpose of the present paper is to discuss the property of decomposition of resolvent of infinitesimal generator of eventually norm continuous semigroup and its perturbation. By the decomposition of resolvent, we give the characterization of eventually norm continuous semigroup, and study the bounded perturbation of its infinitesimal generator. The main results of this paper are as follows. We prove firstly the following result.

**Theorem 1.** *Let  $T(t)$  be a  $C_0$  semigroup on a Hilbert space  $\mathcal{H}$  with infinitesimal generator  $\mathcal{A}$ . Then the following statements are equivalent:*

- (1) *There exists  $t_0 \geq 0$  such that  $T(t)$  is norm continuous for  $t > t_0$ ;*
- (2) *For  $\lambda = \sigma + i\tau$  with  $\sigma > \omega(\mathcal{A})$ , the resolvent of  $\mathcal{A}$  can be written into the form*

$$R(\lambda; \mathcal{A}) = R_1(\lambda) + R_2(\lambda),$$

where  $R_j(\lambda)$ ,  $j = 1, 2$ , satisfy the following properties:

- (a)  $R_1(\lambda)$  is an entire function of exponential type at most  $t_0$ , and for any  $\sigma > \omega(\mathcal{A})$ ,  $x \in \mathcal{H}$ , it holds that

$$\int_{-\infty}^{\infty} \|R_1(\sigma + i\tau)x\|^2 d\tau < \infty$$

and

$$\lim_{\sigma \rightarrow +\infty} \|R_1(\sigma + i\tau)\| = 0;$$

(b)  $R_2(\lambda)$  is analytic in half plane  $\operatorname{Re} \lambda > \omega(\mathcal{A})$ , and there is an integer  $k$  such that

$$\lim_{\operatorname{Im} \lambda \rightarrow \pm\infty} \left\| \frac{d^n}{d\lambda^n} R_2(\lambda) \right\| = 0, \quad \operatorname{Re} \lambda > \omega(\mathcal{A}), \quad n \geq k \geq 1.$$

Applying the result of Blasco and Martinez [4], we give a sufficient condition for eventually norm continuous semigroup.

**Theorem 2.** Let  $T(t)$  be a  $C_0$  semigroup on a Hilbert space  $\mathcal{H}$  with generator  $\mathcal{A}$ . If we can find a decomposition of  $R(\lambda; \mathcal{A})$ ,

$$R(\lambda; \mathcal{A}) = R_1(\lambda) + R_2(\lambda),$$

with property that there exists constant  $C > 0$  such that

$$\limsup_{\tau \rightarrow \pm\infty} \|n! R_1^n(\sigma + i\tau)\|^{1/n} \leq C, \quad \forall n \geq 1,$$

and for some integer  $k$  such that

$$R^n(\lambda; \mathcal{A}) = R_1^n(\lambda) + Q_n(\lambda), \quad n \geq k,$$

and

$$\lim_{\tau \rightarrow \pm\infty} \|Q_n(\sigma + i\tau)\| = 0,$$

then  $T(t)$  is eventually norm continuous and the time of norm continuity is determined by

$$t_0 = \limsup_{n \rightarrow \infty} \limsup_{\tau \rightarrow \pm\infty} \|n! R_1^n(\sigma + i\tau)\|^{1/n}.$$

Basing on Theorem 2, we study the bounded perturbation of infinitesimal generator of  $T(t)$ . Define the subset  $\mathcal{E}_A$  of  $\mathcal{L}(\mathcal{H})$  for  $\operatorname{Re} \lambda > \omega(\mathcal{A})$  by

$$\begin{aligned} \mathcal{E}_A &= \left\{ \mathcal{B} \in \mathcal{L}(\mathcal{H}) \mid \lim_{\operatorname{Im} \lambda \rightarrow \pm\infty} \|R(\lambda; \mathcal{A}) \mathcal{B} R^2(\lambda; \mathcal{A})\| \right. \\ &\quad \left. = \lim_{\operatorname{Im} \lambda \rightarrow \pm\infty} \|R^2(\lambda; \mathcal{A}) \mathcal{B} R(\lambda; \mathcal{A})\| = 0 \right\}. \end{aligned}$$

We obtain following perturbation theorem in Hilbert space  $\mathcal{H}$ .

