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Global existence of small amplitude solution for the generalized IMBq equation

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Abstract

In this paper, we reconsider the problem discussed in [G.W. Chen, S.B. Wang, Small amplitude solutions of the generalized IMBq equation, J. Math. Anal. Appl. 274 (2002) 846–866]. The proof of global existence presented in [G.W. Chen, S.B. Wang, Small amplitude solutions of the generalized IMBq equation, J. Math. Anal. Appl. 274 (2002) 846–866] is very simple in form, but it is a pity that the authors overlooked the bad behavior of low frequency part of $B(t)\psi$ which causes trouble in L^∞ and H^s estimates. In this paper, we will give out a new proof of the global existence under an additional condition on the initial data.
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1. Introduction

In paper [2], global existence of small amplitude solution and scattering theory for the Cauchy problem of the generalized IMBq equation

$$\begin{cases} u_{tt} - \Delta u_{tt} - \Delta u = \Delta f(u), \\ u(0) = \varphi(x), \quad u_t(0) = \psi(x) \end{cases} \tag{1.1}$$

was considered, where $u(x,t)$ denotes the unknown function, $f(u)$ is the given nonlinear function, $\varphi(x)$ and $\psi(x)$ are the given initial data, subscript t indicates the partial derivative with respect to t, Δ denotes the Laplace operator in $\mathbb{R}^n$. For the physic background and researches of the IMBq equation, one may refer the readers to [1–3] and the references therein.

Through combining the factors with functions, the authors used the special case of Lemma 2.2 with $F \equiv 1$ to deal with three terms of the right-hand side of (2.2) below to get the estimates for linearized equation. Together with the estimates for the nonlinearity $f(u)$, global existence of small amplitude solution to (1.1) followed by the usage of Banach fixed point theorem.

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It is a pity that the authors overlooked the infinity of the factor $\frac{\sqrt{1+|\xi|^2}}{|\xi|}$ $(\xi \to 0)$ in dealing with the low frequency part of $B(t)\psi$, using

$$\left\|(I-\Delta)^{\frac{1}{2}}(-\Delta)^{-\frac{1}{2}}\psi\right\|_{L^1} + \left\|(I-\Delta)^{\frac{1}{2}}(-\Delta)^{-\frac{1}{2}}\psi\right\|_{H^s} \leqslant C\big(\|\psi\|_{L^1} + \|\psi\|_{H^s}\big) \tag{1.2}$$

which is not correct to prove Lemma 2.4 of [2].

In this paper, we deduce Lemma 2.2 for general radial function $F(\xi) = F(|\xi|) = F(r)$ instead of $F \equiv 1$ on the basis of Van der Corput lemma. Through the different choices of $F(\xi)$, we deal with three terms in the right-hand side of (2.2) differently to get the estimates for linearized equation. Since the infinity of factor $\frac{\sqrt{1+|\xi|^2}}{|\xi|}$ $(\xi \to 0)$, we should need an additional condition $\psi \in DL^2 = \{f\colon\ D^{-1}f = (-\Delta)^{-1/2}f \in L^2\}$, one will see which in the proofs of Lemma 2.4 and Proposition 2.7 below. Together with the estimates obtained in [2] for $f(u)$, we finally get the result of the present paper.

For the convenience of comparison and contrast, the structure of this paper is similar to [2]. Since the difference between two methods lies almost all in estimates for linearized equation, we describe which in Section 2 in detail. In Section 3, some necessary nonlinear estimates are given without proof except that there are misprints in the original paper [2] since they are just the same. The main result will be concluded in Section 4 in brief.

2. Estimates for linearized equation

Firstly, we consider the linearized equation

$$\begin{cases} u_{tt} - \Delta u_{tt} - \Delta u = \Delta g(x,t), \\ u(0) = \varphi(x), \quad u_t(0) = \psi(x) \end{cases} \tag{2.1}$$

of (1.1). By Duhamel's principle, the solution of (2.1) can be written as

$$u(x,t) = \partial_t B(t)\varphi(x) + B(t)\psi(x) + \int_0^t T(t-\tau)g(x,\tau)\,\mathrm{d}\tau, \tag{2.2}$$

where $T(t) = B(t)(I-\Delta)^{-1}\Delta$,

$$\partial_t B(t)\varphi(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\cos\frac{t|\xi|}{\sqrt{1+|\xi|^2}}\hat{\varphi}(\xi)\,\mathrm{d}\xi, \tag{2.3}$$

$$B(t)\psi(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{\sqrt{1+|\xi|^2}}{|\xi|}\sin\frac{t|\xi|}{\sqrt{1+|\xi|^2}}\hat{\psi}(\xi)\,\mathrm{d}\xi, \tag{2.4}$$

$$\int_0^t T(t-\tau)g(x,\tau)\,\mathrm{d}\tau = -(2\pi)^{-n}\int_0^t\left[\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{|\xi|}{\sqrt{1+|\xi|^2}}\sin\frac{(t-\tau)|\xi|}{\sqrt{1+|\xi|^2}}\hat{g}(\xi,\tau)\,\mathrm{d}\xi\right]\mathrm{d}\tau \tag{2.5}$$

with $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)\,\mathrm{d}x$.

Lemma 2.1 *(Van der Corput). (See [6,8].) Let $h(r)$ be either convex or concave twice differentiable function and $F(r)$ be continuously differentiable function on $[a,b]$ with $-\infty \leqslant a < b \leqslant \infty$. Then*

$$\left|\int_a^b e^{ih(r)}F(r)\,\mathrm{d}r\right| \leqslant 4\Big\{\min_{[a,b]}\big|h''(r)\big|\Big\}^{-\frac{1}{2}}\left[\big|F(b)\big| + \int_a^b \big|F'(r)\big|\,\mathrm{d}r\right]$$

for $h''(r) \neq 0$ in $[a,b]$.

To overcome the shortcoming of Lemma 2.2 of [2] that finally broke up the result, we need the following lemma with $F(\xi) = F(|\xi|)$ instead of the one with $F(\xi) \equiv 1$ in [2]. As everyone will see it plays an important role in this paper.

Lemma 2.2. *For $N, t > 1$ and $0 < \varepsilon < \frac{1}{2}$, we have*

$$\sup_{x\in\mathbb{R}^n}\left|\int\limits_{\varepsilon\leqslant|\xi|\leqslant N} e^{it(\frac{x\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})}F(\xi)\,\mathrm{d}\xi\right| \leqslant C\max\bigl((2\varepsilon)^{-\frac{1}{2}}, N^2\bigr)t^{-\frac{1}{2}}N^{n-1}\left[\|F\|_{L^\infty[\varepsilon,N]}+\int\limits_{\varepsilon}^{N}\bigl|F'(r)\bigr|\,\mathrm{d}r\right], \tag{2.6}$$

where $F(\xi) = F(|\xi|) = F(r)$.

Proof. *Step* 1. $n = 1$.

Let

$$h_1(\xi) = \frac{x\cdot\xi}{t} \pm \frac{\xi}{\sqrt{1+\xi^2}} \quad \text{for } \xi > 0, \tag{2.7}$$

and

$$h_2(\xi) = \frac{x\cdot\xi}{t} \mp \frac{\xi}{\sqrt{1+\xi^2}} \quad \text{for } \xi < 0. \tag{2.8}$$

By direct computation, we have

$$\frac{\partial h_1}{\partial\xi}(\xi) = \frac{x}{t} \pm \bigl(1+\xi^2\bigr)^{-\frac{3}{2}}, \qquad \frac{\partial h_2}{\partial\xi}(\xi) = \frac{x}{t} \mp \bigl(1+\xi^2\bigr)^{-\frac{3}{2}}, \tag{2.9}$$

$$\frac{\partial^2 h_1}{\partial\xi^2}(\xi) = \mp 3\xi\bigl(1+\xi^2\bigr)^{-\frac{5}{2}}, \qquad \frac{\partial^2 h_2}{\partial\xi^2}(\xi) = \pm 3\xi\bigl(1+\xi^2\bigr)^{-\frac{5}{2}}, \tag{2.10}$$

$$\frac{\partial^3 h_1}{\partial\xi^3}(\xi) = \pm 3(2\xi-1)(2\xi+1)\bigl(1+\xi^2\bigr)^{-\frac{7}{2}}, \qquad \frac{\partial^3 h_2}{\partial\xi^3}(\xi) = \mp 3(2\xi-1)(2\xi+1)\bigl(1+\xi^2\bigr)^{-\frac{7}{2}}. \tag{2.11}$$

Therefore

(i) $\frac{\partial^2 h_1}{\partial\xi^2}(\xi) < 0$ (> 0) for $\xi > 0$;

(ii) $\frac{\partial^2 h_2}{\partial\xi^2}(\xi) < 0$ (> 0) for $\xi < 0$;

(iii) for $0 < \varepsilon < \frac{1}{2}$, $N > 1$,

$$\min_{[\varepsilon,N]}\left|\frac{\partial^2 h_1}{\partial\xi^2}(\xi)\right| = \min\left\{\left|\frac{\partial^2 h_1}{\partial\xi^2}(\varepsilon), \frac{\partial^2 h_1}{\partial\xi^2}(N)\right|\right\} \geqslant \frac{3}{2^{\frac{7}{2}}}\min\bigl(2\varepsilon, N^{-4}\bigr);$$

