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Dynamics of smooth essentially strongly order-preserving semiflows with application to delay differential equations ☆

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Abstract

In this paper, we introduce a class of smooth essentially strongly order-preserving semiflows and improve the limit set dichotomy for essentially strongly order-preserving semiflows. Generic convergence and stability properties of this class of smooth essentially strongly order-preserving semiflows are then developed. We also establish the generalized Krein–Rutman Theorem for a compact and eventually essentially strongly positive linear operator. By applying the main results of this paper to essentially cooperative and irreducible systems of delay differential equations, we obtain some results on generic convergence and stability, the linearized stability of an equilibrium and the existence of the most unstable manifold in these systems. The obtained results improve some corresponding ones already known.

Keywords: Convergence; Delay differential equations; Krein–Rutman Theorem; Essentially strongly order-preserving semiflow; Stability

1. Introduction

It is well known that precompact orbits of monotone dynamical systems have a strong tendency to converge to the set of equilibria. Hirsch [4] showed that most orbits of a strongly monotone semiflows on a strongly ordered space tend to the set of equilibria in the sense of topology, Gaussian measure and cardinality, which extends earlier work of Hirsch [2,3] for ordinary differential equations to infinite-dimensional semiflows. Matano [7,8] announced similar results. Combining the ideas of Hirsch and Matano, Smith and Thieme [16] established generic quasi-convergence principle for a strongly order-preserving semiflow. This result was later improved by Smith and Thieme [15], which was inspired by earlier work of Poláčik [9] assuming less compactness but more smoothness. Hirsch and Smith [5] established the generic quasi-convergence principle for strongly order-preserving semiflows by replacing the strong

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compactness assumptions on the semiflows with the assumption that compact invariant sets have an infimum and supremum in the state space. For related results, we refer to [6,14,18,21] and references therein.

Recently, Yi and Huang [19] have pointed out that when the abstract generic quasi-convergence principles developed in the above-mentioned work are applied to quasimonotone systems of delay differential equations and reaction–diffusion equations with delay, there are some drawbacks not to be ignored such as: the requirements of the delicate choice of state space and the technical *ignition* assumption. In [19], the generic quasi-convergence principle for essentially strongly order-preserving semiflows have been established to overcome the aforementioned drawbacks. Somewhat more precisely, let X be an ordered metric space with metric d and a closed partial order relation $\leqslant$. Let Φ be a continuous monotone semiflow on X, that is, Φ is a continuous semiflow on X and whenever $x, y \in X$ with $x \leqslant y$, we have $\Phi_t(x) \leqslant \Phi_t(y)$ for all $t \geqslant 0$. For any $x, y \in X$ and some constant $t_0 \geqslant 0$, we write $x \preccurlyeq_{t_0} y$ iff there exist $\tilde{x}, \tilde{y} \in X$ with $\tilde{x} \leqslant \tilde{y}$ such that $\Phi_{t_0}(\tilde{x}) = x$ and $\Phi_{t_0}(\tilde{y}) = y$. We shall write "$\preccurlyeq$" for "$\preccurlyeq_{t_0}$" when no confusion results. We also write $x \prec y$ iff $x \preccurlyeq y$ and $x \neq y$.

Yi and Huang [19] introduced the following definition.

Definition 1.1. The semiflow Φ is said to be essentially strongly order-preserving if for any $x, y \in X$ with $x \prec y$, there exist open sets U and V, and some constant $t_0 \geqslant 0$ such that $x \in U$, $y \in V$ and $\Phi_{t_0}(U) \leqslant \Phi_{t_0}(V)$.

Yi and Huang [19] then established the following limit set dichotomy principle for essentially strongly order-preserving semiflows.

Theorem 1.1. *Let $x, y \in X$ satisfy $x \prec y$. Then one of the following holds*:

(i) $\omega(x) < \omega(y)$;
(ii) $\omega(x) = \omega(y) \subset E$, *moreover,* $\lim_{t\to\infty} d(\Phi_t(x), \Phi_t(y)) = 0$.

Armed with Theorem 1.1, Yi and Huang [19] derived a series of convergence, quasi-convergence and stability results for essentially strongly order-preserving semiflows. But they have not established the generic convergence principle for essentially strongly order-preserving semiflows. It is naturally necessary to obtain general structural conditions that guarantee the generic convergence principle. For example, the generic convergence principle for a strongly order-preserving semiflow was established by Smith and Thieme [15], who imposed stronger monotonicity and smoothness assumptions on the semiflow with an additional spectral hypothesis. Inspired by the work of [15], we will impose the similar conditions on essentially strongly order-preserving semiflows to obtain the generic convergence principle.

In this paper, by appealing to Proposition 2.1 in Smith and Thieme [15], we establish the improved limit set dichotomy for a class of smooth essentially strongly order-preserving semiflows, from which a series of results including the generic convergence and stability principle are then derived. We also establish the generalized Krein–Rutman Theorem for a compact and eventually essentially strongly positive linear operator in Appendix A. As pointed out by Smith and Thieme [15], the required spectral condition for our main results usually follows from the same considerations which establish the strengthened essentially strongly order-preserving properties and the compactness of the semiflow.

The organization of the rest of this paper is as follows. In Section 2, we establish the improved limit set dichotomy and sequential limit set trichotomy. In Section 3, we derive several results including a generic convergence and stability principle. In Section 4, an application of the generic convergence and stability principle in Section 3 is made to essentially cooperative and irreducible systems of delay differential equations. Armed with the generalized Krein–Rutman Theorem developed in Appendix A, we consider the linearized stability of an equilibrium and the existence of the most unstable manifold in essentially cooperative and irreducible systems of delay differential equations.