**Theorem 3.** Let  $\mathcal{A}$  be the infinitesimal generator of norm continuous semigroup  $T(t)$  for  $t > t_0 \geq 0$  and  $\mathcal{B} \in \mathcal{E}_A$ . Then the semigroup  $S(t)$  generated by  $\mathcal{A} + \mathcal{B}$  is norm continuous for  $t > t_0$ .

In the next section we shall give the proofs of main results of the present paper.

## 2. The proofs of main results

In this section we prove the main results of present paper. In the sequel we always denote by  $\mathcal{H}$  the Hilbert space, by  $\mathcal{L}(\mathcal{H})$  the set of all bounded linear operator and by  $\mathbb{R}$  the real axis.

**Lemma 1.** Let  $\phi \in L^2([-A, A], \mathcal{H})$ . Then

$$f(z) = \int_{-A}^A e^{izt} \phi(t) dt$$

is an entire function of exponential type at most  $A$ , and satisfies

$$\int_{-\infty}^{\infty} \|f(s)\|^2 ds < \infty.$$

The following lemma is a version of Paley–Wiener theorem in Hilbert space.

**Lemma 2** (Paley–Wiener). Let  $f(z)$  be an entire function valued in  $\mathcal{H}$  such that

$$\|f(z)\| \leq C e^{A|z|}$$

for positive constants  $A$  and  $C$  and all values of  $z$ , and

$$\int_{-\infty}^{\infty} \|f(s)\|^2 ds < \infty.$$

Then there exists a function  $\phi$  in  $L^2([-A, A]; \mathcal{H})$  such that

$$f(z) = \int_{-A}^A e^{izt} \phi(t) dt$$

(cf. [6]).

**Proposition 3.** Let  $T(t)$  be a  $C_0$  semigroup on  $\mathcal{H}$  generated by  $\mathcal{A}$ . Assume that  $T(t)$  is norm continuous for  $t > t_0 \geq 0$ . Then there exists a decomposition of the resolvent of  $\mathcal{A}$ ,

$$R(\lambda; \mathcal{A}) = R_1(\lambda) + R_2(\lambda), \quad \operatorname{Re} \lambda > \omega(\mathcal{A}),$$

such that  $R_j(\lambda)$ ,  $j = 1, 2$ , satisfy the following properties:

- (a)  $R_1(\lambda)$  is an entire function of exponential type at most  $t_0$ , which is Fourier transform of an operator-valued function  $\phi(t)$ , and satisfies for any  $x \in \mathcal{H}$ ,

$$\int_{-\infty}^{\infty} \|R_1(\sigma + i\tau)x\|^2 d\tau < \infty$$

and

$$\lim_{\sigma \rightarrow +\infty} \|R_1(\sigma + i\tau)\| = 0;$$

(b)  $R_2(\lambda)$  is analytic in half plane  $\operatorname{Re} \lambda > \omega(\mathcal{A})$ , and

$$\lim_{\operatorname{Im} \lambda \rightarrow \pm \infty} \left\| \frac{d^n}{d\lambda^n} R_2(\lambda) \right\| = 0, \quad \forall n \geq 1.$$

**Proof.** Let  $\omega > \omega(\mathcal{A})$ . Then there exists a constant  $M(\omega)$  such that

$$\|T(t)\| \leq M(\omega)e^{\omega t}.$$

Now for  $\lambda = \sigma + i\tau$  with  $\sigma > \omega$ , it holds that

$$R(\sigma + i\tau; \mathcal{A})x = \int_0^{+\infty} e^{-(\sigma+i\tau)t} T(t)x \, dt, \quad \forall x \in \mathcal{H}.$$

Since  $\mathcal{H}$  is Hilbert space, we have  $\|R(\sigma + i\cdot; \mathcal{A})x\| \in L^2(\mathbb{R})$ .

