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A note on three-parameter families and generalized convex functions

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Abstract

The concept of generalized convex functions introduced by Beckenbach [E.F. Beckenbach, Generalized convex functions, Bull. Amer. Math. Soc. 43 (1937) 363–371] is extended to the two-dimensional case. Using three-parameter families, we define generalized convex (midconvex, M-convex) functions $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ and show some continuity properties of them.

Keywords: Three-parameter families; Generalized convex functions

1. Introduction

By the standard definition, a function $f : I \to \mathbb{R}$ is called convex if, for any two distinct points on the graph of f, the segment joining these points lies above the corresponding part of the graph. In [1] Beckenbach generalized this classical definition by replacing the segments by some curves—the graphs of functions belonging to a two-parameter family. The generalized convex functions obtained in such a way have many properties similar to those of convex functions (cf.,

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e.g., [1–5]). However, all these results are formulated for functions defined on real intervals. In this note we extend the idea given by Beckenbach to two-variable functions. Using a three-parameter family $\mathcal{F}$, we define $\mathcal{F}$-convex, $\mathcal{F}$-midconvex, and $(\mathcal{F}, M)$-convex functions and show their important properties. In particular, we prove that $\mathcal{F}$-convex functions defined on a convex set $D \subset \mathbb{R}^2$ are continuous on $\operatorname{int} D$ and lower semicontinuous $\mathcal{F}$-midconvex and $(\mathcal{F}, M)$-convex functions are also $\mathcal{F}$-convex.

2. Three-parameter families

Let $\mathcal{F}$ be a family of continuous real functions defined on a convex set $D \subseteq \mathbb{R}^2$ with nonempty interior. We say that $\mathcal{F}$ is a *three-parameter family* on D if, for any three non-collinear points $x_1, x_2, x_3 \in D$ and for any $t_1, t_2, t_3 \in \mathbb{R}$, there exists exactly one $\varphi \in \mathcal{F}$ such that

$$\varphi(x_i) = t_i \quad \text{for } i = 1, 2, 3.$$

The unique function $\varphi \in \mathcal{F}$ determined by the points $x_1, x_2, x_3 \in D$ and values $t_1, t_2, t_3 \in \mathbb{R}$ will be denoted by $\varphi_{(x_1,t_1)(x_2,t_2)(x_3,t_3)}$.

Example 2.1. Consider the following classes of functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$:

(1) $\mathcal{F}_1 = \{\varphi(u, v) = au + bv + c\colon\ a, b, c \in \mathbb{R}\}$;
(2) $\mathcal{F}_2 = \{\varphi(u, v) = u^2 + v^2 + au + bv + c\colon\ a, b, c \in \mathbb{R}\}$;
(3) $\mathcal{F}_3 = \{\varphi(u, v) = h(u, v) + au + bv + c\colon\ a, b, c \in \mathbb{R}\}$, where $h : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary continuous (but not necessarily differentiable) function;
(4) $\mathcal{F}_4 = \{\varphi(u, v) = m(u, v)(au + bv + c)\colon\ a, b, c \in \mathbb{R}\}$, where $m : \mathbb{R}^2 \to \mathbb{R} \setminus \{0\}$ is an arbitrary continuous function;
(5) $\mathcal{F}_5 = \{\varphi(u, v) = g(au + bv + c)\colon\ a, b, c \in \mathbb{R}\}$, where $g : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous injection.

The above classes of functions are examples of three-parameter families on $\mathbb{R}^2$. One can check it easily by use of Cramer's theorem and the fact that points $(u_1, v_1), (u_2, v_2), (u_3, v_3) \in \mathbb{R}^2$ are non-collinear iff

$$\det \begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ u_3 & v_3 & 1 \end{bmatrix} \neq 0.$$

