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Periodic boundary value problem for non-Lipschitzian impulsive functional differential equations ☆

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Abstract

We consider a class of first-order impulsive functional differential equations, where the functional dependence is not necessarily a Lipschitzian function. The new maximum principle improves and extends previous results and uniqueness of solution between a lower and an upper solution for a particular nonlinear problem is presented. We give conditions for existence of extremal solutions in an interval delimited by a lower and an upper solution.
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1. Preliminaries

Impulsive functional differential equations have been considered, for example, in [15] and Refs. [3–8] therein and, recently, in [4,6,7,10,22,23]. Some particular nonimpulsive

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functional problems have been studied in [13,16–18]. General references about impulsive ordinary differential equations are [11,21] and the bases of the monotone iterative technique and the method of the lower and upper solutions for ordinary differential equations are exposed in [9].

Usually, it is easier to obtain existence results when the nonlinearities involved are Lipschitzian. The most classical condition guaranteeing existence and uniqueness for ordinary differential equations is the famous Picard–Lipschitz theorem. But, of course, when the nonlinearity is non-Lipschitzian, it is possible to find conditions which guarantee existence [1].

However the study of non-Lipschitzian nonlinearities is important. For instance, in [5], we can find a generalization of the Poincaré method to non-Lipschitzian dynamics. Also, for example, see [2,3,8,12,14,19,20].

As far as we know, in the treatment of impulsive functional differential equations, it has been always supposed that the functional dependence in the equation is given by mappings or operators satisfying a Lipschitz condition. Now we present more general results, showing applications to examples where functions providing the functional dependence are not necessarily Lipschitzian.

Here we consider the problem

$$\begin{cases} u'(t) = f(t, u(t), [\psi_k u_k](t)), \quad t \in \operatorname{int}(J_k),\ k = 1, 2, \ldots, p+1, \\ u(t_k^+) = I_k([\phi_k u_k](t_k)), \quad k = 1, \ldots, p, \\ u(0) = u(T), \end{cases} \tag{1}$$

where $J = [0, T]$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $J_k = [t_{k-1}, t_k]$ for $k = 1, \ldots, p+1$, for $u : [0, T] \to \mathbb{R}$, $u \in C(J)$, the function $u_k : J_k \to \mathbb{R}$ defined by

$$u_k(t) = u(t), \quad t \in (t_{k-1}, t_k], \qquad u_k(t_{k-1}) = u\big(t_{k-1}^+\big),$$

is such that $u_k \in C(J_k)$; $\psi_k : C(J_k) \to C(J_k)$ is continuous, $k = 1, \ldots, p+1$, $\phi_k : C(J_k) \to C(J_k)$, $I_k : \mathbb{R} \to \mathbb{R}$ continuous, $k = 1, 2, \ldots, p$ and

$$f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

continuous in $J' \times \mathbb{R} \times \mathbb{R}$, where $J' = J \setminus \{t_1, \ldots, t_p\}$ and such that there exist the limits $\lim_{t \to t_k^-} f(t, x, y) = f(t_k, x, y)$, $\lim_{t \to t_k^+} f(t, x, y)$, for $k \in \mathbb{N}$, $1 \leqslant k \leqslant p$, $x, y \in \mathbb{R}$.

Note that the impulses at the instants t_k depend on $u(t)$, for $t \in (t_{k-1}, t_k]$. In particular, $[\phi_k u_k](t_k) = u(t_k)$, for $k = 1, \ldots, p$, but, in general, we have functional dependence for the impulses.

Definition 1. A solution of (1) is a function $u \in PC^1(J)$ satisfying the conditions in (1).

We remark that

$$\begin{aligned} PC(J) = \big\{ & u : J \to \mathbb{R} \colon u \text{ is continuous in } J \setminus \{t_1, \ldots, t_p\}; \\ & \text{and } \exists\, u(0^+), u(T^-), u\big(t_k^+\big), u\big(t_k^-\big) = u(t_k),\ k = 1, \ldots, p \big\}, \\ PC^1(J) = \big\{ & u \in PC(J) \colon u \text{ is } C^1 \text{ in } J \setminus \{t_1, \ldots, t_p\}; \\ & \text{and } \exists u'(0^+), u'(T^-), u'\big(t_k^+\big), u'\big(t_k^-\big),\ k = 1, \ldots, p \big\} \end{aligned}$$

are complete spaces under the norms

$$\|u\|_{PC(J)} = \sup\{|u(t)|: t \in J\}, \qquad \|u\|_{PC^1(J)} = \|u\|_{PC(J)} + \|u'\|_{PC(J)}.$$

Remark 1. $PC(J)$ and $\prod_{k=1}^{p+1} C(J_k)$ are equivalent Banach spaces, taking the supremum norm in $C(J_k)$.

Firstly, we prove a maximum principle and then we study existence and uniqueness of solution for a quasi-linear problem related to (1), this makes it possible to develop the monotone iterative technique for (1) and approximate the extremal solutions of this problem between a lower and an upper solutions.

2. Maximum principle

Denote by $\mathbf{0}$ the function $\mathbf{0}(t) = 0$, for all $t \in J$, and $J_k = [t_{k-1}, t_k]$, for $k = 1, \ldots, p+1$.

Theorem 1. *Let $u \in PC^1(J)$, $M > 0$, $N \geqslant 0$, such that*

$$\begin{cases} u'(t) + Mu(t) + N[\psi_k(\max\{u_k, \mathbf{0}\})](t) \geqslant 0, \quad t \in (t_{k-1}, t_k),\ k = 1, \ldots, p+1, \\ u(t_k^+) \geqslant 0, \quad k = 1, \ldots, p, \\ u(0) \geqslant u(T), \end{cases} \tag{2}$$

where the following condition is verified for all $k = 1, \ldots, p+1$:

$$N \int_a^b \big[\psi_k\big(\max\{w, \mathbf{0}\}\big)\big](s) e^{M(s-t_{k-1})}\, ds \leqslant \max_{s\in[t_{k-1},b]} \big\{w(s)e^{M(s-t_{k-1})}\big\},$$

$$\text{for } a < b \in J_k, \text{ and } w \in C(J_k) \text{ with } \max_{s\in[t_{k-1},b]} \big\{w(s)e^{M(s-t_{k-1})}\big\} \geqslant 0. \tag{3}$$

Then, $u \geqslant 0$ on J.

Proof. We prove that the continuous functions

$$u_k(t) = \begin{cases} u(t), & \text{if } t \in (t_{k-1}, t_k], \\ u(t_{k-1}^+), & \text{if } t = t_{k-1}, \end{cases}$$

verify that $u_k \geqslant 0$ on J_k, for $k = 2, \ldots, p+1$. Let $v_k(t) = u_k(t)e^{M(t-t_{k-1})}$, $t \in J_k$, and note that v_k has the same sign than u_k on J_k. Also,

$$\begin{aligned} v_k'(t) &= u_k'(t)e^{M(t-t_{k-1})} + Mu_k(t)e^{M(t-t_{k-1})} \\ &\geqslant -N\big[\psi_k\big(\max\{u_k, \mathbf{0}\}\big)\big](t)e^{M(t-t_{k-1})}, \quad t \in J_k, \end{aligned} \tag{4}$$

and $v_k(t_{k-1}) = u(t_{k-1}^+) \geqslant 0$, for $k = 2, \ldots, p+1$. To prove that $v_k \geqslant 0$ on J_k (with $k \in \{2, \ldots, p+1\}$), suppose that there exists $t_0 \in (t_{k-1}, t_k]$ such that $v_k(t_0) = \min_{[t_{k-1},t_k]} v_k < 0$. Now, let $\xi \in [t_{k-1}, t_0)$ such that

$$v_k(\xi) = \max_{s\in[t_{k-1},t_0]} v_k(s) \geqslant 0.$$

Integrating (4) between ξ and t_0 and using (3), we obtain that

$$-v_k(\xi) > v_k(t_0) - v_k(\xi) \geqslant -N \int_{\xi}^{t_0} \psi_k\big(\max\{u_k, \mathbf{0}\}\big)(s) e^{M(s-t_{k-1})}\, ds$$

$$\geqslant - \max_{s\in[t_{k-1},t_0]} \big\{u_k(s) e^{M(s-t_{k-1})}\big\} = - \max_{s\in[t_{k-1},t_0]} v_k(s) = -v_k(\xi),$$

and this is absurd. So that $v_k \geqslant 0$ on J_k, and $u \geqslant 0$ on J_k, for $k = 2, \ldots, p+1$. This implies that $u(T) \geqslant 0$ but $u(0) \geqslant u(T) \geqslant 0$ so that, repeating the reasoning for $k = 1$, we obtain that $u \geqslant 0$ on J. □