Because  $T(t)$  is norm continuous for  $t > t_0$ , we can define operators  $R_1(\lambda)$  and  $R_2(\lambda)$  by

$$R_1(\lambda) = \int_0^{t_0} e^{-\lambda t} T(t) \, dt \tag{1}$$

and

$$R_2(\lambda) = \int_{t_0}^{+\infty} e^{-\lambda t} T(t) \, dt, \tag{2}$$

the integral at right-hand side of Eq. (2) exists in the sense of operator topology. Obviously,

$$R(\lambda; \mathcal{A}) = R_1(\lambda) + R_2(\lambda).$$

According to Lemma 1, we know that  $R_1(\lambda)$  is an entire function of exponential type at most  $t_0$ , and for any  $\sigma > \omega(\mathcal{A})$ ,  $x \in \mathcal{H}$ ,

$$\int_{-\infty}^{\infty} \|R_1(\sigma + i\tau)x\|^2 \, d\tau = 2\pi \int_0^{t_0} e^{-2\sigma t} \|T(t)x\|^2 \, dt$$

and

$$\lim_{\sigma \rightarrow +\infty} \|R_1(\sigma + i\tau)\| = \lim_{\sigma \rightarrow +\infty} \|R(\sigma + i\tau; \mathcal{A})[I - e^{-(\sigma+i\tau)t_0} T(t_0)]\| = 0.$$

From Eq. (2) we see that  $R_2(\lambda) = e^{-\lambda t_0} T(t_0) R(\lambda; \mathcal{A})$ ,  $\operatorname{Re} \lambda > \omega(\mathcal{A})$ . Therefore, for  $\sigma > \omega(\mathcal{A})$ , we have

$$\lim_{\tau \rightarrow \pm \infty} \|R_2(\sigma + i\tau)\| = 0, \quad \lim_{\tau \rightarrow \pm \infty} \left\| \frac{d^n}{d\lambda^n} R_2(\sigma + i\tau) \right\| = 0.$$

The proof is then complete.  $\square$

**Proposition 4.** Let  $\mathcal{A}$  be the infinitesimal generator of a  $C_0$  semigroup  $T(t)$ . If for  $\operatorname{Re} \lambda > \omega(\mathcal{A})$ , the resolvent of  $\mathcal{A}$  has decomposition

$$R(\lambda; \mathcal{A}) = R_1(\lambda) + R_2(\lambda),$$

where  $R_1(\lambda)$  is an operator-valued entire function of exponential type at most  $t_0$  with property that

$$\int_{-\infty}^{\infty} \|R_1(\sigma + i\tau)x\|^2 d\tau < \infty, \quad \forall x \in \mathcal{H},$$

and

$$\lim_{\sigma \rightarrow +\infty} \|R_1(\sigma + i\tau)\| = 0,$$

and there is an integer  $k$  such that

$$\lim_{\tau \rightarrow \pm\infty} \left\| \frac{d^n}{d\lambda^n} R_2(\sigma + i\tau) \right\| = 0, \quad \sigma > \omega(\mathcal{A}), \quad \forall n \geq k,$$

then  $T(t)$  is norm continuous for  $t > t_0$ .

**Proof.** According to the assumption on  $R_1(\lambda)$ , we can obtain from Paley–Wiener theorem (see Lemma 2) that there is an operator-valued function  $\phi(t)$  such that

$$R_1(\lambda)x = \int_0^{t_0} e^{-\lambda t} \phi(t)x dt.$$

We can assume without loss of generality that  $\operatorname{Re} \lambda = \sigma > 0$ . Therefore, there is a constant  $M > 0$  such that

$$\int_0^{t_0} \|\phi(t)x\|^2 dt \leq M^2 \|x\|^2$$

and

$$\left\| \frac{d^n}{d\lambda^n} R_1(\lambda) \right\| \leq M \left( \frac{t_0^{2n+1}}{2n+1} \right)^{1/2}.$$

Obviously,  $R_2(\lambda) = R(\lambda; \mathcal{A}) - R_1(\lambda)$  is analytic in half plane  $\operatorname{Re} \lambda > \omega(\mathcal{A})$ , and

$$\frac{d^n}{d\lambda^n} R_2(\lambda) = \frac{d^n}{d\lambda^n} R(\lambda; \mathcal{A}) - \frac{d^n}{d\lambda^n} R_1(\lambda) = (-1)^n n! R^{n+1}(\lambda; \mathcal{A}) - \frac{d^n}{d\lambda^n} R_1(\lambda).$$

