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Multilinear commutators of Calderón–Zygmund operators on Hardy-type spaces with non-doubling measures ☆

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Abstract

Under the assumption that μ is a non-negative Radon measure on $\mathbb{R}^d$ which only satisfies some growth condition, the authors obtain the boundedness in some Hardy-type spaces of multilinear commutators generated by Calderón–Zygmund operators or fractional integrals with $RBMO(\mu)$ functions, where the Hardy-type spaces are some appropriate subspaces, associated to the considered $RBMO(\mu)$ functions, of the Hardy space $H^1(\mu)$ of Tolsa.

Keywords: Multilinear commutator; Calderón–Zygmund operator; Fractional integral; Hardy-type space; *RBMO*(μ); Non-doubling measure

1. Introduction

We will work on the d-dimensional Euclidean space $\mathbb{R}^d$ with a non-negative Radon measure μ which only satisfies the following growth condition: there exists a constant $C_0 > 0$ such that

$$\mu\big(B(x,r)\big) \leqslant C_0 r^n$$

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for all $x \in \mathbb{R}^d$ and $r > 0$, where $B(x,r) = \{y \in \mathbb{R}^d\colon\ |y - x| < r\}$, n is a fixed number and $0 < n \leqslant d$. Since the measure μ is not necessary to satisfy the doubling condition that there exists a constant $C > 0$ such that $\mu(B(x,2r)) \leqslant C\mu(B(x,r))$ for all $x \in \operatorname{supp}(\mu)$ and $r > 0$, we call such $\mathbb{R}^d$ a non-homogeneous space. It is well known that the doubling condition is a key assumption in the analysis on spaces of homogeneous type. However, in recent years, a lot of papers focus on the study of function spaces and the boundedness of Calderón–Zygmund operators in non-homogeneous spaces and indicate that many classical results still hold in non-homogeneous spaces; see [4–7,10–13] and their references. The analysis on non-homogeneous spaces was proved to play a striking role in solving the long open Painlevé's problem by Tolsa in [14]; see also [15] for more background.

The main purpose of this paper is to establish the boundedness in some Hardy-type spaces of multilinear commutators generated by Calderón–Zygmund operators or fractional integrals with $RBMO(\mu)$ functions. Before stating our results, we first recall some necessary notation and definitions.

Let K be a function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y)\colon\ x = y\}$ satisfying that for $x \neq y$,

$$\left|K(x,y)\right| \leqslant C|x-y|^{-n}, \tag{1.1}$$

and for $|x-y| \geqslant 2|x-x'|$,

$$\left|K(x,y) - K(x',y)\right| + \left|K(y,x) - K(y,x')\right| \leqslant C\frac{|x-x'|^{\delta}}{|x-y|^{n+\delta}}, \tag{1.2}$$

where $\delta \in (0,1]$ and $C > 0$ is a constant. The Calderón–Zygmund operator associated to the above kernel K and the measure μ is formally defined by

$$T(f)(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,d\mu(y). \tag{1.3}$$

This integral may be not convergent for many functions. Thus we consider the truncated operators T_ϵ for $\epsilon > 0$ defined by

$$T_\epsilon(f)(x) = \int_{|x-y|>\epsilon} K(x,y)f(y)\,d\mu(y).$$

We say that T is bounded on $L^2(\mu)$ if the operators T_ϵ are bounded on $L^2(\mu)$ uniformly on $\epsilon > 0$.

By a cube $Q \subset \mathbb{R}^d$ we mean a closed cube whose sides parallel to the axes and we denote its side length by $l(Q)$. Let α and β_d be positive constants such that $\alpha > 1$ and $\beta_d > \alpha^n$. For a cube Q, we say that Q is (α,β)-doubling if $\mu(\alpha Q) \leqslant \beta\mu(Q)$, where αQ denotes the cube concentric with Q and having side length $\alpha l(Q)$. In what follows, for definiteness, if α and β are not specified, by a doubling cube we mean a $(2, 2^{d+1})$-doubling cube. Especially, for any given cube Q, we denote by $\widetilde{Q}$ the smallest doubling cube in the family $\{2^k Q\}_{k \geqslant 0}$. For two cubes $Q_1 \subset Q_2$, set

$$K_{Q_1,Q_2} = 1 + \sum_{k=1}^{N_{Q_1,Q_2}} \frac{\mu(2^k Q_1)}{l(2^k Q_1)^n},$$

where N_{Q_1,Q_2} is the first positive integer k such that $l(2^k Q_1) \geqslant l(Q_2)$; see [10] for some basic properties of K_{Q_1,Q_2}.

Definition 1.1. Let $\rho > 1$ be some fixed constant. We say that a function $b \in L^1_{\mathrm{loc}}(\mu)$ belongs to the space $RBMO(\mu)$ if there is a constant $B > 0$ such that

$$\sup_Q \frac{1}{\mu(\rho Q)} \int_Q \left|b(x) - m_{\widetilde{Q}}(b)\right| d\mu(x) \leqslant B < \infty, \tag{1.4}$$

and if $Q_1 \subset Q_2$ are doubling cubes,

$$\left|m_{Q_1}(b) - m_{Q_2}(b)\right| \leqslant B K_{Q_1,Q_2}, \tag{1.5}$$

where the supremum is taken over all cubes centered at some point of $\operatorname{supp}\mu$ and $m_{\widetilde{Q}}(b)$ is the mean value of b on $\widetilde{Q}$, namely,

$$m_{\widetilde{Q}}(b) = \frac{1}{\mu(\widetilde{Q})} \int_{\widetilde{Q}} b(x)\, d\mu(x).$$

The minimal constant B in (1.4) and (1.5) is defined to be the $RBMO(\mu)$ norm of b and is denoted by $\|b\|_*$.

The space $RBMO(\mu)$ was introduced by Tolsa [11] and he proved there that the definition of $RBMO(\mu)$ is independent of the choices of numbers ρ. In the sequel, we will choose $\rho = 2$.

Let T be the Calderón–Zygmund operator defined by (1.3). In what follows, we will always assume that T is bounded on $L^2(\mu)$. We now fix a $\widetilde{T}$, which is a weak limit as $\epsilon \to 0$ of some subsequence of the uniformly $L^2(\mu)$ bounded operators T_ϵ on $\epsilon > 0$; see [11, p. 141]. It is easy to deduce that $\widetilde{T}$ is still bounded on $L^2(\mu)$; moreover, for $f \in L^2(\mu)$ whose support is not all of $\mathbb{R}^d$,

$$\widetilde{T}(f)(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\, d\mu(y)$$

with the same K as in T, which satisfies (1.1) and (1.2). For such a $\widetilde{T}$, $m \in \mathbb{N}$ and $b_i \in RBMO(\mu)$, $i = 1, 2, \ldots, m$, we formally define the multilinear commutator $T_{\vec{b}}$ by

$$T_{\vec{b}}(f)(x) = \left[b_m, \left[b_{m-1}, \ldots, [b_1, \widetilde{T}]\cdots\right]\right](f)(x). \tag{1.6}$$

A such type of multilinear commutators when μ is the d-dimensional Lebesgue measure was first introduced by Pérez and Trujillo-González in [9]. When μ is a non-doubling measure, it was proved in [10] for $m = 1$ and in [3] for $m > 1$ that if T is bounded on $L^2(\mu)$ and $b_i \in RBMO(\mu)$ for $i = 1, \ldots, m$, then $T_{\vec{b}}$ is bounded on $L^p(\mu)$ for $1 < p < \infty$. But $T_{\vec{b}}$ is not bounded from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ even when μ is the d-dimensional Lebesgue measure and $m = 1$; see [8]. In this paper, motivated by [8], we will first prove that for any $m \in \mathbb{N}$, $T_{\vec{b}}$ is bounded from some subspace of $H^1(\mu)$ associated with $\vec{b}$ into $L^1(\mu)$, in analogy with the result established by Pérez in [8] with μ being the d-dimensional Lebesgue measure.