2. The improved limit set dichotomy and sequential limit set trichotomy

Let X be an ordered metric space with metric d and order relation $\leqslant$. Let Φ be a continuous monotone semiflow on X, that is, Φ is a continuous semiflow on X and whenever $x, y \in X$ with $x \leqslant y$, we have $\Phi_t(x) \leqslant \Phi_t(y)$ for all $t \geqslant 0$. If $x \in X$, we assume that $O(x)$ has compact closure in X. It is also assumed that (Y, Y_+) is ordered Banach space with the norm $\|\cdot\|$, where $\|\cdot\|$ denotes the norm on the Banach space Y and $\operatorname{Int} Y_+ \neq \phi$. The subset $Z \subseteq X \cap Y$ satisfies the following conditions:

(i) The inclusion $i : (Z, d_Y) \to (Z, d)$ is continuous, where d_Y is induced by the norm $\|\cdot\|$ on Y.
(ii) $x, y \in Z$ with $x \leqslant y$ if and only if $x \leqslant_Y y$, where the ordering $\leqslant_Y$ is induced by the order cone Y_+.

Before introducing the following assumptions, we set out some notation. For $x \in X$, let $O(x) = \{\Phi_t(x)\colon t \geqslant 0\}$. If $\overline{O(x)}$ is compact, we define

$$\omega(x) = \bigcap_{t \geqslant 0} \overline{O\big(\Phi_t(x)\big)}.$$

As is well known, $\omega(x)$ is nonempty, compact, connected, and invariant. Let $E = \{x \in X\colon \Phi_t(x) = x,\ t \geqslant 0\}$ be the set of equilibria of Φ. The set of convergent points is denoted by $C = \{x \in X\colon \omega(x)$ is a singleton set$\}$. Assume that $t_0 \geqslant 0$ is a constant. For any $x, y \in X$, we write $x \preccurlyeq_{t_0} y$ iff there exist $\tilde{x}, \tilde{y} \in X$ with $\tilde{x} \leqslant \tilde{y}$ such that $\Phi_{t_0}(\tilde{x}) = x$ and $\Phi_{t_0}(\tilde{y}) = y$. We shall write "$\preccurlyeq$" for "$\preccurlyeq_{t_0}$" when no confusion results. We also write $x \prec y$ iff $x \preccurlyeq y$ and $x \neq y$.

We tacitly assume throughout the rest of this paper that the topology on Z is induced by d_Y, and let τ be a given constant.

Now, we are in the position to introduce the following assumptions.

(I) $\Phi_t(Z) \subseteq Z$, $t \geqslant 0$.
(J) $\Phi_\tau(X) \subseteq Z$ and $\Phi_\tau : X \to Z$ is continuous.
(M) If $x, y \in X$ satisfy $x \prec y$, then $\Phi_\tau(x) \ll \Phi_\tau(y)$, where the ordering $\ll$ is induced by $\operatorname{Int} Y_+$.
(D) For any $e \in E$, there exists a neighborhood U of e in Z which is order convex in Y such that Φ_τ is continuously differentiable on U and $\Phi'_\tau \in C(U, L_+(Y))$, where $L_+(Y)$ denotes the set of all positive linear operators on Y mapping Y_+ into itself.
(Σ) For any $e \in E$ satisfying $\rho(e) = \rho(\Phi'_\tau(e)) \geqslant 1$, $\rho(e)$ is a pole of the resolvent of $\Phi'_\tau(e)$ with finite rank and there exists $v \in \operatorname{Int} Y_+$ such that $N(\rho(e)I - \Phi'_\tau(e)) = \operatorname{span}\{v\}$.

In the following, we always assume that (I), (J), (M), (D), and (Σ) hold.

Lemma 2.1. *We have the following*:

(i) *$\Phi_t : X \to Z$ is continuous for all $t \geqslant \tau$*;
(ii) *For any $x, y \in X$ with $x \prec y$, $\Phi_t(x) \ll \Phi_t(y)$ for all $t \geqslant \tau$*;
(iii) *If K is a compact invariant set in X, then K is also compact and invariant in Z*;
(iv) *Φ is essentially strongly order-preserving on X.*

Proof. (i) follows easily from $\Phi_t = \Phi_\tau(\Phi_{t-\tau})$ and (J).

(ii) Assume that $t \geqslant \tau$. From $x \prec y$ and (M), it follows that $\Phi_{t-\tau}(x) \prec \Phi_{t-\tau}(y)$. From (M) again, we obtain $\Phi_t(x) \ll \Phi_t(y)$.

(iii) Since K is invariant, we have from (J) that $K \subseteq Z$. From (J) again, we have that $\Phi_\tau : X \to Z$ is continuous, and hence K is a compact connected set in Z.