Corollary 1. *Let $M > 0$, $N \geqslant 0$, θ_k continuous functions for $k = 1, \ldots, p+1$, verifying $\theta_k : [t_{k-1}, t_k] \to [t_{k-1}, t_k]$, $\theta_k(t) \leqslant t$, $t \in J_k$, and $u \in PC^1(J)$ such that*

$$\begin{cases} u'(t) + Mu(t) + N \max\{u_k, \mathbf{0}\}(\theta(t)) \geqslant 0, \quad t \in (t_{k-1}, t_k),\ k = 1, \ldots, p+1, \\ u(t_k^+) \geqslant 0, \quad k = 1, \ldots, p, \\ u(0) \geqslant u(T), \end{cases}$$

and

$$N \int_{t_{k-1}}^{t_k} e^{M(s-\theta_k(s))}\, ds \leqslant 1, \quad k = 1, \ldots, p+1. \tag{5}$$

Then $u \geqslant 0$ on J.

Proof. This is easily deduced from Theorem 1 with $[\psi_k w](t) = w(\theta_k(t))$, $t \in J_k$ and $k = 1, \ldots, p+1$. To prove (3), consider $k \in \{1, \ldots, p+1\}$, $a < b \in J_k$, and $w \in C(J_k)$ with $\max_{[t_{k-1},b]} w \geqslant 0$, then

$$N \int_a^b \big(\max\{w, \mathbf{0}\}\big)\big(\theta_k(s)\big) e^{M(s-t_{k-1})}\, ds$$

$$= N \int_a^b \max\big\{w\big(\theta_k(s)\big) e^{M(\theta_k(s)-t_{k-1})} e^{-M(\theta_k(s)-t_{k-1})}, 0\big\} e^{M(s-t_{k-1})}\, ds$$

$$\leqslant \max_{s\in[t_{k-1},b]} \big\{w(s) e^{M(s-t_{k-1})}\big\} N \int_{t_{k-1}}^{t_k} e^{M(s-\theta_k(s))}\, ds$$

$$\leqslant \max_{s\in[t_{k-1},b]} \big\{w(s) e^{M(s-t_{k-1})}\big\},$$

where we have used that $\theta_k(t) \leqslant t$, for $t \in J_k$. □

The new estimate (5) provides a specific condition adapted to each particular delay function and improves, for the case of impulsive delayed equations, the one given in [4]. Compare (5) with condition (iii) obtained in [18, Theorem 2.4] for the nonimpulsive case.

Let $\sigma = \max\{t_k - t_{k-1},\ k = 1, \ldots, p+1\}$.

Corollary 2. *Let $M > 0$, $N \geqslant 0$, and $u \in PC^1(J)$ such that* (2) *holds and*

$$[\psi_k w](t) \leqslant \max_{[t_{k-1},t]} w, \quad \text{for } t \in J_k,\ w \in C(J_k),\ k = 1, 2, \ldots, p+1, \tag{6}$$

$$\frac{N}{M}\big(e^{M\sigma} - 1\big) \leqslant 1. \tag{7}$$

Then $u \geqslant 0$ on J.

Proof. It is evident that (3) is valid. For $a < b \in J_k$, and $w \in C(J_k)$ with $\max_{[t_{k-1},b]} w \geqslant 0$, we can easily prove that

$$\begin{aligned}
&N \int_a^b \big[\psi_k\big(\max\{w, \mathbf{0}\}\big)\big](s) e^{M(s-t_{k-1})}\, ds \\
&\quad \leqslant N \int_a^b \max_{r\in[t_{k-1},s]} \big(\max\big\{w(r), 0\big\}\big) e^{M(s-t_{k-1})}\, ds \\
&\quad \leqslant N \max_{r\in[t_{k-1},b]} \big\{w(r) e^{M(r-t_{k-1})}\big\} \int_a^b \Big(\max_{r\in[t_{k-1},s]} e^{-M(r-t_{k-1})}\Big) e^{M(s-t_{k-1})}\, ds \\
&\quad \leqslant N \max_{r\in[t_{k-1},b]} \big\{w(r) e^{M(r-t_{k-1})}\big\} \int_{t_{k-1}}^{t_k} e^{M(s-t_{k-1})}\, ds \\
&\quad = \frac{N}{M}\big(e^{M(t_k-t_{k-1})} - 1\big) \max_{s\in[t_{k-1},b]} \big\{w(s) e^{M(s-t_{k-1})}\big\} \\
&\quad \leqslant \frac{N}{M}\big(e^{M\sigma} - 1\big) \max_{s\in[t_{k-1},b]} \big\{w(s) e^{M(s-t_{k-1})}\big\} \leqslant \max_{s\in[t_{k-1},b]} \big\{w(s) e^{M(s-t_{k-1})}\big\},
\end{aligned}$$

for $k = 1, \ldots, p+1$. □

Compare this estimate (7) with $\frac{N}{M}(e^{MT} - 1) < 1$. This last one was obtained in [17, Theorem 5] for a nonimpulsive problem in $[0, T]$.

If we suppose condition (H_2) in [4], i.e., existence of $L > 0$ with

$$\big[\psi_k(w)\big](t) \leqslant L \max_{[t_{k-1},t]} w, \quad \text{for } t \in J_k,\ w \in C(J_k),$$

and all $k = 1, \ldots, p+1$, we would have obtained the estimate $\frac{LN}{M}(e^{M\sigma} - 1) \leqslant 1$, that is sharper than condition $NL\sigma e^{M\sigma} \leqslant 1$, given in [4, Lemma 3.2].

Corollary 3. *Let $M > 0$, $N \geqslant 0$, and $u \in PC^1(J)$ such that conditions in* (2) *are verified, and suppose that the following conditions hold*:

$$[\psi_k w](t) \leqslant \int_{t_{k-1}}^{t} w(s)\,ds, \quad \text{for } t \in J_k,\ w \in C(J_k),\ k = 1, 2, \ldots, p+1,$$

$$\frac{N}{M^2}\big(e^{M\sigma} - M\sigma - 1\big) \leqslant 1. \tag{8}$$

Then $u \geqslant 0$ on J.