In the sequel, for $1 \leqslant i \leqslant m$, we denote by C_i^m the family of all finite subsets $\sigma = \{\sigma(1), \ldots, \sigma(i)\}$ of $\{1, 2, \ldots, m\}$ with i different elements. For any $\sigma \in C_i^m$, the complementary sequence σ' is given by $\sigma' = \{1, 2, \ldots, m\} \setminus \sigma$. Let $\vec{b} = (b_1, b_2, \ldots, b_m)$ be a finite family of locally integrable functions. For all $1 \leqslant i \leqslant m$ and $\sigma = \{\sigma(1), \ldots, \sigma(i)\} \in C_i^m$, we define

$$\left[b(x) - b(y)\right]_\sigma = \left[b_{\sigma(1)}(x) - b_{\sigma(1)}(y)\right] \cdots \left[b_{\sigma(i)}(x) - b_{\sigma(i)}(y)\right],$$
$$\left[b(x) - m_Q(b)\right]_\sigma = \left[b_{\sigma(1)}(x) - m_Q(b_{\sigma(1)})\right] \cdots \left[b_{\sigma(i)}(x) - m_Q(b_{\sigma(i)})\right],$$

and

$$\left[m_R(b)-m_Q(b)\right]_\sigma=\left[m_R(b_{\sigma(1)})-m_Q(b_{\sigma(1)})\right]\cdots\left[m_R(b_{\sigma(i)})-m_Q(b_{\sigma(i)})\right],$$

where Q and R are cubes in $\mathbb{R}^d$ and $x,y\in\mathbb{R}^d$. With this notation, we write

$$\|\vec{b}_\sigma\|_*=\|b_{\sigma(1)}\|_*\cdots\|b_{\sigma(i)}\|_*.$$

If $\sigma=\{1,\dots,m\}$, we simply write

$$\|\vec{b}\|_*=\|b_1\|_*\cdots\|b_m\|_*.$$

Definition 1.2. Let $\rho>1$, $1<p\leqslant\infty$ and $\gamma,\tau\in\mathbb{N}$. Suppose $b_i\in RBMO(\mu)$ for $i=1,2,\dots,\tau$. A function $h\in L^1_{\mathrm{loc}}(\mu)$ is called a $(\vec{b},p,\tau,\gamma)$-atomic block if

(a) there exists some cube R such that $\operatorname{supp}(h)\subset R$;
(b) $\int_{\mathbb{R}^d}h(y)\,d\mu(y)=0$;
(c) $\int_{\mathbb{R}^d}h(y)b_\sigma(y)\,d\mu(y)=0$ for all $1\leqslant i\leqslant\tau$ and $\sigma\in C_i^\tau$;
(d) for $j=1,2$, there are functions a_j supported on cube $Q_j\subset R$ and numbers $\lambda_j\in\mathbb{R}$ such that $h=\lambda_1a_1+\lambda_2a_2$, and

$$\|a_j\|_{L^p(\mu)}\leqslant\left[\mu(\rho Q_j)\right]^{1/p-1}K_{Q_j,R}^{-\gamma}.$$

Then we denote

$$|h|_{H^{1,p}_{\vec{b},\tau,\gamma}(\mu)}=|\lambda_1|+|\lambda_2|.$$

We say that $f\in H^{1,p}_{\vec{b},\tau,\gamma}(\mu)$ if there are $(\vec{b},p,\tau,\gamma)$-atomic blocks $\{h_k\}_{k\in\mathbb{N}}$ such that

$$f=\sum_{k=1}^\infty h_k$$

with $\sum_{k=1}^\infty|h_k|_{H^{1,p}_{\vec{b},\tau,\gamma}(\mu)}<\infty$. The $H^{1,p}_{\vec{b},\tau,\gamma}(\mu)$ norm of f is defined by

$$\|f\|_{H^{1,p}_{\vec{b},\tau,\gamma}(\mu)}=\inf\left\{\sum_k|b_k|_{H^{1,p}_{\vec{b},\tau,\gamma}(\mu)}\right\},$$

where the infimum is taken over all the possible decompositions of f in $(\vec{b},p,\tau,\gamma)$-atomic blocks.

Remark 1.1. By an argument similar to that in [10], we easily see that the above definition is independent of the chosen constant $\rho>1$. If $\tau=0$, the space $H^{1,p}_{\vec{b},\tau,\gamma}(\mu)$ is just the atomic Hardy space introduced by Tolsa in [10] when $\gamma=1$ and when $\gamma>1$ by the authors in [2], which was proved in [2,10] to be the Hardy space $H^1(\mu)$ of Tolsa in [13] with equivalent norms; see also Definition 1.4 below. However, it is still open if the spaces $H^{1,p}_{\vec{b},\tau,\gamma}(\mu)$ are equivalent for any fixed $\tau\in\mathbb{N}$ and different $\gamma\in\mathbb{N}$ and $1<p\leqslant\infty$. But we have the following obvious properties that for any $\tau\in\mathbb{N}$, $1<p\leqslant\infty$ and $\gamma_1,\gamma_2\in\mathbb{N}$ with $1\leqslant\gamma_1<\gamma_2$,

$$H^{1,p}_{\vec{b},\tau,\gamma_2}(\mu)\subset H^{1,p}_{\vec{b},\tau,\gamma_1}(\mu)\subset H^1(\mu),$$

and for any $\tau, \gamma \in \mathbb{N}$ and $1 < p_1 < p_2 \leqslant \infty$,

$$H^{1,\infty}_{\vec{b},\tau,\gamma}(\mu) \subset H^{1,p_2}_{\vec{b},\tau,\gamma}(\mu) \subset H^{1,p_1}_{\vec{b},\tau,\gamma}(\mu) \subset H^1(\mu).$$

Here is one of the main results of this paper.

Theorem 1.1. *Let $1 < p \leqslant \infty$, $m \in \mathbb{N}$ and $b_i \in RBMO(\mu)$ for $i = 1, 2, \ldots, m$. Let T and $T_{\vec{b}}$ be as in* (1.3) *and* (1.6)*, respectively. Suppose that T is bounded on $L^2(\mu)$. Then the multilinear commutator $T_{\vec{b}}$ is bounded from $H^{1,p}_{\vec{b},m,m+1}(\mu)$ into $L^1(\mu)$.*

Remark 1.2. Let us consider the multilinear commutator of the fractional integral operator, $I_{\alpha;\vec{b}}$, defined by

$$I_{\alpha;\vec{b}}(f)(x) = \int_{\mathbb{R}^d} \prod_{i=1}^{m} \big[b_i(x) - b_i(y)\big] \frac{f(y)}{|x-y|^{n-\alpha}} \, d\mu(y), \tag{1.7}$$

where $m \in \mathbb{N}$, $b_i \in RBMO(\mu)$ for $i = 1, 2, \ldots, m$ and $0 < \alpha < n$. By an argument similar to the proof of Theorem 1.1, we can prove the following result for $I_{\alpha;\vec{b}}$ and we omit the details by similarity.