We will use the following notation. For $a, b \in \mathbb{R}^2$, $a \neq b$, we denote by $[a, b]$ the closed segment joining a and b (i.e., $[a, b] = \{(1 - t)a + tb\colon\ 0 \leqslant t \leqslant 1\}$), and by $L(a, b)$—the line spanned by a, b (i.e., $L(a, b) = \{(1 - t)a + tb\colon\ t \in \mathbb{R}\}$). For non-collinear points $a, b, c \in \mathbb{R}^2$, $L(a, b)c$ stands for the open half-plane determined by $L(a, b)$ containing c. By $\triangle_{abc}$ we denote the closed triangle with vertices a, b, c, and by $\triangle^a_{bc}$—the extension of this triangle via a, that is,

$$\triangle^a_{bc} = \bigl\{x \in \mathbb{R}^2\colon\ a \in [x, z] \text{ for some } z \in \triangle_{abc},\ z \neq a\bigr\}.$$

The sets $\triangle_{abc}$ and $\triangle^a_{bc}$ can also be expressed in terms of barycentric coordinates, namely

$$\triangle_{abc} = \{ra + sb + tc\colon\ r, s, t \geqslant 0,\ r + s + t = 1\},$$
$$\triangle^a_{bc} = \{ra + sb + tc\colon\ s, t \leqslant 0,\ r + s + t = 1\}.$$

In the sequel $D \subset \mathbb{R}^2$ will be a convex set with non-empty interior and $\mathcal{F}$ be a fixed three-parameter family on D.

Lemma 2.2. *Let $\varphi_1, \varphi_2 \in \mathcal{F}$ and x_1, x_2, x_3 be non-collinear points in D. If $\varphi_1(x_1) = \varphi_2(x_1)$, $\varphi_1(x_2) = \varphi_2(x_2)$ and $\varphi_1(x_3) \leqslant \varphi_2(x_3)$, then $\varphi_1(x) \leqslant \varphi_2(x)$ for every $x \in L(x_1, x_2)x_3$.*

Proof. If $\varphi_1(x_3) = \varphi_2(x_3)$ then, by the fact that $\mathcal{F}$ is a three-parameter family, we have $\varphi_1 = \varphi_2$, which finishes the proof. Now, assume that $\varphi_1(x_3) < \varphi_2(x_3)$ and suppose there exists $x_4 \in L(x_1, x_2)x_3$ such that $\varphi_1(x_4) > \varphi_2(x_4)$. By the continuity of φ_1, φ_2 there exists $x_0 \in [x_3, x_4]$ such that $\varphi_1(x_0) = \varphi_2(x_0)$. Indeed, consider $\psi : [0, 1] \to \mathbb{R}$ defined by

$$\psi(t) = \varphi_1\big((1-t)x_3 + tx_4\big) - \varphi_2\big((1-t)x_3 + tx_4\big), \quad t \in [0, 1].$$

Since ψ is continuous and $\psi(0) < 0$, $\psi(1) > 0$, there exists $t_0 \in (0, 1)$ such that $\psi(t_0) = 0$. Then for $x_0 = (1 - t_0)x_3 + t_0x_4$ we have $\varphi_1(x_0) = \varphi_2(x_0)$. Since $\mathcal{F}$ is a three-parameter family and x_1, x_2, x_0 are non-collinear, this implies that $\varphi_1 = \varphi_2$, which contradicts the assumption $\varphi_1(x_3) < \varphi_2(x_3)$. $\square$

Lemma 2.3. *Let $\varphi_1, \varphi_2 \in \mathcal{F}$ and x_1, x_2, x_3 be non-collinear points in D. If $\varphi_1(x_i) \leqslant \varphi_2(x_i)$, $i = 1, 2, 3$, then $\varphi_1(x) \leqslant \varphi_2(x)$ for all $x \in \triangle_{x_1x_2x_3}$.*

Proof. Consider $\varphi_{11}, \varphi_{12} \in \mathcal{F}$ such that

$$\varphi_{11}(x_i) = \varphi_1(x_i), \quad i = 1, 2, \quad \text{and} \quad \varphi_{11}(x_3) = \varphi_2(x_3),$$
$$\varphi_{12}(x_1) = \varphi_1(x_1) \quad \text{and} \quad \varphi_{12}(x_i) = \varphi_2(x_i), \quad i = 2, 3.$$