(iv) for $0 < \varepsilon < \frac{1}{2}$, $N > 1$,

$$\min_{[-N,-\varepsilon]}\left|\frac{\partial^2 h_2}{\partial\xi^2}(\xi)\right| = \min\left\{\left|\frac{\partial^2 h_2}{\partial\xi^2}(-\varepsilon), \frac{\partial^2 h_2}{\partial\xi^2}(-N)\right|\right\} \geqslant \frac{3}{2^{\frac{7}{2}}}\min\bigl(2\varepsilon, N^{-4}\bigr).$$

In fact, (i) and (ii) are obvious. Since (iii) and (iv) are similar, we take (iii) as an example. For $\xi > 0$, let $f(\xi) = |\frac{\partial^2 h_1}{\partial\xi^2}(\xi)|$, from (2.10) and (2.11), one gets

$$f'(\xi) = -3(2\xi-1)(2\xi+1)\bigl(1+\xi^2\bigr)^{-\frac{7}{2}}$$

which implies that

$$\begin{cases} f'(\xi) > 0, & \varepsilon \leqslant \xi < \frac{1}{2}, \\ f'(\xi) < 0, & \frac{1}{2} < \xi \leqslant N. \end{cases}$$

Together with (i), we have $\max_{[\varepsilon,N]} f(\xi) = f(\frac{1}{2})$ and

$$\begin{aligned}\min_{[\varepsilon,N]}\left|\frac{\partial^2 h_1}{\partial\xi^2}(\xi)\right| &= \min_{[\varepsilon,N]} f(\xi) = \min\bigl(f(\varepsilon), f(N)\bigr)\\ &= \min\left(\left|\frac{\partial^2 h_1}{\partial\xi^2}(\varepsilon)\right|, \left|\frac{\partial^2 h_1}{\partial\xi^2}(N)\right|\right)\\ &= \min\left(\frac{3\varepsilon}{(1+\varepsilon^2)^{\frac{5}{2}}}, \frac{3N}{(1+N^2)^{\frac{5}{2}}}\right)\\ &\geqslant \min\left(\frac{3\varepsilon}{(1+1^2)^{\frac{5}{2}}}, \frac{3N}{(N^2+N^2)^{\frac{5}{2}}}\right)\\ &\geqslant \frac{3}{2^{\frac{7}{2}}}\min\bigl(2\varepsilon, N^{-4}\bigr).\end{aligned}$$

This complete the proof of (iii).

By Lemma 2.1, we have

$$\begin{aligned}\mathrm{I}_1 &= \left|\int_\varepsilon^N e^{ith_1(\xi)}F(\xi)\,\mathrm{d}\xi\right|\\ &\leqslant 4\Bigl\{\min_{[\varepsilon,N]}\bigl|th_1''(\xi)\bigr|\Bigr\}^{-\frac{1}{2}}\left[\bigl|F(N)\bigr| + \int_\varepsilon^N \bigl|F'(\xi)\bigr|\,\mathrm{d}\xi\right]\\ &\leqslant 4t^{-\frac{1}{2}}\Bigl\{\min_{[\varepsilon,N]}\bigl|h_1''(\xi)\bigr|\Bigr\}^{-\frac{1}{2}}\left[\|F\|_{L^\infty[\varepsilon,N]} + \int_\varepsilon^N \bigl|F'(r)\bigr|\,\mathrm{d}r\right]\\ &\leqslant 4t^{-\frac{1}{2}}\left(\frac{3}{2^{\frac{7}{2}}}\right)^{-\frac{1}{2}}\max\bigl((2\varepsilon)^{-\frac{1}{2}}, N^2\bigr)\left[\|F\|_{L^\infty[\varepsilon,N]} + \int_\varepsilon^N \bigl|F'(r)\bigr|\,\mathrm{d}r\right]\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}\mathrm{I}_2 &= \left|\int_{-N}^{-\varepsilon} e^{ith_2(\xi)}F(\xi)\,\mathrm{d}\xi\right|\\ &\leqslant 4\Bigl\{\min_{[-N,-\varepsilon]}\bigl|th_2''(\xi)\bigr|\Bigr\}^{-\frac{1}{2}}\left[\bigl|F(-\varepsilon)\bigr| + \int_{-N}^{-\varepsilon} \bigl|F'(\xi)\bigr|\,\mathrm{d}\xi\right]\\ &\leqslant 4t^{-\frac{1}{2}}\left(\frac{3}{2^{\frac{7}{2}}}\right)^{-\frac{1}{2}}\max\bigl((2\varepsilon)^{-\frac{1}{2}}, N^2\bigr)\left[\|F\|_{L^\infty[\varepsilon,N]} + \int_\varepsilon^N \bigl|F'(r)\bigr|\,\mathrm{d}r\right].\end{aligned}\tag{2.13}$$

So

$$\begin{aligned}\left|\int_{\varepsilon\leqslant|\xi|\leqslant N} e^{it(\frac{x\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})}F(\xi)\,\mathrm{d}\xi\right| &\leqslant \left|\int_\varepsilon^N e^{ith_1(\xi)}F(\xi)\,\mathrm{d}\xi\right| + \left|\int_{-N}^{-\varepsilon} e^{ith_2(\xi)}F(\xi)\,\mathrm{d}\xi\right|\\ &\leqslant Ct^{-\frac{1}{2}}\max\bigl((2\varepsilon)^{-\frac{1}{2}}, N^2\bigr)\left[\|F\|_{L^\infty[\varepsilon,N]} + \int_\varepsilon^N \bigl|F'(r)\bigr|\,\mathrm{d}r\right],\end{aligned}\tag{2.14}$$

where $C = 8(3/2^{\frac{7}{2}})^{-\frac{1}{2}}$.

Step 2. $n \geqslant 2$.

Using spherical polar coordinate and taking the polar axis along x direction, thus $x \cdot \xi = |x| r \cos\theta_1$, and from $\int_0^\pi \sin^k\theta \,\mathrm{d}\theta = \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})}\pi^{\frac{1}{2}}$, we have

$$\begin{aligned}
&\int\limits_{\varepsilon\leqslant|\xi|\leqslant N} e^{it(\frac{x\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})} F(\xi)\,\mathrm{d}\xi \\
&= \int\limits_{\varepsilon}^{N}\int\limits_{0}^{2\pi}\int\limits_{0}^{\pi}\cdots\int\limits_{0}^{\pi} e^{it(\frac{|x|\cos\theta_1}{t}r\pm\frac{r}{\sqrt{1+r^2}})} F(r) r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}\,\mathrm{d}\theta_{n-1}\,\mathrm{d}\theta_{n-2}\cdots\mathrm{d}\theta_2\,\mathrm{d}\theta_1\,\mathrm{d}r \\
&= \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int\limits_0^{2\pi}\left[\int\limits_{\varepsilon}^{N} e^{it(\frac{|x|\cos\theta_1}{t}r\pm\frac{r}{\sqrt{1+r^2}})} F(r) r^{n-1}\,\mathrm{d}r\right]\sin^{n-2}\theta_1\,\mathrm{d}\theta_1. \qquad (2.15)
\end{aligned}$$

Claim.

$$\left|\int\limits_{\varepsilon}^{N} e^{it(\frac{|x|\cos\theta_1}{t}r\pm\frac{r}{\sqrt{1+r^2}})} F(r) r^{n-1}\,\mathrm{d}r\right| \leqslant C\max\big((2\varepsilon)^{-\frac{1}{2}}, N^2\big)t^{-\frac{1}{2}}N^{n-1}\left[\|F\|_{L^\infty[\varepsilon,N]} + \int\limits_{\varepsilon}^{N}\big|F'(r)\big|\,\mathrm{d}r\right].$$

Proof of the claim. Let $h(r) = \frac{|x|\cos\theta_1}{t}r \pm \frac{r}{\sqrt{1+r^2}}$, similar to Step 1, one gets

$$\min_{[\varepsilon,N]}\left|\frac{\partial^2 h}{\partial r^2}(r)\right| \geqslant \frac{3}{2^{\frac{7}{2}}}\min\big(2\varepsilon, N^{-4}\big),$$

thus by Lemma 2.1, we have

$$\begin{aligned}
&\left|\int\limits_{\varepsilon}^{N} e^{it(\frac{|x|\cos\theta_1}{t}r\pm\frac{r}{\sqrt{1+r^2}})} F(r) r^{n-1}\,\mathrm{d}r\right| \\
&= \left|\int\limits_{\varepsilon}^{N} e^{ith(r)} F(r) r^{n-1}\,\mathrm{d}r\right| \\
&\leqslant Ct^{-\frac{1}{2}}\max\big((2\varepsilon)^{-\frac{1}{2}}, N^2\big)\left[\big|F(N)\big|N^{n-1} + \int\limits_{\varepsilon}^{N}\big|r^{n-1}F'(r)\big|\,\mathrm{d}r + \int\limits_{\varepsilon}^{N}\big|\big(r^{n-1}\big)'F(r)\big|\,\mathrm{d}r\right] \\
&\leqslant Ct^{-\frac{1}{2}}\max\big((2\varepsilon)^{-\frac{1}{2}}, N^2\big)\left[\big|F(N)\big|N^{n-1} + N^{n-1}\int\limits_{\varepsilon}^{N}\big|F'(r)\big|\,\mathrm{d}r + \|F\|_{L^\infty[\varepsilon,N]}\big(N^{n-1}-\varepsilon^{n-1}\big)\right] \\
&\leqslant Ct^{-\frac{1}{2}}\max\big((2\varepsilon)^{-\frac{1}{2}}, N^2\big)N^{n-1}\left[\|F\|_{L^\infty[\varepsilon,N]} + \int\limits_{\varepsilon}^{N}\big|F'(r)\big|\,\mathrm{d}r\right].
\end{aligned}$$

