



Summability of formal solutions to the n -dimensional inhomogeneous heat equation

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ABSTRACT

The Cauchy problem for n -dimensional inhomogeneous complex heat equation is considered. The Borel summability of formal solutions is characterised in terms of analytic continuation with an appropriate growth condition of some function connected with the inhomogeneity and the Cauchy data.

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1. Introduction

Recently the theory of the summability of the formal power series solutions to differential equations has been developed. In particular, it was proved that every formal solution to an ordinary differential equation with irregular singular point is multisummable (see B.L.J. Braaksma [5]).

The first result in that direction for partial differential equations was obtained by Lutz, Miyake and Schäfke [7]. They showed that the formal solution to the Cauchy problem for the 1-dimensional homogeneous complex heat equation is 1-summable in a direction θ if and only if the Cauchy data $\varphi(z)$ can be analytically continued to infinity in some sectors in directions $\theta/2$ and $\pi + \theta/2$ and the continuation is of exponential growth of order at most 2. Analogous result for more general initial data was given by W. Balser [2]. The multidimensional homogeneous heat equation was investigated by Balser and Malek [1] and by S. Michalik [9].

On the other hand, similar result to [7] for the inhomogeneous case was obtained by W. Balser [4]. He proved that the formal solution to the Cauchy problem for 1-dimensional inhomogeneous complex heat equation is 1-summable in a direction θ if and only if the inhomogeneity is 1-summable in a direction θ and moreover the function $\tilde{\varphi}(\tau)$ (defined by (17) below), connected with the inhomogeneity and the Cauchy data, is analytically continued to some infinite sectors in directions $\theta/2$ and $\pi + \theta/2$ and the continuation is of exponential growth of order at most 2.

In this article we generalise the result of W. Balser [4] to the higher spatial dimensions as well the result of S. Michalik [9] to the inhomogeneous equation.

Namely, we consider the Cauchy problem for the n -dimensional inhomogeneous complex heat equation

$$\partial_\tau u(\tau, z) - \Delta_z u(\tau, z) = \hat{f}(\tau, z), \quad u(0, z) = \varphi(z), \quad (1)$$

where $\tau \in \mathbb{C}$, $z \in \mathbb{C}^n$, $\Delta_z := \sum_{i=1}^n \partial_{z_i}^2$, $\hat{f}(\tau, z) = \sum_{j=0}^\infty f_j(z) \tau^j$ is a formal power series, $f_j(z)$ and $\varphi(z)$ are analytic in a complex neighbourhood of the origin.

To formulate our result, let us define a function

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$$\hat{g}(\tau, z) := \int_0^\tau \hat{f}(s, z) ds + \varphi(z). \quad (2)$$

Then $\hat{g}(\tau, z) = \sum_{j=0}^\infty g_j(z) \tau^j$, where $g_0(z) = \varphi(z)$, $g_j(z) = \frac{f_{j-1}(z)}{j}$ and the unique formal solution to the Cauchy problem (1) has the form

$$\hat{u}(\tau, z) = \sum_{j=0}^\infty u_j(z) \tau^j \quad \text{with } u_j(z) := \sum_{\substack{m, k \geq 0 \\ m+k=j}} \frac{m! \Delta^k g_m(z)}{j!}, \quad (3)$$

which diverges for a general data.

Our main result can be formulate as follows (for the precise formulation see Corollary 1):

The formal solution (3) to the Cauchy problem (1) is 1-summable in a direction θ if and only if the formal series $\hat{g}(\tau, z)$ defined by (2) is 1-summable in a direction θ and the function $\tilde{\Phi}_n(\tau, z)$ defined by (18) is analytically continued to some infinite sectors in directions $\theta/2$ and $\pi + \theta/2$ (with respect to τ) and to a ball with a centre at origin (with respect to z) and this continuation is of exponential growth of order at most two as $\tau \rightarrow \infty$.