Then, using Lemma 2.2 three times, we get

$$\varphi_1(x) \leqslant \varphi_{11}(x) \leqslant \varphi_{12}(x) \leqslant \varphi_2(x)$$

for all $x \in \triangle_{x_1x_2x_3}$. $\square$

3. $\mathcal{F}$-convex functions

A function $f : D \to \mathbb{R}$ is said to be $\mathcal{F}$-*convex* if for any non-collinear $a, b, c \in D$,

$$f(x) \leqslant \varphi_{(a,f(a))(b,f(b))(c,f(c))}(x), \quad x \in \triangle_{abc}.$$

Lemma 3.1. *Let $f : D \to \mathbb{R}$ be $\mathcal{F}$ -convex and $a, b, c \in D$ be non-collinear. Then*

$$f(x) \geqslant \varphi_{(a,f(a))(b,f(b))(c,f(c))}(x), \quad x \in \big(\triangle^a_{bc} \cup \triangle^b_{ac} \cup \triangle^c_{ab}\big) \cap D.$$

Proof. Fix, for instance, $x_0 \in \triangle^c_{ab} \cap D$. Then $c \in \triangle_{abx_0}$. Let $\varphi = \varphi_{(a,f(a))(b,f(b))(c,f(c))}$ and $\varphi_1 = \varphi_{(a,f(a))(b,f(b))(x_0,f(x_0))}$. By the $\mathcal{F}$-convexity of f we have

$$\varphi(c) = f(c) \leqslant \varphi_1(c).$$

Hence, by Lemma 2.2,

$$\varphi(x) \leqslant \varphi_1(x) \quad \text{for every } x \in L(a, b)c.$$

In particular,

$$\varphi(x_0) \leqslant \varphi_1(x_0) = f(x_0),$$

which completes the proof. □

Theorem 3.2. *If $f: D \to \mathbb{R}$ is $\mathcal{F}$-convex, then it is continuous on* $\operatorname{int} D$.

Proof. Fix $x_0 \in \operatorname{int} D$ and take $a, b, c \in D$ such that $x_0 \in \operatorname{int} \triangle_{abc}$. Consider the triangles $\triangle_1 = \triangle_{abx_0}$, $\triangle_2 = \triangle_{acx_0}$ and $\triangle_3 = \triangle_{bcx_0}$, and take

$$\varphi_1 = \varphi_{(a,f(a))(b,f(b))(x_0,f(x_0))}, \qquad \varphi_2 = \varphi_{(a,f(a))(c,f(c))(x_0,f(x_0))},$$
$$\varphi_3 = \varphi_{(b,f(b))(c,f(c))(x_0,f(x_0))}.$$

By the $\mathcal{F}$-convexity of f, we have

$$f(x) \leqslant \varphi_i(x) \quad \text{for all } x \in \triangle_i,\ i = 1, 2, 3.$$

Hence

$$f(x) \leqslant \max\{\varphi_i(x):\ i = 1, 2, 3\}, \quad x \in \triangle_{abc}. \tag{1}$$

On the other hand, by Lemma 3.1, we get

$$f(x) \geqslant \varphi_i(x) \quad \text{for all } x \in \triangle_i^{x_0} \cap D,\ i = 1, 2, 3$$

(here $\triangle_i^{x_0}$ denotes the extension of $\triangle_i$ via x_0) and hence

$$f(x) \geqslant \min\{\varphi_i(x):\ i = 1, 2, 3\}, \quad x \in \triangle_{abc}. \tag{2}$$

Since φ_i are continuous and $\varphi_i(x_0) = f(x_0)$, $i = 1, 2, 3$, therefore inequalities (1) and (2) imply that f is continuous at x_0, which was to be proved. □