This complete the proof of the claim. □

By the claim we just proved, we can easily get from (2.15) that

$$\left|\int\limits_{\varepsilon\leqslant|\xi|\leqslant N} e^{it(\frac{x\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})} F(\xi)\,\mathrm{d}\xi\right|$$

$$\leqslant Ct^{-\frac{1}{2}}\max\bigl((2\varepsilon)^{-\frac{1}{2}}, N^2\bigr)N^{n-1}\left[\|F\|_{L^\infty[\varepsilon,N]}+\int_\varepsilon^N \bigl|F'(r)\bigr|\,\mathrm{d}r\right]\int_0^\pi \sin^{n-2}\theta_1\,\mathrm{d}\theta_1$$

$$\leqslant Ct^{-\frac{1}{2}}\max\bigl((2\varepsilon)^{-\frac{1}{2}}, N^2\bigr)N^{n-1}\left[\|F\|_{L^\infty[\varepsilon,N]}+\int_\varepsilon^N \bigl|F'(r)\bigr|\,\mathrm{d}r\right],$$

thus we done. □

Lemma 2.3. *Let* $F_1=1$, $F_2=\frac{\sqrt{1+r^2}}{r}$, $F_3=\frac{r}{\sqrt{1+r^2}}$, $F_4=\frac{r^2}{1+r^2}$. *If we set*

$$G_i=\|F_i\|_{L^\infty[\varepsilon,N]}+\int_\varepsilon^N \bigl|F_i'(r)\bigr|\,\mathrm{d}r, \tag{2.16}$$

then

$$G_i\leqslant C\varepsilon^{-1}\quad \textit{for } i=1,2,3,4. \tag{2.17}$$

Proof. (i)

$$\begin{aligned}G_1&=\|F_1\|_{L^\infty[\varepsilon,N]}+\int_\varepsilon^N \bigl|F_1'(r)\bigr|\,\mathrm{d}r\\&=\|1\|_{L^\infty[\varepsilon,N]}+\int_\varepsilon^N |1'|\,\mathrm{d}r\\&=1.\end{aligned} \tag{2.18}$$

(ii)

$$\begin{aligned}G_2&=\|F_2\|_{L^\infty[\varepsilon,N]}+\int_\varepsilon^N \bigl|F_2'(r)\bigr|\,\mathrm{d}r\\&=\left\|\frac{\sqrt{1+r^2}}{r}\right\|_{L^\infty[\varepsilon,N]}+\int_\varepsilon^N\left|\left(\frac{\sqrt{1+r^2}}{r}\right)'\right|\mathrm{d}r\\&=\left\|\frac{\sqrt{1+r^2}}{r}\right\|_{L^\infty[\varepsilon,N]}+\int_\varepsilon^N\left|\frac{-1}{r^2\sqrt{1+r^2}}\right|\mathrm{d}r\\&\leqslant\frac{\sqrt{1+\varepsilon^2}}{\varepsilon}+\int_\varepsilon^N\frac{\mathrm{d}r}{r^2}\\&\leqslant\frac{\sqrt{2}}{\varepsilon}+\left(\frac{1}{\varepsilon}-\frac{1}{N}\right)\\&\leqslant C\varepsilon^{-1}.\end{aligned} \tag{2.19}$$

(iii)

$$G_3=\left\|\frac{r}{\sqrt{1+r^2}}\right\|_{L^\infty[\varepsilon,N]}+\int_\varepsilon^N\left|\left(\frac{r}{\sqrt{1+r^2}}\right)'\right|\mathrm{d}r$$

$$\begin{aligned}
&\leqslant 1+\int_{\varepsilon}^{N}\left|\frac{1}{(1+r^2)^{\frac{3}{2}}}\right|\mathrm{d}r\\
&=1+\int_{\varepsilon}^{N}\left|\frac{1}{(1+r^2)\sqrt{1+r^2}}\right|\mathrm{d}r\\
&\leqslant 1+\int_{\varepsilon}^{N}\frac{\mathrm{d}r}{r^2\sqrt{1+r^2}}\\
&\leqslant C\varepsilon^{-1}.
\end{aligned}\tag{2.20}$$

(iv)

$$\begin{aligned}
G_4&=\left\|\frac{r^2}{1+r^2}\right\|_{L^\infty[\varepsilon,N]}+\int_{\varepsilon}^{N}\left|\left(\frac{r^2}{1+r^2}\right)'\right|\mathrm{d}r\\
&\leqslant 1+\int_{\varepsilon}^{N}\frac{2r}{(1+r^2)^2}\,\mathrm{d}r\\
&\leqslant 1+2\int_{\varepsilon}^{N}\frac{\sqrt{1+r^2}}{(1+r^2)(1+r^2)}\,\mathrm{d}r\\
&=1+2\int_{\varepsilon}^{N}\frac{1}{(1+r^2)\sqrt{(1+r^2)}}\,\mathrm{d}r\\
&\leqslant C\varepsilon^{-1}. \qquad \square
\end{aligned}\tag{2.21}$$

Lemma 2.4. *Let $s>\frac{n}{2}$ and $\psi\in H^s\cap L^1\cap DL^2$. Then*

$$\left\|B(t)\psi\right\|_{L^\infty}\leqslant C(1+t)^{-\theta}\left(\|\psi\|_{L^1}+\|\psi\|_{H^s}+\|\psi\|_{DL^2}\right),\tag{2.22}$$

where $DL^2=\{f\colon D^{-1}f=(-\Delta)^{-\frac{1}{2}}f\in L^2\}$, and

$$\theta=\begin{cases}\frac{2s-n}{2n+20+4s} & \text{for } \frac{n}{2}<s<\frac{5n}{2},\\ \frac{n}{3n+5} & \text{for } s\geqslant\frac{5n}{2}.\end{cases}$$

Proof. By direct computation, we have

$$\begin{aligned}
B(t)\psi&=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\frac{\sqrt{1+|\xi|^2}}{|\xi|}\sin\frac{t|\xi|}{\sqrt{1+|\xi|^2}}\hat{\psi}(\xi)\,\mathrm{d}\xi\\
&=C\Bigg(\int_{|\xi|<\varepsilon}+\int_{\varepsilon\leqslant|\xi|\leqslant N}+\int_{|\xi|>N}\Bigg)e^{ix\cdot\xi}\frac{\sqrt{1+|\xi|^2}}{|\xi|}\sin\frac{t|\xi|}{\sqrt{1+|\xi|^2}}\hat{\psi}(\xi)\,\mathrm{d}\xi\\
&\triangleq C(\mathrm{I}+\mathrm{II}+\mathrm{III}),
\end{aligned}\tag{2.23}$$

where $0<\varepsilon<\frac{1}{2}$ and $N>1$ are constants to be determined later.

Case 1. $t > 1$.

By Hölder inequality, Hausdorff–Young inequality, Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
|\mathrm{I}| &= \left| \int\limits_{|\xi|<\varepsilon} e^{ix\cdot\xi} \frac{\sqrt{1+|\xi|^2}}{|\xi|} \sin \frac{t|\xi|}{\sqrt{1+|\xi|^2}} \hat{\psi}(\xi)\,\mathrm{d}\xi \right| \\
&\leqslant C \int\limits_{|\xi|<\varepsilon} \left| |\xi|^{-1} \hat{\psi}(\xi) \right| \mathrm{d}\xi \\
&\leqslant C \Big(\int\limits_{|\xi|<\varepsilon} 1\,\mathrm{d}\xi \Big)^{\frac{1}{2}} \big\| |\xi|^{-1} \hat{\psi}(\xi) \big\|_{L^2} \\
&\leqslant C\varepsilon^{\frac{n}{2}} \|\psi\|_{DL^2}, \\
|\mathrm{II}| &= \left| \int\limits_{\varepsilon\leqslant|\xi|\leqslant N} e^{ix\cdot\xi} \frac{\sqrt{1+|\xi|^2}}{|\xi|} \sin \frac{t|\xi|}{\sqrt{1+|\xi|^2}} \hat{\psi}(\xi)\,\mathrm{d}\xi \right| \\
&= \left| \int\limits_{\mathbb{R}^n} \Big(\int\limits_{\varepsilon\leqslant|\xi|\leqslant N} e^{i(x-y)\cdot\xi} \frac{(e^{it\frac{|\xi|}{\sqrt{1+|\xi|^2}}} - e^{-it\frac{|\xi|}{\sqrt{1+|\xi|^2}}})}{2i} \frac{\sqrt{1+|\xi|^2}}{|\xi|}\,\mathrm{d}\xi \Big) \psi(y)\,\mathrm{d}y \right| \\
&\leqslant C \sum_{\pm} \int\limits_{\mathbb{R}^n} \left| \int\limits_{\varepsilon\leqslant|\xi|\leqslant N} e^{it(\frac{(x-y)\cdot\xi}{t} \pm \frac{|\xi|}{\sqrt{1+|\xi|^2}})} \frac{\sqrt{1+|\xi|^2}}{|\xi|}\,\mathrm{d}\xi \right| \big|\psi(y)\big|\,\mathrm{d}y \\
&\leqslant C t^{-\frac{1}{2}} N^{n-1} \varepsilon^{-1} \max\big((2\varepsilon)^{-\frac{1}{2}}, N^2\big) \|\psi\|_{L^1},
\end{aligned}$$