(iv) If $x, y \in X$ satisfy $x \prec y$, then from (M), we have $\Phi_\tau(x) \ll \Phi_\tau(y)$. So there exist open subsets $\widetilde{U}$, $\widetilde{V}$ in Z such that $\widetilde{U} \leqslant \widetilde{V}$, $\Phi_\tau(x) \in \widetilde{U}$, and $\Phi_\tau(y) \in \widetilde{V}$. Let $U = (\Phi_\tau)^{-1}(\widetilde{U})$ and $V = (\Phi_\tau)^{-1}(\widetilde{V})$. Then, by (J), U and V are open subsets of X with $x \in U$, $y \in V$ and $\Phi_\tau(U) \leqslant \Phi_\tau(V)$ Therefore, Φ is essentially strongly order-preserving on X. $\square$

Theorem 2.1. *If $x, y \in X$ and $x \prec y$, then either*

(i) *$\omega(x) < \omega(y)$, or*
(ii) *$\omega(x) = \omega(y) = \{p\}$ for some $p \in E$.*

Proof. By Lemma 2.1(iv) and Theorem 1.1, we have either $\omega(x) < \omega(y)$ or $\omega(x) = \omega(y) \subseteq E$. If the former holds, then the proof is complete. If the latter holds, we let $K = \omega(x)$. Then, by Lemma 2.1(iii), K is a compact connected set of Y. Thus, by [19, Proposition 3.1] and the fact that $K \subseteq E$, K contains no pair of order-related points. Let $u_n = \Phi_{n\tau}(x)$, $v_n = \Phi_{n\tau}(y)$, and $S = \Phi_\tau : Z \to Z$. Then K is a compact connected set of fixed points of S, $u_{n+1} = Su_n$, $v_{n+1} = Sv_n$ and $u_n \ll v_n$. Moreover, $\mathrm{dist}_Y(u_n, K) = \inf_{y\in K} \|u_n - y\| \to 0$ and $\mathrm{dist}_Y(v_n, K) = \inf_{y\in K} \|v_n - y\| \to 0$ as $n \to \infty$. (D) and (Σ) imply that S satisfies the conditions of Proposition 1.2 of [15], so by Proposition 1.2 of [15], K is a singleton. □

In the applications, the assumption (Σ) above may also be replaced by the assumption (S) below.

(S) For any $e \in E$, $\Phi'_\tau(e)$ denotes the Frechét derivative of Φ_τ at e. $\Phi'_\tau(e)$ is a compact and essentially strongly positive operator, i.e. $T(X_+) \subseteq \mathrm{Int}\, X_+ \cup \{0\}$.

Lemma 2.2. *If* (S) *holds, then* (Σ) *holds.*

Proof. Assume that $e \in E$. Since $\Phi'_\tau(e)$ is compact, it follows from Corollary 10.3 of [17] that $\rho(e)$ is a pole of the resolvent of $\Phi'_\tau(e)$ with finite rank. Therefore, by $\rho(e) \geqslant 1$ and Corollary A.1 in Appendix A, there exists $v \in \mathrm{Int}\, Y_+$ such that $N(\rho(e)I - \Phi'_\tau(e)) = \mathrm{span}\{v\}$. □

Now we are ready to present the improvement of the sequential limit set trichotomy.

The following two definitions can be found in [19].

Definition 2.1. If $x \in X$, then x has the property ω_- (ω_+) if there exists $\{x_n\}_{n=1}^\infty$ such that $\{x_n\}_{n=1}^\infty$ essentially approximates x from below (above) and $\overline{\bigcup_{n\geqslant 1} \omega(x_n)}$ is compact. We also use $\omega_-[x]$ $(\omega_+[x])$ to denote x has the property ω_- (ω_+).

Definition 2.2. Assume that $x \in X$ and that there exists a sequence x_n in X such that $\{x_n\}_{n=1}^\infty$ essentially approximates x from below (above), and $\overline{\bigcup_{n\geqslant 1} \omega(x_n)}$ is compact. The point $x \in X$ has the property A_- (A_+) if there exists $p \in E$ such that $\omega(x_n) < \omega(x_{n+1}) < p = \omega(x_0)$ $(\omega(x_0) = p < \omega(x_{n+1}) < \omega(x_n))$, $n \geqslant 1$ and $\lim_{n\to+\infty} \mathrm{dist}(\omega(x_n), p) = \lim_{n\to+\infty} \inf_{y\in\omega(x_n)} d(y, p) = 0$. The point x has the property B_- (B_+) if there exists $p \in E$ such that $\omega(x_n) = p < \omega(x_0)$ $(\omega(x_0) > p = \omega(x_n))$, $n \geqslant 1$ and if $q \in E$ and $q < \omega(x_0)$ $(q > \omega(x_0))$ then $q \leqslant p$ $(p \leqslant q)$. The point x has the property C_- (C_+) if $\omega(x_n) = \omega(x_0) \subset E$, $n \geqslant 1$ and $\lim_{t\to\infty} d(\Phi_t(x_1), \Phi_t(x_0)) = 0$.

Below we will give a new definition.

Definition 2.3. Assume that $x \in X$ and that there exists a sequence x_n in X such that $\{x_n\}_{n=1}^\infty$ essentially approximates x from below (above), and $\overline{\bigcup_{n\geqslant 1} \omega(x_n)}$ is compact. The point x has the property D_- (D_+) if there exists $p \in E$ such that $\omega(x_n) = \omega(x_0) = \{p\}$, $n \geqslant 1$ and $\lim_{t\to\infty} d(\Phi_t(x_1), \Phi_t(x_0)) = 0$.

As an application of Theorem 2.1, we obtain the following improved sequential limit set trichotomy.

Proposition 2.1. *Let* $x_0 \in X$. *If* $\omega_-[x_0]$ *holds, then* $A_-[x_0] \cup B_-[x_0] \cup D_-[x_0]$ *holds, that is,* x_0 *has one of the properties* A_-, B_- *or* D_-.