Using the assumption on  $R_2(\lambda)$ , we get that, for  $\sigma > \omega$  and  $n \geq k$ ,

$$\limsup_{\tau \rightarrow \pm\infty} \|n! R^{n+1}(\sigma + i\tau; \mathcal{A})\| \leq \limsup_{\tau \rightarrow \pm\infty} \left\| \frac{d^n}{d\lambda^n} R_1(\lambda) \right\|_{\sigma + i\tau} \leq M \left( \frac{t_0^{2n+1}}{2n+1} \right)^{1/2} < \infty.$$

Since for  $\operatorname{Re} \lambda = \sigma > \omega(\mathcal{A})$  it holds that

$$\|R(\sigma + i\cdot; \mathcal{A})x\|, \|R(\sigma + i\cdot; \mathcal{A}^*)y\| \in L^2(\mathbb{R}), \quad \forall x, y \in \mathcal{H},$$

there exists a constant  $M_\sigma > 0$  such that

$$\int_{\mathbb{R}} \|R(\sigma + i\tau; \mathcal{A})x\|^2 d\tau \leq M_\sigma \|x\|^2,$$

$$\int_{\mathbb{R}} \|R(\sigma + i\tau; \mathcal{A}^*)y\|^2 d\tau \leq M_\sigma \|y\|^2.$$

Thus we have

$$\begin{aligned} \rho_n &= \lim_{k \rightarrow \infty} \sup_{\|x\| \leq 1, \|y\| \leq 1} \int_{|\tau| \geq k} (n+1)! |R^{n+2}(\sigma + i\tau; \mathcal{A})x, y| d\tau \\ &\leq \limsup_{\tau \rightarrow \pm\infty} \{ \|(n+1)! R^n(\sigma + i\tau; \mathcal{A})\| \} \\ &\quad \times \sup_{\|x\| \leq 1, \|y\| \leq 1} \int_{\mathbb{R}} \|R(\sigma + i\tau; \mathcal{A})x\| \|R(\sigma - i\tau; \mathcal{A}^*)y\| d\tau \\ &\leq M_\sigma \limsup_{\tau \rightarrow \pm\infty} \{ (n+1)! \|R^n(\sigma + i\tau; \mathcal{A})\| \}, \end{aligned}$$

i.e.,

$$\rho_n \leq M_\sigma \limsup_{\tau \rightarrow \pm\infty} \{ (n+1)! \|R^n(\sigma + i\tau; \mathcal{A})\| \}. \quad (3)$$

Hence for any  $t > t_0$ ,

$$\lim_{n \rightarrow \infty} \frac{\rho_n}{t^n} \leq \lim_{n \rightarrow \infty} M_\sigma M(n+1)n \left( \frac{t_0}{2n+1} \right)^{1/2} \left( \frac{t_0}{t} \right)^n = 0.$$

Theorem 5 in [4] reads that  $T(t)$  is norm continuous semigroup for  $t > t_0$ . The proof is then finished.  $\square$

**Proof of Theorem 1.** Theorem 1 immediately follows from Propositions 3 and 4.  $\square$

**Proof of Theorem 2.** Let  $\mathcal{A}$  be the infinitesimal generator of a  $C_0$  semigroup  $T(t)$ . Assume that resolvent of  $\mathcal{A}$  has decomposition

$$R(\lambda; \mathcal{A}) = R_1(\lambda) + R_2(\lambda),$$

where  $R_1(\lambda)$  and  $R_2(\lambda)$  are analytic in half plane  $\operatorname{Re} \lambda = \sigma > \omega(\mathcal{A})$ , satisfy

$$\limsup_{\tau \rightarrow \pm\infty} \|n! R_1^n(\sigma + i\tau)\|^{1/n} < \infty,$$

and for some integer  $k$ ,

$$R^n(\sigma + i\tau; \mathcal{A}) = R_1^n(\sigma + i\tau) + Q_n(\sigma + i\tau), \quad n \geq k,$$

where  $\lim_{\tau \rightarrow \pm\infty} \|Q_n(\sigma + i\tau)\| = 0$ . Thus, we have

$$\limsup_{\tau \rightarrow \pm\infty} \|n! R^n(\sigma + i\tau; \mathcal{A})\|^{1/n} \leq \limsup_{\tau \rightarrow \pm\infty} \|n! R_1^n(\sigma + i\tau)\|^{1/n} < \infty.$$

The result in [4] reads that  $T(t)$  is eventually norm continuous.