Proof. Let $k \in \{1, \ldots, p+1\}$, $a < b \in J_k$ and $w \in C(J_k)$ with $\max_{[t_{k-1},b]} w \geqslant 0$, then, by hypotheses,

$$\begin{aligned}
&N \int_a^b \big[\psi_k\big(\max\{w, \mathbf{0}\}\big)\big](s) e^{M(s-t_{k-1})}\,ds \\
&\quad \leqslant N \int_a^b \int_{t_{k-1}}^{s} \max\big\{w(r), 0\big\}\,dr\, e^{M(s-t_{k-1})}\,ds \\
&\quad \leqslant N \max_{r\in[t_{k-1},b]} \big\{w(r)e^{M(r-t_{k-1})}\big\} \int_a^b \int_{t_{k-1}}^{s} e^{-M(r-t_{k-1})}\,dr\, e^{M(s-t_{k-1})}\,ds \\
&\quad \leqslant \frac{N}{M} \max_{r\in[t_{k-1},b]} \big\{w(r)e^{M(r-t_{k-1})}\big\} \int_{t_{k-1}}^{t_k} \big(e^{M(s-t_{k-1})} - 1\big)\,ds \\
&\quad = \frac{N}{M} \max_{r\in[t_{k-1},b]} \big\{w(r)e^{M(r-t_{k-1})}\big\} \frac{e^{M(t_k-t_{k-1})} - M(t_k - t_{k-1}) - 1}{M} \\
&\quad \leqslant \max_{s\in[t_{k-1},b]} \big\{w(s)e^{M(s-t_{k-1})}\big\} \frac{N}{M^2}\big(e^{M\sigma} - M\sigma - 1\big) \\
&\quad \leqslant \max_{s\in[t_{k-1},b]} \big\{w(s)e^{M(s-t_{k-1})}\big\},
\end{aligned}$$

and (3) holds. □

Example 1. Let $r_k > 0$, $[\psi_k x](t) = r_k - \sqrt{r_k^2 - (x(t))^2}$, for $x \in C(J_k)$, $-r_k \leqslant x(t) \leqslant r_k$, $t \in J_k$, $k = 1, 2, \ldots, p+1$ and suppose that $N\sigma \leqslant 1$.

Let $a < b \in J_k$, $w \in C(J_k)$, $w \in [-r_k, r_k]$, with $\max_{[t_{k-1},b]} w \geqslant 0$, and analyze the expression $r_k - \sqrt{r_k^2 - (\max\{w(s), 0\})^2}$, for $s \in [a, b]$. If $s \in [a, b]$ is such that $w(s) < 0$, then $r_k - \sqrt{r_k^2 - (\max\{w(s), 0\})^2} = r_k - \sqrt{r_k^2} = 0$ and, if $s \in [a, b]$ verifies $r_k \geqslant w(s) \geqslant 0$, then $0 \leqslant r_k - w(s) \leqslant r_k + w(s)$, which implies that $(r_k - w(s))^2 \leqslant r_k^2 - (w(s))^2$, so that

$$r_k - \sqrt{r_k^2 - \big(\max\big\{w(s), 0\big\}\big)^2} = r_k - \sqrt{r_k^2 - \big(w(s)\big)^2} \leqslant w(s).$$

This leads to

$$
\begin{aligned}
&\big[\psi_k\big(\max\{w,\mathbf{0}\}\big)\big](s)e^{M(s-t_{k-1})}\\
&\quad=\Big(r_k-\sqrt{r_k^2-\big(\max\big\{w(s),0\big\}\big)^2}\Big)e^{M(s-t_{k-1})}\\
&\quad\leqslant\begin{cases} w(s)e^{M(s-t_{k-1})}, & \text{if } w(s)\geqslant 0,\\ 0, & \text{if } w(s)<0\end{cases}\ \leqslant \max_{s\in[t_{k-1},b]}\big\{w(s)e^{M(s-t_{k-1})}\big\},
\end{aligned}
$$

for $s\in[a,b]$, therefore

$$
\begin{aligned}
&N\int_a^b\big[\psi_k\big(\max\{w,\mathbf{0}\}\big)\big](s)e^{M(s-t_{k-1})}\,ds\\
&\quad\leqslant N\int_a^b\max_{s\in[t_{k-1},b]}\big\{w(s)e^{M(s-t_{k-1})}\big\}\,ds\\
&\quad\leqslant N(t_k-t_{k-1})\max_{s\in[t_{k-1},b]}\big\{w(s)e^{M(s-t_{k-1})}\big\}\\
&\quad\leqslant N\sigma\max_{s\in[t_{k-1},b]}\big\{w(s)e^{M(s-t_{k-1})}\big\}\leqslant\max_{s\in[t_{k-1},b]}\big\{w(s)e^{M(s-t_{k-1})}\big\}.
\end{aligned}
$$

If ψ_k was a Lipschitzian function in $[-r_k,r_k]$, there would exist $L>0$ such that

$$
\sup_{t\in J_k}\Big|-\sqrt{r_k^2-\big(x(t)\big)^2}+\sqrt{r_k^2-\big(y(t)\big)^2}\,\Big|\leqslant L\sup_{t\in J_k}\big|x(t)-y(t)\big|,
$$

for every $x,y\in C(J_k)$, with $-r_k\leqslant x(t),y(t)\leqslant r_k$, $t\in J_k$.

Taking $x_n(t)=r_k-1/n$, $n\in\mathbb{N}$, and $y(t)=r_k$, for all $t\in J_k$,

$$
\big\|[\psi_k x_n]-[\psi_k y]\big\|=\sqrt{r_k^2-\left(r_k-\frac{1}{n}\right)^2}=\sqrt{\frac{2r_k}{n}-\frac{1}{n^2}}\leqslant L\frac{1}{n},\quad\forall n\in\mathbb{N},
$$

which implies the contradiction $2nr_k-1\leqslant L^2$, $\forall n\in\mathbb{N}$.

So that, we have obtained a very general maximum principle that can be applied in cases where the functional dependence is not given by a Lipschitzian function.

Example 2. Let $M_k\in\mathbb{R}$, $[\psi_k x](t)=x(t)+e^{M_k}-e^{M_k-x(t)}$, for $x\in C(J_k)$, $t\in J_k$ and $k=1,2,\ldots,p+1$. Suppose that $N(e^{M_k}+1)(t_k-t_{k-1})\leqslant 1$, for all k. We prove that again ψ_k verifies condition (3) in Theorem 1. Consider $a<b\in J_k$ and $w\in C(J_k)$ with $\max_{[t_{k-1},b]}w\geqslant 0$. Using that

$$
z+e^{M_k}-e^{M_k-z}\leqslant\big(e^{M_k}+1\big)z,\quad\text{for } z\geqslant 0,
$$

we have, for $s\in[a,b]$, that

$$
\begin{aligned}
&\big[\psi_k\big(\max\{w,\mathbf{0}\}\big)\big](s)e^{M(s-t_{k-1})}\\
&\quad=\begin{cases}(w(s)+e^{M_k}-e^{M_k-w(s)})e^{M(s-t_{k-1})}, & \text{if } w(s)\geqslant 0,\\ 0, & \text{if } w(s)<0\end{cases}\\
&\quad\leqslant\begin{cases}(e^{M_k}+1)w(s)e^{M(s-t_{k-1})}, & \text{if } w(s)\geqslant 0,\\ 0, & \text{if } w(s)<0\end{cases}
\end{aligned}
$$

$$\leqslant \left(e^{M_k}+1\right) \max_{s\in[t_{k-1},b]} \left\{w(s)e^{M(s-t_{k-1})}\right\}$$

and, in consequence,

$$\begin{aligned} N &\int_a^b \left[\psi_k\left(\max\{w,\mathbf{0}\}\right)\right](s)e^{M(s-t_{k-1})}\,ds \\ &\leqslant N\left(e^{M_k}+1\right)(t_k-t_{k-1}) \max_{s\in[t_{k-1},b]} \left\{w(s)e^{M(s-t_{k-1})}\right\} \\ &\leqslant \max_{s\in[t_{k-1},b]} \left\{w(s)e^{M(s-t_{k-1})}\right\}. \end{aligned}$$

If ψ_k was a Lipschitzian function, there would exist $L>0$ such that

$$\sup_{t\in J_k}\left|x(t)-e^{M_k-x(t)}-y(t)+e^{M_k-y(t)}\right| \leqslant L \sup_{t\in J_k}\left|x(t)-y(t)\right|.$$

For $x_n(t)=-n+M_k$, $n\in\mathbb{N}$ and $y(t)=M_k$, $t\in J_k$, we get

$$\left\|[\psi_k x_n]-[\psi_k y]\right\| = \sup_{t\in J_k}\left|-n+M_k-e^n-M_k+1\right| \leqslant Ln, \quad \forall n\in\mathbb{N},$$

or $|-n-e^n+1| = e^n+n-1 \leqslant Ln,\ \forall n\in\mathbb{N}$, which leads to the contradiction $e^n-1\leqslant (L-1)n,\ \forall n\in\mathbb{N}$.