Proof. Since x_0 has the property ω_-, it follows from [19, Proposition 4.1] that $A_-[x_0] \cup B_-[x_0] \cup C_-[x_0]$ holds. If x_0 has either the property A_- or B_-, then the proof is complete. If x_0 has the property C_-, then $x_n \prec x_0$ and $\omega(x_n) = \omega(x_0) \subseteq E$ for $n \geqslant 1$ by the definition of C_-. Hence, by Theorem 2.1, there exists $p \in E$ such that $\omega(x_n) = \omega(x_0) = \{p\}$ for $n \geqslant 1$. Therefore, x_0 has the property D_-. This completes the proof. □

Similarly, we can prove the following result.

Proposition 2.2. *Let* $x_0 \in X$. *If* $\omega_+[x_0]$ *holds, then* $A_+[x_0] \cup B_+[x_0] \cup D_+[x_0]$.

3. Generic convergence and stability

In this section, we always assume that (I), (J), (M), (D) and (Σ) of Section 2 hold. In order to obtain results concerning stability, we also need the following definitions which can be found in [19].

Definition 3.1. If $x \in X$, then x is a stable point if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\Phi_t(x), \Phi_t(y)) < \varepsilon$ for $t \geqslant 0$ whenever $y \in X$ and $d(x, y) < \delta$. We let S be set of all stable points of X. A point x is an asymptotically stable point if there is a neighborhood V of x with the property that for every $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that $d(\Phi_t(x), \Phi_t(y)) < \varepsilon$ if $t \geqslant T_\varepsilon$ and $y \in V$. We let A denote the set of all asymptotically stable points of X.

Definition 3.2. If $x \in X$, then Φ is said to be locally uniformly normally ordered at x if there exist an open neighborhood M of x and $T_0 > 0$ such that $\overline{\Phi(M \times [T_0, \infty))}$ is a normally ordered metric subspace of the ordered metric space X. Φ is said to be locally uniformly normally ordered if for every $x \in X$, Φ is locally uniformly normally ordered at x.

Definition 3.3. If $x \in X$, then Φ is said to be locally uniformly compact at x if there exist an open neighborhood M of x and $T_0 > 0$ such that $\overline{\Phi(M \times [T_0, \infty))}$ is a compact subspace of the ordered metric space X. Φ is called locally uniformly compact if for every $x \in X$, Φ is locally uniformly compact at x.

As the properties D_+ and D_- imply the properties C_+ and C_-, respectively, the results of Propositions 4.3 and 4.4 in [19] remain valid if we replace C_+ and C_- by D_+ and D_-, respectively. More precisely, we have the following.

Proposition 3.1. *Assume that $x_0 \in X$ and Φ is locally uniformly normally ordered at x_0. Then we have the following*:

(i) *If $B_-[x_0] \cup B_+[x_0]$, then $x_0 \in \overline{A}$*;
(ii) *If $D_-[x_0] \cup D_+[x_0]$, then $x_0 \in \overline{A}$*;
(iii) *If $A_-[x_0] \cap A_+[x_0]$, then $x_0 \in S$*;
(iv) *If $D_-[x_0] \cap D_+[x_0]$, then $x_0 \in A$.*

Proposition 3.2. *Assume that $x_0 \in X$ and Φ is locally uniformly compact at x_0. Then we have the following*:

(i) *If $B_-[x_0] \cup B_+[x_0]$, then $x_0 \in \overline{A}$*;
(ii) *If $D_-[x_0] \cup D_+[x_0]$, then $x_0 \in \overline{A}$*;
(iii) *If $A_-[x_0] \cap A_+[x_0]$, then $x_0 \in S$*;
(iv) *If $D_-[x_0] \cap D_+[x_0]$, then $x_0 \in A$.*

Theorem 3.1. *If $\omega_+ \cup \omega_-$ holds, then $X = \overline{\operatorname{Int} C}$.*

Proof. Let $x_0 \in X \setminus \operatorname{Int} C$. Then there exists $y_n \in X \setminus C$ such that $y_n \to x_0$ as $n \to \infty$. Assume, without loss of generality, that $\omega_-[y_n]$ holds for all $n \geqslant 1$. Then, from $y_n \notin C$ and Proposition 2.1, it follows that $B_-[y_n]$ holds for all $n \geqslant 1$. So, by Proposition 4.3 in [19], we have $y_n \in \overline{\operatorname{Int} C}$ and hence $x_0 \in \overline{\operatorname{Int} C}$. Therefore, we obtain $X = \overline{\operatorname{Int} C}$. This completes the proof. □

Proposition 3.3. *If $\omega_- \cup \omega_+$ holds, then $S \subseteq C$.*

Proof. Proposition 3.3 follows immediately from the definitions of the properties $A_\pm$, $B_\pm$ and $D_\pm$ and Propositions 3.1 and 3.2. □

Theorem 3.2. *Assume that Φ is either locally uniformly compact or locally uniformly normally ordered. If $\omega_- \cup \omega_+$ holds, then $X = \overline{A \cup \operatorname{Int} C}$. Furthermore, if there exists an open and dense subset $X_0 \subseteq X$ such that $\omega_-[X_0] \cap \omega_+[X_0]$ holds, then $X = \overline{\operatorname{Int}(S \cup C)}$ and hence, $X = \overline{\operatorname{Int} S}$.*