Set

$$t_0 = \limsup_{n \rightarrow \infty} \limsup_{\tau \rightarrow \pm\infty} \|n! R_1^n(\sigma + i\tau)\|^{1/n}.$$

For any  $t > t_0$ , taking  $0 < \varepsilon < t - t_0$ , there exists an integer  $N$  such that

$$\limsup_{\tau \rightarrow \pm\infty} \|n! R_1^n(\sigma + i\tau)\|^{1/n} \leq (t_0 + \varepsilon), \quad \forall n \geq N.$$

Using inequality (3) again, we obtain

$$\lim_{n \rightarrow \infty} \frac{\rho_n}{t^n} \leq \lim_{n \rightarrow \infty} M_\sigma(n+1) \frac{(t_0 + \varepsilon)^n}{t^n} = 0.$$

So the time of norm continuity is  $t_0$ . The proof is then complete.  $\square$

Now we are in the position to investigate the bounded perturbation of infinitesimal generator  $\mathcal{A}$  of  $T(t)$ . Let  $T(t)$  be norm continuous for  $t > t_0 \geq 0$  and  $\mathcal{E}_A$  be given by

$$\begin{aligned} \mathcal{E}_A &= \left\{ \mathcal{B} \in \mathcal{L}(\mathcal{H}) \mid \lim_{\operatorname{Im} \lambda \rightarrow \pm\infty} \|R(\lambda; \mathcal{A}) \mathcal{B} R^2(\lambda; \mathcal{A})\| \right. \\ &\quad \left. = \lim_{\operatorname{Im} \lambda \rightarrow \pm\infty} \|R^2(\lambda; \mathcal{A}) \mathcal{B} R(\lambda; \mathcal{A})\| = 0 \right\}. \end{aligned}$$

Clearly,  $\mathcal{E}_A$  is a closed subspace of  $\mathcal{L}(\mathcal{H})$ . Furthermore,  $\mathcal{E}_A$  is independent of  $\operatorname{Re} \lambda > \omega(\mathcal{A})$ . In order to characterize the set  $\mathcal{E}_A$ , we introduce following concept.

**Definition 4.** Let  $\mathcal{H}$  be a Hilbert space,  $B_1 \subset \mathcal{H}$  be the unit ball of  $\mathcal{H}$  and  $\{\Psi(\tau), \tau \in \mathbb{R}\}$  be a family of bounded linear operators. The family  $\{\Psi(\tau), \tau \in \mathbb{R}\}$  is said to be compact at infinity, if

$$\lim_{\tau \rightarrow \pm\infty} \|\Psi(\tau)x\| = 0, \quad \forall x \in \mathcal{H},$$

and there is a compact subset  $\Omega$  of  $\mathcal{H}$  such that

$$\lim_{\tau \rightarrow \pm\infty} \sup_{x \in B_1} \operatorname{dist}(\Psi(\tau)x, \Omega) = 0.$$

**Proposition 5.** If family  $\{\Psi(\tau), \tau \in \mathbb{R}\}$  is compact at infinity, then for any uniformly bounded family  $\{\Phi(\tau), \tau \in \mathbb{R}\}$  satisfying  $\lim_{\tau \rightarrow \pm\infty} \|\Phi(\tau)x\| = 0$  for each  $x \in \mathcal{H}$ , we have  $\lim_{\tau \rightarrow \pm\infty} \|\Phi(\tau)\Psi(\tau)\| = 0$ .

**Proof.** Since  $\Omega$  is compact set, for arbitrary  $\varepsilon > 0$ , there exist  $x_1, x_2, \dots, x_k$  such that balls  $O(x_j, \varepsilon)$ ,  $j = 1, 2, \dots, k$ , cover set  $\Omega$ , i.e.,

$$\Omega \subset \bigcup_{j=1}^k O(x_j, \varepsilon).$$

Since  $\Psi(\tau)$  is uniformly bounded and compact at infinity, for  $\varepsilon > 0$ , there exists a constant  $N$  such that

$$\sup_{x \in B_1} \operatorname{dist}(\Psi(\tau)x, \Omega) < \varepsilon, \quad |\tau| \geq N.$$

