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A remark on the ODE with two discrete delays

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Abstract

The aim of this paper is to outline a formal framework for the analytical analysis of the Hopf bifurcations in the delay differential equations with two independent time delays. Some results for the differential–difference equations with two delays, when the both of the coefficients of linearized equation are negative were obtained in [X. Li, S. Ruan, J. Wei, Stability and bifurcation in delay-differential equations with two delays, J. Math. Anal. Appl. 236 (1999) 254–280]. In the paper we present some remarks on the case studied in [X. Li, S. Ruan, J. Wei, Stability and bifurcation in delay-differential equations with two delays, J. Math. Anal. Appl. 236 (1999) 254–280] and also two other cases, namely when the coefficients of linearized equation have different signs and when coefficients are both positive.

Keywords: Delays; Stability analysis; Hopf bifurcation; Stability switches

1. Introduction

There have arisen several papers related to ordinary differential equations with two delays in last twenty years. One of them focus on investigation the stable region for the two delay differential equation [11,13], other obtain some results for equation in special form such as the logistic equation with two delays [4,9] or motor control equation [1–3].

Some results for ordinary differential equations with two delays, when the coefficients of linearized equation are negative were obtained in [14]. We realized that there are some mistakes

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in [14], which were reflected in lemmas and theorems formulated in [14], so they were needed to be correct. However the main ideas of proofs presented in [14] are correct and we follow them in this paper. Therefore, we omit these parts of the proofs which are similar to those in [14].

In presented paper we consider the case studied in [14] and also two other cases, namely when the coefficients of linearized equation are positive or have different signs. Following [14], we choose one of delays as a bifurcation parameter and investigate the possibility of the Hopf bifurcation occurance.

It is important to realize that in this paper we assume that both of the delays are equally important, so we cannot scale time to obtain one of the delays equal to 1, as it was done in [4].

Consider the equation

$$\frac{d}{dt}x(t) = f\big(x(t-\tau_1), x(t-\tau_2)\big), \tag{1}$$

with an initial continuous function $x^0 : [-\tilde{\tau}, 0] \to \mathbb{R}_+ \cup \{0\}$, where τ_1, τ_2 and $\tilde{\tau} = \max(\tau_1, \tau_2)$ are the nonnegative constants. Suppose that Eq. (1) has the trivial stationary solution, i.e., $f(0,0) = 0$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable.

It is easy to see, [10], that if $x^0 : [-\tilde{\tau}, 0] \to \mathbb{R}_+ \cup \{0\}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, finding the solution to Eq. (1) for $t \geqslant t_0$ satisfying $x(t) = x^0(t)$, $t \in [-\tilde{\tau}, 0]$ is equivalent to finding a solution to the integral equation $x(t) = x^0(0) + \int_0^t f(x(s-\tau_1), x(s-\tau_2))\,ds$, $t \geqslant 0$ with $x(t) = x^0(t)$ for $t \in [-\tilde{\tau}, 0]$.

Using the step method (see, e.g., [7,10]) we can show that if there exists a solution for $t \in [(n-1)\bar{\tau}, n\bar{\tau}]$, then the solution for $t \in [n\bar{\tau}, (n+1)\bar{\tau}]$, where $\bar{\tau} = \min\{\tau_1, \tau_2\}$, is defined by the formula

$$x(t) = x(n\bar{\tau}) + \int_{n\bar{\tau}}^{t} f\big(x(s-\tau_1), x(s-\tau_2)\big)\,ds. \tag{2}$$

We see that the right-hand side of Eq. (2) is well defined because $s-\tau_1, s-\tau_2 \in [(n-1)\bar{\tau}, n\bar{\tau}]$. Notice that the step method gives the existence of unique solution to Eq. (1) for all $t \geqslant 0$.

Linearizing Eq. (1) around the trivial solution we obtain

$$\frac{d}{dt}x(t) = -A_1 x(t-\tau_1) - A_2 x(t-\tau_2), \tag{3}$$

where $-A_1$, $-A_2$ are the first derivatives of f with respect to appropriate co-ordinates. The characteristic equation related to Eq. (3) has the form

$$z = -A_1 e^{-z\tau_1} - A_2 e^{-z\tau_2}. \tag{4}$$

It is obvious that if $A_1 = A_2 = 0$ or $A_1 < 0$, $A_2 > 0$ and $|A_1| = A_2$ (i.e., $A = 1$), then the assumptions of the linearization theorem are not satisfied for Eq. (1) and because of that for those cases we have to apply other technique to examine the stability of the trivial solution. In this paper we use the linearization and therefore, we exclude the case from our analysis.

If $A_1 = 0$ and $A_2 \neq 0$ or $A_2 = 0$ and $A_1 \neq 0$, then Eq. (3) becomes the equation with one discrete delay. The theory of stability of solutions and Hopf bifurcations for the class of ordinary differential equations with one discrete delay is well known (see, e.g., [7,8] or [10]), so we not look after such cases.

Let from now on: $A_1 \neq 0$, $A_2 > 0$, $\lambda = \frac{z}{|A_1|}$, $A = \frac{A_2}{|A_1|}$, $\tau_1 = \frac{r_1}{|A_1|}$, $\tau_2 = \frac{r_2}{|A_1|}$.

Note that under assumption $A_1 > 0$ Eq. (4) has the form

$$\lambda = -e^{-\lambda r_1} - Ae^{-\lambda r_2}, \tag{5}$$

whereas for $A_1 < 0$

$$\lambda = e^{-\lambda r_1} - Ae^{-\lambda r_2}. \tag{6}$$

It is easy to see that if $\tau_1 = \tau_2 = 0$, then for $A_1 + A_2 > 0$ the trivial solution to Eq. (3) is asymptotically stable, while for $A_1 + A_2 < 0$ it is unstable. By continuity, for sufficiently small $\tau_1, \tau_2 > 0$, the trivial solution to Eq. (3) is asymptotically stable under the assumption $A_1 + A_2 > 0$ and unstable in the case $A_1 + A_2 < 0$.