Proof. We proceed by contradiction. Suppose $X \setminus \overline{A \cup \operatorname{Int} C} \neq \phi$. Then there exists an open subset $U \subseteq X$ such that $U \cap A = \phi$ and $U \cap \operatorname{Int} C = \phi$. Let $x \in U$. Without loss of generality, we may assume that $\omega_-[x]$ holds. As $x \in U$, we have $x \notin \overline{A}$. Also, Propositions 2.1 and 3.1 or Proposition 3.2 imply that $A_-[x]$ holds. Therefore, we have $x \in C$, that is, $U \subseteq C$, a contradiction to $U \cap \operatorname{Int} C = \phi$. Now, we will prove the second assertion. By way of contradiction, suppose there exists an open subset $U \subseteq X$ such that $U \cap A = \phi$ and $U \cap \operatorname{Int}(S \cap C) = \phi$. Let $U_0 = U \cap X_0 \neq \phi$. By the argument in the proof of the first assertion, $U_0 \subseteq C$ and $A_-[U_0] \cap A_+[U_0]$ holds. Hence, by Proposition 3.1, we obtain $U_0 \subseteq S$, which is a contradiction to $U_0 \cap \operatorname{Int}(S \cap C) = \phi$. This completes the proof. □

Theorem 3.2 is a sharpened versions of Theorems 5.1 and 5.3 in [19]. In particular, Theorem 3.2 implies that the interior of the set of stable points is dense.

Remark 3.1. By Lemma 2.2, if (Σ) is replaced by (S), then the results of Theorems 3.1 and 3.2 continue to hold.

4. An application

In this section, we apply the generic convergence and stability principle in Section 3 to essentially cooperative and irreducible systems of delay differential equations.

Let $r > 0$ be given and let $C = C([-r, 0], R^n)$ be the Banach space of continuous mappings from $[-r, 0]$ into R^n, equipped with the usual supremum norm. Define $C_+ = C([-r, 0], R^n_+)$. Note that C_+ is an order cone in C and induces the usual pointwise ordering.

If $\sigma > 0$ and $x \in C([-r, \sigma], R^n)$, then for any $t \in [0, \sigma]$, we let $x_t \in C([-r, \sigma], R^n)$ be defined by $x_t(\theta) = x(t + \theta)$, $-r \leqslant \theta \leqslant 0$.

Consider the following system of delay differential equations

$$x'(t) = f(x_t), \tag{4.1}$$

where $f : C \to R^n$ is continuously differentiable. Assume for simplicity that solutions of the initial value problem (4.1) exist and are unique on R^1_+. Smith [11] gave sufficient conditions for (1.1) to generate an eventually strongly monotone semiflow and showed that choosing a state space properly is necessary for eventual strong monotonicity. Recently, Yi and Huang [19] established a generic quasi-convergence principle for essentially strongly order-preserving semiflows and applied this principle to quasimonotone system of delay differential equations, which shows that the new order-preserving property does not require the delicate choice of state space and the technical *ignition* assumption required in classical work. More precisely, Yi and Huang [19] introduced the following assumptions:

(K) For all $\psi \in C$ and $\varphi \in C_+$ with $\varphi_i(0) = 0$, $f_i'(\psi)(\varphi) \geqslant 0$.
(I) For each $\psi \in C$, the $n \times n$ matrix $(f'(\psi)\hat{e}_1, \ldots, f'(\psi)\hat{e}_n)$ is irreducible.

According to [19], if f satisfies (K) and (I) on C, then f is said to be essentially cooperative and irreducible in C.

In order to insure that (4.1) generates a global semiflow with appropriate compactness hypotheses we assume the following:

(T) For each $\varphi \in C$, $\sigma_\varphi = +\infty$. Moreover, for each compact subset $A \subset C$, there exists a closed and bounded subset $B = B(A) \subset C$ such that $O(x_{T_0}(A)) \subset B$ for some $T_0 > 0$.

The following result comes from Theorem 2.1 in [19].

Theorem 4.1. *Let f satisfy* (K), (I) *and* (T). *If $\varphi, \psi \in C$, and $\varphi < \psi$, then either $x_t(\varphi) = x_t(\psi)$ for $t \geqslant (n+2)r$ or $x_t(\varphi) \ll x_t(\psi)$ for $t \geqslant (n+2)r$. Hence, $x_t(\cdot)$ is an essentially strongly order-preserving semiflow. Moreover, ω_+ and ω_- also hold. If $e \in C$ is an equilibrium point of* (4.1), *then $\{T_e(t) \equiv \Phi_t'(e)\}_{t \geqslant 0}$ is a strongly continuous semigroup and $y_t = \Phi_t'(e)\phi$ satisfies*

$$y'(t) = L(y_t), \quad L(\psi) = f'(e)\psi, \quad y_0 = \phi.$$

Moreover, $T_e(t)$ is a compact and essentially strongly positive linear operator for $t \geqslant (n+2)r$ (*see Appendix* A *for more details on this definition*).

The following result implies that most precompact orbits of essentially cooperative and irreducible systems of delay differential equations are stable and converge to equilibrium.

Theorem 4.2. *Let f satisfy* (K), (I) *and* (T). *Then C contains an open and dense set of stable convergent points.*

Proof. Theorem 4.2 follows easily from Theorems 3.2 and 4.1. □

Armed with Theorem 4.1 and Corollary A.1 in Appendix A, we can obtain the following result by applying the similar arguments in [11, Theorem 3.1 and Corollary 3.2].