Remark 2. If $[\psi_k w](t)\leqslant 0$, $\forall w\in C(J_k)$, $w\geqslant 0$, then condition (3) in Theorem 1 is trivially valid.

3. Quasi-linear problem associated to (1)

Let $M>0$, $N\geqslant 0$, $b\in PC(J)$ and consider the problem

$$\begin{cases} u'(t)+Mu(t)+N[\psi_k u_k](t)=b(t), \quad t\in(t_{k-1},t_k),\ k=1,\dots,p+1, \\ u(t_k^+)=c_k, \quad k=1,\dots,p, \\ u(0)=u(T). \end{cases} \tag{9}$$

Definition 2. A solution of (9) is a function $u\in PC^1(J)$ satisfying conditions in (9).

Definition 3. A function $\alpha\in PC^1(J)$ is a lower solution for problem (9) if

$$\begin{cases} \alpha'(t)+M\alpha(t)+N[\psi_k\alpha_k](t)\leqslant b(t), \quad t\in(t_{k-1},t_k),\ k=1,\dots,p+1, \\ \alpha(t_k^+)\leqslant c_k, \quad k=1,\dots,p, \\ \alpha(0)\leqslant\alpha(T). \end{cases}$$

Analogously, $\beta\in PC^1(J)$ is an upper solution for (9) if it verifies similar conditions for the inequalities reversed.

For two functions $y_1, y_2 \in PC(J)$, we say that $y_1 \leqslant y_2$ if $y_1(t) \leqslant y_2(t)$, for $t \in J$, that is, if $(y_1)_k \leqslant (y_2)_k$ on J_k, for $k = 1, \ldots, p+1$. In these conditions, we denote $[y_1, y_2] := \{u \in PC(J):\ y_1 \leqslant u \leqslant y_2\}$.

For $\alpha, \beta \in PC(J)$, $\alpha \leqslant \beta$, we introduce the maps $q_k : C(J_k) \to C(J_k)$, $k = 1, \ldots, p+1$, given by

$$[q_k w](t) = \max\bigl\{\alpha_k(t), \min\bigl\{w(t), \beta_k(t)\bigr\}\bigr\} = \begin{cases} \alpha_k(t), & \text{if } w(t) < \alpha_k(t), \\ w(t), & \text{if } \alpha_k(t) \leqslant w(t) \leqslant \beta_k(t), \\ \beta_k(t), & \text{if } w(t) > \beta_k(t), \end{cases}$$

for $w \in C(J_k)$, $t \in J_k$, where $\alpha_k(t) = \alpha(t)$, $t \in (t_{k-1}, t_k]$, $\alpha_k(t_{k-1}) = \alpha(t_{k-1}^+)$, and analogously for β_k.

Theorem 2. *If $\alpha, \beta \in PC^1(J)$ are, respectively, lower and upper solutions for* (9) *with $\alpha \leqslant \beta$, and ψ_k satisfy, for all $k = 1, \ldots, p+1$,* (3)*, and*

$$\psi_k \text{ maps bounded sets into bounded sets}, \tag{10}$$

$$\psi_k\bigl(\max\{w - \alpha_k, \mathbf{0}\}\bigr) \geqslant \psi_k\bigl(q_k(w)\bigr) - \psi_k(\alpha_k) \quad \text{on } J_k, \text{ for } w \in C(J_k), \tag{11}$$

$$\psi_k\bigl(\max\{\beta_k - w, \mathbf{0}\}\bigr) \geqslant \psi_k(\beta_k) - \psi_k\bigl(q_k(w)\bigr) \quad \text{on } J_k, \text{ for } w \in C(J_k), \tag{12}$$

$$\psi_k\bigl(\max\{f - g, \mathbf{0}\}\bigr) \geqslant \psi_k(f) - \psi_k(g) \quad \text{on } J_k, \text{ for } f, g \in C(J_k),\ f, g \in [\alpha_k, \beta_k], \tag{13}$$

then there exists exactly one solution to (9) *in* $[\alpha, \beta]$.

Proof. Firstly, we prove that problem (9) has, at most, one solution in $[\alpha, \beta]$. Suppose that $u_1, u_2 \in [\alpha, \beta]$ are solutions to (9) and consider $m_1 = u_1 - u_2$, $m_2 = u_2 - u_1 \in PC^1(J)$. Then, by (13), for $t \in (t_{k-1}, t_k)$, $k = 1, \ldots, p+1$, we obtain

$$\begin{aligned} &m_1'(t) + Mm_1(t) + N\bigl[\psi_k\bigl(\max\bigl\{(m_1)_k, \mathbf{0}\bigr\}\bigr)\bigr](t) \\ &\quad = u_1'(t) + Mu_1(t) - u_2'(t) - Mu_2(t) + N\bigl[\psi_k\bigl(\max\bigl\{(u_1)_k - (u_2)_k, \mathbf{0}\bigr\}\bigr)\bigr](t) \\ &\quad = b(t) - N\bigl[\psi_k(u_1)_k\bigr](t) - b(t) + N\bigl[\psi_k(u_2)_k\bigr](t) \\ &\qquad + N\bigl[\psi_k\bigl(\max\bigl\{(u_1)_k - (u_2)_k, \mathbf{0}\bigr\}\bigr)\bigr](t) \geqslant 0, \\ &m_1(0) = u_1(0) - u_2(0) = u_1(T) - u_2(T) = m_1(T), \end{aligned}$$

and, for $k = 1, 2, \ldots, p$, $m_1(t_k^+) = u_1(t_k^+) - u_2(t_k^+) = c_k - c_k = 0$. Analogously, we prove that

$$\begin{aligned} &m_2'(t) + Mm_2(t) + N\bigl[\psi_k\bigl(\max\{m_2, \mathbf{0}\}\bigr)\bigr](t) \geqslant 0, \quad t \in (t_{k-1}, t_k),\ k = 1, \ldots, p+1, \\ &m_2(0) = m_2(T), \qquad m_2\bigl(t_k^+\bigr) = 0, \quad k = 1, 2, \ldots, p. \end{aligned}$$

By Theorem 1, we can affirm that $u_1 = u_2$ on J. Note that, if hypothesis (13) is true for $f, g \in C(J_k)$, then we are in conditions to prove uniqueness of solution of (9) and not only uniqueness in $[\alpha, \beta]$.