2. The Hopf bifurcation analysis

The occurance of the Hopf bifurcation and investigation of the type of this bifurcation for the case $A_1, A_2 > 0$ was studied in [14]. It is important to realize that there are some mistakes in [14], which need to be correct.

In this section we shall study the Hopf bifurcation of Eq. (1) under the assumption $A_1, A_2 > 0$ as well as cases when A_1 and A_2 have different signs and have the same signs but $A_1, A_2 < 0$. Following the ideas presented in [14] we choose r_2 as a bifurcation parameter. We focus on the improvement of the results from the original paper [14] and give remarks about the corresponding results.

2.1. Case $A_1, A_2 > 0$

To prove the existence of the Hopf bifurcation (see, e.g., [12]), we need to investigate the existence of a pair of purely imaginary roots and location of the rest complex roots of characteristic equation associated with Eq. (3). After that we check the condition under which the purely imaginary roots are able to pass through the imaginary axis as the bifurcation parameter changes.

Equation (5) has purely imaginary roots $\pm i\omega$, where $\omega > 0$, if the following system of equations is satisfied

$$\begin{aligned} &\cos(\omega r_1) = -A\cos(\omega r_2),\\ &\omega - \sin(\omega r_1) = A\sin(\omega r_2). \end{aligned} \tag{7}$$

Squaring and adding up both equations, we obtain

$$\sin(\omega r_1) = \frac{\omega^2 + 1 - A^2}{2\omega}. \tag{8}$$

Denote

$$g(\omega) = \frac{\omega^2 + 1 - A^2}{2\omega}, \quad \omega \in (0, +\infty). \tag{9}$$

Because properties of the function $g(\omega)$ strongly depends on A, so we shall consider three cases: $A \in (0, 1)$, $A = 1$ and $A > 1$.

It is easy to see that $\omega \in [1 - A, 1 + A]$ for $A \in (0, 1)$, $\omega \in (0, 2]$ for $A = 1$ and $\omega \in [-1 + A, 1 + A]$ for $A > 1$.

2.1.1. Subcase $A > 1$

In [14] there was formulated Lemma 3.1 which says that if $A_1, A_2 > 0$, $A > 1$ and $r_1 < \frac{5\pi}{2(A+1)}$, then Eq. (8) has a unique solution in $[A - 1, A + 1]$. It occurs that the range of r_1 stated in this lemma is too large. Consider the example with $A = 2$ and $r_1 = \frac{5\pi - 0.1}{6}$. Figure 1 shows

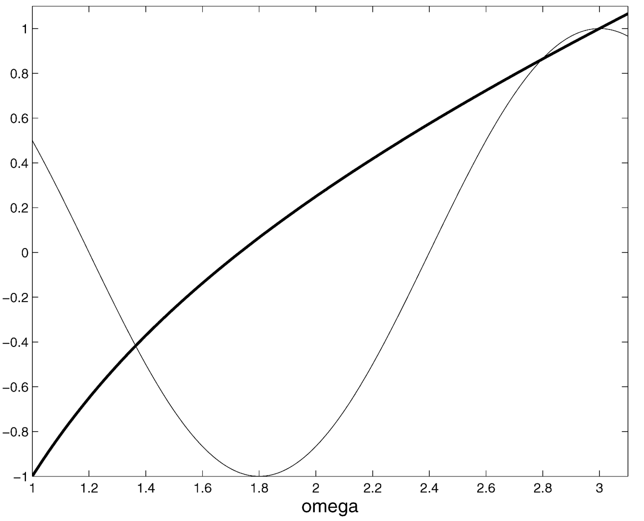

Fig. 1. The graph of $g(\omega)$ (width —) and $\sin(\frac{5\pi-0.1}{6}\omega)$ (—) for $\omega \in [1, 3]$.

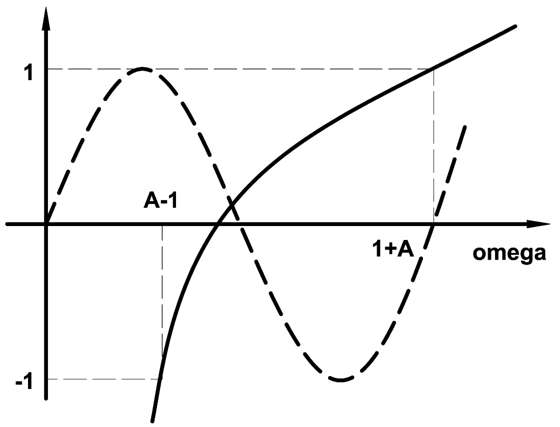

Fig. 2. The graph of $g(\omega)$ (—) and $\sin(\frac{2\omega\pi}{1+A})$ (– –) for $A > 1$.

that Eq. (8) has three solutions in the interval $[1, 3]$. The consequences of this false lemma are reflected in others lemmas and theorems formulated in [14]. Instead of Lemma 3.1 in [14] we propose the following one.

Lemma 1. *If $A > 1$, then for every arbitrary $r_1 > 0$ the functions $\sin(\omega r_1)$ and $g(w)$ intersect at least once, moreover for $r_1 \in (0, \frac{2\pi}{1+A}]$, $\sin(\omega r_1)$ and $g(w)$ intersect only once for $\omega \in [-1 + A, 1 + A]$, compare Fig. 2.*

It is clear that for any $r_1 > 0$ Eq. (8) has the finite number ($m \in \mathbb{Z}_+$) of solutions, denoted by $\omega_1, \ldots, \omega_m$.