Theorem 4.3. *Let f satisfy* (K) *and* (H). *Then the equilibria of* (4.1) *have the same stability type as the associated ordinary differential equation obtained by ignoring the delay.*

If f satisfies the hypotheses of Theorem 4.3 and the equilibrium $\hat{v}$ of system (4.1) is linearly unstable, i.e., $s(d\hat{f}(v)) > 0$, where $\hat{f} : R^n \to R^n$ is given such that $\hat{f}(v) = f(\hat{v})$ and $s(d\hat{f}(v))$ denotes the maximum of the real parts of the eigenvalues in the matrix $d\hat{f}(v)$, we can obtain the following "most unstable" manifold of $\hat{v}$ by applying Theorem 1.1 in [12] and Corollary A.1 in Appendix A and by making simple modifications of the arguments in [12, Theorem 2.1] and [13, Theorem 2.8].

Theorem 4.4. *Let f be essentially cooperative and irreducible. Suppose $f(\hat{v}) = 0$, $s = s(d\hat{f}(v)) > 0$, and suppose df is Lipschitz continuous in a neighborhood of $\hat{v}$. Suppose $\hat{v} + C_+$ belongs to the domain of f and f is bounded on bounded subsets of U. Then there exist $\bar{u} \in \operatorname{Int} C_+$ and a unique C^1 function $y : [0, \infty) \to \hat{v} + C_+$ satisfying*:

(1) $y(\tau) = \hat{v} + \tau\bar{u} + o(\tau)$ *as* $\tau \to 0$.
(2) $x_t(y(\tau)) = y(e^{st}\tau)$, $t \geqslant 0$, $\tau \geqslant 0$.
(3) $0 \leqslant \tau_1 \leqslant \tau_2$ *implies* $y(\tau_1) \leqslant y(\tau_2)$.
(4) *Either* (a) $\lim_{\tau\to\infty} \|y(\tau)\| = \infty$ *or* (b) $\lim_{\tau\to\infty} \|y(\tau)\| = \hat{w}$, *where* $w \in R^n$, $w \gg v$, $f(\hat{w}) = 0$ *and* $s(d\hat{f}(w)) \leqslant 0$.
(5) *If* (4a) *holds, then for all* $\varphi \gg \hat{v}$, $\|x_t(\varphi)\| \to \infty$ *as t tends to the right-hand limit of the maximal interval of existence of* $x_t(\phi)$. *If* (4b) *holds, then for all* φ, $\hat{v} \ll \varphi \leqslant \hat{w}$, $x_t(\varphi) \to \hat{w}$ *as* $t \to \infty$.

Remark 4.1. Hirsch and Smith [5] considered quasimonotone systems of delay differential equations with respect to a class of nonstandard order cones and provided sufficient conditions for these systems to generate an eventually strongly monotone semiflow. Arguing as above in this section, we can obtain the improvement of Theorem 4.12 in Chapter 4 of Hirsch and Smith [5] by appealing to the main results of this paper.

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Appendix A. The Generalized Krein–Rutman Theorem

It is well known that the classical Krein–Rutman Theorem is important in applied mathematics. See [1,10,20] for more details. In Appendix A, we will establish the generalized Krein–Rutman Theorem.

Let X be a real Banach space with an order cone X_+ having a nonempty interior. Then X_+ induces a closed partial order relation on X. For any $x, y \in X$, we write $x \leqslant y$ iff $x - y \in X_+$, $x < y$ iff $x \leqslant y$ and $x \neq y$, $x \ll y$ iff $x - y \in \operatorname{Int} X_+$.

The following lemma is taken from [20, Lemma 7.31].

Lemma A.1. *Let $e \gg 0$. Then for every $x \in X \setminus X_+$, there is a uniquely determined number $\alpha_e(x) > 0$ such that*

(i) $0 \leqslant \alpha \leqslant \alpha_e(x)$ *implies* $e + \alpha x \geqslant 0$;
(ii) $\alpha > \alpha_e(x)$ *implies* $e + \alpha x \in X \setminus X_+$.

An important consequence, which we shall use frequently, is

$$e + \alpha x \gg 0 \quad \text{and} \quad \alpha > 0 \quad \text{implies } \alpha < \alpha_e(x).$$

Several concepts of positive operator will be given below.

Definition A.1. Let T be a positive operator on X, that is, $TX_+ \subseteq X_+$. Then

(i) T is said to be a strongly positive operator if $T(X_+ \setminus \{0\}) \subseteq \operatorname{Int} X_+$;
(ii) T is said to be an essentially strongly positive operator if $T(X_+) \subseteq \operatorname{Int} X_+ \cup \{0\}$;
(iii) T is said to be an eventually essentially strongly positive operator if for any $x \in X_+$, there exists $k = k_x > 0$ such that $T^k x \in \operatorname{Int} X_+ \cup \{0\}$.

Obviously, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). But (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) do not necessarily hold. For example, Definition A.1(ii) does not imply Definition A.1(i). In fact, if $X = R^2$, $X_+ = R^2_+$, and

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

then T satisfies Definition A.1(ii), but T is not a strongly positive operator.

The following lemma will ensure that $\rho(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} > 0$, where $\rho(T)$ denotes the spectral radius of the positive operator T.