Following W. Balser (see [1,2]) we shall use the modified Borel transform of $\hat{u}(\tau, z)$ instead of its Borel transform. Analogously to [9], it appears that after appropriate change of variables this transform satisfies the wave equation (see Lemma 2 below). In this way one can reduce the investigation of summability to the study of the solution to the wave equation. In fact, one can express the summability of the formal solution to the heat equation in terms of analytic continuation with an appropriate growth condition of the solution to the wave equation.

2. Notation

The complex (resp. real) ball with a centre at $z_0 \in \mathbb{C}^n$ (resp. $x_0 \in \mathbb{R}^n$) and a radius $r > 0$ is denoted by $D^n(z_0, r) := \{z \in \mathbb{C}^n : |z - z_0| < r\}$ (resp. $B^n(x_0, r) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$), where $|\cdot|$ is the Euclidean norm $|z| := \sqrt{z_1 \bar{z}_1 + \dots + z_n \bar{z}_n}$ in \mathbb{C}^n (resp. $|x| := \sqrt{x_1^2 + \dots + x_n^2}$ in \mathbb{R}^n). To simplify notation we write $D(z_0, r)$ (resp. $B(x_0, r)$) for $n = 1$, $D^n(r)$ and $D(r)$ (resp. $B^n(r)$ and $B(r)$) for $z_0 = 0$ (resp. $x_0 = 0$).

The mean values of a function f over a ball $B^n(r)$ and over a sphere $\partial B^n(r)$ are denoted by

$$\oint_{B^n(r)} f(x) dx := \frac{1}{\alpha(n)r^n} \int_{B^n(r)} f(x) dx$$

and

$$\oint_{\partial B^n(r)} f(y) dS(y) := \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B^n(r)} f(y) dS(y),$$

where $\alpha(n) := \frac{\pi^{n/2}}{\Gamma(n/2+1)}$ is the volume of the unit ball $B^n(1)$ and $n\alpha(n)$ is the surface measure of the unit sphere $\partial B^n(1)$.

For $\theta \in \mathbb{R}$, $\varepsilon > 0$ and $\delta > 0$ we set $E_+(\theta, \varepsilon) := \{s \in \mathbb{C} : \text{dist}(s, e^{i\theta}\mathbb{R}_+) < \varepsilon\}$, $E_+^2(\theta, \varepsilon) := \{s \in \mathbb{C} : s = \tau^2, \text{dist}(\tau, e^{i\theta/2}\mathbb{R}) < \sqrt{\varepsilon}\}$, $\Delta(\theta/2, \delta) := \{z \in \mathbb{C} : \text{dist}(z^2, e^{i\theta}\mathbb{R}_+) < \delta^2\}$ and $\Omega(\theta/2, \delta) := \{z \in \mathbb{C} : \text{dist}(z, e^{i\theta/2}\mathbb{R}) < \delta\}$.

A sector in the universal covering space $\tilde{\mathbb{C}}$ of $\mathbb{C} \setminus \{0\}$ is denoted by

$$S(\theta, \alpha, T) := \{z \in \tilde{\mathbb{C}} : z = re^{i\varphi}, \theta - \alpha/2 < \varphi < \theta + \alpha/2, 0 < r < T\}$$

for $\theta \in \mathbb{R}$, $\alpha > 0$ and $0 < T \leq +\infty$. In case $T = +\infty$ we denote it briefly by $S(\theta, \alpha)$. A sector S' is called a *proper subsector* of $S(\theta, \alpha, T)$ if its closure in $\tilde{\mathbb{C}}$ is contained in $S(\theta, \alpha, T)$.

By $\mathcal{O}(D)$ we denote the space of analytic functions on a domain $D \subseteq \mathbb{C}^n$. The Banach space of analytic functions on $D^n(r)$, continuous on its closure and equipped with the norm $\|\varphi\|_r := \max_{|z| \leq r} |\varphi(z)|$ is denoted by $\mathbb{E}_n(r)$.

3. Gevrey formal power series and Borel summability

Following [3] and [7] we recall some fundamental facts about the Borel summability.