Remark 3.3. We can give another definition of generalized convex functions involving only two points. Namely, let Φ be a family of continuous functions defined on straight lines in $\mathbb{R}^2$. Φ is called a *two-parameter family* if for any $a, b \in \mathbb{R}^2$, $a \neq b$, and for any $t_1, t_2 \in \mathbb{R}$ there exists exactly one $\varphi \in \Phi$ such that $\varphi: L(a, b) \to \mathbb{R}$ and

$$\varphi(a) = t_1, \qquad \varphi(b) = t_2.$$

The unique function $\varphi \in \Phi$ determined by the points $a, b \in \mathbb{R}^2$ and $t_1, t_2 \in \mathbb{R}$ will be denoted by $\varphi_{(a,t_1)(b,t_2)}$. We say that a function $f: D \to \mathbb{R}$ is *Φ-convex* if for any $a, b \in D$, $a \neq b$,

$$f(x) \leqslant \varphi_{(a,f(a))(b,f(b))}(x), \quad x \in [a, b].$$

At the first sight such a definition may seem to be a better counterpart of the classical one. However, the properties of Φ-convex functions are not as good as those of $\mathcal{F}$-convex functions. For instance, Φ-convex functions need not be continuous as it is shown by the next example.

Example 3.4. Let $h: \mathbb{R}^2 \to \mathbb{R}$ be a function whose restrictions to every straight line in $\mathbb{R}^2$ are continuous. Consider the family Φ_h of all functions $\varphi: L(a, b) \to \mathbb{R}$ of the form

$$\varphi\big((1-t)a + tb\big) = h\big((1-t)a + tb\big) + \alpha t + \beta, \quad t \in \mathbb{R},$$

where $a, b \in \mathbb{R}^2$, $a \neq b$, and $\alpha, \beta \in \mathbb{R}$. It is easy to verify that Φ_h is a two-parameter family on $\mathbb{R}^2$ and the function h itself is Φ_h-convex. Now, take $h_0 : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$h_0(x) = \begin{cases} \frac{x_1^2 x_2}{x_1^4 + x_2^2}, & \text{if } x = (x_1, x_2) \neq (0,0), \\ 0, & \text{if } x = (0,0). \end{cases}$$

Then h_0 is Φ_{h_0}-convex, but it is not continuous at $x = (0,0)$.

4. $\mathcal{F}$-midconvex and $(\mathcal{F}, M)$-convex functions

A function $f : D \to \mathbb{R}$ is said to be $\mathcal{F}$-*midconvex* if for all non-collinear points $a, b, c \in D$,

$$f\left(\frac{a+b+c}{3}\right) \leqslant \varphi_{(a,f(a))(b,f(b))(c,f(c))}\left(\frac{a+b+c}{3}\right).$$

To replace the arithmetic mean in the above definition by a more general mean, we introduce the notion of strict means on D as follows. A three variable function $M : D^3 \to D$ is called a *strict mean on D* if, for all non-collinear points $a, b, c \in D$,

$$M(a,b,c) \in \operatorname{int} \triangle_{abc}.$$

Given a strict mean $M : D^3 \to D$, a subset $C \subseteq D$ is called M-*convex* if $M(x,y,z) \in C$ for all non-collinear elements $x, y, z \in C$. A function $f : D \to \mathbb{R}$ is said to be $(\mathcal{F}, M)$-*convex* if for all non-collinear points $a, b, c \in D$,

$$f\big(M(a,b,c)\big) \leqslant \varphi_{(a,f(a))(b,f(b))(c,f(c))}\big(M(a,b,c)\big).$$

A set $C \subseteq D$ is called *two-dimensional* if it cannot be covered by a line, i.e., it contains at least three non-collinear points. The next two lemmas establish the equivalence of convexity and M-convexity of two-dimensional closed sets under certain conditions on the mean M. In the first one we assume that M is contractive in its third variable, whereas in the second one that M is a weighted arithmetic mean with the weights separated from 0.