It is obvious that for every arbitrary chosen $r_1 > 0$ and for each ω_k we have infinite number of r_2 such that $\cos(\omega_k r_1) = -A\cos(\omega_k r_2)$. For all $k \in \{1,\dots,m\}$ we define r_2^k in following way $r_2^k = \min\{r_2 \in \mathbb{R}_+\colon \cos(\omega_k r_1) = -A\cos(\omega_k r_2)\}$. Let

$$r_2^0 = \min\{r_2^k\colon k \in \{1,\dots,m\}\}. \tag{10}$$

It should be noticed that the definitions of the sequence r_2^k and the values r_2^0 are slightly different than in [14], i.e., without reversing of the cosine function. Therefore, we do not loose the information about the whole sequence and can discuss stability switches (compare the last section). We see that for some $k \in \{1,\dots,m\}$ there is $r_2^k = r_2^0$, so for that k we have define $\omega_0 = \omega_k$.

From literature (e.g., [8] or [10]) it is known that if $A > 1$, then for $r_2 = 0$ and any $r_1 \geqslant 0$ all roots of Eq. (5) have strictly negative real parts.

Because we know the localization of roots of Eq. (5) for $r_2 = 0$, so we are able to deduce (by using Rouché's Theorem, [6]) the localization of the roots of Eq. (5) on the complex plane for $r_2 \in [0, r_2^0)$ and $r_2 = r_2^0$. For details see [14].

Let $\Omega_1 = \bigcup_{l\in\mathbb{N}}[2l\pi, \frac{\pi}{2} + 2l\pi]$, $\Omega_2 = \bigcup_{l\in\mathbb{N}}[(2l+1)\pi, \frac{3}{2}\pi + 2l\pi]$ and $\Omega = \Omega_1 \cup \Omega_2$, where $\mathbb{N}$ are the natural numbers with zero.

Lemma 2. *Let $A_1, A_2 > 0$, $A > 1$ and Ω defined as above.*

(1) *If $r_2 \in [0, r_2^0)$, then Eq.* (5) *has all roots with strictly negative real parts.*
(2) *If $r_2 = r_2^0$ and $\omega_0 r_2^0 \in \Omega$, then Eq.* (5) *has a pair of simple purely imaginary roots and all other roots have strictly negative real parts.*

Instead of Lemma 3.5 postulated in [14], which assumes that $\frac{\pi}{2r_1} < \sqrt{A^2 - 1} < \frac{3\pi}{2r_1}$, we propose more general lemma guaranteeing that the purely imaginary roots to Eq. (5) (and Eq. (6) as well) pass through the imaginary axis at $\pm i\omega_0$.

Let now $\lambda(r_2) = \Re(\lambda(r_2)) + i\Im(\lambda(r_2))$ be a root of Eq. (5) (or Eq. (6)) such that there exists some $r_2 = r_2^0 > 0$ for which $\Re(\lambda(r_2^0)) = 0$ and $\Im(\lambda(r_2^0)) = \omega_0 > 0$. By simple calculations we are able to prove the following lemma.

Lemma 3.

(1) *If $\omega_0 r_2^0 \in \Omega_1$, then $\frac{d}{dr_2}\Re(\lambda(r_2))|_{r_2=r_2^0} > 0$.*
(2) *If $\omega_0 r_2^0 \in \Omega_2$, then $\frac{d}{dr_2}\Re(\lambda(r_2))|_{r_2=r_2^0} < 0$.*

Lemma 2 and statement (1) of Lemma 3 imply that if $\omega_0 r_2^0 \in \Omega_1$, then the purely imaginary roots of Eq. (5) must pass through the imaginary axis from the negative to the positive half-plane as r_2 increases. Moreover, Lemma 2 and statement (2) of Lemma 3 tell as that if $\omega_0 r_2^0 \in \Omega_2$, then roots cross the imaginary axis at $\pm i\omega_0$ from the right to the left half-plane as r_2 increases.

In Lemmas 2 and 3 we have checked all assumptions of the Hopf bifurcation theorem (see, e.g., [12]) and we are able to show the existence of periodic solutions to Eq. (1).

Theorem 1. *Let $A_2 > A_1 > 0$ and $\tau_2^0 = \frac{r_2^0}{A_1}$, where r_2^0 is defined by* (10).

(1) *For $\tau_2 \in [0, \tau_2^0)$ the trivial solution to Eq.* (1) *is asymptotically stable.*

(2) *If* $\omega_0\tau_2^0 A_1 \in \Omega_1$, *then the Hopf bifurcation occurs at* $\tau_2 = \tau_2^0$ *for Eq.* (1).

(3) *If* $\omega_0\tau_2^0 A_1 \in \Omega_2$, *then there is no Hopf bifurcation at* $\tau_2 = \tau_2^0$ *for Eq.* (1).

Corollary 1. *If* $A_2 > A_1 > 0$, $\tau_1 < \frac{2\pi}{A_1+A_2}$ *and* $\frac{\pi}{2\tau_1} < \sqrt{A_2^2 - A_1^2} < \frac{\pi}{\tau_1}$, *then there exists* τ_2^0 *such that for* $\tau_2 \in [0, \tau_2^0)$ *the trivial solution to Eq.* (1) *is asymptotically stable and for* $\tau_2 = \tau_2^0$ *the Hopf bifurcation occurs.*

Corollary 2. *If* $A_2 > A_1 > 0$ *and* $\frac{\pi}{A_1+A_2} < \tau_1 < \frac{\pi}{\sqrt{A_2^2-A_1^2}}$, *then there exists* τ_2^0 *such that for* $\tau_2 \in [0, \tau_2^0)$ *the trivial solution to Eq.* (1) *is asymptotically stable and for* $\tau_2 = \tau_2^0$ *the Hopf bifurcation occurs.*

2.1.2. Subcase $A = 1$

For $A = 1$ instead of Eqs. (7) and (8) we obtain

$$\begin{aligned} &\cos(\omega r_1) = -\cos(\omega r_2),\\ &\omega - \sin(\omega r_1) = \sin(\omega r_2), \end{aligned} \tag{11}$$

and

$$\sin(\omega r_1) = \frac{\omega}{2}, \tag{12}$$

respectively. Denote $g(\omega) = \frac{\omega}{2}$ for $\omega \in (0, +\infty)$.