Theorem 1.2. *Let $0 < \alpha < n$, $1 < p \leqslant \infty$, $m \in \mathbb{N}$, $b_i \in RBMO(\mu)$ for $i = 1, 2, \ldots, m$, and $I_{\alpha;\vec{b}}$ be as in* (1.7)*. Then $I_{\alpha;\vec{b}}$ is bounded from $H^{1,p}_{\vec{b},m,m+1}(\mu)$ into $L^{n/(n-\alpha)}(\mu)$.*

In [1], the authors proved that $\widetilde{T}$ is bounded on the Hardy space $H^1(\mu)$ if $\widetilde{T}^*(1) = 0$. Motivated by this, we now consider the boundedness of $T_{\vec{b}}$ from $H^{1,p}_{\vec{b},m,m+2}(\mu)$ into $H^1(\mu)$ with the assumption that $T^*_{\vec{b}}(1) = 0$. Here, by $T^*_{\vec{b}}(1) = 0$, we mean that for any bounded function h with compact support satisfying (b) and (c) of Definition 1.2,

$$\int_{\mathbb{R}^d} T_{\vec{b}}(h)(x) \, d\mu(x) = 0. \tag{1.8}$$

We point out that for a such function h, it is easy to see that $h \in H^{1,p}_{\vec{b},m,m+2}(\mu)$ and therefore, $T_{\vec{b}}(h) \in L^1(\mu)$ by Theorem 1.1. Also, if $T_{\vec{b}}(h) \in H^1(\mu)$, then $T_{\vec{b}}(h)$ should satisfy (1.8) by the definition of the Hardy space $H^1(\mu)$; see [10,13] or Definition 1.4 below. Thus, in some sense, condition (1.8) is also necessary. We remark that this result is new even when μ is the d-dimensional Lebesgue measure.

Definition 1.3. Given $f \in L^1_{\mathrm{loc}}(\mu)$, we set

$$M_\Phi f(x) = \sup_{\varphi \sim x} \left| \int_{\mathbb{R}^d} f\varphi \, d\mu \right|,$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d)$ and satisfies

(i) $\|\varphi\|_{L^1(\mu)} \leqslant 1$,
(ii) $0 \leqslant \varphi(y) \leqslant |y-x|^{-n}$ for all $y \in \mathbb{R}^d$, and
(iii) $|\nabla\varphi(y)| \leqslant |y-x|^{-(n+1)}$ for all $y \in \mathbb{R}^d$, where $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$.

Based on Theorem 1.2 of Tolsa in [13], we define the Hardy space $H^1(\mu)$ as follows.

Definition 1.4. The Hardy space $H^1(\mu)$ is the set of all functions $f \in L^1(\mu)$ satisfying that $\int_{\mathbb{R}^d} f\,d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. Moreover, we define the norm of $f \in H^1(\mu)$ by

$$\|f\|_{H^1(\mu)} = \|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}.$$

Another main result of this paper is as follows.

Theorem 1.3. *Let $1 < p \leqslant \infty$, $m \in \mathbb{N}$ and $b_i \in RBMO(\mu)$ for $i = 1, 2, \ldots, m$. Let T and $T_{\vec{b}}$ be as in* (1.3) *and* (1.6)*, respectively. Suppose that T is bounded on $L^2(\mu)$ and $T^*_{\vec{b}}(1) = 0$ as in* (1.8)*. Then $T_{\vec{b}}$ is bounded from $H^{1,p}_{\vec{b},m,m+2}(\mu)$ into $H^1(\mu)$.*

In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$.

2. Proof of Theorem 1.1

We begin with some necessary lemmas.

Lemma 2.1. *Let $1 < p < \infty$, $m \in \mathbb{N}$ and $b_i \in RBMO(\mu)$ for $i = 1, 2, \ldots, m$. Let T and $T_{\vec{b}}$ be as in* (1.3) *and* (1.6)*, respectively. Suppose that T is bounded on $L^2(\mu)$. Then $T_{\vec{b}}$ is bounded on $L^p(\mu)$ with the norm no more than $C\|\vec{b}\|_*$, where $C > 0$ is a constant.*

Lemma 2.1 was proved by Tolsa for the case $m = 1$ in [10] and by the authors for the cases $m > 1$ in [3]. The following Lemma 2.2 is a simple corollary of the John–Nirenberg inequality with non-doubling measure; see [10].

Lemma 2.2. *Let $m \in \mathbb{N}$, $b_i \in RBMO(\mu)$ for $i = 1, 2, \ldots, m$, $\rho > 1$ and $1 \leqslant p < \infty$. Then there exists a constant $C > 0$ such that for any cube Q,*

$$\left\{\frac{1}{\mu(\rho Q)} \int_Q \prod_{i=1}^m \left|b_i(x) - m_{\widetilde{Q}}(b_i)\right|^p d\mu(x)\right\}^{1/p} \leqslant C \prod_{i=1}^m \|b_i\|_*.$$

Proof of Theorem 1.1. By a standard argument, it suffices to verify that for any $(\vec{b}, p, m, m+1)$-atomic block h as in Definition 1.2 with $\rho = 4$, $T_{\vec{b}}(h)$ is in $L^1(\mu)$ with norm no more than $C|h|_{H^{1,p}_{\vec{b},m,m+1}(\mu)}$, where $C > 0$ is a constant independent of h. Let all the notation be the same as in Definition 1.2. By our choices, a_j, $j = 1, 2$, now satisfies the following size condition:

$$\|a_j\|_{L^p(\mu)} \leqslant \mu(4Q_j)^{1/p-1}[K_{Q_j,R}]^{-(m+1)}. \tag{2.1}$$

Write

$$\int_{\mathbb{R}^d} \left|T_{\vec{b}}(h)(x)\right| d\mu(x) = \int_{2R} \left|T_{\vec{b}}(h)(x)\right| d\mu(x) + \int_{\mathbb{R}^d\setminus 2R} \left|T_{\vec{b}}(h)(x)\right| d\mu(x) = \mathrm{M} + \mathrm{N}.$$

To estimate M, we further decompose

$$\mathrm{M} \leqslant \sum_{j=1}^{2} |\lambda_j| \int_{2Q_j} \left|T_{\vec{b}}(a_j)(x)\right| d\mu(x) + \sum_{j=1}^{2} |\lambda_j| \int_{2R_j\setminus 2Q_j} \left|T_{\vec{b}}(a_j)(x)\right| d\mu(x) = \mathrm{M}_1 + \mathrm{M}_2.$$

From the Hölder inequality, Lemma 2.1 and (2.1), it follows that

$$\begin{aligned}\mathrm{M}_1 &\leqslant \sum_{j=1}^{2} |\lambda_j| \left\|T_{\vec{b}}(a_j)\right\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \\ &\leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j|.\end{aligned}$$