Definition 1. We say that a function $u(t, z) \in \mathcal{O}(S(\theta, \varepsilon) \times D^n(r))$ (resp. $u(t, z) \in \mathcal{O}(E_+(\theta, \varepsilon) \times D^n(r))$, $u(t, z) \in \mathcal{O}(E_+^2(\theta, \varepsilon) \times D^n(r))$, $u(t, z) \in \mathcal{O}(\Delta(\theta/2, \varepsilon) \times D^n(r))$, $u(t, z) \in \mathcal{O}(\Omega(\theta/2, \varepsilon) \times D^n(r))$) is of *exponential growth of order at most k as $t \rightarrow \infty$ in $S(\theta, \varepsilon)$ (resp. in $E_+(\theta, \varepsilon)$, in $E_+^2(\theta, \varepsilon)$, in $\Delta(\theta/2, \varepsilon)$, in $\Omega(\theta/2, \varepsilon)$)* if and only if for any $r_1 \in (0, r)$ and any $\varepsilon_1 \in (0, \varepsilon)$ there exist positive constants C and B such that

$$\max_{|z| \leq r_1} |u(t, z)| < C e^{B|t|^k}$$

for every $t \in S(\theta, \varepsilon_1)$ (resp. $t \in E_+(\theta, \varepsilon_1)$, $t \in E_+^2(\theta, \varepsilon_1)$, $t \in \Delta(\theta, \varepsilon_1)$, $t \in \Omega(\theta, \varepsilon_1)$). If $k = 1$, we say for short that $u(t, z)$ is of *exponential growth* as $t \rightarrow \infty$ in $S(\theta, \varepsilon)$ (resp. in $E_+(\theta, \varepsilon)$, in $E_+^2(\theta, \varepsilon)$, in $\Delta(\theta, \varepsilon)$, in $\Omega(\theta, \varepsilon)$).

Analogously, we say that a function $\varphi(z) \in \mathcal{O}(S(\theta, \varepsilon))$ (resp. $\varphi(z) \in \mathcal{O}(\Omega(\theta, \varepsilon))$, $\varphi(z) \in \mathcal{O}(\Delta(\theta, \varepsilon))$) is of *exponential growth of order at most k* as $z \rightarrow \infty$ in $S(\theta, \varepsilon)$ (resp. in $\Omega(\theta, \varepsilon)$, in $\Delta(\theta, \varepsilon)$) if and only if for any $\varepsilon_1 \in (0, \varepsilon)$ there exist positive constants C and B such that

$$|\varphi(z)| < C e^{B|z|^k} \quad \text{for every } z \in S(\theta, \varepsilon_1) \text{ (resp. } z \in \Omega(\theta, \varepsilon_1), z \in \Delta(\theta, \varepsilon_1)).$$

If $k = 1$, we say for short that $\varphi(z)$ is of *exponential growth* as $z \rightarrow \infty$ in $S(\theta, \varepsilon)$ (resp. in $\Omega(\theta, \varepsilon)$, in $\Delta(\theta, \varepsilon)$).

Definition 2. We say that a formal power series

$$\hat{u}(\tau, z) := \sum_{k=0}^{\infty} u_k(z) \tau^k \quad \text{with } u_k(z) \in \mathbb{E}_n(r) \quad (4)$$

is *1-Gevrey formal power series* if its coefficients satisfy

$$\max_{|z| \leq r} |u_k(z)| \leq A B^k k! \quad \text{for } k = 0, 1, \dots$$

with some positive constants A and B .

The set of 1-Gevrey formal power series in τ over $\mathbb{E}_n(r)$ is denoted by $\mathbb{E}_n(r)[[\tau]]_1$. We also set $\mathbb{E}_n[[\tau]]_1 := \bigcup_{r>0} \mathbb{E}_n(r)[[\tau]]_1$.