To prove existence of solution of (9) in $[\alpha, \beta]$, we consider the auxiliary problem

$$\begin{cases} u'(t) + Mu(t) + N[\psi_k q_k u_k](t) = b(t), \quad t \in (t_{k-1}, t_k),\ k = 1, \ldots, p+1, \\ u(t_k^+) = c_k, \quad k = 1, \ldots, p, \\ u(0) = u(T). \end{cases} \tag{14}$$

We prove that every solution of (14) belongs to $[\alpha, \beta]$, and that there exists a solution of (14). Thus, (14) has a solution $u \in [\alpha, \beta]$, and therefore it is a solution of (9), since $q_k u_k = u_k$, for all $k = 1, \ldots, p+1$. Let us prove that every solution u of (14) is in $[\alpha, \beta]$. Set $m_1 = u - \alpha \in PC^1(J)$ and $m_2 = \beta - u \in PC^1(J)$. By (11) and (12), we obtain, for $k = 1, \ldots, p+1$ and $t \in (t_{k-1}, t_k)$,

$$\begin{aligned} &m_1'(t) + Mm_1(t) + N\big[\psi_k\big(\max\big\{(m_1)_k, \mathbf{0}\big\}\big)\big](t) \\ &\quad = u'(t) + Mu(t) - \alpha'(t) - M\alpha(t) + N\big[\psi_k\big(\max\{u_k - \alpha_k, \mathbf{0}\}\big)\big](t) \geqslant b(t) \\ &\qquad - N[\psi_k q_k u_k](t) - b(t) + N[\psi_k \alpha_k](t) + N\big[\psi_k\big(\max\{u_k - \alpha_k, \mathbf{0}\}\big)\big](t) \geqslant 0, \\ &m_2'(t) + Mm_2(t) + N\big[\psi_k\big(\max\big\{(m_2)_k, \mathbf{0}\big\}\big)\big](t) \geqslant 0, \\ &m_1(0) = u(0) - \alpha(0) \geqslant u(T) - \alpha(T) = m_1(T), \\ &m_2(0) = \beta(0) - u(0) \geqslant \beta(T) - u(T) = m_2(T), \end{aligned}$$

and, for $k = 1, 2, \ldots, p$,

$$\begin{aligned} &m_1\big(t_k^+\big) = u\big(t_k^+\big) - \alpha\big(t_k^+\big) \geqslant c_k - c_k = 0, \\ &m_2\big(t_k^+\big) = \beta\big(t_k^+\big) - u\big(t_k^+\big) \geqslant c_k - c_k = 0. \end{aligned}$$

From application of Theorem 1 we get $m_1 \geqslant 0$, and $m_2 \geqslant 0$, that is, $\alpha \leqslant u \leqslant \beta$.

Now, we prove existence of solution for problem (14). Consider the operators $T_k : C(J_k) \to C(J_k)$, $k = 1, \ldots, p+1$, given by

$$[T_k w](t) = c_{k-1} e^{-M(t - t_{k-1})} + \int_{t_{k-1}}^{t} \big\{b(s) - N[\psi_k q_k w](s)\big\} e^{M(s-t)}\, ds, \tag{15}$$

for $t \in J_k$. Since ψ_k and q_k are continuous, we can prove that the operators T_k are continuous. Also, T_k is completely continuous, for all k. Indeed, let $k \in \{1, 2, \ldots, p+1\}$, and $S \subseteq C(J_k)$ a bounded set ($\|u\| \leqslant r_1$, for $u \in S$). Then for $w \in S$, $t \in J_k$,

$$\begin{aligned} \big|(T_k w)(t)\big| &\leqslant |c_{k-1}| e^{-M(t - t_{k-1})} + \int_{t_{k-1}}^{t} \big(\big|b(s)\big| + N\big|[\psi_k q_k w](s)\big|\big) e^{M(s-t)}\, ds \\ &\leqslant |c_{k-1}| + \big(\|b\| + N R_k\big) \int_{t_{k-1}}^{t} e^{M(s-t)}\, ds \\ &\leqslant |c_{k-1}| + \big(\|b\| + N R_k\big) \frac{1 - e^{-M(t - t_{k-1})}}{M} \\ &\leqslant |c_{k-1}| + \big(\|b\| + N R_k\big) \frac{1 - e^{-M(t_k - t_{k-1})}}{M} = \delta_k, \end{aligned}$$

where we have used that ψ_k maps bounded sets into bounded sets:

$$\|q_k w\| \leqslant \max\{\|\alpha_k\|, \|\beta_k\|\} \quad \Rightarrow \quad \|\psi_k q_k w\| \leqslant R_k,$$

and, also, since $T_k w$ given by (15) verifies equation

$$u'(t) + Mu(t) = b(t) - N[\psi_k q_k w](t), \quad t \in (t_{k-1}, t_k),$$

then

$$\begin{aligned}\left|(T_k w)'(t)\right| &= \left|b(t) - N[\psi_k q_k w](t) - M(T_k w)(t)\right| \\ &\leqslant \|b\| + N\|\psi_k q_k w\| + M\delta_k \leqslant \|b\| + NR_k + M\delta_k, \quad t \in J_k,\end{aligned}$$

so that T_k is completely continuous. On the other hand, let $u \in C(J_k)$ and $\lambda \in (0, 1)$, such that $u = \lambda T_k u$. Then $\|u\| = \|\lambda T_k u\| \leqslant \|T_k u\| \leqslant \delta_k$. Applying Schaefer's theorem, we obtain existence of a fixed point $u_k \in C(J_k)$ for T_k ($u_k \in C^1(J_k)$). Note that the initial value for the equation in $[0, t_1]$ (relative to operator T_1) is chosen once we have proved existence of $u_2, \ldots, u_{p+1}$ (fixed points for $T_2, \ldots, T_{p+1}$, respectively) as $c_0 = u_{p+1}(T)$. Thus, we have obtained that function $u \in PC^1(J)$ given by $u(t) = u_k(t)$, for $t \in (t_{k-1}, t_k]$, $k = 2, \ldots, p+1$ and $u(t) = u_1(t)$, for $t \in [0, t_1]$, is a solution to (14). Since every solution of (14) belongs to $[\alpha, \beta]$, then u is a solution to (9) in $[\alpha, \beta]$ (in fact, u is the unique solution to (9) in this functional interval). □

Conditions in Theorem 2 are quite restrictive but, on the other hand, we have to take into account that it guarantees uniqueness of solution for a nonlinear functional problem where the functional dependence is not necessarily Lipschitzian, under the assumption of existence of appropriate upper and lower solutions. The comparison result and the existence and uniqueness result for the quasilinear problem (9) are very useful to develop the monotone method for problem (1). Anyway, our results are interesting by themselves. The maximum principle (Theorem 1) extends previous results and allows a general formulation and the existence and uniqueness theorem (Theorem 2) makes reference to equations given by an operator not necessarily linear nor Lipschitzian in presence of upper and lower solutions and it is valid without specifying any fixed expression for ψ_k.

Remark 3. Note that if

$$\begin{aligned}&\psi_k\big(\max\{w - \alpha_k, \mathbf{0}\}\big) \geqslant \psi_k(w) - \psi_k(\alpha_k) \quad \text{on } J_k, \text{ for } w \in C(J_k), \\ &\psi_k\big(\max\{\beta_k - w, \mathbf{0}\}\big) \geqslant \psi_k(\beta_k) - \psi_k(w) \quad \text{on } J_k, \text{ for } w \in C(J_k),\end{aligned}$$

hold, then every solution of (9) is in $[\alpha, \beta]$ and, therefore, is a solution of (14).