Let $N_{j,1} = N_{2Q_j,2R_j}$ for $j = 1, 2$. With the aid of the formula that for $x, y \in \mathbb{R}^d$,

$$\prod_{i=1}^{m} \left[b_i(x) - b_i(y)\right] = \sum_{i=0}^{m} \sum_{\sigma\in C_i^m} \left[b(x) - m_{\widetilde{Q}_j}(b)\right]_\sigma \left[m_{\widetilde{Q}_j}(b) - b(y)\right]_{\sigma'}, \tag{2.2}$$

where if $i = 0$, then $\sigma' = \{1, 2, \dots, m\}$ and $\sigma = \emptyset$, by (1.1), the Hölder inequality, Lemma 2.2 and (2.1), we easily obtain

$$\begin{aligned}\mathrm{M}_2 &\leqslant \sum_{j=1}^{2} |\lambda_j| \sum_{k=1}^{N_{j,1}} \int_{2^{k+1}Q_j\setminus 2^k Q_j} \left| \int_{Q_j} \prod_{i=1}^{m} \left[b_i(x) - b_i(y)\right] K(x, y) a_j(y)\, d\mu(y) \right| d\mu(x) \\ &\leqslant C \sum_{j=1}^{2} |\lambda_j| \sum_{i=0}^{m} \sum_{\sigma\in C_i^m} \sum_{k=1}^{N_{j,1}} \int_{2^{k+1}Q_j\setminus 2^k Q_j} \left|\left[b(x) - m_{\widetilde{Q}_j}(b)\right]_\sigma\right| \\ &\quad \times \int_{Q_j} \left|\left[m_{\widetilde{Q}_j}(b) - b(y)\right]_{\sigma'}\right| \frac{|a_j(y)|}{|x-y|^n}\, d\mu(y)\, d\mu(x) \\ &\leqslant C \sum_{j=1}^{2} |\lambda_j| \sum_{i=0}^{m} \sum_{\sigma\in C_i^m} \left\{ \int_{Q_j} \left|a_j(y)\right|^p d\mu(y) \right\}^{1/p} \left\{ \int_{Q_j} \left|\left[m_{\widetilde{Q}_j}(b) - b(y)\right]_{\sigma'}\right|^{p'} d\mu(y) \right\}^{1/p'} \\ &\quad \times \left\{ \sum_{l=0}^{i} \sum_{\eta\in C_l^i} \sum_{k=1}^{N_{j,1}} \frac{1}{l(2^k Q_j)^n} \right. \\ &\quad \left. \times \int_{2^{k+1}Q_j} \left|\left[b(x) - m_{\widetilde{2^{k+1}Q_j}}(b)\right]_\eta \left[m_{\widetilde{2^{k+1}Q_j}}(b) - m_{\widetilde{Q}_j}(b)\right]_{\eta'}\right| d\mu(x) \right\}\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \sum_{k=1}^{N_{j,1}} \frac{\mu(2^{k+2}Q_j)}{l(2^k Q_j)^n} \big[K_{\widetilde{Q}_j, \widetilde{2^{k+1}Q_j}}\big]^i \\
&\leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} [K_{Q_j,R}]^{m+1} \\
&\leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j|,
\end{aligned}
$$

where we have used the fact that for any $1 \leqslant k \leqslant N_{j,1}$, $K_{\widetilde{Q}_j, \widetilde{2^{k+1}Q_j}} \leqslant CK_{Q_j,R}$; see [10].

Now we turn to estimate N. Denote the center of R by x_R. The propositions (b) and (c) of Definition 1.2, (2.2), (1.2), the Hölder inequality, Lemma 2.2 and (2.1) lead to

$$
\begin{aligned}
\mathrm{N} &= \int_{\mathbb{R}^d \setminus 2R} \left| \int_R \prod_{i=1}^{m} \big[b_i(x) - b_i(y)\big]\big[K(x,y) - K(x,x_R)\big] h(y)\, d\mu(y) \right| d\mu(x) \\
&\leqslant C \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \sum_{k=1}^{\infty} \int_R \big|\big[b(y) - m_{\widetilde{R}}(b)\big]_{\sigma'}\big| \big|h(y)\big| \\
&\quad \times \int_{2^{k+1}R \setminus 2^k R} \big|\big[b(x) - m_{\widetilde{R}}(b)\big]_{\sigma}\big| \frac{|x - x_R|^\delta}{|x-y|^{n+\delta}}\, d\mu(x)\, d\mu(y) \\
&\leqslant C \sum_{j=1}^{2} |\lambda_j| \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \left\{ \sum_{l=0}^{m-i} \sum_{\eta \in C_l^{m-i}} \int_{Q_j} \big|\big[b(y) - m_{\widetilde{Q}_j}(b)\big]_\eta \big[m_{\widetilde{Q}_j}(b) - m_{\widetilde{R}}(b)\big]_{\eta'} \right. \\
&\quad \left. \times \big|\big|a_j(y)\big|\, d\mu(y) \right\} \\
&\quad \times \left\{ \sum_{s=0}^{i} \sum_{\theta \in C_s^i} \sum_{k=1}^{\infty} \frac{l(R)^\delta}{l(2^k Q_j)^{n+\delta}} \right. \\
&\quad \left. \times \int_{2^{k+1}R} \big|\big[b(x) - m_{\widetilde{2^{k+1}R}}(b)\big]_\theta \big[m_{\widetilde{2^{k+1}R}}(b) - m_{\widetilde{R}}(b)\big]_{\theta'}\big|\, d\mu(x) \right\} \\
&\leqslant C \sum_{j=1}^{2} |\lambda_j| \|a_j\|_L^p(\mu) \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \|\vec{b}_\sigma\|_* \sum_{k=1}^{\infty} \frac{\mu(2^{k+2}R) l(R)^\delta}{l(2^k Q_j)^{n+\delta}} \big[K_{\widetilde{R}, \widetilde{2^{k+1}R}}\big]^i \\
&\quad \times \sum_{l=0}^{m-i} \sum_{\eta \in C_l^{m-i}} \left\{ \int_{Q_j} \big|\big[b(y) - m_{\widetilde{Q}_j}(b)\big]_\eta \big[m_{\widetilde{Q}_j}(b) - m_{\widetilde{R}}(b)\big]_{\eta'}\big|^{p'} d\mu(y) \right\}^{1/p'} \\
&\leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} [K_{Q_j,R}]^m
\end{aligned}
$$

$$\leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j|.$$

Combining the estimates for M and N, we have finished the proof of Theorem 1.1. □

3. Proof of Theorem 1.3

In order to prove Theorem 1.3, we first recall the following basic fact in [1].