and

$$\begin{aligned}
|\mathrm{III}| &= \left| \int\limits_{|\xi|>N} e^{ix\cdot\xi} \frac{\sqrt{1+|\xi|^2}}{|\xi|} \sin \frac{t|\xi|}{\sqrt{1+|\xi|^2}} |\xi|^{-s} |\xi|^s \hat{\psi}(\xi)\,\mathrm{d}\xi \right| \\
&\leqslant \sqrt{2} \Big(\int\limits_{|\xi|>N} |\xi|^{-2s}\,\mathrm{d}\xi \Big)^{\frac{1}{2}} \big\| |\xi|^s \hat{\psi}(\xi) \big\|_{L^2} \\
&= \sqrt{2} \big\| (-\Delta)^{\frac{s}{2}} \psi \big\|_{L^2} \left\{ \int\limits_N^\infty \int\limits_{\sum^{n-1}} r^{-2s} r^{n-1}\,\mathrm{d}r\,\mathrm{d}\sigma \right\}^{\frac{1}{2}} \\
&\leqslant C \|\psi\|_{\dot{H}^s} N^{\frac{n-2s}{2}}.
\end{aligned}$$

Thus we obtain

$$\big\| B(t)\psi \big\|_{L^\infty} \leqslant C\big(\|\psi\|_{L^1} + \|\psi\|_{\dot{H}^s} + \|\psi\|_{DL^2}\big)\big[\varepsilon^{\frac{n}{2}} + t^{-\frac{1}{2}} N^{n-1} \varepsilon^{-1} \max\big((2\varepsilon)^{-\frac{1}{2}}, N^2\big) + N^{\frac{n-2s}{2}}\big]. \tag{2.24}$$

Setting $(2\varepsilon)^{-\frac{1}{2}} = N^2$ and $N = t^k$ for $k > 0$ in (2.24), then $N > 1$, $0 < \varepsilon = t^{-4k}/2 < \frac{1}{2}$, and (2.24) becomes

$$\big\| B(t)\psi \big\|_{L^\infty} \leqslant C\big(\|\psi\|_{L^1} + \|\psi\|_{\dot{H}^s} + \|\psi\|_{DL^2}\big)\big[t^{-2nk} + t^{-\frac{1}{2}+(n+5)k} + t^{\frac{n-2s}{2}k}\big]. \tag{2.25}$$

(a) If we set

$$-2nk = -\frac{1}{2} + (n+5)k, \tag{2.26}$$

then

$$k = \frac{1}{6n+10}. \tag{2.27}$$

Thus

$$-\frac{1}{2} + (n+5)k = -2nk = \frac{-n}{3n+5} < 0, \tag{2.28}$$

$$\frac{n-2s}{2}k = \frac{n-2s}{12n+20} < 0. \tag{2.29}$$

Note that $\frac{-n}{3n+5} = \frac{n-2s}{12n+20}$ implies $s = \frac{5n}{2}$, we have

$$\theta_1 = \begin{cases} \frac{2s-n}{12n+20} & \text{for } \frac{n}{2} < s < \frac{5n}{2}, \\ \frac{n}{3n+5} & \text{for } s \geqslant \frac{5n}{2}. \end{cases} \tag{2.30}$$

(b) If we set

$$-\frac{1}{2} + (n+5)k = \frac{n-2s}{2}k, \tag{2.31}$$

then

$$k = \frac{1}{n+10+2s}. \tag{2.32}$$

Thus

$$-\frac{1}{2} + (n+5)k = \frac{n-2s}{2}k = \frac{n-2s}{2n+20+4s} < 0, \tag{2.33}$$

$$-2nk = \frac{-2n}{n+10+2s} < 0. \tag{2.34}$$

Note that $\frac{n-2s}{2n+20+4s} = \frac{-2n}{n+10+2s}$ implies $s = \frac{5n}{2}$, we have

$$\theta_2 = \begin{cases} \frac{2s-n}{2n+20+4s} & \text{for } \frac{n}{2} < s < \frac{5n}{2}, \\ \frac{2n}{n+10+2s} & \text{for } s \geqslant \frac{5n}{2}. \end{cases} \tag{2.35}$$

All in all, since

$$0 < \frac{2s-n}{12n+20} < \frac{2s-n}{2n+20+4s} \quad \Leftrightarrow \quad \frac{n}{2} < s < \frac{5n}{2} \tag{2.36}$$

and

$$\frac{n}{3n+5} \geqslant \frac{2n}{n+10+2s} \quad \Leftrightarrow \quad s \geqslant \frac{5n}{2}, \tag{2.37}$$

we have

$$\theta = \begin{cases} \frac{2s-n}{2n+20+4s} & \text{for } \frac{n}{2} < s < \frac{5n}{2}, \\ \frac{n}{3n+5} & \text{for } s \geqslant \frac{5n}{2}. \end{cases} \tag{2.38}$$

Thus we conclude for $t > 1$ that

$$\begin{aligned} \big\| B(t)\psi \big\|_{L^\infty} &\leqslant Ct^{-\theta}\big(\|\psi\|_{L^1} + \|\psi\|_{\dot{H}^s} + \|\psi\|_{DL^2}\big) \\ &= C(2t)^{-\theta}2^{\theta}\big(\|\psi\|_{L^1} + \|\psi\|_{\dot{H}^s} + \|\psi\|_{DL^2}\big) \\ &\leqslant C(1+t)^{-\theta}\big(\|\psi\|_{L^1} + \|\psi\|_{\dot{H}^s} + \|\psi\|_{DL^2}\big). \end{aligned} \tag{2.39}$$

Case 2. $0 < t \leqslant 1$.

In this case, one can easily get

$$\begin{aligned}\big\|B(t)\psi\big\|_{L^\infty} &\leqslant C\left|\int\limits_{|\xi|\leqslant 1} e^{ix\cdot\xi}\frac{\sin\frac{t|\xi|}{\sqrt{1+|\xi|^2}}}{\frac{t|\xi|}{\sqrt{1+|\xi|^2}}}t\hat{\psi}(\xi)\,\mathrm{d}\xi\right| + C\left|\int\limits_{|\xi|>1} e^{ix\cdot\xi}\frac{\sqrt{1+|\xi|^2}}{|\xi|}\sin\frac{t|\xi|}{\sqrt{1+|\xi|^2}}|\xi|^{-s}|\xi|^s\hat{\psi}(\xi)\,\mathrm{d}\xi\right| \\ &\leqslant C\big(\|\psi\|_{L^1}+\|\psi\|_{\dot{H}^s}\big) \\ &= C2^\theta(1+1)^{-\theta}\big(\|\psi\|_{L^1}+\|\psi\|_{\dot{H}^s}\big) \\ &\leqslant C(1+t)^{-\theta}\big(\|\psi\|_{L^1}+\|\psi\|_{\dot{H}^s}+\|\psi\|_{DL^2}\big). \end{aligned} \tag{2.40}$$

The combination of (2.39) and (2.40) yields (2.22). □

Lemma 2.5. *Let $s > \frac{n}{2}$ and $\varphi \in H^s \cap L^1$. Then*

$$\big\|\partial_t B(t)\varphi\big\|_{L^\infty} \leqslant C(1+t)^{-\theta}\big(\|\varphi\|_{L^1}+\|\varphi\|_{H^s}\big) \tag{2.41}$$

with θ defined by (2.38).