Hence for every  $x \in B_1$ , there exists some  $x_j$  such that

$$\|\Psi(\tau)x - x_j\| < 2\varepsilon, \quad |\tau| \geq N.$$

Let  $\Phi(\tau)$  be any uniformly bounded family satisfying  $\lim_{\tau \rightarrow \infty} \|\Phi(\tau)x\| = 0$  for each  $x \in \mathcal{H}$ . Setting

$$M = \sup_{\tau \in \mathbb{R}} \|\Phi(\tau)\| < \infty,$$

we have

$$\|\Phi(\tau)\Psi(\tau)x - \Phi(\tau)x_j\| \leq M\|\Psi(\tau)x - x_j\| \leq 2M\varepsilon, \quad \forall |\tau| \geq N.$$

Now for each  $x_j$  there exists  $N_j > N$  such that

$$\|\Phi(\tau)x_j\| < \varepsilon, \quad |\tau| \geq N_j, \quad j = 1, 2, \dots, k.$$

Set  $\hat{N} = \max\{N_j \mid j = 1, 2, \dots, k\}$ . When  $|\tau| \geq \hat{N}$ , it holds that

$$\|\Phi(\tau)\Psi(\tau)x\| \leq \|\Phi(\tau)\Psi(\tau)x - \Phi(\tau)x_j\| + \|\Phi(\tau)x_j\| \leq 2M\varepsilon + \varepsilon, \quad \forall x \in B_1.$$

So we have

$$\|\Phi(\tau)\Psi(\tau)\| = \sup_{x \in B_1} \|\Phi(\tau)\Psi(\tau)x\| \leq 2M\varepsilon + \varepsilon, \quad |\tau| \geq \hat{N}.$$

The desired result follows. The proof is then complete.  $\square$

**Corollary 6.** If  $\mathcal{B} \in \mathcal{L}(\mathcal{H})$  makes that  $\mathcal{B}R(\lambda; \mathcal{A})$  is compact at infinity, then  $\mathcal{B} \in \mathcal{E}_A$ .

**Proof.** Set  $\Psi(\tau) = \mathcal{B}R(\sigma + i\tau; \mathcal{A})$  and  $\Phi(\tau) = R(\sigma + i\tau; \mathcal{A})$ . Proposition 6 reads  $\lim_{\tau \rightarrow \infty} \|R(\sigma + i\tau; \mathcal{A})\mathcal{B}R(\sigma + i\tau; \mathcal{A})\| = 0$ . The assertion holds true.  $\square$

**Proof of Theorem 3.** Let  $\mathcal{A}$  be the infinitesimal generator of norm continuous semigroup  $T(t)$  for  $t > t_0 \geq 0$ , satisfying  $\|T(t)\| \leq Me^{\omega t}$ , and  $\mathcal{B} \in \mathcal{E}_A$ .  $S(t)$  is the semigroup generated by  $\mathcal{A} + \mathcal{B}$ . For  $\sigma > \omega(\mathcal{A}) + M\|\mathcal{B}\|$ , we have

$$R(\sigma + i\tau; \mathcal{A} + \mathcal{B}) = R(\sigma + i\tau; \mathcal{A}) + \sum_{k=1}^{\infty} R(\sigma + i\tau; \mathcal{A})[\mathcal{B}R(\sigma + i\tau; \mathcal{A})]^k.$$

Set

$$R_1(\sigma + i\tau) = R(\sigma + i\tau; \mathcal{A})$$

and

$$R_2(\sigma + i\tau) = \sum_{k=1}^{\infty} R(\sigma + i\tau; \mathcal{A})[\mathcal{B}R(\sigma + i\tau; \mathcal{A})]^k.$$

For  $\mathcal{B} \in \mathcal{E}_A$ , it is easily to verify

$$\begin{aligned} \lim_{\tau \rightarrow \pm\infty} \|R_1(\sigma + i\tau)R_2(\sigma + i\tau)\| &= 0, \\ \lim_{\tau \rightarrow \pm\infty} \|R_2(\sigma + i\tau)R_1(\sigma + i\tau)\| &= 0, \end{aligned}$$

and

$$\lim_{\tau \rightarrow \pm\infty} \|R_2(\sigma + i\tau)R_2(\sigma + i\tau)\| = 0.$$