Lemma 3.1. *Let M_Φ be as in Definition* 1.3 *and* $1 < p < \infty$. *Then M_Φ is bounded on $L^p(\mu)$.*

Proof of Theorem 1.3. Just as in the proof of Theorem 1.1, we only need to verify that for any $(\vec{b}, p, m, m+2)$-atomic block h as in Definition 1.2 with $\rho = 4$, $T_{\vec{b}}(h)$ is in $H^1(\mu)$ with norm no more than $C|h|_{H^{1,p}_{\vec{b},m,m+2}(\mu)}$, where $C > 0$ is a constant independent of h. By our choices, for $j = 1, 2$, a_j satisfies the following size condition that

$$\|a_j\|_{L^p(\mu)} \leqslant \mu(4Q_j)^{1/p-1}[K_{Q_j,R}]^{-(m+2)}. \tag{3.1}$$

By the assumption that $T^*_{\vec{b}}(1) = 0$, Theorem 1.1 and Definition 1.4, we deduce that the proof of Theorem 1.3 can be reduced to proving that

$$\big\|M_\Phi\big[T_{\vec{b}}(h)\big]\big\|_{L^1(\mu)} \leqslant C\|\vec{b}\|_* |h|_{H^{1,p}_{\vec{b},m,m+2}(\mu)}. \tag{3.2}$$

Write

$$\big\|M_\Phi\big[T_{\vec{b}}(h)\big]\big\|_{L^1(\mu)} = \int_{4R} M_\Phi\big[T_{\vec{b}}(h)\big](x)\,d\mu(x) + \int_{\mathbb{R}^d\setminus 4R} M_\Phi\big[T_{\vec{b}}(h)\big](x)\,d\mu(x) = \mathrm{I} + \mathrm{II}.$$

Noting that M_Φ is sublinear, we can control I by

$$\mathrm{I} \leqslant \int_{4R} M_\Phi\big\{\big[T_{\vec{b}}(h)\big]\chi_{8R}\big\}(x)\,d\mu(x) + \int_{4R} M_\Phi\big\{\big[T_{\vec{b}}(h)\big]\chi_{\mathbb{R}^d\setminus 8R}\big\}(x)\,d\mu(x) = \mathrm{I}_1 + \mathrm{I}_2.$$

From the fact that for $j = 1, 2$, $Q_j \subset R$, it follows that for any $z \in Q_j$ and any $y \in 2^{k+1}R \setminus 2^kR$, $k \geqslant 3$, $|y - z| \geqslant l(2^{k-2}R)$. By this fact, (ii) of Definition 1.3, (2.2), (1.1), Lemma 2.2 and (3.1), we obtain

$$\begin{aligned}
\mathrm{I}_2 &\leqslant \int_{4R} \sup_{\varphi\sim x}\bigg[\int_{\mathbb{R}^d\setminus 8R} \big|T_{\vec{b}}(h)(y)\big|\varphi(y)\,d\mu(y)\bigg]\,d\mu(x) \\
&\leqslant \sum_{j=1}^{2}|\lambda_j| \sum_{k=3}^{\infty} \int_{4R} \int_{2^{k+1}R\setminus 2^kR} \bigg|\int_{Q_j} \prod_{i=1}^{\infty}\big[b_i(y) - b_i(z)\big]K(y,z)a_j(z)\,d\mu(z)\bigg| \\
&\quad \times \frac{1}{|x-y|^n}\,d\mu(y)\,d\mu(x) \\
&\leqslant C\sum_{j=1}^{2}|\lambda_j| \sum_{i=0}^{m} \sum_{\sigma\in C_i^m} \sum_{k=3}^{\infty} \int_{4R} \int_{2^{k+1}R\setminus 2^kR} \frac{|[b(y) - m_{\widetilde{Q}_j}(b)]_\sigma|}{|x-y|^n}
\end{aligned}$$

$$
\begin{aligned}
&\quad\times \int_{Q_j} \frac{|[m_{\widetilde{Q}_j}(b)-b(z)]_{\sigma'}||a_j(z)|}{|y-z|^n}\,d\mu(z)\,d\mu(y)\,d\mu(x)\\
&\leqslant C\sum_{j=1}^{2}|\lambda_j|\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\sum_{k=3}^{\infty}\frac{\mu(4R)}{l(2^kR)^n}\frac{\|a_j\|_{L^p(\mu)}}{l(2^kR)^n}\left\{\int_{Q_j}\left|\left[m_{\widetilde{Q}_j}(b)-b(z)\right]_{\sigma'}\right|^{p'}d\mu(z)\right\}^{1/p'}\\
&\quad\times\int_{2^{k+1}R}\left|\left[b(y)-m_{\widetilde{2^{k+1}R}}(b)+m_{\widetilde{2^{k+1}R}}(b)-m_{\widetilde{R}}(b)+m_{\widetilde{R}}(b)-m_{\widetilde{Q}_j}(b)\right]_{\sigma}\right|d\mu(y)\\
&\leqslant C\|\vec{b}\|_*\sum_{j=1}^{2}|\lambda_j|\|a_j\|_{L^p(\mu)}\mu(2Q_j)^{1/p'}\\
&\quad\times\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\sum_{k=3}^{\infty}\frac{\mu(4R)\mu(2^{k+2}R)}{l(2^kR)^n l(2^kR)^n}\left\{\left[K_{\widetilde{R},\widetilde{2^{k+1}R}}\right]^i+\left[K_{\widetilde{Q}_j,\widetilde{R}}\right]^i\right\}\\
&\leqslant C\|\vec{b}\|_*\sum_{j=1}^{2}|\lambda_j|.
\end{aligned}
$$

To estimate I_1, we write

$$
\begin{aligned}
\mathrm{I}_1 &\leqslant \sum_{j=1}^{2}|\lambda_j|\int_{4Q_j} M_\Phi\left\{\left[T_{\vec{b}}(a_j)\right]\chi_{8R}\right\}(x)\,d\mu(x)\\
&\quad+\sum_{j=1}^{2}|\lambda_j|\int_{4R\setminus 4Q_j} M_\Phi\left\{\left[T_{\vec{b}}(a_j)\right]\chi_{2Q_j}\right\}(x)\,d\mu(x)\\
&\quad+\sum_{j=1}^{2}|\lambda_j|\int_{4R\setminus 4Q_j} M_\Phi\left\{\left[T_{\vec{b}}(a_j)\right]\chi_{8R\setminus 2Q_j}\right\}(x)\,d\mu(x)\\
&=\mathrm{I}_{11}+\mathrm{I}_{12}+\mathrm{I}_{13}.
\end{aligned}
$$

The Hölder inequality, Lemma 3.1, Lemma 2.1 and (3.1) lead to

$$
\begin{aligned}
\mathrm{I}_{11} &\leqslant \sum_{j=1}^{2}|\lambda_j|\mu(4Q_j)^{1/p'}\left\|M_\Phi\left\{\left[T_{\vec{b}}(a_j)\right]\chi_{8R}\right\}\right\|_{L^p(\mu)}\\
&\leqslant C\sum_{j=1}^{2}|\lambda_j|\mu(4Q_j)^{1/p'}\left\|T_{\vec{b}}(a_j)\right\|_{L^p(\mu)}\\
&\leqslant C\|\vec{b}\|_*\sum_{j=1}^{2}|\lambda_j|\|a_j\|_{L^p(\mu)}\mu(4Q_j)^{1/p'}\\
&\leqslant C\|\vec{b}\|_*\sum_{j=1}^{2}|\lambda_j|.
\end{aligned}
$$