Lemma 4.1. *Let $M : D^3 \to D$ be a strict mean which is contractive in its third variable, i.e., for all $x, y, z', z'' \in D$ with $z' \neq z''$,*

$$\big|M(x,y,z') - M(x,y,z'')\big| < |z' - z''|.$$

Furthermore, assume that, for $x \neq y$ in D,

$$M(x,y,x),\ M(x,y,y) \notin \{x,y\}. \tag{3}$$

Then any two-dimensional closed M-convex set $C \subseteq D$ is also convex.

Proof. Assume, on the contrary, that C is not convex. Then there exist two points of C, say p and q, such that the segment $[p,q]$ is not contained in C. Due to the closedness of C, there exists a subsegment $[x,y]$ of $[p,q]$ such that

$$x, y \in C \quad \text{and} \quad]x,y[\cap C = \big\{tx + (1-t)y\colon\ 0 < t < 1\big\} \cap C = \emptyset.$$

Since C is two-dimensional, there exists a point $z \in C$ which is not on the line $L(x,y)$. Then x, y, and z are non-collinear. Define the sequence (z_n) by the recursion

$$z_1 := z, \qquad z_{n+1} := M(x,y,z_n) \quad (n \in \mathbb{N}).$$

By our contractivity assumption on M, the map $u \mapsto M(x, y, u)$ is a contractive selfmap (a Piccard iteration) of the triangle $\triangle_{xyz}$ (which is a compact metric space). Therefore, by a well-known result of iteration theory, the sequence converges to a limit point $w \in \triangle_{xyz}$ which is the unique fixed point of the map $u \mapsto M(x, y, u)$, i.e., $w = M(x, y, w)$. If x, y and w were non-collinear, then, by the strict mean value property, the point $w = M(x, y, w)$ is in the interior of the triangle $\triangle_{xyw}$, which is impossible. Thus, w is an element of the segment $[x, y]$. The equalities $w = x$ and $w = y$ contradict (3), therefore $w \in]x, y[$.

On the other hand, C is M-convex, therefore, by induction, it follows that the sequence (z_n) is contained in C. In view of the closedness, the limit point w is also in C. This however yields the contradiction $w \in]x, y[\cap C$. Thus the proof is complete. □

Lemma 4.2. *Let $\lambda_1, \lambda_2, \lambda_3 : D^3 \to (0, 1)$ be given functions such that*

$$\lambda_1(x, y, z) + \lambda_2(x, y, z) + \lambda_3(x, y, z) = 1, \quad x, y, z \in D, \tag{4}$$

and, for all non-collinear points $x, y, z \in D$,

$$\inf_{u \in \triangle_{xyz}} \lambda_i(x, y, u) > 0, \quad i \in \{1, 2\}. \tag{5}$$

Let

$$M(x, y, z) = \lambda_1(x, y, z)x + \lambda_2(x, y, z)y + \lambda_3(x, y, z)z, \quad x, y, z \in D. \tag{6}$$

Then any two-dimensional closed M-convex set $C \subseteq D$ is also convex.

Note that condition (5) is easily satisfied if λ_1 and λ_2 are continuous function on D^3 because $\triangle_{xyz}$ is compact.

Proof. Suppose, on the contrary, that C is not convex. Then, as in the proof of the previous lemma, there exist points $x, y \in C$ such that $]x, y[\cap C = \emptyset$ and a point $z \in C$ such that x, y, z are non-collinear. Consider the sequence (z_n) defined by

$$z_1 := z, \qquad z_{n+1} := M(x, y, z_n) \quad (n \in \mathbb{N}).$$

Since all $z_n \in \triangle_{xyz}$, there exists a subsequence (z_{n_k}) of (z_n) convergent to a point $w \in \triangle_{xyz}$. We will show that $w \in]x, y[$. Using the representation (6) $(n_k - 1)$ times, we can express z_{n_k} in the form

$$z_{n_k} = r_k x + s_k y + t_k z, \tag{7}$$

where the barycentric coordinates r_k, s_k, t_k depend on values of $\lambda_1, \lambda_2, \lambda_3$ at the points (x, y, z_i), $i = 1, \ldots, n_k - 1$. It is easily seen that