Proof. Using Euler formula, we have

$$\begin{aligned}\partial_t B(t)\varphi &= \frac{1}{2}(2\pi)^{-n}\int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}\big(e^{it\frac{|\xi|}{\sqrt{1+|\xi|^2}}}+e^{-it\frac{|\xi|}{\sqrt{1+|\xi|^2}}}\big)\hat{\varphi}(\xi)\,\mathrm{d}\xi \\ &= C\sum_{\pm}\int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}e^{\pm it\frac{|\xi|}{\sqrt{1+|\xi|^2}}}\hat{\varphi}(\xi)\,\mathrm{d}\xi \\ &= C\sum_{\pm}\left(\int\limits_{|\xi|<\varepsilon}+\int\limits_{\varepsilon\leqslant|\xi|\leqslant N}+\int\limits_{|\xi|>N}\right)e^{it(\frac{x\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})}\hat{\varphi}(\xi)\,\mathrm{d}\xi \\ &\triangleq C\sum_{\pm}(\mathrm{I}+\mathrm{II}+\mathrm{III}). \end{aligned} \tag{2.42}$$

By Hölder inequality, Lemmas 2.2 and 2.3, we have for $t > 1$ that

$$\begin{aligned}|\mathrm{I}| &= \left|\int\limits_{\mathbb{R}^n}\left(\int\limits_{|\xi|<\varepsilon} e^{it(\frac{(x-y)\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})}\,\mathrm{d}\xi\right)\varphi(y)\,\mathrm{d}y\right| \\ &\leqslant \int\limits_{\mathbb{R}^n}\left|\int\limits_{|\xi|<\varepsilon} e^{it(\frac{(x-y)\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})}\,\mathrm{d}\xi\right|\big|\varphi(y)\big|\,\mathrm{d}y \\ &\leqslant \varepsilon^n\|\varphi\|_{L^1}, \\ |\mathrm{II}| &= \left|\int\limits_{\mathbb{R}^n}\left(\int\limits_{\varepsilon\leqslant|\xi|\leqslant N} e^{it(\frac{(x-y)\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})}\,\mathrm{d}\xi\right)\varphi(y)\,\mathrm{d}y\right| \\ &\leqslant C\max\big((2\varepsilon)^{-\frac{1}{2}},N^2\big)t^{-\frac{1}{2}}N^{n-1}\varepsilon^{-1}\|\varphi\|_{L^1}, \end{aligned}$$

and

$$\begin{aligned}|\mathrm{III}| &= \left|\int\limits_{|\xi|>N} e^{it(\frac{x\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})}|\xi|^{-s}|\xi|^s\hat{\varphi}(\xi)\,\mathrm{d}\xi\right| \\ &\leqslant \left(\int\limits_{|\xi|>N}|\xi|^{-2s}\,\mathrm{d}\xi\right)^{\frac{1}{2}}\big\||\xi|^s\hat{\varphi}\big\|_{L^2} \\ &\leqslant C\|\varphi\|_{\dot{H}^s}N^{\frac{n-2s}{2}}. \end{aligned}$$

On the other hand, if $t \leqslant 1$, one can easily get

$$\begin{aligned}\left\|\partial_t B(t)\varphi\right\|_{L^\infty} &\leqslant (2\pi)^{-n}\left|\int\limits_{|\xi|\leqslant 1} e^{ix\cdot\xi}\cos\frac{t|\xi|}{\sqrt{1+|\xi|^2}}\hat{\varphi}(\xi)\,\mathrm{d}\xi\right| \\ &\quad + (2\pi)^{-n}\left|\int\limits_{|\xi|>1} e^{ix\cdot\xi}\cos\frac{t|\xi|}{\sqrt{1+|\xi|^2}}|\xi|^{-s}|\xi|^s\hat{\varphi}(\xi)\,\mathrm{d}\xi\right| \\ &\leqslant C\big(\|\varphi\|_{L^1} + \|\varphi\|_{\dot{H}^s}\big). \end{aligned} \tag{2.43}$$

Since $\varepsilon^n \leqslant \varepsilon^{\frac{n}{2}}$ for $0 < \varepsilon < \frac{1}{2}$, arguing as in Lemma 2.4, we done. □

Lemma 2.6. *Let $s > \frac{n}{2}$ and $g \in H^s \cap L^1$. Then*

$$\left\|\int\limits_0^t T(t-\tau)g(\cdot,\tau)\,\mathrm{d}\tau\right\|_{L^\infty} \leqslant C\int\limits_0^t (1+t-\tau)^{-\theta}\big(\left\|g(\cdot,\tau)\right\|_{L^1} + \left\|g(\cdot,\tau)\right\|_{H^s}\big)\,\mathrm{d}\tau \tag{2.44}$$

with θ defined by (2.38).

Proof. Note that

$$\begin{aligned} T(t)g &= (2\pi)^{-n}\int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{|\xi|}{\sqrt{1+|\xi|^2}}\frac{e^{it\frac{|\xi|}{\sqrt{1+|\xi|^2}}} - e^{-it\frac{|\xi|}{\sqrt{1+|\xi|^2}}}}{2i}\hat{g}(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{2i}(2\pi)^{-n}\sum_{\pm}(\pm 1)\left(\int\limits_{|\xi|<\varepsilon} + \int\limits_{\varepsilon\leqslant|\xi|\leqslant N} + \int\limits_{|\xi|>N}\right)e^{it(\frac{x\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})}\frac{|\xi|}{\sqrt{1+|\xi|^2}}\hat{g}(\xi)\,\mathrm{d}\xi \\ &\triangleq \frac{1}{2i}(2\pi)^{-n}\sum_{\pm}(\pm 1)(\mathrm{I}+\mathrm{II}+\mathrm{III}), \end{aligned} \tag{2.45}$$

by Hölder inequality, Lemmas 2.2 and 2.3, we have for $t > 1$ that

$$\begin{aligned} |\mathrm{I}| &= \left|\int\limits_{|\xi|<\varepsilon} e^{it(\frac{x\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})}\frac{|\xi|}{\sqrt{1+|\xi|^2}}\hat{g}(\xi)\,\mathrm{d}\xi\right| \\ &\leqslant C\varepsilon^n\|g\|_{L^1}, \end{aligned}$$

$$\begin{aligned} |\mathrm{II}| &= \left|\int\limits_{\mathbb{R}^n}\left(\int\limits_{\varepsilon\leqslant|\xi|\leqslant N} e^{it(\frac{(x-y)\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})}\frac{|\xi|}{\sqrt{1+|\xi|^2}}\,\mathrm{d}\xi\right)g(y)\,\mathrm{d}y\right| \\ &\leqslant Ct^{-\frac{1}{2}}N^{n-1}\varepsilon^{-1}\max\big((2\varepsilon)^{-\frac{1}{2}}, N^2\big)\|g\|_{L^1}, \end{aligned}$$

and

$$\begin{aligned} |\mathrm{III}| &= \left|\int\limits_{|\xi|>N} e^{it(\frac{x\cdot\xi}{t}\pm\frac{|\xi|}{\sqrt{1+|\xi|^2}})}\frac{|\xi|}{\sqrt{1+|\xi|^2}}|\xi|^{-s}|\xi|^s\hat{g}(\xi)\,\mathrm{d}\xi\right| \\ &\leqslant C\|g\|_{\dot{H}^s}N^{\frac{n-2s}{2}}. \end{aligned}$$

On the other hand, if $t \leqslant 1$, one can easily get

$$\begin{aligned}\big\|T(t)g\big\|_{L^\infty} &\leqslant (2\pi)^{-n}\left|\int\limits_{|\xi|\leqslant 1} e^{ix\cdot\xi}\frac{|\xi|}{\sqrt{1+|\xi|^2}}\sin\frac{t|\xi|}{\sqrt{1+|\xi|^2}}\hat{g}(\xi)\,\mathrm{d}\xi\right| \\ &\quad + (2\pi)^{-n}\left|\int\limits_{|\xi|>1} e^{ix\cdot\xi}\frac{|\xi|}{\sqrt{1+|\xi|^2}}\sin\frac{t|\xi|}{\sqrt{1+|\xi|^2}}|\xi|^{-s}|\xi|^s\hat{g}(\xi)\,\mathrm{d}\xi\right| \\ &\leqslant C\big(\|g\|_{L^1}+\|g\|_{\dot{H}^s}\big). \end{aligned} \tag{2.46}$$

Since $\varepsilon^n \leqslant \varepsilon^{\frac{n}{2}}$ for $0<\varepsilon<\frac{1}{2}$, arguing as in Lemma 2.4, we done. □

Proposition 2.7. *Let $s>\frac{n}{2}$, $\varphi\in H^s\cap L^1, \psi\in H^s\cap L^1\cap DL^2$ and $g\in L^2([0,T];H^s\cap L^1)$. Then* (2.1) *has a unique generalized solution $u(x,t)\in C^2([0,T];H^s)$. Moreover, we have*

$$\begin{aligned}\big\|u(\cdot,t)\big\|_{L^\infty} &\leqslant C_0(1+t)^{-\theta}\big(\|\varphi\|_{L^1}+\|\varphi\|_{H^s}+\|\psi\|_{L^1}+\|\psi\|_{H^s}+\|\psi\|_{DL^2}\big) \\ &\quad + C_0\int\limits_0^t (1+t-\tau)^{-\theta}\big(\big\|g(\cdot,\tau)\big\|_{L^1}+\big\|g(\cdot,\tau)\big\|_{H^s}\big)\,\mathrm{d}\tau, \end{aligned} \tag{2.47}$$

$$\big\|u(\cdot,\tau)\big\|_{H^s} \leqslant C_0\left(\|\varphi\|_{H^s}+\|\psi\|_{DL^2}+\|\psi\|_{H^s}+\int\limits_0^t\big\|g(\cdot,\tau)\big\|_{H^s}\,\mathrm{d}\tau\right), \tag{2.48}$$

$$\begin{aligned}\big\|u_t(\cdot,t)\big\|_{L^\infty} &\leqslant C_0(1+t)^{-\theta}\big(\|\varphi\|_{L^1}+\|\varphi\|_{H^s}+\|\psi\|_{L^1}+\|\psi\|_{H^s}\big) \\ &\quad + C_0\int\limits_0^t (1+t-\tau)^{-\theta}\big(\big\|g(\cdot,\tau)\big\|_{L^1}+\big\|g(\cdot,\tau)\big\|_{H^s}\big)\,\mathrm{d}\tau, \end{aligned} \tag{2.49}$$

$$\big\|u_t(\cdot,\tau)\big\|_{H^s} \leqslant C_0\left(\|\varphi\|_{H^s}+\|\psi\|_{H^s}+\int\limits_0^t\big\|g(\cdot,\tau)\big\|_{H^s}\,\mathrm{d}\tau\right). \tag{2.50}$$

Proof. We can get the existence and uniqueness of the solution $u(x,t)$ for the problem (2.1) by means of Garlerkin's method (cf. [5]). And (2.47) follows immediately from Lemmas 2.4–2.6.