It is easy to show that if $A_1, A_2 > 0$, $A = 1$ and $r_1^0 = \frac{1}{2}$, then for $r_1 \in (0, \frac{1}{2}]$ the functions $\sin(\omega r_1)$ and $g(w)$ do not intersect, and for $r_1 > \frac{1}{2}$ they intersect at least once for $\omega \in (0, 2]$ (compare Fig. 3). We see that for any $r_1 > r_1^0$ Eq. (12) has finite number ($m \in \mathbb{Z}_+$) of solutions, we denote them by $\omega_1, \ldots, \omega_m$.

Using similar procedure as in case $A_1, A_2 > 0$ and $A > 1$ for every arbitrary chosen $r_1 > r_1^0$ and for each ω_k we define $r_2^k = \min\{r_2 \in \mathbb{R}_+\colon \cos(\omega_k r_1) = -\cos(\omega_k r_2)\}$, and

$$r_2^0 = \min\bigl\{r_2^k\colon k \in \{1, \ldots, m\}\bigr\}. \tag{13}$$

It is important to realize that in [14] authors claimed that if $\omega_k r_2^k \in (0, \pi]$, then $\omega_0 r_2^0 \in (0, \frac{\pi}{2}]$. We give an example when $\omega_0 r_2^0 \notin (0, \frac{\pi}{2}]$. Let $r_1 = \frac{\pi}{5}$. Then Eq. (12) has exactly one solution $\omega_0 \in (0, 2]$ (see Fig. 4). Moreover, $\omega_0 - \sin(\omega_0 r_1), \cos(\omega_0 r_1) > 0$ and from Eqs. (11) we obtain that $\omega_0 r_2^0 \in (\frac{\pi}{2}, \pi)$. This implies that Lemma 3.14 and Theorem 3.16 in [14] are false.

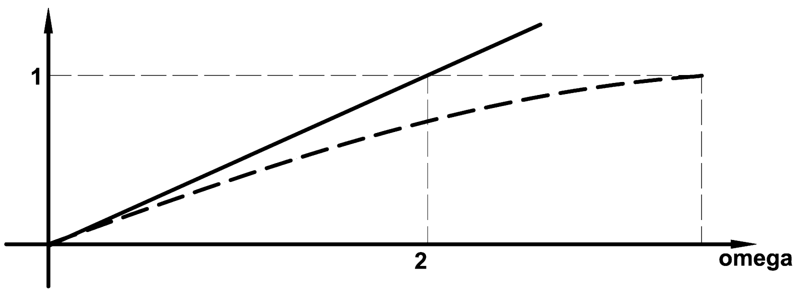

Fig. 3. The graph of $g(\omega)$ (—) and $\sin(\frac{\omega}{2})$ (– –) for $A = 1$.

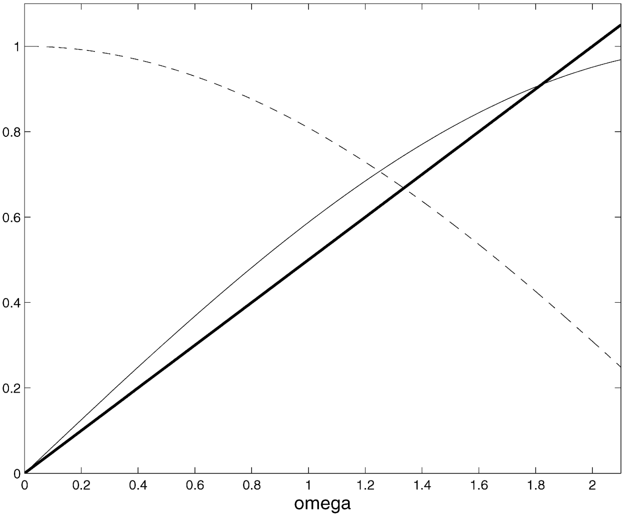

Fig. 4. The graph of $g(\omega)$ (width —) and $\sin(\frac{\pi}{5}\omega)$ (—) and $\cos(\frac{\pi}{5}\omega)$ (– –) for $A = 1$.

Once again knowing that if $A = 1$, then for $r_2 = 0$ and any $r_1 \geqslant 0$ all roots of Eq. (5) have strictly negative real parts we can prove lemma which gives as information about localization of roots of Eq. (5) for positive r_2.

Lemma 4. *Let $A_1, A_2 > 0$ and $A = 1$.*

(1) *If $r_1 \in [0, r_1^0]$ and $r_2 \geqslant 0$, or $r_1 > r_1^0$ and $r_2 \in [0, r_2^0)$, then Eq.* (5) *has all roots with strictly negative real parts.*
(2) *If $r_1 > r_1^0$ and $r_2 = r_2^0$ and $\omega_0 r_2^0 \in \Omega$, then Eq.* (5) *has a pair of simply purely imaginary roots and all other roots have strictly negative real parts.*

Instead of Theorem 3.16 in [14], by collecting together Lemmas 3 and 4, we formulate the following theorem.

Let

$$\tau_1^0 = \frac{r_1^0}{A_1} = \frac{1}{2A_1}, \qquad \tau_2^0 = \frac{r_2^0}{A_1}, \tag{14}$$

where r_2^0 is defined by (13).

Theorem 2. *Let $A_1 = A_2 > 0$ and τ_1^0 and τ_2^0 are defined by* (14).

(1) *For $\tau_1 \in [0, \tau_1^0]$ and $\tau_2 > 0$ or $\tau_1 > \tau_1^0$ and $\tau_2 \in [0, \tau_2^0)$ the trivial solution to Eq.* (1) *is asymptotically stable.*
(2) *If $\tau_1 > \tau_1^0$ and $\omega_0 \tau_2^0 A_1 \in \Omega_1$, then the Hopf bifurcation occurs at $\tau_2 = \tau_2^0$ for Eq.* (1).
(3) *If $\tau_1 > \tau_1^0$ and $\omega_0 \tau_2^0 A_1 \in \Omega_2$, then there is no Hopf bifurcation at $\tau_2 = \tau_2^0$ for Eq.* (1).