Therefore, for  $n \geq 2$ , we have

$$R^n(\sigma + i\tau; \mathcal{A} + \mathcal{B}) = R_1^n(\sigma + i\tau; \mathcal{A}) + Q_n(\sigma + i\tau),$$

where

$$\lim_{\tau \rightarrow \pm\infty} \|Q_n(\sigma + i\tau)\| = 0.$$

Since  $\mathcal{A}$  is the infinitesimal generator of eventually norm continuous semigroup  $T(t)$ , we have

$$\lim_{\tau \rightarrow \pm\infty} \|n!R_1^n(\sigma + i\tau)\|^{1/n} = \lim_{\tau \rightarrow \pm\infty} \|n!R^n(\sigma + i\tau; \mathcal{A})\|^{1/n} < \infty.$$

Thus  $R_1(\lambda)$  and  $R_2(\lambda)$  satisfy the conditions required in Theorem 2. The desired result follows from Theorem 2. The proof is complete.  $\square$

**Corollary 7.** *Let  $\mathcal{A}$  be the infinitesimal generator of norm continuous semigroup for  $t > t_0$ , and  $\mathcal{B} \in \mathcal{E}_A$ . Then for any  $\alpha \in \mathbb{R}$ , the semigroup  $S(t)$  generated by  $\mathcal{A} + \alpha I + \mathcal{B}$  also is norm continuous for  $t > t_0$ .*

**Proof.** Set  $\mathcal{A}_\alpha = \mathcal{A} + \alpha I$ ; then  $\mathcal{A}_\alpha$  is the infinitesimal generator of semigroup  $T_\alpha(t) = e^{\alpha t}T(t)$ . Obviously,  $T_\alpha(t)$  is norm continuous for  $t > t_0$ . So we need only to prove that  $\mathcal{B} \in \mathcal{E}_{\mathcal{A}_\alpha}$ . Noting the fact that

$$R(\lambda; \mathcal{A}_\alpha) = R(\lambda - \alpha; \mathcal{A}), \quad \forall \operatorname{Re} \lambda > \alpha + \omega(\mathcal{A}),$$

and the set  $\mathcal{E}_A$  is independent of  $\operatorname{Re} \lambda > \omega(\mathcal{A})$ , we have  $\mathcal{B} \in \mathcal{E}_{\mathcal{A}_\alpha}$ . Therefore, Theorem 3 ensures that  $S(t)$  is norm continuous for  $t > t_0$ . The proof is then complete.  $\square$

**Remark.** In the proof of Theorem 3, we mainly apply the result of Theorem 2. Under the same decomposition of  $R(\lambda; \mathcal{A} + \mathcal{B})$ , we can verify that for  $n \geq 2$ ,

$$\lim_{\tau \rightarrow \pm\infty} \left\| \frac{d^n}{d\lambda^n} R_2(\sigma + i\tau) \right\| = 0.$$

Since  $\mathcal{A}$  is infinitesimal generator of eventually norm continuous semigroup  $T(t)$ , according to Theorem 1,  $R(\lambda; \mathcal{A})$  has decomposition

$$R(\lambda; \mathcal{A}) = H_1(\lambda) + H_2(\lambda),$$

where  $H_j(\lambda)$ ,  $j = 1, 2$ , satisfy the conditions required in Theorem 1, thus the decomposition

$$R(\lambda; \mathcal{A} + \mathcal{B}) = H_1(\lambda) + [H_2(\lambda) + R_2(\lambda)] = \hat{R}_1(\lambda) + \hat{R}_2(\lambda)$$

satisfies the conditions in Theorem 1. The result of Theorem 3 follows from Theorem 1.

In Theorem 3, we show that the property of the semigroup  $T(t)$  is preserved by bounded perturbation in class  $\mathcal{E}_A$ . Applying entire similar manner, we can investigate the bounded perturbation belonging to more extension operator class such as  $\mathcal{E}_A^n$  defined by

$$\mathcal{E}_A^n = \left\{ B \in \mathcal{L}(\mathcal{H}) \mid \lim_{\tau \rightarrow \pm\infty} \|R(\sigma + i\tau; \mathcal{A})[BR(\sigma + i\tau; \mathcal{A})]^n\| = 0 \right\}.$$

One question is whether any one of eventually norm continuous semigroup  $T(t)$  can be regarded as a perturbation of infinitesimal generator of a nilpotent semigroup in some operator class. The problem of generation of eventually norm continuous seems still to be open.

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