Note that

$$\|f\|_{H^{s,p}} = \big\|(I-\Delta)^{\frac{s}{2}}f\big\|_{L^p} = \big\|\mathcal{F}^{-1}\big[\big(1+|\xi|^2\big)^{\frac{s}{2}}\hat{f}(\xi)\big]\big\|_{L^p}, \tag{2.51}$$

we have

$$\|f\|_{H^s} = \big\|\mathcal{F}^{-1}\big[\big(1+|\xi|^2\big)^{\frac{s}{2}}\hat{f}(\xi)\big]\big\|_{L^2} = \big\|\big(1+|\xi|^2\big)^{\frac{s}{2}}\hat{f}(\xi)\big\|_{L^2}. \tag{2.52}$$

Therefore

$$\begin{aligned}\big\|u(\cdot,t)\big\|_{H^s} &\leqslant \big\|\partial_t B(t)\varphi(\cdot)\big\|_{H^s}+\big\|B(t)\psi(\cdot)\big\|_{H^s}+\int\limits_0^t\big\|T(t-\tau)g(\cdot,\tau)\big\|_{H^s}\,\mathrm{d}\tau \\ &= \big\|\big(1+|\xi|^2\big)^{\frac{s}{2}}\mathcal{F}\big[\partial_t B(t)\varphi\big](\xi)\big\|_{L^2}+\big\|\big(1+|\xi|^2\big)^{\frac{s}{2}}\mathcal{F}\big[B(t)\psi\big](\xi)\big\|_{L^2} \\ &\quad + \int\limits_0^t\big\|\big(1+|\xi|^2\big)^{\frac{s}{2}}\mathcal{F}\big[T(t-\tau)g(\cdot,\tau)\big](\xi)\big\|_{L^2}\,\mathrm{d}\tau \\ &= \left\|\big(1+|\xi|^2\big)^{\frac{s}{2}}\cos\frac{t|\xi|}{\sqrt{1+|\xi|^2}}\hat{\varphi}(\xi)\right\|_{L^2}+\left\|\big(1+|\xi|^2\big)^{\frac{s}{2}}\frac{\sqrt{1+|\xi|^2}}{|\xi|}\sin\frac{t|\xi|}{\sqrt{1+|\xi|^2}}\hat{\psi}(\xi)\right\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \left\| \left(1+|\xi|^2\right)^{\frac{s}{2}} \frac{|\xi|}{\sqrt{1+|\xi|^2}} \sin\frac{(t-\tau)|\xi|}{\sqrt{1+|\xi|^2}} \hat{g}(\xi) \right\|_{L^2} \mathrm{d}\tau \\
&\leqslant \left\| \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{\varphi}(\xi) \right\|_{L^2} + \left\| \left(1+|\xi|^2\right)^{\frac{s}{2}} \frac{\sqrt{1+|\xi|^2}}{|\xi|} \sin\frac{t|\xi|}{\sqrt{1+|\xi|^2}} \hat{\psi}(\xi) \right\|_{L^2(|\xi|>1)} \\
&\quad + \left\| \left(1+|\xi|^2\right)^{\frac{s}{2}} \frac{\sqrt{1+|\xi|^2}}{|\xi|} \sin\frac{t|\xi|}{\sqrt{1+|\xi|^2}} \hat{\psi}(\xi) \right\|_{L^2(|\xi|\leqslant 1)} + \int_0^t \left\| \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{g}(\xi) \right\|_{L^2} \mathrm{d}\tau \\
&\leqslant C\left(\|\varphi\|_{H^s} + \|\psi\|_{H^s} + \int_0^t \left\| g(\cdot,\tau) \right\|_{H^s} \mathrm{d}\tau \right) + C \left\| |\xi|^{-1} \hat{\psi}(\xi) \right\|_{L^2(|\xi|\leqslant 1)} \\
&\leqslant C_0\left(\|\varphi\|_{H^s} + \|\psi\|_{DL^2} + \|\psi\|_{H^s} + \int_0^t \left\| g(\cdot,\tau) \right\|_{H^s} \mathrm{d}\tau \right),
\end{aligned}
$$

and (2.48) follows.

Since

$$
\begin{aligned}
u_t(x,t) = &-(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{|\xi|}{\sqrt{1+|\xi|^2}} \sin\frac{t|\xi|}{\sqrt{1+|\xi|^2}} \hat{\varphi}(\xi)\,\mathrm{d}\xi + (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \cos\frac{t|\xi|}{\sqrt{1+|\xi|^2}} \hat{\psi}(\xi)\,\mathrm{d}\xi \\
&- (2\pi)^{-n} \int_0^t \left[\int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{|\xi|^2}{1+|\xi|^2} \cos\frac{(t-\tau)|\xi|}{\sqrt{1+|\xi|^2}} \hat{g}(\xi,\tau)\,\mathrm{d}\xi \right] \mathrm{d}\tau, \qquad (2.53)
\end{aligned}
$$

we can establish Lemmas 2.2–2.6 similarly to obtain (2.49) and (2.50). □

3. Nonlinear estimates

Since the proofs of this part are almost the same as those in [2], for the convenience of reading and the completeness of the paper, we just state them without proofs except that there exist misprints in it.

Lemma 3.1. *Assume that $f(u) \in C^k(\mathbb{R})$, $f(0) = 0$, $u \in L^\infty \cap H^s$ and $k = [s] + 1$ with $s \geqslant 0$. Then*

$$
\left\| f(u) \right\|_{H^s} \leqslant C(M) \|u\|_{H^s} \qquad (3.1)
$$

for $\|u\|_{L^\infty} \leqslant M$, where $C(M)$ is a constant depending on M.

Proof. There are some misprints in the original paper, so we prove this lemma in detail.

Case 1. $s = 0$.

From $f(0) = 0$ we have

$$
f(u) = \int_0^1 f'(\lambda u) u\,\mathrm{d}\lambda;
$$

it follows that

$$
\left\| f(u) \right\|_{L^2} \leqslant C(M) \|u\|_{L^2}. \qquad (3.2)
$$

Case 2. $s > 0$ is an integer.

Since

$$
\left\| f(u) \right\|_{H^s} \leqslant C\left(\left\| f(u) \right\|_{L^2} + \sum_{i=1}^n \left\| \frac{\partial^s f(u)}{\partial x_i^s} \right\|_{L^2} \right) \qquad (3.3)
$$

and

$$\left\| \frac{\partial^s f(u)}{\partial x_i^s} \right\|_{L^2} \leqslant \sum_{l=1}^{s} \sum_{\alpha} \left\| f^{(l)}(u) \frac{\partial^{\alpha_1} u}{\partial {x_i}^{\alpha_1}} \frac{\partial^{\alpha_2} u}{\partial {x_i}^{\alpha_2}} \cdots \frac{\partial^{\alpha_l} u}{\partial {x_i}^{\alpha_l}} \right\|_{L^2}$$
$$\leqslant \sum_{l=1}^{s} \sum_{\alpha} \left\| f^{(l)}(u) \right\|_{L^\infty} \left\| \frac{\partial^{\alpha_1} u}{\partial {x_i}^{\alpha_1}} \right\|_{L^{p_1}} \left\| \frac{\partial^{\alpha_2} u}{\partial {x_i}^{\alpha_2}} \right\|_{L^{p_2}} \cdots \left\| \frac{\partial^{\alpha_l} u}{\partial {x_i}^{\alpha_l}} \right\|_{L^{p_l}},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$, $\alpha_k \geqslant 1$, $\alpha_1 + \alpha_2 + \cdots + \alpha_l = s$ and $p_k = \frac{2s}{\alpha_k}$ $(k = 1, 2, \ldots, l)$.

Applying Gagliardo–Nirenberg's inequality [7] we have

$$\left\| \frac{\partial^{\alpha_k} u}{\partial {x_i}^{\alpha_k}} \right\|_{L^{p_k}} \leqslant C \|u\|_{L^\infty}^{1-\frac{\alpha_k}{s}} \left\| \frac{\partial^s u}{\partial {x_i}^s} \right\|_{L^2}^{\frac{\alpha_k}{s}};$$

it follows that

$$\left\| \frac{\partial^s f(u)}{\partial x_i^s} \right\|_{L^2} \leqslant C(M) \left\| \frac{\partial^s u}{\partial {x_i}^s} \right\|_{L^2}. \tag{3.4}$$

Combining (3.2)–(3.4) we get (3.1).