Corollary 3. *If $A_1 = A_2 > 0$ and $\frac{\pi}{2A_1} < \tau_1 \leqslant \frac{\pi}{A_1}$, then there exists τ_2^0 such that for $\tau_2 \in [0, \tau_2^0)$ the trivial solution to Eq.* (1) *is asymptotically stable and for $\tau_2 = \tau_2^0$ the Hopf bifurcation occurs.*

2.1.3. Subcase $A \in (0, 1)$

In this case we again use the function $g(\omega)$ defined by Eq. (9).

In the case $A \in (0, 1)$ there is also a mistake in [14], namely the authors defined $r_1^0 = \min\{r_1\colon \sin(\omega r_1)$ intersects $g(\omega)\}$ and suggest that if $r_1 > r_1^0$, then $\sin(\omega r_1)$ and $g(\omega)$ intersect at least twice. We present the example which shows that it is not truth. Let $A = \frac{1}{3}$. We see that for $r_1 = \frac{3}{8}\pi > r_1^0$, $\sin(\omega r_1)$ and $g(\omega)$ intersect but for $r_1 = \frac{3}{2}\pi > r_1^0$ they do not intersect, see Fig. 5.

We propose the following observation. If $A_1, A_2 > 0$ and $A \in (0, 1)$, then for $r_1 < \frac{1}{2}$ the functions $\sin(\omega r_1)$ and $g(w)$ do not intersect, and for $r_1 \geqslant \frac{\pi}{A} > \pi$, $\sin(\omega r_1)$ and $g(w)$ intersect at least twice for $\omega \in [1 - A, 1 + A]$, compare Fig. 6.

We know that for $r_1 > \frac{\pi}{A}$ the function $\sin(\omega r_1)$ intersects $g(\omega)$ at least once, but we need (the careful reader will see later why) a better estimation for r_1 such that $\sin(\omega r_1)$ and $g(\omega)$ intersect at least once. Let $\hat{r}_1 = \frac{\pi}{A}$. We consider points $\hat{\omega}_n$ ($n \in \mathbb{N}$) such that $\sin(\hat{\omega}_n \hat{r}_1) = 1$. Therefore,

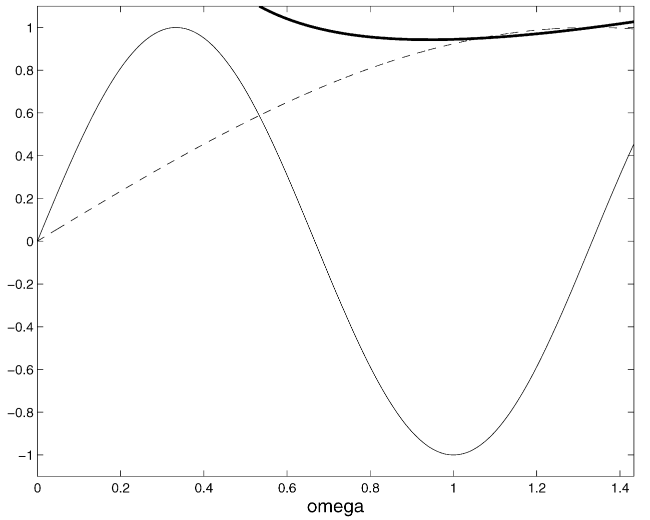

Fig. 5. The graph of $g(\omega)$ (width —) and $\sin(\frac{3}{2}\omega\pi)$ (—) and $\sin(\frac{3}{8}\omega\pi)$ (– –).

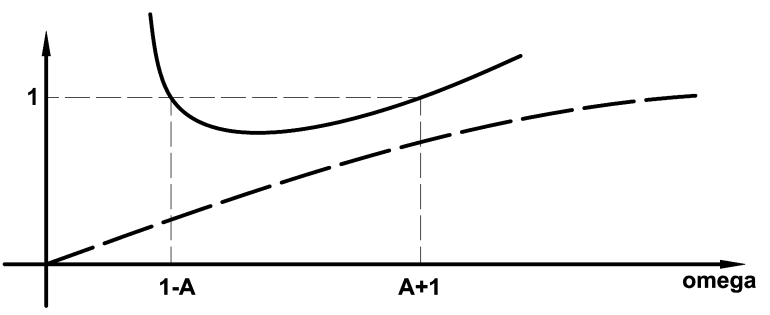

Fig. 6. The graph of $g(\omega)$ (—) and $\sin(\frac{\omega}{2})$ (width – –) for $A \in (0, 1)$.

$\hat{\omega}_n = \frac{1}{\hat{r}_1}(\frac{\pi}{2} + 2n\pi)$. We know that the period of $\sin(\omega\hat{r}_1)$ is equal to $2A$, so the graph of $\sin(\omega\hat{r}_1)$ and the graph of $g(\omega)$ intersect at least twice for $\omega \in (1 - A, 1 + A]$. Hence, there exists at least one n such that $1 - A < \hat{\omega}_n \leqslant 1 + A$. Therefore, we obtain $\frac{1}{2A} - \frac{3}{4} < n \leqslant \frac{1}{2A} + \frac{1}{4}$. Hence, $n = [\frac{2+A}{4A}]$, where $[x]$ denotes entier of x. Let r_1^* be such that $\sin(r_1^*(1 + A)) = 1$ and

$$r_1^*(1 + A) = \frac{\pi}{2} + 2n\pi. \tag{15}$$

Then $\sin(\omega r_1)$ intersects $g(\omega)$ at least twice for $r_1 \geqslant r_1^*$ for $\omega \in [1 - A, 1 + A]$.

It is easy to see that if $A \in (0, 1)$, then $[\frac{2+A}{4A}] = k$ ($k \in \mathbb{N}$) iff $\frac{2}{4k+3} < A \leqslant \frac{2}{4k-1}$ and $[\frac{2+A}{4A}] = 0$ iff $A \in (\frac{2}{3}, 1)$. Therefore, r_1^* as a function of A is not continuous, moreover $r_1^* < 1$ for $A \in (\frac{2}{3}, 1)$.