Proposition 1. *Observe that, for every $k = 1, 2, \ldots, p+1$,*

$$\max\{w - \alpha_k, \mathbf{0}\} \geqslant q_k(w) - \alpha_k \quad \textit{on } J_k, \textit{ for } w \in C(J_k),$$

and

$$\max\{\beta_k - w, \mathbf{0}\} \geqslant \beta_k - q_k(w) \quad \textit{on } J_k, \textit{ for } w \in C(J_k).$$

Proof. If $w \in C(J_k)$, and $t \in J_k$,

$$\max\{w-\alpha_k, \mathbf{0}\}(t) = \begin{cases} 0, & \text{if } w(t) \leqslant \alpha_k(t), \\ w(t)-\alpha_k(t), & \text{if } w(t) > \alpha_k(t) \end{cases}$$

$$\geqslant \begin{cases} \alpha_k(t)-\alpha_k(t), & \text{if } w(t) \leqslant \alpha_k(t), \\ q_k(w)(t)-\alpha_k(t), & \text{if } w(t) > \alpha_k(t) \end{cases} = \big(q_k(w)-\alpha_k\big)(t).$$

Similarly, it can be proved the second assertion. □

Proposition 2. *If ψ_k is nondecreasing and*

$$\psi_k(f-g) \geqslant \psi_k f - \psi_k g \quad \text{on } J_k, \text{ for } f, g \in C(J_k),\ f, g \in [\alpha_k, \beta_k],$$

then hypotheses (11)–(13) *hold.*

Proof. Let $w \in C(J_k)$, then, by hypotheses and Proposition 1,

$$\psi_k\big(\max\{w-\alpha_k, \mathbf{0}\}\big) \geqslant \psi_k\big(q_k(w)-\alpha_k\big) \geqslant \psi_k\big(q_k(w)\big) - \psi_k(\alpha_k) \quad \text{on } J_k,$$

$$\psi_k\big(\max\{\beta_k-w, \mathbf{0}\}\big) \geqslant \psi_k\big(\beta_k-q_k(w)\big) \geqslant \psi_k(\beta_k) - \psi_k\big(q_k(w)\big) \quad \text{on } J_k,$$

and, for $f, g \in C(J_k)$, $f, g \in [\alpha_k, \beta_k]$,

$$\psi_k\big(\max\{f-g, \mathbf{0}\}\big) \geqslant \psi_k(f-g) \geqslant \psi_k(f) - \psi_k(g) \quad \text{on } J_k.$$

In consequence, (11)–(13) hold. □

Example 3. Observe that a function ψ_k in the hypotheses of Proposition 2 is not necessarily Lipschitzian. For instance, take $\psi_k(f) = \sqrt{|f|}$, $f \in C(J_k)$.

If $0 \leqslant f \leqslant g$, then

$$\psi_k(f) = \sqrt{|f|} = \sqrt{f} \leqslant \sqrt{g} = \sqrt{|g|} = \psi_k(g) \quad \text{on } J_k,$$

so that ψ_k is nondecreasing for nonnegative functions.

Now, let $f, g \in C(J_k)$, $f, g \geqslant 0$ on J_k, and calculate

$$\big[\psi_k(f-g)\big](t) = \sqrt{|f-g|(t)}.$$

If $f(t) \geqslant g(t) \geqslant 0$, then $2\sqrt{f(t)g(t)} \geqslant 2\sqrt{(g(t))^2} = 2\,g(t)$ and, thus,

$$\big(\sqrt{f(t)} - \sqrt{g(t)}\big)^2 = f(t) + g(t) - 2\sqrt{f(t)g(t)} \leqslant f(t) + g(t) - 2g(t)$$

$$= f(t) - g(t) = \big(\sqrt{f(t)-g(t)}\big)^2.$$

Since, in this case, $\sqrt{f(t)} - \sqrt{g(t)} \geqslant 0$, it is obtained that

$$\big[\psi_k(f-g)\big](t) = \sqrt{|f-g|(t)} = \sqrt{f(t)-g(t)} \geqslant \sqrt{f(t)} - \sqrt{g(t)}$$

$$= \big[\psi_k(f)\big](t) - \big[\psi_k(g)\big](t).$$

If $0 \leqslant f(t) < g(t)$, then

$$\big[\psi_k(f-g)\big](t) = \sqrt{|f-g|(t)} = \sqrt{g(t)-f(t)} > 0$$

$$> \sqrt{|f(t)|} - \sqrt{|g(t)|} = \big[\psi_k(f)\big](t) - \big[\psi_k(g)\big](t).$$

Nevertheless, ψ_k is not a Lipschitzian function. If there existed $L \geqslant 0$ with

$$\big\|\psi_k(f) - \psi_k(g)\big\| \leqslant L\|f - g\|, \quad \text{for all } f, g \in C(J_k),\ f, g \geqslant 0,$$

then, taking $g \equiv 0$, $\|\psi_k(f)\| \leqslant L\|f\|$, for all $f \in C(J_k)$, $f \geqslant 0$. In particular, for $f_n(t) = 1/n$ constant in J_k, $n \in \mathbb{N}$, we obtain that $\sqrt{1/n} \leqslant L(1/n)$, $n \in \mathbb{N}$, and we get the contradiction $n \leqslant L^2$, $\forall n \in \mathbb{N}$.

Remark 4. Conditions (10)–(13) in Theorem 2 and conditions in Proposition 2 are verified in the cases where functions ψ_k involve a delayed argument,

$$[\psi_k w](t) = w\big(\theta_k(t)\big), \quad \text{for } t \in J_k,$$

where $\theta_k : J_k \to J_k$ is continuous and $\theta_k(t) \leqslant t$, for all $t \in J_k$, a maximum function

$$[\psi_k w](t) = \max_{[t_{k-1},t]} w, \quad \text{for } t \in J_k,$$

or an integral

$$[\psi_k w](t) = \int_{t_{k-1}}^{t} w(s)\,ds, \quad \text{for } t \in J_k.$$

This fact is exposed in the following corollaries.

Corollary 4. *Let $\alpha \leqslant \beta \in PC^1(J)$ be, respectively, lower and upper solutions of*

$$\begin{cases} u'(t) + Mu(t) + Nu(\theta_k(t)) = b(t), \quad t \in (t_{k-1}, t_k),\ k = 1, \ldots, p+1, \\ u(t_k^+) = c_k, \quad k = 1, \ldots, p, \\ u(0) = u(T), \end{cases} \tag{16}$$

where $M > 0$, $N \geqslant 0$, $b \in PC(J)$, $\theta_k : J_k \to J_k$ continuous, with $\theta_k(t) \leqslant t$ for $t \in J_k$. Suppose that (5) *holds. Then, there exists exactly one solution to* (16) *in* $[\alpha, \beta]$.

Proof. The functionals defined by $[\psi_k w](t) = w(\theta_k(t))$, $t \in J_k$, verify hypotheses (10)–(13) in Theorem 2. Indeed, ψ_k are continuous, (10) is verified, and (11)–(13) hold by Proposition 2 since ψ_k are nondecreasing and, for $f, g \in C(J_k)$, $s \in J_k$,

$$\big[\psi_k(f - g)\big](s) = f\big(\theta_k(s)\big) - g\big(\theta_k(s)\big) = \big[\psi_k(f)\big](s) - \big[\psi_k(g)\big](s).$$

Finally, validity of condition (3) is obtained from (5). □

Corollary 5. *If $\alpha \leqslant \beta \in PC^1(J)$ are, respectively, lower and upper solutions to*

$$\begin{cases} u'(t) + Mu(t) + N \max_{[t_{k-1},t]} u_k = b(t), \quad t \in (t_{k-1}, t_k),\ k = 1, \ldots, p+1, \\ u(t_k^+) = c_k, \quad k = 1, \ldots, p, \\ u(0) = u(T), \end{cases} \tag{17}$$

where $M > 0$, $N \geqslant 0$, $b \in PC(J)$, and (7) *holds, then problem* (17) *has exactly one solution in* $[\alpha, \beta]$.