Case 3. $s > 0$ is not an integer.

Let $m = [s]$, from the above proof, we have

$$\left\| f(u) \right\|_{H^m} \leqslant C(M) \|u\|_{H^m} \tag{3.5}$$

and

$$\left\| f(u) \right\|_{H^{m+1}} \leqslant C(M) \|u\|_{H^{m+1}}. \tag{3.6}$$

Applying interpolation between (3.5) and (3.6) yields

$$\left\| f(u) \right\|_{H^s} \leqslant C(M) \|u\|_{H^s}.$$

This complete the proof of the lemma. □

Lemma 3.2. *If $s > 0$, then $H^s \cap L^\infty$ is an algebra. Moreover,*

$$\|fg\|_{H^s} \leqslant C\big(\|f\|_{L^\infty} \|g\|_{H^s} + \|g\|_{L^\infty} \|f\|_{H^s}\big)$$

for $f, g \in H^s \cap L^\infty$.

Proof. See [4, Appendix, Lemma X4]. □

Lemma 3.3. *(See [2].) Suppose that $f \in C^k$, $k = [s] + 1$ and $f(u) = O(|u|^{1+\alpha})$ as $u \to 0$, $\alpha \geqslant 1$ be a positive integer. If $u \in H^s \cap L^\infty$ and $\|u\|_{L^\infty} \leqslant M$, then*

$$\left\| f(u) \right\|_{H^s} \leqslant C(M) \|u\|_{H^s} \|u\|_{L^\infty}^{\alpha}, \tag{3.7}$$
$$\left\| f(u) \right\|_{L^1} \leqslant C(M) \|u\|_{L^2}^2 \|u\|_{L^\infty}^{\alpha-1}, \tag{3.8}$$

where $C(M)$ is a constant dependent on M.

Lemma 3.4. *(See [2].) Suppose that $f \in C^k$, $k = [s] + 1$ and $f(u) = O(|u|^{1+\alpha})$ as $u \to 0$, $\alpha \geqslant 1$ be a positive integer. If $u, v \in H^s \cap L^\infty$ and $\|u\|_{L^\infty} \leqslant M$, $\|v\|_{L^\infty} \leqslant M$, then*

$$\left\| f(u) - f(v) \right\|_{H^s} \leqslant C(M)\big[\|u - v\|_{L^\infty}\big(\|u\|_{H^s} + \|v\|_{H^s}\big)\big(\|u\|_{L^\infty} + \|v\|_{L^\infty}\big)^{\alpha-1}$$
$$+ \|u - v\|_{H^s}\big(\|u\|_{L^\infty} + \|v\|_{L^\infty}\big)^{\alpha}\big], \tag{3.9}$$
$$\left\| f(u) - f(v) \right\|_{L^1} \leqslant C(M)\big(\|u\|_{L^\infty} + \|v\|_{L^\infty}\big)^{\alpha-1}\big(\|u\|_{L^2} + \|v\|_{L^2}\big)\|u - v\|_{L^2}, \tag{3.10}$$

where $C(M)$ is a constant dependent on M.

Lemma 3.5. *If a, b are constants, $b \geqslant a \geqslant 0$, then*

$$\int_0^t (1+t-\tau)^{-a}(1+\tau)^{-b}\,\mathrm{d}\tau \leqslant C(1+t)^{-a}\int_0^t (1+\tau)^{-b}\,\mathrm{d}\tau. \tag{3.11}$$

Proof. Note that

$$\begin{aligned}
\mathrm{I}_1 &= \int_0^{\frac{t}{2}} (1+t-\tau)^{-a}(1+\tau)^{-b}\,\mathrm{d}\tau \\
&\leqslant \left(1+\frac{t}{2}\right)^{-a}\int_0^{\frac{t}{2}} (1+\tau)^{-b}\,\mathrm{d}\tau \\
&\leqslant C(1+t)^{-a}\int_0^t (1+\tau)^{-b}\,\mathrm{d}\tau
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
\mathrm{I}_2 &= \int_{\frac{t}{2}}^t (1+t-\tau)^{-a}(1+\tau)^{-b}\,\mathrm{d}\tau \\
&\leqslant \left(1+\frac{t}{2}\right)^{-b}\int_{\frac{t}{2}}^t (1+t-\tau)^{-a}\,\mathrm{d}\tau \\
&= 2^b(2+t)^{-b}\int_{t-\frac{t}{2}}^{t-t} (1+s)^{-a}(-\mathrm{d}s) \\
&\leqslant 2^b(1+t)^{-b}\int_0^{\frac{t}{2}} (1+\tau)^{-a}\,\mathrm{d}\tau \\
&\leqslant 2^b(1+t)^{-a}\int_0^t \frac{(1+t)^a}{(1+t)^b}\frac{(1+\tau)^b}{(1+\tau)^a}(1+\tau)^{-b}\,\mathrm{d}\tau \\
&\leqslant 2^b(1+t)^{-a}\int_0^t \left(\frac{1+\tau}{1+t}\right)^{b-a}(1+\tau)^{-b}\,\mathrm{d}\tau \\
&\leqslant 2^b(1+t)^{-a}\int_0^t (1+\tau)^{-b}\,\mathrm{d}\tau.
\end{aligned} \tag{3.13}$$

The combination of (3.12) and (3.13) gives (3.11). □

4. Main result

After all the preparations of above two sections, our main result can be stated as:

Theorem 4.1. *Let $f \in C^k$, with k an integer, $k \geqslant s > \frac{n}{2}$, satisfies $f(u) = O(|u|^{\alpha+1})$ as $u \to 0$ with α an integer. Suppose that $(\alpha-1)\theta > 1$ with θ defined by* (2.38). *Then there exists a constant $\delta > 0$ such that for any $\varphi \in H^s \cap L^1$, $\psi \in H^s \cap L^1 \cap DL^2$ satisfying*

$$\|\varphi\|_{H^s} + \|\varphi\|_{L^1} + \|\psi\|_{H^s} + \|\psi\|_{L^1} + \|\psi\|_{DL^2} \leqslant \delta, \tag{4.1}$$

problem (1.1) *has a unique solution $u(x,t) \in C^1([0,\infty); H^s)$. Moreover,*

$$\sup_{0\leqslant t<\infty} \big\{(1+t)^{\theta}\big[\big\|u(\cdot,t)\big\|_{L^\infty} + \big\|u_t(\cdot,t)\big\|_{L^\infty}\big] + \big\|u(\cdot,t)\big\|_{H^s} + \big\|u_t(\cdot,t)\big\|_{H^s}\big\} \leqslant C\delta, \tag{4.2}$$

where the constant C depends only on f and initial data.

Proof. Define a metric space

$$X = \big\{u(x,t) \in C^2\big([0,\infty); H^s\big)\colon\ |||u||| \leqslant 5C_0\delta\big\} \tag{4.3}$$

with the norm

$$|||u||| = \sup_{t\geqslant 0}\big\{\big\|u(\cdot,t)\big\|_{H^s} + \big\|u_t(\cdot,t)\big\|_{H^s} + (1+t)^{-\theta}\big[\big\|u(\cdot,t)\big\|_{L^\infty} + \big\|u_t(\cdot,t)\big\|_{L^\infty}\big]\big\}, \tag{4.4}$$

where $\delta > 0$ satisfies (4.1) and C_0 is the constant in Proposition 2.7. It is easy to prove that X is a complete metric space.

Let

$$N(u) = \partial_t B(t)\varphi + B(t)\psi + \int_0^t T(t-\tau) f\big[u(\tau)\big]\,\mathrm{d}\tau. \tag{4.5}$$

We should prove that $N : X \to X$ is strictly contractive if δ is small enough.