Denote

$$r_1^0 = \min\{r_1 \in \mathbb{R}_+ \colon \sin(\omega r_1) \text{ intersects } g(\omega) \text{ exactly once, } \omega \in [1 - A, 1 + A]\}. \tag{16}$$

It means that for $0 < r_1 < r_1^0$ the functions $\sin(\omega r_1)$ and $g(\omega)$ do not intersect. It is clear that for any $r_1 \geqslant r_1^*$ Eq. (8) has a finite number ($m \in \mathbb{Z}_+$) of solutions, denoted by $\omega_1, \ldots, \omega_m$. Similarly as before we see that for every arbitrary chosen $r_1 \geqslant r_1^*$ and for each ω_k we have infinite number of r_2 such that $\cos(\omega_k r_1) = -A\cos(\omega_k r_2)$. As in the previous cases for all $k \in \{1, \ldots, m\}$ we define $r_2^k = \min\{r_2 \in \mathbb{R}_+ \colon \cos(\omega_k r_1) = -A\cos(\omega_k r_2)\}$, and

$$r_2^0 = \min\{r_2^k \colon k \in \{1, \ldots, m\}\}. \tag{17}$$

Let

$$\bar{r}_1 = \frac{\arcsin(\sqrt{1 - A^2})}{\sqrt{1 - A^2}} = \frac{\arccos(-A)}{\sqrt{1 - A^2}}. \tag{18}$$

From the theory of ordinary differential equations with one delay (for details see [5,10] or [8]) it is known that if $r_2 = 0$ and $A \in (0, 1)$, then for $r_1 \in [0, \bar{r}_1)$ all roots of Eq. (5) have strictly negative real parts; for $r_1 = \bar{r}_1$ Eq. (5) has a pair of purely imaginary roots and all other roots have strictly negative real parts; for $r_1 > \bar{r}_1$ Eq. (5) has at least one root with positive real part.

It is obvious that if $A \in (0, 1)$ and we treat $\bar{r}_1$ and r_1^* as functions of A, then for $A \in (0, 1)$ we have $\bar{r}_1 \in (1, \frac{\pi}{2})$. Furthermore, if $A \in (\frac{2}{3}, 1)$, then $\bar{r}_1 > 1 > r_1^*$ (see Fig. 7). Because for $A \in (0, 1)$ there is $\frac{\pi}{A} > \pi > \bar{r}_1$, so now it is clear why we needed better estimation for r_1 such that Eq. (8) has at least one solution for $\omega \in [1 - A, A + 1]$.

Collecting together all those information about the roots of Eq. (4) we formulate the following lemma and theorem.

Lemma 5. *Let* $A_1, A_2 > 0$ *and* $A \in (\frac{2}{3}, 1)$.

(1) *If* $r_1 \in [0, r_1^0)$ *and* $r_2 > 0$ *or* $r_1 \in [r_1^*, \bar{r}_1)$ *and* $r_2 \in [0, r_2^0)$, *then Eq.* (5) *has all roots with strictly negative real parts.*
(2) *If* $r_1 \in [r_1^*, \bar{r}_1)$, $r_2 = r_2^0$ *and* $\omega_0 r_2^0 \in \Omega$, *then Eq.* (5) *has a pair of simply purely imaginary roots and all other roots have strictly negative real parts.*

Let

$$\tau_1^0 = \frac{r_1^0}{A_1}, \qquad \tau_1^* = \frac{r_1^*}{A_1}, \qquad \bar{\tau}_1 = \frac{\bar{r}_1}{A_1}, \qquad \tau_2^0 = \frac{r_2^0}{A_1}, \tag{19}$$

where r_1^*, r_1^0, r_2^0 and $\bar{r}_1$ are defined by (15), (16), (17), (18), respectively.

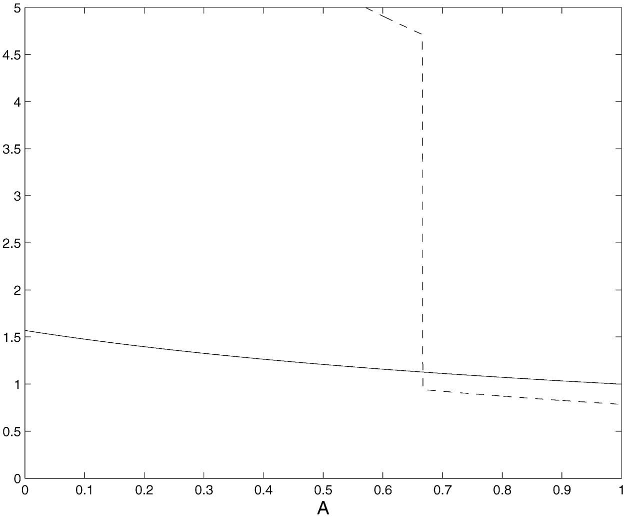

Fig. 7. The graph of r_1^* (– –) and $\bar{r}_1$ (—) for $A \in (0,1)$.

Theorem 3. *Let* $\frac{2}{3}A_1 < A_2 < A_1$, τ_1^*, τ_1^0, τ_2^0, $\bar{\tau}_1$ *are defined by* (19).

(1) *If* $\tau_1 \in [0, r_1^0)$ *and* $\tau_2 > 0$ *or* $\tau_1 \in [\tau_1^*, \bar{r}_1)$ *and* $\tau_2 \in [0, \tau_2^0)$, *then the trivial solution to Eq.* (1) *is asymptotically stable.*
(2) *If* $\tau_1 \in [\tau_1^*, \bar{r}_1)$ *and* $\tau_2 = \tau_2^0$ *and* $\omega_0 \tau_2^0 A_1 \in \Omega_1$, *then the Hopf bifurcation occurs at* $\tau_2 = \tau_2^0$ *for Eq.* (1).
(3) *If* $\tau_1 \in [\tau_1^*, \bar{r}_1)$ *and* $\tau_2 = \tau_2^0$ *and* $\omega_0 \tau_2^0 A_1 \in \Omega_2$, *then there is no Hopf bifurcation at* $\tau_2 = \tau_2^0$ *for Eq.* (1).