For $j = 1, 2$, denote $N_{Q_j,4R}$ simply by $N_{j,2}$. By (ii) of Definition 1.3, the Hölder inequality, Lemma 2.1 and (3.1), we have

$$\begin{aligned}
\mathrm{I}_{12} &\leqslant \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{j,2}} \int\limits_{2^{k+1}Q_j \setminus 2^k Q_j} \sup_{\varphi \sim x} \left| \int\limits_{2Q_j} T_{\vec{b}}(a_j)(y)\varphi(y)\, d\mu(y) \right| d\mu(x) \\
&\leqslant C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{j,2}} \int\limits_{2^{k+1}Q_j \setminus 2^k Q_j} \frac{1}{l(2^k Q_j)^n}\, d\mu(x) \int\limits_{2Q_j} \left| T_{\vec{b}}(a_j)(y) \right| d\mu(y) \\
&\leqslant C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{j,2}} \frac{\mu(2^{k+1}Q_j)}{l(2^k Q_j)^n} \left\| T_{\vec{b}}(a_j) \right\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \\
&\leqslant C \|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} K_{Q_j,R} \\
&\leqslant C \|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j|.
\end{aligned}$$

For I_{13}, we further decompose it into

$$\begin{aligned}
\mathrm{I}_{13} &\leqslant \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{j,2}} \int\limits_{2^{k+1}Q_j \setminus 2^k Q_j} M_\Phi\left[\left| T_{\vec{b}}(a_j) \right| \chi_{2^{k+2}Q_j \setminus 2^{k-1}Q_j} \right](x)\, d\mu(x) \\
&\quad + \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{j,2}} \int\limits_{2^{k+1}Q_j \setminus 2^k Q_j} M_\Phi\left[\left| T_{\vec{b}}(a_j) \right| \chi_{\max\{2^{k+2}Q_j, 8R\} \setminus 2^{k+2}Q_j} \right](x)\, d\mu(x) \\
&\quad + \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{j,2}} \int\limits_{2^{k+1}Q_j \setminus 2^k Q_j} M_\Phi\left[\left| T_{\vec{b}}(a_j) \right| \chi_{2^{k-1}Q_j \setminus 2Q_j} \right](x)\, d\mu(x) \\
&= \mathrm{E} + \mathrm{F} + \mathrm{G}.
\end{aligned}$$

The Hölder inequality, Lemma 3.1, (2.2), (1.1), Lemma 2.2 and (3.1) tell us that

$$\begin{aligned}
\mathrm{E} &\leqslant \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{j,2}} \mu\big(2^{k+1}Q_j\big)^{1/p'} \left\| M_\Phi\left[\left| T_{\vec{b}}(a_j) \right| \chi_{2^{k+2}Q_j \setminus 2^{k-1}Q_j} \right] \right\|_{L^p(\mu)} \\
&\leqslant C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{j,2}} \mu\big(2^{k+1}Q_j\big)^{1/p'} \left\{ \int\limits_{2^{k+2}Q_j \setminus 2^{k-1}Q_j} \left| T_{\vec{b}}(a_j) \right|^p d\mu(y) \right\}^{1/p} \\
&\leqslant C \sum_{j=1}^{2} |\lambda_j| \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \sum_{k=2}^{N_{j,2}} \mu\big(2^{k+1}Q_j\big)^{1/p'} \left\{ \int\limits_{2^{k+2}Q_j \setminus 2^{k-1}Q_j} \left| \left[b(y) - m_{\widetilde{Q}_j}(b) \right]_\sigma \right|^p \right.
\end{aligned}$$

$$\times \left| \int_{Q_j} \left[m_{\widetilde{Q}_j}(b) - b(z)\right]_{\sigma'} K(y,z) a_j(z)\, d\mu(z) \right|^p d\mu(y) \Biggr\}^{1/p}$$

$$\leqslant C \sum_{j=1}^{2} |\lambda_j| \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \sum_{k=2}^{N_{j,2}} \frac{\mu(2^{k+1}Q_j)^{1/p'}}{l(2^k Q_j)^n} \int_{Q_j} \left|\left[m_{\widetilde{Q}_j}(b) - b(z)\right]_{\sigma'}\right| \left|a_j(z)\right| d\mu(z)$$

$$\times \Biggl\{ \int_{2^{k+2}Q_j} \left|\left[b(y) - m_{\widetilde{2^{k+2}Q_j}}(b) + m_{\widetilde{2^{k+2}Q_j}}(b) - m_{\widetilde{Q}_j}(b)\right]_\sigma\right|^p d\mu(y) \Biggr\}^{1/p}$$

$$\leqslant C \sum_{j=1}^{2} |\lambda_j| \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \|\vec{b}_\sigma\|_* \sum_{k=2}^{N_{j,2}} \frac{\mu(2^{k+3}Q_j)}{l(2^k Q_j)^n} \left[K_{\widetilde{Q}_j, \widetilde{2^{k+2}Q_j}}\right]^i$$

$$\times \Biggl\{ \int_{Q_j} \left|a_j(z)\right|^p d\mu(z) \Biggr\}^{1/p} \Biggl\{ \int_{Q_j} \left|\left[m_{\widetilde{Q}_j}(b) - b(z)\right]_{\sigma'}\right|^{p'} d\mu(z) \Biggr\}^{1/p'}$$

$$\leqslant C \|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} [K_{Q_j,R}]^{m+1}$$

$$\leqslant C \|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j|.$$

By (ii) of Definition 1.3, (2.2), (1.1), the Hölder inequality, Lemma 2.2 and (3.1), we easily see

$$\mathrm{G} \leqslant \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{j,2}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \sup_{\varphi \sim x} \Biggl[\int_{2^{k-1}Q_j \setminus 2Q_j} \left|T_{\vec{b}}(a_j)(y)\right| \varphi(y)\, d\mu(y) \Biggr] d\mu(x)$$

$$\leqslant \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{j,2}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \sum_{l=1}^{k-2} \int_{2^{l+1}Q_j \setminus 2^l Q_j} \frac{|T_{\vec{b}}(a_j)(y)|}{|y-x|^n}\, d\mu(y)\, d\mu(x)$$

$$\leqslant C \sum_{j=1}^{2} |\lambda_j| \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \sum_{k=2}^{N_{j,2}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \sum_{l=1}^{k-2} \int_{2^{l+1}Q_j \setminus 2^l Q_j} \frac{|[b(y) - m_{\widetilde{Q}_j}(b)]_\sigma|}{|y-x|^n}$$

$$\times \left| \int_{Q_j} \left[m_{\widetilde{Q}_j}(b) - b(z)\right]_{\sigma'} K(y,z) a_j(z)\, d\mu(z) \right| d\mu(y)\, d\mu(x)$$

$$\leqslant C \sum_{j=1}^{2} |\lambda_j| \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \sum_{k=2}^{N_{j,2}} \frac{\mu(2^{k+1}Q)}{l(2^k Q_j)^n} \sum_{l=1}^{k-2} \frac{1}{l(2^l Q_j)^n} \int_{2^{l+1}Q_j} \left|\left[b(y) - m_{\widetilde{Q}_j}(b)\right]_\sigma\right| d\mu(y)$$

$$\times \Biggl\{ \int_{Q_j} \left|a_j(z)\right|^p d\mu(z) \Biggr\}^{1/p} \Biggl\{ \int_{Q_j} \left|\left[m_{\widetilde{Q}_j}(b) - b(z)\right]_{\sigma'}\right|^{p'} d\mu(z) \Biggr\}^{1/p'}$$

$$\leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j| \sum_{i=0}^{m} \sum_{\sigma\in C_i^m} \sum_{k=2}^{N_{j,2}} \frac{\mu(2^{k+1}Q)}{l(2^k Q_j)^n} \sum_{l=1}^{k-2} \frac{\mu(2^{l+2}Q_j)}{l(2^l Q_j)^n} \big[K_{\widetilde{Q}_j, \widetilde{2^{l+1}Q_j}}\big]^i$$

$$\leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} [K_{Q_j,R}]^{m+2}$$

$$\leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j|.$$

An argument similar to the estimate for G can also give us a desired estimate for F. The estimates for E, F and G lead us a desired estimate for I_{13}. Combining the estimates for I_{11}, I_{12}, I_{13} and I_2 yield

$$\mathrm{I} = \int_{4R} M_\Phi\big[T_{\vec{b}}(h)\big](x)\,d\mu(x) \leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j| = C\|\vec{b}\|_* |h|_{H^{1,p}_{\vec{b},m,m+2}(\mu)}. \tag{3.3}$$