Lemma A.2. *Let T be a positive operator. If there exist $x \gg 0$ and $k > 0$ such that $T^k x \gg 0$, then $\rho(T) > 0$.*

Proof. By assumption, there exists $\beta > 0$ such that $T^k x \gg \beta x$, and hence $T^{nk} x \geqslant \beta^n x$. Again since $x \gg 0$, there exists $\delta > 0$ such that $\|T^{nk}\| \geqslant \delta \beta^n$. It then follows that

$$\rho(T^k) = \lim_{n\to+\infty} \|T^{nk}\|^{\frac{1}{n}} \geqslant \beta \lim_{n\to+\infty} \delta^{\frac{1}{n}} = \beta > 0.$$

Therefore, from [17], we have $\rho(T) = (\rho(T^k))^{\frac{1}{k}} \geqslant \beta^{\frac{1}{k}} > 0$. This completes the proof. $\square$

Theorem A.1. *Let T be a compact and eventually essentially strongly positive operator with $\rho(T) > 0$. Then the following are true.*

(i) *There exists $e \gg 0$ such that $Te = \rho(T)e$;*
(ii) $\bigcup_{k \geqslant 1} N((\rho(T)I - T)^k) = \operatorname{span}\{e\}$;
(iii) *If $\upsilon > 0$ is an eigenvector associated to a nonzero eigenvalue for T, then there exists $s > 0$ such that $\upsilon = se$;*
(iv) *If $\lambda \in \sigma(T) \setminus \{\rho(T)\}$, then $|\lambda| < \rho(T)$.*

Proof. (i) By [20, Proposition 7.26], there exists $e > 0$ such that $Te = \rho(T)e$. Since T is an eventually essentially strongly positive operator, there exists $k = k_e > 0$ such that either $T^k e \gg 0$ or $T^k e = 0$. Thus, from $T^k e = (\rho(T))^k e > 0$, we obtain $e \gg 0$. This proves (i).

(ii) Let us show first that $N(\rho(T)I - T) = \operatorname{span}\{e\}$. On the contrary, suppose that $x \in N(\rho(T)I - T) \setminus \operatorname{span}\{e\}$. Without loss of generality, we may assume that $x \notin X_+$. Then $y = e + \alpha_e(x)x > 0$. Hence, there exists $k = k_y > 0$ such that $T^k y \subseteq \operatorname{Int} X_+ \cup \{0\}$. It follows from $T^k y = \rho^k(T) y \neq 0$ that $T^k y = (\rho(T))^k y \gg 0$, that is, $y \gg 0$. Thus, by Lemma A.1, we get $\alpha_e(x) < \alpha_e(x)$, a contradiction.

Next we will show that $N((\rho(T)I - T)^2) = \operatorname{span}\{e\}$. Otherwise, we have $x \in N((\rho(T)I - T)^2) \setminus N(\rho(T)I - T)$. Then, $\rho(T)x - Tx \in N(\rho(T)I - T)$. From the above discussion, we know that there exists $s \in R^1$ such that $\rho(T)x - Tx = se$. Without loss of generality, we may assume that $s > 0$. We next distinguish two cases to finish the proof.

Case 1. $x \notin X_+$. In this case, by Lemma A.1, we have $y = e + \alpha_e(x)x > 0$. Again by the assumptions of T, there exists $k = k_y > 0$ such that $T^k y \in \operatorname{Int} X_+ \cup \{0\}$. Using $Tx = \rho(T)x - se$ and calculating, we can get

$$T^k y = \left((\rho(T))^k - k(\rho(T))^{k-1}\alpha_e(x)s\right)e + \alpha_e(x)(\rho(T))^k x.$$

Thus,

$$\big(\rho(T) - ks\alpha_e(x)\big)e + \alpha_e(x)\rho(T)x \gg 0.$$

Therefore, we obtain $\rho(T) - ks\alpha_e(x) > 0$. It then follows from Lemma A.1 that

$$\frac{\alpha_e(x)\rho(T)}{\rho(T) - ks\alpha_e(x)} < \alpha_e(x),$$

which yields a contradiction.

Case 2. $x > 0$. In this case, since T is a positive operator, we get $\rho(T)x - se = Tx \geqslant 0$, and hence $x \gg 0$. We have from Lemma A.1 that

$$y = x + \alpha_x(-e)(-e) > 0.$$

So, there exists $k = k_y > 0$ such that $T^k y \subseteq \operatorname{Int} X_+ \cup \{0\}$. Again by calculating, we obtain

$$T^k y = \big(\rho(T)\big)^{k-1}\big(\rho(T)x - \big(ks + \alpha_x(-e)\rho(T)\big)e\big).$$

Thus, $\rho(T)x + (\rho(T)\alpha_x(-e) + ks)(-e) \gg 0$, and hence

$$\frac{\rho(T)\alpha_x(e) + ks}{\rho(T)} < \alpha_x(-e),$$

which yields a contradiction. This proves (ii).

(iii) Let $\upsilon > 0$ and $\lambda \neq 0$ be chosen such that $T\upsilon = \lambda\upsilon$. Since T is a positive operator, we obtain $\lambda > 0$. Since T is also an eventually essentially strongly positive operator, it easily follows that $\upsilon \gg 0$. From Lemma A.1, we then have

$$\text{either} \quad \upsilon + \alpha_\upsilon(-e)(-e) = 0 \quad \text{or} \quad \upsilon + \alpha_\upsilon(-e)(-e) > 0.$$

If $\upsilon + \alpha_\upsilon(-e)(-e) = 0$, then the proof is complete. Otherwise, we have $y = \upsilon + \alpha_\upsilon(-e)(-e) > 0$. Since T is an eventually essentially strongly positive operator, there exists $k = k_y > 0$ such that $T^k y \subseteq \operatorname{Int} X_+ \cup \{0\}$. Again by calculating, we get $T^k y = \lambda^k\upsilon + \alpha_\upsilon(-e)(\rho(T))^k(-e) \gg 0$. Hence,

$$\frac{\alpha_\upsilon(-e)(\rho(T))^k}{\lambda^k} < \alpha_\upsilon(-e).$$

Therefore, $\rho(T) < \lambda$, a contradiction.