2.2. *Case* $A_1 < 0$, $A_2 > 0$

In this section we investigate the occurance of the Hopf bifurcation for Eq. (1) under the assumption $A_1 < 0$ and $A_2 > 0$. As before we choose r_2 as a bifurcation parameter.

For Eq. (6) instead of condition (7), which guarantees the existence of purely imaginary roots $\pm i\omega$ (where $\omega > 0$), we have the following one

$$\begin{aligned} &\cos(\omega r_1) = A\cos(\omega r_2), \\ &\omega + \sin(\omega r_1) = A\sin(\omega r_2). \end{aligned} \tag{20}$$

By squaring and adding up both equations, we have

$$\sin(\omega r_1) = \frac{A^2 - \omega^2 - 1}{2\omega}. \tag{21}$$

Let

$$g(\omega) = \frac{-\omega^2 - 1 + A^2}{2\omega}, \quad \omega \in (0, +\infty). \tag{22}$$

As before the properties of the function $g(\omega)$ strongly depends on A, however we shall consider only two cases: $A \in (0,1)$ and $A > 1$. We do not consider $A = 1$, because in this case the assumptions of the linearization theorem are not satisfied for Eq. (1). It is easy to see that the inequality $|\sin(\omega r_1)| \leqslant 1$ implies: $\omega \in [1 - A, 1 + A]$ for $A \in (0, 1)$ and $\omega \in [-1 + A, 1 + A]$ for $A > 1$.

2.2.1. Subcase $A > 1$

It is easy to see that if $A_1 < 0$, $A_2 > 0$ and $A > 1$, then for every arbitrary $r_1 > 0$ the functions $\sin(\omega r_1)$ and $g(w)$ intersect at least once, moreover for $r_1 \in (0, \frac{\pi}{1+A}]$, $\sin(\omega r_1)$ and $g(w)$ intersect only once for $\omega \in [-1 + A, 1 + A]$, see Fig. 8.

In the case when $A_1 \cdot A_2 < 0$ and $A > 1$ most of the lemmas and theorems are analogous to lemmas and theorems for the case $A_1, A_2 > 0$ and $A > 1$ even though the fact that there are some changes in the definitions of r_2^0 and $g(\omega)$. Hence, we only formulate the necessary definitions, lemmas and final theorem.

It is clear that for any $r_1 > 0$ Eq. (21) has finite number ($m \in \mathbb{Z}_+$) of solutions $\omega_1, \ldots, \omega_m$. For every arbitrary chosen $r_1 > 0$ and for each ω_k we have an infinite number of r_2 such that $\cos(\omega_k r_1) = A\cos(\omega_k r_2)$. As before for all $k \in \{1, \ldots, m\}$ we define $r_2^k = \min\{r_2 \in \mathbb{R}_+ \colon \cos(\omega_k r_1) = A\cos(\omega_k r_2)\}$. Let

$$r_2^0 = \min\{r_2^k \colon k \in \{1, \ldots, m\}\}. \tag{23}$$

Because we know [5,10] that if $A > 1$, then for $r_2 = 0$ and any $r_1 \geqslant 0$ all roots of Eq. (6) have strictly negative real parts, so we are able to prove the lemma below.

Lemma 6. *Let $A_1 < 0$, $A_2 > 0$ and $A > 1$.*

(1) *If $r_2 \in [0, r_2^0)$, then Eq.* (6) *has all roots with strictly negative real parts.*
(2) *If $r_2 = r_2^0$ and $\omega_0 r_2^0 \in \Omega$, then Eq.* (6) *has a pair of simply purely imaginary roots and all other roots have strictly negative real parts.*

Let $\lambda(r_2) = \Re(\lambda(r_2)) + i\Im(\lambda(r_2))$ be the root of Eq. (6) such that there exists some $r_2 = r_2^0 > 0$ for which $\Re(\lambda(r_2^0)) = 0$ and $\Im(\lambda(r_2^0)) = \omega_0 > 0$.

Collecting Lemmas 3 and 6 together we formulate the final theorem.

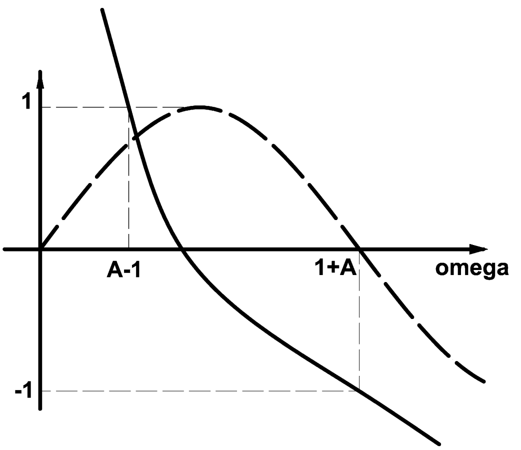

Fig. 8. The graph of $g(\omega)$ (—) and $\sin(\frac{\omega\pi}{1+A})$ (– –) for $A > 1$.

Theorem 4. *Let $A_1 < 0$, $A_2 > |A_1|$ and $\tau_2^0 = \frac{r_2^0}{|A_1|}$, where r_2^0 is defined by* (23).

(1) *For $\tau_2 \in [0, \tau_2^0)$ the trivial solution to Eq.* (1) *is asymptotically stable.*
(2) *If $\omega_0 \tau_2^0 |A_1| \in \Omega_1$, then the Hopf bifurcation occurs at $\tau_2 = \tau_2^0$ for Eq.* (1).
(3) *If $\omega_0 \tau_2^0 |A_1| \in \Omega_2$, then there is no Hopf bifurcation at $\tau_2 = \tau_2^0$ for Eq.* (1).