Now we turn to the estimate for II. Invoking that $T^*_{\vec{b}}(1) = 0$, we obtain

$$\begin{aligned}
\mathrm{II} &= \int_{\mathbb{R}^d\setminus 4R} \sup_{\varphi\sim x} \left| \int_{\mathbb{R}^d} T_{\vec{b}}(h)(y)\big[\varphi(y) - \varphi(x_R)\big]\,d\mu(y) \right| d\mu(x) \\
&\leqslant \int_{\mathbb{R}^d\setminus 4R} \sup_{\varphi\sim x} \left| \int_{2R} T_{\vec{b}}(h)(y)\big[\varphi(y) - \varphi(x_R)\big]\,d\mu(y) \right| d\mu(x) \\
&\quad + \int_{\mathbb{R}^d\setminus 4R} \sup_{\varphi\sim x} \left| \int_{\mathbb{R}^d\setminus 2R} T_{\vec{b}}(h)(y)\big[\varphi(y) - \varphi(x_R)\big]\,d\mu(y) \right| d\mu(x) \\
&= \mathrm{II}_1 + \mathrm{II}_2.
\end{aligned}$$

Note that for any $z \in 2R$, $x \in 2^{k+1}R \setminus 2^k R$, $k \geqslant 2$, $|x - z| \geqslant l(2^{k-2}R)$. This together with (iii) of Definition 1.3 and the mean value theorem leads to

$$\big|\varphi(y) - \varphi(x_R)\big| \leqslant C \frac{l(R)}{l(2^{k-2}R)^{n+1}} \tag{3.4}$$

for $y \in 2R$. By (3.4), (2.2), (1.1), the Hölder inequality, Lemmas 2.1 and 2.2, and (3.1), we have

$$\begin{aligned}
\mathrm{II}_1 &\leqslant \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \sup_{\varphi\sim x} \left[\int_{2R\setminus 2Q_j} \big|T_{\vec{b}}(a_j)(y)\big| \big|\varphi(y) - \varphi(x_R)\big|\,d\mu(y) \right] d\mu(x) \\
&\quad + \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \sup_{\varphi\sim x} \left[\int_{2Q_j} \big|T_{\vec{b}}(a_j)(y)\big| \big|\varphi(y) - \varphi(x_R)\big|\,d\mu(y) \right] d\mu(x) \\
&\leqslant C \sum_{j=1}^{2} |\lambda_j| \sum_{i=0}^{m} \sum_{\sigma\in C_i^m} \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \frac{l(R)}{l(2^k R)^{n+1}} \sum_{l=1}^{N_{j,2}-1} \int_{2^{l+1}Q_j\setminus 2^l Q_j} \big|\big[b(y) - m_{\widetilde{Q}_j}(b)\big]_\sigma\big|
\end{aligned}$$

$$
\begin{aligned}
&\times \int\limits_{Q_j} \bigl|\bigl[m_{\widetilde{Q}_j}(b) - b(z)\bigr]_{\sigma'}\bigr| \frac{|a_j(z)|}{|y-z|^n}\, d\mu(z)\, d\mu(y)\, d\mu(x) \\
&+ C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{\infty} \int\limits_{2^{k+1}R\setminus 2^k R} \frac{l(R)}{l(2^k R)^{n+1}} \bigl\|\bigl[T_{\vec{b}}(a_j)\bigr]\chi_{2Q_j}\bigr\|_{L^1(\mu)}\, d\mu(x) \\
\leqslant\ & C \sum_{j=1}^{2} |\lambda_j| \sum_{i=0}^{m} \sum_{\sigma\in C_i^m} \sum_{k=2}^{\infty} \frac{\mu(2^{k+1}R) l(R)}{l(2^k R)^{n+1}} \\
&\times \sum_{l=1}^{N_{j,2}-1} \frac{1}{l(2^l Q_j)^n} \int\limits_{2^{l+1}Q_j} \bigl|\bigl[b(y) - m_{\widetilde{Q}_j}(b)\bigr]_\sigma\bigr|\, d\mu(y) \\
&\times \int\limits_{Q_j} \bigl|a_j(z)\bigr| \bigl|\bigl[m_{\widetilde{Q}_j}(b) - b(z)\bigr]_{\sigma'}\bigr|\, d\mu(z) \\
&+ C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{\infty} \frac{\mu(2^{k+1}R) l(R)}{l(2^k R)^{n+1}} \bigl\|\bigl[T_{\vec{b}}(a_j)\bigr]\chi_{2Q_j}\bigr\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \\
\leqslant\ & C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p} \sum_{k=2}^{\infty} \frac{\mu(2^{k+1}R) l(R)}{l(2^k R)^{n+1}} \\
&\times \sum_{l=1}^{N_{j,2}-1} \frac{\mu(2^{l+2}Q_j)}{l(2^l Q_j)^n} \bigl[K_{\widetilde{Q}_j, \widetilde{2^{l+1}Q_j}}\bigr]^m \\
&+ C \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \\
\leqslant\ & C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} [K_{Q_j,R}]^{m+1} \\
\leqslant\ & C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j|.
\end{aligned}
$$

We further estimate II_2 by

$$
\begin{aligned}
\mathrm{II}_2 &= \sum_{k=2}^{\infty} \int\limits_{2^{k+1}R\setminus 2^k R} \sup_{\varphi\sim x} \Biggl| \int\limits_{\mathbb{R}^d\setminus 2R} T_{\vec{b}}(h)(y)\bigl[\varphi(y) - \varphi(x_R)\bigr]\, d\mu(y)\Biggr|\, d\mu(x) \\
&\leqslant \sum_{k=2}^{\infty} \int\limits_{2^{k+1}R\setminus 2^k R} M_\Phi\bigl[\bigl|T_{\vec{b}}(h)\bigr|\chi_{2^{k+2}R\setminus 2^{k-1}R}\bigr](x)\, d\mu(x) \\
&\quad + \sum_{k=2}^{\infty} \int\limits_{2^{k+1}R\setminus 2^k R} \sup_{\varphi\sim x} \Biggl[\ \int\limits_{2^{k+2}R\setminus 2^{k-1}R} \bigl|T_{\vec{b}}h(y)\bigr|\varphi(x_R)\, d\mu(y)\Biggr]\, d\mu(x)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=2}^{\infty}\int\limits_{2^{k+1}R\setminus 2^kR}\sup_{\varphi\sim x}\biggl[\int\limits_{\mathbb{R}^d\setminus 2^{k+2}R}\bigl|T_{\vec{b}}(h)(y)\bigr|\bigl\{\varphi(y)+\varphi(x_R)\bigr\}\,d\mu(y)\biggr]\,d\mu(x)\\
&+\sum_{k=2}^{\infty}\int\limits_{2^{k+1}R\setminus 2^kR}\sup_{\varphi\sim x}\biggl[\int\limits_{2^{k-1}R\setminus 2R}\bigl|T_{\vec{b}}(h)(y)\bigr|\bigl\{\varphi(y)+\varphi(x_R)\bigr\}\,d\mu(y)\biggr]\,d\mu(x)\\
&=\mathrm{II}_{21}+\mathrm{II}_{22}+\mathrm{II}_{23}+\mathrm{II}_{24}.
\end{aligned}
$$