(iv) Suppose that $\lambda \in \sigma(T) \setminus \{\rho(T)\}$. If $\lambda = 0$, then the proof is complete. Below, we assume that $\lambda \neq 0$. We next distinguish three cases to finish the proof.

Case 1. $\lambda > 0$. In this case, owing to the compactness of T and (iii), there exists $x \notin X_+ \cup (-X_+)$ such that $Tx = \lambda x$. By Lemma A.1, we have $e + \alpha_e(x)x > 0$. Since T is an eventually essentially strongly positive operator, there exists $k = k_{e+\alpha_e(x)x} > 0$ such that

$$T^k\big(e + \alpha_e(x)x\big) \subseteq \operatorname{Int} X_+ \cup \{0\}.$$

Hence, $\rho^k e + \alpha_e(x)\lambda^k x \gg 0$. It then follows from Lemma A.1 that

$$\frac{\lambda^k\alpha_e(x)}{\rho^k} < \alpha_e(x),$$

that is, $\lambda < \rho$.

Case 2. $\lambda < 0$. In this case, set $S = T^2$. Then it is easily shown that S is an eventually essentially strongly positive operator. Note that $\lambda^2 \in \sigma(S) \setminus \rho(S)$. So, similar to the proof of Case 1, we have $\lambda^2 < \rho(S) = (\rho(T))^2$, that is, $|\lambda| < \rho(T)$.

Case 3. $\lambda \notin R^1$. In this case, we consider the complexification $X_C = X + iX$ of X. Denote by $\widetilde{T}$ the complexified operator of T. Then there exists $z \in X_C$ such that $\widetilde{T}z = \lambda z$, where $z = x + iy$, $x, y \in X$, $\lambda = \alpha + i\beta$, $\alpha, \beta \in R^1$. Thus, we have

$$\begin{cases} Tx = \alpha x - \beta y, \\ Ty = \alpha y + \beta x. \end{cases}$$

It easily follows that x and y are linearly independent. Let

$$\Pi = \{ax + by\colon a, b \in R^1\} \subseteq X \quad \text{and} \quad \Pi_+ = \Pi \cap X_+.$$

Then $T(\Pi) \subseteq \Pi$, and Π_+ is an order cone on Π. Since $T|_\Pi \neq 0$, we obtain $\rho(T|_\Pi) > 0$. Next we will show that $\Pi_+ = \{0\}$. Assume, by contradiction, that $\Pi_+ \neq \{0\}$. In this case, we can conclude that $\operatorname{Int} \Pi_+ \neq \phi$. So $T|_\Pi$ is an eventually essentially strongly positive operator on Π. From (i), we know that there exists $\upsilon \in \operatorname{Int} \Pi_+$ such that

$$T\upsilon = (T|_\Pi)(\upsilon) = \rho(T|_\Pi)\upsilon.$$

In view of (iii), there exists $s > 0$ such that $\upsilon = se$, from which we can conclude that $\beta = 0$, a contradiction. Therefore, $\Pi_+ = \{0\}$. Let

$$S = \frac{T|_\Pi}{\sqrt{\alpha^2 + \beta^2}}.$$

Then S is a linear operator on Π. Also, let $B = \{ax + by\colon a^2 + b^2 = 1,\ a, b \in R^1\}$. Then $S(B) \subseteq B$ and B is compact. Assume that $b \in B$. Then we have $b \notin X_+$. By Lemma A.1, α_e is well defined and continuous on B. Hence, there exists $z = a_0 x + b_0 y \in B$ such that $\alpha_e(z) = \max_{b \in B} \alpha_e(b)$. Again since $z \notin X_+ \cup (-X_+)$, we have $u = e + \alpha_e(z)z > 0$. Thus, there exists $k = k_u > 0$ such that $T^k u \subseteq \{0\} \cup \operatorname{Int} X_+$. It then follows that

$$\frac{(\rho(T))^k}{(\sqrt{\alpha^2 + \beta^2})^k} e + \alpha_e(z) S^k(z) \gg 0.$$

Therefore,

$$\frac{(\sqrt{\alpha^2 + \beta^2})^k}{(\rho(T))^k} \alpha_e(z) < \alpha_e\bigl(S^k(z)\bigr) \leqslant \alpha_e(z),$$

which implies that $|\lambda| = \sqrt{\alpha^2 + \beta^2} < \rho(T)$. This completes the proof. □

Corollary A.1. *Let T be a compact and essentially strongly positive operator. If $\rho(T) > 0$, then we have the following*:

(i) *There exists $e \gg 0$ such that $Te = \rho(T)e$*;
(ii) $\bigcup_{k \geqslant 1} N((\rho(T)I - T)^k) = \operatorname{span}\{e\}$;
(iii) *If $\upsilon > 0$ is an eigenvector associated to a nonzero eigenvalue for T, then there exists $s > 0$ such that $\upsilon = se$*;
(iv) *If $\lambda \in \sigma(T) \setminus \{\rho(T)\}$, then $|\lambda| < \rho(T)$.*

If T is a strongly positive operator, that is, $T(X_+ \setminus \{0\}) \subseteq \operatorname{Int} X_+$, then $\rho(T) > 0$ by Lemma A.2*, and hence the conclusions of Corollary* A.1 *are valid, which yields the well-known Krein–Rutman Theorem.*

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