In fact, by (2.47) and (2.49) in Proposition 2.7 and Lemma 3.3, we have

$$\begin{aligned}
&\big\|N(u)\big\|_{L^\infty} + \big\|N(u)_t\big\|_{L^\infty}\\
&\quad\leqslant 2C_0(1+t)^{-\theta}\big(\|\varphi\|_{L^1} + \|\varphi\|_{H^s} + \|\psi\|_{L^1} + \|\psi\|_{H^s} + \|\psi\|_{DL^2}\big)\\
&\qquad + 2C_0\int_0^t (1+t-\tau)^{-\theta}\big\{\big\|f\big[u(\tau)\big]\big\|_{L^1} + \big\|f\big[u(\tau)\big]\big\|_{H^s}\big\}\,\mathrm{d}\tau\\
&\quad\leqslant 2C_0(1+t)^{-\theta}\delta + C\int_0^t (1+t-\tau)^{-\theta}\big[\big\|u(\tau)\big\|_{L^2}^2\big\|u(\tau)\big\|_{L^\infty}^{\alpha-1} + \big\|u(\tau)\big\|_{H^s}\big\|u(\tau)\big\|_{L^\infty}^{\alpha}\big]\,\mathrm{d}\tau\\
&\quad\leqslant 2C_0(1+t)^{-\theta}\delta + C\int_0^t (1+t-\tau)^{-\theta}\big[(1+\tau)^{-(\alpha-1)\theta} + (1+\tau)^{-\alpha\theta}\big]\,\mathrm{d}\tau\ |||u|||^{\alpha+1}.
\end{aligned}$$

Since $(\alpha-1)\theta > 1 > \theta > 0$, it follows from Lemma 3.5 that

$$(1+t)^{\theta}\big[\big\|N(u)\big\|_{L^\infty} + \big\|N(u)_t\big\|_{L^\infty}\big] \leqslant 2C_0\delta + C|||u|||^{\alpha+1}. \tag{4.6}$$

On the other hand, since $\theta\alpha > 1$, by (2.48), (2.50) and Lemma 3.3, we have

$$\begin{aligned}
\big\|N(u)\big\|_{H^s} + \big\|N(u)_t\big\|_{H^s} &\leqslant 2C_0\big(\|\varphi\|_{H^s} + \|\psi\|_{DL^2} + \|\psi\|_{H^s}\big) + 2C_0\int_0^t \big\|f\big[u(\tau)\big]\big\|_{H^s}\,\mathrm{d}\tau\\
&\leqslant 2C_0\delta + C\int_0^t \big\|u(\tau)\big\|_{H^s}\big\|u(\tau)\big\|_{L^\infty}^{\alpha}\,\mathrm{d}\tau
\end{aligned}$$

$$\leqslant 2C_0\delta + C\int_0^t (1+\tau)^{-\theta\alpha}\,\mathrm{d}\tau\,|||u|||^{\alpha+1}$$
$$\leqslant 2C_0\delta + C|||u|||^{\alpha+1}. \tag{4.7}$$

Combing (4.6) with (4.7) yields

$$\big|\big|\big|N(u)\big|\big|\big| \leqslant 4C_0\delta + C|||u|||^{\alpha+1}.$$

Taking δ small enough such that $C(5c_0\delta)^\alpha < \frac{1}{5}$, we know that N maps X into X.

Set $u, v \in X$, it follows from (4.5) that

$$N(u) - N(v) = \int_0^t T(t-\tau)\big\{f\big[u(\tau)\big] - f\big[v(\tau)\big]\big\}\,\mathrm{d}\tau. \tag{4.8}$$

Using (2.47), (2.49) and Lemma 3.4, we get

$$\begin{aligned}
&\big\|N(u) - N(v)\big\|_{L^\infty} + \big\|N(u)_t - N(v)_t\big\|_{L^\infty}\\
&\quad\leqslant 2C_0\int_0^t (1+t-\tau)^{-\theta}\big\{\big\|f\big[u(\tau)\big] - f\big[v(\tau)\big]\big\|_{H^s} + \big\|f\big[u(\tau)\big] - f\big[v(\tau)\big]\big\|_{L^1}\big\}\,\mathrm{d}\tau\\
&\quad\leqslant C\int_0^t (1+t-\tau)^{-\theta}\big[\|u-v\|_{L^\infty}\big(\|u\|_{H^s} + \|v\|_{H^s}\big)\big(\|u\|_{L^\infty} + \|v\|_{L^\infty}\big)^{\alpha-1}\\
&\qquad + \|u-v\|_{H^s}\big(\|u\|_{L^\infty} + \|v\|_{L^\infty}\big)^{\alpha} + \big(\|u\|_{L^\infty} + \|v\|_{L^\infty}\big)^{\alpha-1}\big(\|u\|_{L^2} + \|v\|_{L^2}\big)\|u-v\|_{L^2}\big]\,\mathrm{d}\tau\\
&\quad\leqslant C\int_0^t (1+t-\tau)^{-\theta}\big[(1+\tau)^{-\theta\alpha} + (1+\tau)^{-\theta(\alpha-1)}\big]\,\mathrm{d}\tau\,\big(|||u||| + |||v|||\big)^{\alpha}|||u-v|||\\
&\quad\leqslant C(1+t)^{-\theta}\big(|||u||| + |||v|||\big)^{\alpha}|||u-v|||
\end{aligned}$$

since $\theta(\alpha-1) > 1 > \theta > 0$. Hence

$$(1+t)^{-\theta}\big[\big\|N(u) - N(v)\big\|_{L^\infty} + \big\|N(u)_t - N(v)_t\big\|_{L^\infty}\big] \leqslant C\big(|||u||| + |||v|||\big)^{\alpha}|||u-v|||. \tag{4.9}$$

On the other hand, using (2.48), (2.50) and Lemma 3.4, we get

$$\begin{aligned}
&\big\|N(u) - N(v)\big\|_{H^s} + \big\|N(u)_t - N(v)_t\big\|_{H^s}\\
&\quad\leqslant 2C_0\int_0^t \big\|f\big[u(\tau)\big] - f\big[v(\tau)\big]\big\|_{H^s}\,\mathrm{d}\tau\\
&\quad\leqslant C\int_0^t \big[\|u-v\|_{L^\infty}\big(\|u\|_{H^s} + \|v\|_{H^s}\big)\big(\|u\|_{L^\infty} + \|v\|_{L^\infty}\big)^{\alpha-1} + \|u-v\|_{H^s}\big(\|u\|_{L^\infty} + \|v\|_{L^\infty}\big)^{\alpha}\big]\,\mathrm{d}\tau\\
&\quad\leqslant C\int_0^t (1+\tau)^{-\theta\alpha}\,\mathrm{d}\tau\big(|||u||| + |||v|||\big)^{\alpha}|||u-v|||\\
&\quad\leqslant C\big(|||u||| + |||v|||\big)^{\alpha}|||u-v|||
\end{aligned} \tag{4.10}$$

since $\theta\alpha > 1$. Combing (4.9) with (4.10) yields

$$\left|\!\left|\!\left| N(u) - N(v) \right|\!\right|\!\right| \leqslant C\big(|\!|\!| u |\!|\!| + |\!|\!| v |\!|\!|\big)^{\alpha} |\!|\!| u - v |\!|\!|. \tag{4.11}$$

Taking δ small enough such that $C(10C_0\delta)^{\alpha} \leqslant \beta < 1$, we know that $N : X \to X$ is strictly contractive from (4.11).

Using contractive mapping principle, $N(u)$ has a unique fixed point $u(x,t) \in C^2([0,\infty); H^s)$ on X and $u(x,t)$ is the solution of (1.1).

We claim that the solution $u(x,t)$ of (1.1) is also unique in $C^2([0,\infty); H^s)$.

In fact, let u_1, u_2 be any two solutions of the problem (1.1) and $u_1, u_2 \in C^2([0,\infty); H^s)$. Let $u = u_1 - u_2$, then

$$u_{tt} - \Delta u_{tt} - \Delta u = \Delta\big[f(u_1) - f(u_2)\big].$$

Multiplying the above equation by $(-\Delta)^{-1}u_t$ and integrating with respect to x, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\big\|(-\Delta)^{-\frac{1}{2}}u_t\big\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u\|_{L^2}^2\big] = -\int_{\mathbb{R}^n}\big[f(u_1) - f(u_2)\big]u_t\,\mathrm{d}x.$$

For any $T > 0$, $u_i(x,t) \in C^2([0,\infty); H^s)$, $s > \frac{n}{2}$, Sobolev imbedding theorem implies that $\|u_i(t)\|_{L^\infty} \leqslant C$ for $i = 1, 2$ and $0 \leqslant t \leqslant T$. Thus

$$\begin{aligned}
\left|\int_{\mathbb{R}^n}\big[f(u_1) - f(u_2)\big]u_t\,\mathrm{d}x\right| &\leqslant \big\|f(u_1) - f(u_2)\big\|_{L^2}\|u_t\|_{L^2} \\
&\leqslant \left\|\int_0^1 f'\big[\lambda u_1 + (1-\lambda)u_2\big]\,\mathrm{d}\lambda\,(u_1 - u_2)\right\|_{L^2}\|u_t\|_{L^2} \\
&\leqslant \int_0^1 \big\|f'\big[\lambda u_1 + (1-\lambda)u_2\big]\big\|_{L^\infty}\,\mathrm{d}\lambda\,\|u_1 - u_2\|_{L^2}\|u_t\|_{L^2} \\
&\leqslant C\big(\|u_1\|_{L^\infty}^{\alpha} + \|u_2\|_{L^\infty}^{\alpha}\big)\|u_1 - u_2\|_{L^2}\|u_t\|_{L^2} \\
&\leqslant C(T)\|u\|_{L^2}\|u_t\|_{L^2}.
\end{aligned}$$

It follows that

$$\big\|(-\Delta)^{-\frac{1}{2}}u_t\big\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u\|_{L^2}^2 \leqslant C(T)\int_0^t\big[\big\|u(\tau)\big\|_{L^2}^2 + \big\|u_t(\tau)\big\|_{L^2}^2\big]\,\mathrm{d}\tau.$$

By Gronwall inequality, we have $\|u_t\|_{L^2}^2 + \|u\|_{L^2}^2 \equiv 0$ for $0 \leqslant t \leqslant T$. Hence $u \equiv 0$ for $0 \leqslant t \leqslant T$. □

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