Proof. We can apply Theorem 2 again, since, for $[\psi_k w](t) = \max_{[t_{k-1},t]} w$, $w \in C(J_k)$, $t \in J_k$, conditions (3) and (10) hold, and the validity of (11)–(13) comes from Proposition 2, since ψ_k is nondecreasing for $k = 1, \ldots, p+1$ and, for $f, g \in C(J_k)$, and $s \in J_k$,

$$\begin{aligned}\big[\psi_k(f-g)\big](s) &= \max_{r\in[t_{k-1},s]}(f-g)(r) \geqslant \max_{r\in[t_{k-1},s]} f(r) - \max_{r\in[t_{k-1},s]} g(r)\\ &= \big[\psi_k(f)\big](s) - \big[\psi_k(g)\big](s). \qquad \square\end{aligned}$$

Corollary 6. *If $\alpha \leqslant \beta \in PC^1(J)$ are, respectively, lower and upper solutions for problem*

$$\begin{cases} u'(t) + Mu(t) + N\int_{t_{k-1}}^{t} u_k(s)\,ds = b(t), \quad t \in (t_{k-1}, t_k),\ k = 1, \ldots, p+1,\\ u(t_k^+) = c_k, \quad k = 1, \ldots, p,\\ u(0) = u(T), \end{cases} \tag{18}$$

where $M > 0$, $N \geqslant 0$, $b \in PC(J)$, and estimate (8) *holds, then problem* (18) *has a unique solution in* $[\alpha, \beta]$.

Proof. Taking $[\psi_k w](t) = \int_{t_{k-1}}^{t} w(s)\,ds$, $w \in C(J_k)$, $t \in J_k$, it is evident that conditions (3) and (10)–(13) in Theorem 2 hold, since ψ_k is nondecreasing for $k = 1, \ldots, p+1$, and

$$\begin{aligned}\big[\psi_k(f-g)\big](s) &= \int_{t_{k-1}}^{s}(f-g)(r)\,dr = \int_{t_{k-1}}^{s} f(r)\,dr - \int_{t_{k-1}}^{s} g(r)\,dr\\ &= \big[\psi_k(f)\big](s) - \big[\psi_k(g)\big](s),\end{aligned}$$

for $f,\ g \in C(J_k)$, $s \in J_k$, and every $k = 1, \ldots, p+1$. $\square$

Remark 5. The maximum principle Theorem 1 is valid replacing $\max\{u_k, \mathbf{0}\}$ by u_k in (2) and $\max\{w, \mathbf{0}\}$ by w in condition (3). Using this information and considering the following hypotheses, for all $k = 1, 2, \ldots, p+1$,

$$\psi_k(f-g) \geqslant \psi_k f - \psi_k g \quad \text{on } J_k, \ \text{for} f, g \in C(J_k), \tag{19}$$

$$\big|[\psi_k w](s)\big| \leqslant R\big|w(s)\big|, \quad \forall w \in C(J_k),\ \forall s \in J_k,\ \text{for some } R > 0, \tag{20}$$

we can prove that problem (9) has a unique solution. Moreover, the unique solution belongs to the functional interval $[\alpha, \beta]$, where α, β are well-ordered lower and upper solutions, respectively. To prove existence of solution, we can proceed similarly to Theorem 2, applying Schaefer's theorem to operators

$$[\tilde{T}_k w](t) = c_{k-1}e^{-M(t-t_{k-1})} + \int_{t_{k-1}}^{t}\big\{b(s) - N[\psi_k w](s)\big\}e^{M(s-t)}\,ds,$$

for $w \in C(J_k)$, $t \in J_k$, and $k = 1, \ldots, p+1$, and considering $C(J_k)$ furnished with the norm $\|w\|_\rho = \sup_{t\in J_k}|w(t)|e^{-\rho(t-t_{k-1})}$, where $\rho > 0$ is such that

$$NR\frac{1 - e^{-(M+\rho)(t_k - t_{k-1})}}{M+\rho} < 1, \quad \forall k = 1, 2, \ldots, p+1.$$

4. Monotone method

In order to obtain the existence of solution of (1), we give some definitions and results.

Definition 4. A function $\alpha \in PC^1(J)$ is a lower solution of (1) if

$$\begin{cases} \alpha'(t) \leqslant f(t, \alpha(t), [\psi_k \alpha_k](t)), \quad t \in \operatorname{int}(J_k),\ k = 1, 2, \ldots, p+1, \\ \alpha(t_k^+) \leqslant I_k([\phi_k \alpha_k](t_k)), \quad k = 1, \ldots, p, \\ \alpha(0) \leqslant \alpha(T). \end{cases} \tag{21}$$

We can define an upper solution $\beta \in PC^1(J)$ in the same terms, but with the inequalities reversed.

Theorem 3. *Let $\alpha,\ \beta \in PC^1(J)$ be, respectively, lower and upper solutions for* (1) *with $\alpha \leqslant \beta$ and suppose that there exist $M > 0$, $N \geqslant 0$ such that, for all $k = 1, 2, \ldots, p+1$,*

$$\begin{aligned} &f\big(t, x(t), [\psi_k x](t)\big) - f\big(t, y(t), [\psi_k y](t)\big) \\ &\quad \geqslant -M\big(x(t) - y(t)\big) - N\big([\psi_k x](t) - [\psi_k y](t)\big), \\ &\quad \text{for } t \in J_k,\ x, y \in C(J_k) \text{ such that } \alpha_k \leqslant y \leqslant x \leqslant \beta_k \text{ on } J_k, \end{aligned} \tag{22}$$

with ψ_k satisfying (3) *and* (10)–(13). *Suppose also that*

$$I_k\big([\phi_k x](t_k)\big) \geqslant I_k\big([\phi_k y](t_k)\big), \quad \text{for } x \geqslant y \in C(J_k), \text{ and } k = 1, \ldots, p \tag{23}$$

(*it is true, for instance, if all I_k and ϕ_k are nondecreasing*). *Then there exist monotone sequences $\{\alpha_n\} \uparrow \rho$, $\{\beta_n\} \downarrow \gamma$ uniformly, where $\alpha_0 = \alpha$, $\beta_0 = \beta$, and ρ, γ are the extremal solutions to* (1) *in the functional interval* $[\alpha, \beta]$.

Proof. For a fixed $\eta \in [\alpha, \beta]$, consider the problem

$$\begin{cases} u'(t) + Mu(t) + N[\psi_k u_k](t) = b_\eta(t), \quad t \in (t_{k-1}, t_k),\ k = 1, \ldots, p+1, \\ u(t_k^+) = I_k([\phi_k \eta_k](t_k)), \quad k = 1, \ldots, p, \\ u(0) = u(T), \end{cases} \tag{24}$$

where $b_\eta(t) = f(t, \eta(t), [\psi_k \eta_k](t)) + M\eta(t) + N[\psi_k \eta_k](t)$, for $t \in (t_{k-1}, t_k)$, $k = 1, \ldots, p+1$, which has a unique solution $u_\eta \in [\alpha, \beta]$, using Theorem 2, since α, β are, respectively, lower and upper solutions of (24). Indeed, using the properties of α, β and (22), we obtain, for $k = 1, \ldots, p+1$, and $t \in (t_{k-1}, t_k)$,

$$\begin{aligned} &\alpha'(t) + M\alpha(t) + N[\psi_k \alpha_k](t) \\ &\quad \leqslant f\big(t, \alpha(t), [\psi_k \alpha_k](t)\big) + M\alpha(t) + N[\psi_k \alpha_k](t) \\ &\quad \leqslant f\big(t, \eta(t), [\psi_k \eta_k](t)\big) + M\eta(t) + N[\psi_k \eta_k](t) = b_\eta(t), \end{aligned}$$

and, analogously,

$$\beta'(t) + M\beta(t) + N[\psi_k \beta_k](t) \geqslant b_\eta(t).$$

Besides, for $k = 1, \ldots, p$,

$$\alpha\big(t_k^+\big) \leqslant I_k\big([\phi_k\alpha_k](t_k)\big) \leqslant I_k\big([\phi_k\eta_k](t_k)\big),$$
$$\beta\big(t_k^+\big) \geqslant I_k\big([\phi_k\beta_k](t_k)\big) \geqslant I_k\big([\phi_k\eta_k](t_k)\big).$$