Corollary 4. *If $A_1 < 0$, $A_2 > |A_1|$ and $\tau_1 \in (0, \frac{\pi}{2\sqrt{A_2^2 - A_1^2}}]$, then there exists τ_2^0 such that for $\tau_2 \in [0, \tau_2^0)$ the trivial solution to Eq.* (1) *is asymptotically stable and for $\tau_2 = \tau_2^0$ the Hopf bifurcation occurs.*

2.2.2. Subcase $A \in (0, 1)$

Now we focus on case $A_1 < 0$ and $0 < A_2 < |A_1|$. It is obvious that because $A \in (0, 1)$, so $A - 1 < 0$. Hence, the trivial stationary solution to Eq. (1) is unstable independently on the values of both delays (for details see [11]). Through that for $A_1 < 0$ and $0 < A_2 < |A_1|$ there is no Hopf bifurcation.

Of course if $A_1 > 0$ and $A_2 < 0$, then we can exchange A_1 for A_2 and τ_1 for τ_2 and apply the derived results.

2.3. Case $A_1, A_2 < 0$

This case is similar to the case $A_1 < 0$ and $0 < A_2 < |A_1|$. As before because $A_1, A_2 < 0$ implies that $A_1 + A_2 < 0$, so zero solution to Eq. (1) is unstable independently on the values of both delays [11].

3. Discussion

In presented paper we considered differential equations with two independent discrete delays. We focus on the existence of the Hopf bifurcation for equations whose linearization around the trivial solution has the form

$$\frac{d}{dt}x(t) = -A_1 x(t - \tau_1) - A_2 x(t - \tau_2). \tag{24}$$

Three main cases, namely $A_1, A_2 > 0$, $A_1 \cdot A_2 < 0$ and $A_1, A_2 < 0$ were presented. As a bifurcation parameter the second delay τ_2 was chosen.

This paper is the first steep to investigation the possibility of existence of the stability switches for the class of differential equations with two independent delays, i.e., existing a sequence of values of τ_2, namely

$$0 < \tau_2^0 = \tau_2^{0,1} < \tau_2^{0,2} < \tau_2^{1,1} < \tau_2^{1,2} < \tau_2^{2,1} < \tau_2^{2,2} < \cdots$$

such that if $\tau_2 = \tau_2^{i,1}$ for $i \in \mathbb{N}$ the stationary stable solution becomes unstable, and if $\tau_2 = \tau_2^{i,2}$ for $i \in \mathbb{N}$ unstable stationary solution becomes stable. It means there is a possibility of repeatedly switching the stability of solution. The different, than in [14], definition of critical value of delay r_2^0 under which the Hopf bifurcation occurs allows us to create the candidate sequence of values of τ_2 which could assure the existing of stability switches. If we assume that the solution to Eq. (24) is stable for all $\tau_1 \geqslant 0$ as $\tau_2 = 0$, then the sufficient conditions to obtain the stability switches have the following form

$$\frac{d}{d\tau_2}\Re\big(\lambda(\tau_2)\big)\big|_{\tau_2=\tau_2^{i,1}} > 0 \quad \text{for } i \in \mathbb{N}, \qquad \frac{d}{d\tau_2}\Re\big(\lambda(\tau_2)\big)\big|_{\tau_2=\tau_2^{i,2}} < 0 \quad \text{for } i \in \mathbb{N},$$

where $\lambda(\tau_2) = \Re(\lambda(\tau_2)) + i\Im(\lambda(\tau_2))$ is a root of linear equation (24) such that there exist $\tau_2 = \tau_2^{i,j} > 0$ for which $\Re(\lambda(\tau_2^{i,j})) = 0$ and $\Im(\lambda(\tau_2^{i,j})) = \omega_{0,i} > 0$ and $i \in \mathbb{N}$, $j \in \{1, 2\}$.

As the reader can see the investigation of stability switches becomes quite complicated.

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References

[1] J. Bélair, S.A. Campbell, Stability and bifurcations of equilibrium in multiple-delayed differential equation, SIAM J. Appl. Math. 54 (1994) 1402–1424.
[2] A. Beuter, J. Bélair, Feedback and delay in neurological diseases: A modeling study using dynamical systems, Bull. Math. Biol. 55 (1993) 525–541.
[3] A. Beuter, D. Larocque, L. Glass, Complex oscillations in a human motor system, J. Motor Behavior 21 (1989) 277–289.
[4] R.D. Braddock, P. van den Driessche, On two lag differential delay equation, J. Aust. Math. Soc. Ser. B 24 (1983) 292–317.
[5] O. Diekmann, S. van Giles, S. Verduyn Lunel, H. Walter, Delay Equation, Springer, New York, 1995.
[6] J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960.
[7] R.D. Driver, Ordinary and Delay Differential Equations, Springer, New York, 1977.
[8] U. Foryś, Biological delay systems and Mikhailov criterion of stability, J. Biol. Syst. 12 (1) (2004) 1–16.
[9] K. Gopalsamy, Global stability in the delay-logistic equation with discrete delays, Houston J. Math. 16 (1990) 347–356.
[10] J.K. Hale, Theory of Functional Differential Equations, Springer, New York, 1977.
[11] J.K. Hale, W. Huang, Global geometry of the stable regions for two delay differential equations, J. Math. Anal. Appl. 178 (1993) 344–362.
[12] B.D. Hassard, N.D. Kazarinoff, Y.-H. Wan, Theory and Applications of Hopf Bifurcation, Cambridge Univ. Press, New York, 1981.
[13] J.M. Mahaffy, P.J. Zak, K.M. Joiner, A geometric analysis of the stability regions for a linear differential equation with two delays, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 5 (1995) 779–796.
[14] X. Li, S. Ruan, J. Wei, Stability and bifurcation in delay-differential equations with two delays, J. Math. Anal. Appl. 236 (1999) 254–280.