From the Hölder inequality, Lemma 3.1, (b) and (c) of Definition 1.2, (2.2), (1.2), Lemma 2.2 and (3.1), we can deduce that

$$
\begin{aligned}
\mathrm{II}_{21}&\leqslant\sum_{k=2}^{\infty}\mu\bigl(2^{k+1}R\bigr)^{1/p'}\bigl\|M_{\Phi}\bigl[\bigl|T_{\vec{b}}(h)\bigr|\chi_{2^{k+2}R\setminus 2^{k-1}R}\bigr]\bigr\|_{L^p(\mu)}\\
&\leqslant C\sum_{k=2}^{\infty}\mu\bigl(2^{k+1}R\bigr)^{1/p'}\Biggl\{\int\limits_{2^{k+2}R\setminus 2^{k-1}R}\Biggl|\int\limits_{R}\prod_{i=1}^{m}\bigl[b_i(y)-b_i(z)\bigr]\bigl[K(y,z)-K(y,x_R)\bigr]\\
&\quad\times h(z)\,d\mu(z)\Biggr|^p\,d\mu(y)\Biggr\}^{1/p}\\
&\leqslant C\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\sum_{k=2}^{\infty}\mu\bigl(2^{k+1}R\bigr)^{1/p'}\Biggl\{\int\limits_{2^{k+2}R\setminus 2^{k-1}R}\bigl|\bigl[b(y)-m_{\widetilde{R}}(b)\bigr]_{\sigma}\bigr|^p\\
&\quad\times\Biggl|\int\limits_{R}\bigl[m_{\widetilde{R}}(b)-b(z)\bigr]_{\sigma'}\bigl[K(y,z)-K(y,x_R)\bigr]h(z)\,d\mu(z)\Biggr|^p\,d\mu(y)\Biggr\}^{1/p}\\
&\leqslant C\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\sum_{k=2}^{\infty}\frac{\mu(2^{k+1}R)^{1/p'}l(R)^{\delta}}{l(2^kR)^{n+\delta}}\Biggl\{\int\limits_{2^{k+2}R}\bigl|\bigl[b(y)-m_{\widetilde{R}}(b)\bigr]_{\sigma}\bigr|^p\,d\mu(y)\Biggr\}^{1/p}\\
&\quad\times\int\limits_{R}\bigl|\bigl[m_{\widetilde{R}}(b)-b(z)\bigr]_{\sigma'}\bigr|\bigl|h(z)\bigr|\,d\mu(z)\\
&\leqslant C\sum_{j=1}^{2}|\lambda_j|\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\|\vec{b}_{\sigma}\|_*\sum_{k=2}^{\infty}\frac{\mu(2^{k+3}R)l(R)^{\delta}}{l(2^kR)^{n+\delta}}\bigl[K_{\widetilde{R},\widetilde{2^{k+2}R}}\bigr]^i\\
&\quad\times\Biggl\{\int\limits_{Q_j}\bigl|a_j(z)\bigr|^p\,d\mu(z)\Biggr\}^{1/p}\Biggl\{\int\limits_{Q_j}\bigl|\bigl[m_{\widetilde{R}}(b)-b(z)\bigr]_{\sigma'}\bigr|^{p'}\,d\mu(z)\Biggr\}^{1/p'}\\
&\leqslant C\|\vec{b}\|_*\sum_{j=1}^{2}|\lambda_j|\|a_j\|_{L^p(\mu)}\mu(2Q_j)^{1/p'}[K_{Q_j,R}]^m\\
&\leqslant C\|\vec{b}\|_*\sum_{j=1}^{2}|\lambda_j|.
\end{aligned}
$$

An argument similar to the estimate for II_{21} can also give us a desired estimate for II_{22}.

Finally, we estimate II_{23}. By (b) and (c) of Definition 1.2, (ii) of Definition 1.3, (2.2), (1.2), the Hölder inequality, Lemma 2.2 and (3.1), we obtain

$$\begin{aligned}
\mathrm{II}_{23} &\leqslant \sum_{k=2}^{\infty} \int_{2^{k+1}R\setminus 2^k R} \sum_{l=k+2}^{\infty} \int_{2^{l+1}R\setminus 2^l R} \int_R \left|K(y,z)-K(y,x_R)\right| \prod_{i=1}^{m} \left|b_i(y)-b_i(z)\right| \\
&\quad \times \left|h(z)\right| d\mu(z) \left\{\frac{1}{|y-x|^n}+\frac{1}{|x_R-x|^n}\right\} d\mu(y)\, d\mu(x) \\
&\leqslant C\sum_{j=1}^{2}|\lambda_j| \sum_{i=0}^{m} \sum_{\sigma\in C_i^m} \sum_{k=2}^{\infty} \sum_{l=k+2}^{\infty} \frac{\mu(2^{k+1}R)}{l(2^kR)^n} \frac{l(R)^{\delta}}{l(2^lR)^{n+\delta}} \int_{2^{l+1}R} \left|\left[b(y)-m_{\widetilde{R}}(b)\right]_{\sigma}\right| d\mu(y) \\
&\quad \times \int_{Q_j} \left|\left[m_{\widetilde{R}}(b)-b(z)\right]_{\sigma'}\right| \left|a_j(z)\right| d\mu(z) \\
&\leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \sum_{i=0}^{m} \sum_{\sigma\in C_i^m} [K_{Q_j R}]^{m-i} \\
&\quad \times \sum_{k=2}^{\infty} \sum_{l=k+2}^{\infty} \frac{\mu(2^{k+1}R)}{l(2^kR)^n} \frac{\mu(2^{l+2}R)l(R)^{\delta}}{l(2^lR)^{n+\delta}} \left[K_{\widetilde{R},\widetilde{2^{l+1}R}}\right]^{i} \\
&\leqslant C\|\vec{b}\|_* \sum_{j=1}^{2} |\lambda_j|.
\end{aligned}$$

An argument similar to the estimate for II_{23} can also give us a desired estimate for II_{24}.

Combining the estimates for II_{21}, II_{22}, II_{23} and II_{24}, we obtain a desired estimate for II_2. The estimates for II_1 and II_2 tell us that

$$\mathrm{II} = \int_{\mathbb{R}^d\setminus 4R} M_{\Phi}\left[T_{\vec{b}}(h)\right](x)\, d\mu(x) \leqslant C|b|_{H^{1,p}_{\vec{b},m,m+2}(\mu)}. \tag{3.5}$$

The estimates (3.3) and (3.5) lead to (3.2) and this completes the proof of our theorem. □

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