At this moment, it is possible to define $A:[\alpha,\beta]\to[\alpha,\beta]$, by $A\eta = u_\eta$. To check the nondecreasing character of A, take $\eta_1,\eta_2\in PC(J)$ such that $\alpha\leqslant\eta_1\leqslant\eta_2\leqslant\beta$, by (13) and (22), we obtain that $m = A\eta_2 - A\eta_1$ satisfies the following properties:

$$\begin{aligned}
&m'(t) + Mm(t) + N\big[\psi_k\big(\max\{m_k,\mathbf{0}\}\big)\big](t)\\
&\quad= (A\eta_2)'(t) + MA\eta_2(t) - (A\eta_1)'(t) - MA\eta_1(t)\\
&\qquad + N\psi_k\big[\max\big\{(A\eta_2)_k - (A\eta_1)_k,\mathbf{0}\big\}\big](t)\\
&\quad= f\big(t,\eta_2(t),\big[\psi_k(\eta_2)_k\big](t)\big) + M\eta_2(t) + N\big[\psi_k(\eta_2)_k\big](t) - N\big[\psi_k(A\eta_2)_k\big](t)\\
&\qquad - f\big(t,\eta_1(t),\big[\psi_k(\eta_1)_k\big](t)\big) - M\eta_1(t) - N\big[\psi_k(\eta_1)_k\big](t) + N\big[\psi_k(A\eta_1)_k\big](t)\\
&\qquad + N\psi_k\big[\max\big\{(A\eta_2)_k - (A\eta_1)_k,\mathbf{0}\big\}\big](t)\geqslant 0,\quad t\in(t_{k-1},t_k),\ k=1,\dots,p+1,\\
&m\big(t_k^+\big) = I_k\big(\big[\phi_k(\eta_2)_k\big](t_k)\big) - I_k\big(\big[\phi_k(\eta_1)_k\big](t_k)\big)\geqslant 0,\quad k=1,\dots,p,
\end{aligned}$$

and $m(0) = m(T)$. Then $A\eta_1\leqslant A\eta_2$, due to the application of Theorem 1.

Define the sequences $\{\alpha_n\},\{\beta_n\}$ such that $\alpha_0=\alpha$, $\beta_0=\beta$, $\alpha_{n+1}=A\alpha_n$ and $\beta_{n+1}=A\beta_n$, for $n\geqslant 1$, then $\{\alpha_n\}$ is nondecreasing, $\{\beta_n\}$ is nonincreasing, and $\alpha=\alpha_0\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_n\leqslant\beta_n\leqslant\cdots\leqslant\beta_1\leqslant\beta_0=\beta$. By a classical reasoning, it can be proved that $\{\alpha_n\}\uparrow\rho$, $\{\beta_n\}\downarrow\gamma$ uniformly on J, where $\rho,\gamma\in[\alpha,\beta]$ are the extremal solutions of (1) in $[\alpha,\beta]$, that is, every solution u of (1) in $[\alpha,\beta]$ verifies that $\rho\leqslant u\leqslant\gamma$ on J. □

Remark 6. Taking into account Remark 5, a similar result can be obtained with the obvious changes in condition (3) and replacing conditions (10)–(13) by (19)–(20).

Theorem 3 allows different types of functional dependence, including delay, maximum and integral functions, as stated in the following corollaries.

Corollary 7. *Let $\alpha,\beta\in PC^1(J)$, $\alpha\leqslant\beta$ be, respectively, lower and upper solutions of*

$$\begin{cases} u'(t) = f(t,u(t),u(\theta_k(t))),\quad t\in(t_{k-1},t_k),\ k=1,\dots,p+1,\\ u(t_k^+) = I_k([\phi_k u_k](t_k)),\quad k=1,\dots,p,\\ u(0) = u(T), \end{cases} \tag{25}$$

where $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}$, and $\theta_k:J_k\to J_k$ with $\theta_k(t)\leqslant t$, $t\in J_k$. Suppose that there exist $M>0$, $N\geqslant 0$ such that, for every $k=1,2,\dots,p+1$,

$$\begin{aligned}
&f\big(t,x(t),x\big(\theta_k(t)\big)\big) - f\big(t,y(t),y\big(\theta_k(t)\big)\big)\\
&\quad\geqslant -M\big(x(t)-y(t)\big) - N\big(x\big(\theta_k(t)\big) - y\big(\theta_k(t)\big)\big),\\
&\quad \text{for } t\in J_k,\ x,y\in C(J_k),\ \alpha_k\leqslant y\leqslant x\leqslant\beta_k \text{ on } J_k,
\end{aligned} \tag{26}$$

and assume that (5) *and* (23) *are also valid. Then there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ with $\alpha_0=\alpha$, $\beta_0=\beta$, uniformly convergent to the extremal solutions of* (25) *in* $[\alpha,\beta]$.

Corollary 8. *Let $\alpha, \beta \in PC^1(J)$, $\alpha \leqslant \beta$ be, respectively, lower and upper solutions to*

$$\begin{cases} u'(t) = f(t, u(t), \max_{s\in[t_{k-1},t]} u_k(s)), \quad t \in (t_{k-1}, t_k),\ k = 1, \ldots, p+1, \\ u(t_k^+) = I_k([\phi_k u_k](t_k)), \quad k = 1, \ldots, p, \\ u(0) = u(T), \end{cases} \tag{27}$$

where $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$. Suppose that there exist $M > 0$, $N \geqslant 0$ such that, for every $k = 1, 2, \ldots, p+1$,

$$\begin{aligned} &f\Big(t, x(t), \max_{[t_{k-1},t]} x\Big) - f\Big(t, y(t), \max_{[t_{k-1},t]} y\Big) \\ &\quad \geqslant -M\big(x(t) - y(t)\big) - N\Big(\max_{[t_{k-1},t]} x - \max_{[t_{k-1},t]} y\Big), \\ &\quad \text{for } t \in J_k,\ x, y \in C(J_k),\ \alpha_k \leqslant y \leqslant x \leqslant \beta_k \text{ on } J_k, \end{aligned} \tag{28}$$

and (7), (23) *are valid. Then, it is possible to construct monotone sequences $\{\alpha_n\}, \{\beta_n\}$, with $\alpha_0 = \alpha$, $\beta_0 = \beta$, uniformly convergent to the extremal solutions of* (27) *in* $[\alpha, \beta]$.

Corollary 9. *Let $\alpha \leqslant \beta \in PC^1(J)$ be, respectively, lower and upper solutions to*

$$\begin{cases} u'(t) = f(t, u(t), \int_{t_{k-1}}^t u_k(s)\,ds), \quad t \in (t_{k-1}, t_k),\ k = 1, \ldots, p+1, \\ u(t_k^+) = I_k([\phi_k u_k](t_k)), \quad k = 1, \ldots, p, \\ u(0) = u(T), \end{cases} \tag{29}$$

where $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$. Suppose that there exist $M > 0$, $N \geqslant 0$ such that, for every $k = 1, 2, \ldots, p+1$,

$$\begin{aligned} &f\left(t, x(t), \int_{t_{k-1}}^t x(s)\,ds\right) - f\left(t, y(t), \int_{t_{k-1}}^t y(s)\,ds\right) \\ &\quad \geqslant -M\big(x(t) - y(t)\big) - N\left(\int_{t_{k-1}}^t x(s)\,ds - \int_{t_{k-1}}^t y(s)\,ds\right), \\ &\quad \text{for } t \in J_k,\ x, y \in C(J_k),\ \alpha_k \leqslant y \leqslant x \leqslant \beta_k \text{ on } J_k, \end{aligned} \tag{30}$$

verifying also (8), (23). *Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$, with $\alpha_0 = \alpha$, $\beta_0 = \beta$, uniformly convergent to the extremal solutions of* (29) *in* $[\alpha, \beta]$.

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