$$r_k > \lambda_1(x, y, z_{n_k - 1}), \quad s_k > \lambda_2(x, y, z_{n_k - 1}) \quad \text{and} \quad t_k = \prod_{i=1}^{n_k - 1} \lambda_3(x, y, z_i),$$

for every $k > 1$. On account of assumptions (4) and (5), we have

$$\sup_{u \in \triangle_{xyz}} \lambda_3(x, y, u) < 1.$$

Consequently, by the above formula on t_k, we obtain $t_k \to 0$. The sequences (r_k) and (s_k) contain subsequences convergent to some points $r, s \in [0, 1]$, respectively. For simplicity assume that $r_k \to r$ and $s_k \to s$. Since

$$r_k > \lambda_1(x, y, z_{n_k - 1}) \geqslant \inf_{u \in \triangle_{xyz}} \lambda_1(x, y, u) > 0,$$

we get $r > 0$. Similarly, $s > 0$. Now, letting $k \to \infty$ in (7), we obtain

$$w = rx + sy.$$

Hence $w \in]x, y[$. On the other hand, since C is closed and $z_{n_k} \in C$ for all $k \in \mathbb{N}$, we have $w \in C$. This contradicts the fact that $]x, y[\cap C = \emptyset$ and finishes the proof. $\square$

The next theorem is an analogy of the well-known result stating that every midconvex function $f : D \to \mathbb{R}$ with closed epigraph (or, equivalently, lower semicontinuous) is continuous on $\operatorname{int} D$.

Theorem 4.3. *Let $M : D^3 \to D$ be a strict mean satisfying the conditions of Lemma* 4.1 *or of Lemma* 4.2*. If $f : D \to \mathbb{R}$ is a lower semicontinuous $(\mathcal{F}, M)$-convex function, then it is also $\mathcal{F}$-convex and continuous on* $\operatorname{int} D$.

Proof. Let a, b, c be arbitrary non-collinear points in D. Define the set $C \subseteq \triangle_{abc}$ as follows:

$$C := \{x \in \triangle_{abc}\colon f(x) \leqslant \varphi_{(a,f(a))(b,f(b))(c,f(c))}(x)\}.$$

Then, obviously $a, b, c \in C$ and thus C is two-dimensional. Due to the lower semicontinuity of f, the set C is also closed.

To show that C is M-convex let x, y, z be arbitrary non-collinear points of C. Then

$$\varphi_{(x,f(x))(y,f(y))(z,f(z))}(u) \leqslant \varphi_{(a,f(a))(b,f(b))(c,f(c))}(u) \tag{8}$$

if $u \in \{x, y, z\}$. Therefore, by Lemma 2.3, this inequality is valid for all $u \in \triangle_{xyz}$. Particularly, (8) is also true if $u = M(x, y, z)$. On the other hand, by the $(\mathcal{F}, M)$-convexity of f, we also have

$$f\big(M(x, y, z)\big) \leqslant \varphi_{(x,f(x))(y,f(y))(z,f(z))}\big(M(x, y, z)\big).$$

This inequality combined with (8) yields that $M(x, y, z) \in C$, i.e., the set C is indeed M-convex.

Using Lemma 4.1 (or Lemma 4.2), it follows that C is convex, hence $C = \triangle_{abc}$ (because $a, b, c \in C$). This yields that, for all $x \in \triangle_{abc}$,

$$f(x) \leqslant \varphi_{(a,f(a))(b,f(b))(c,f(c))}(x).$$

Thus the proof of the $\mathcal{F}$-convexity of f is complete. The continuity of f on $\operatorname{int} D$ follows from Theorem 3.2. $\square$

Obviously, the arithmetic mean satisfies all the requirements of Lemma 4.1 (and also of Lemma 4.2). Therefore, as an immediate consequence of the previous theorem, we obtain the following result.

Corollary 4.4. *If $f : D \to \mathbb{R}$ is a lower semicontinuous $\mathcal{F}$-midconvex function, then it is also $\mathcal{F}$-convex and continuous on* $\operatorname{int} D$.

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