Definition 3. We say that a formal series $\hat{u}(\tau, z) \in \mathbb{E}_n[[\tau]]_1$ defined by (4) is *1-summable* in $S(\theta, \alpha)$ (for some $\theta \in \mathbb{R}$ and $\alpha > \pi$) if its Borel transform

$$\tilde{v}(s, z) := \sum_{k=0}^{\infty} u_k(z) \frac{s^k}{k!}$$

is analytic in $S(\theta, \alpha - \pi) \times D^n(r)$ (for some $r > 0$) and is of exponential growth as $s \rightarrow \infty$ in $S(\theta, \alpha - \pi)$. The 1-sum of $\hat{u}(\tau, z)$ in $S(\theta, \alpha)$ is represented by the Laplace transform of $\tilde{v}(s, z)$

$$u^\varphi(\tau, z) := \frac{1}{\tau} \int_0^{e^{i\varphi}\infty} e^{-s/\tau} \tilde{v}(s, z) ds,$$

where the integration is taken over any ray $e^{i\varphi}\mathbb{R}_+ := \{re^{i\varphi} : r \geq 0\}$ with $\varphi \in (\theta - \alpha + \pi, \theta + \alpha - \pi)$.

In the crucial case of $\alpha = \pi$, we shall replace 1-summability by *fine 1-summability* (see Section 1.4.2 in [8]).

Definition 4. We say that a formal series $\hat{u}(\tau, z) \in \mathbb{E}_n[[\tau]]_1$ is *fine 1-summable in a direction θ* if and only if its Borel transform $\tilde{v}(s, z)$ belongs to $\mathcal{O}(E_+(\theta, \varepsilon) \times D^n(r))$ (for some $\varepsilon > 0$ and $r > 0$) and is of exponential growth as $s \rightarrow \infty$ in $E_+(\theta, \varepsilon)$. The *fine 1-sum* of $\hat{u}(\tau, z)$ in the direction θ is represented by the Laplace transform of $\tilde{v}(s, z)$ in the direction θ .

Since $1/4^k \leq (k!)^2/(2k)! \leq 1$ for every $k \in \mathbb{N}$, according to the general theory of moment summability (see Section 6.5 in [3]) a formal series $\hat{u}(\tau, z) = \sum u_k(z) \tau^k$ is 1-summable in $S(\theta, \alpha)$ (resp. 1-fine summable in a direction θ) if and only if the same holds for the series

$$\sum u_k(z) \frac{(k!)^2}{(2k)!} \tau^k.$$

Consequently, we obtain a characterisation of 1-summability and of fine 1-summability (analogous to Definitions 3 and 4), if we replace the Borel transform by the *modified Borel transform*

$$v(s, z) := \mathcal{B}\hat{u}(s, z) := \sum_{j=0}^{\infty} u_j(z) \frac{j! s^j}{(2j)!} \quad (5)$$

and the Laplace transform by the *Ecalte acceleration operator*

$$u^\theta(\tau, z) = \frac{1}{\sqrt{\tau}} \int_0^{e^{i\theta}\infty} v(s, z) C_2(\sqrt{s/\tau}) d\sqrt{s}. \quad (6)$$

Here integration is taken over the ray $e^{i\theta}\mathbb{R}_+$ and $C_2(\zeta)$ is defined by

$$C_2(\zeta) := \frac{1}{2\pi i} \int_\gamma \frac{e^{u-\zeta\sqrt{u}}}{\sqrt{u}} du$$

with a path of integration γ as in the Hankel integral for the inverse Gamma function (from ∞ along $\arg u = -\pi$ to some $u_0 < 0$, then on the circle $|u| = |u_0|$ to $\arg u = \pi$, and back to ∞ along this ray).

Hence the fine 1-summability is characterised as follows:

Proposition 1. A formal series $\hat{u}(\tau, z) \in \mathbb{E}_n[[\tau]]_1$ is fine 1-summable in a direction θ if and only if its modified Borel transform $v(s, z)$ belongs to $\mathcal{O}(E_+(\theta, \varepsilon) \times D^n(r))$ (for some $\varepsilon > 0$ and $r > 0$) and is of exponential growth as $s \rightarrow \infty$ in $E_+(\theta, \varepsilon)$, where $E_+(\theta, \varepsilon) = \{s \in \mathbb{C}: \text{dist}(s, e^{i\theta}\mathbb{R}_+) < \varepsilon\}$. The fine 1-sum of $\hat{u}(\tau, z)$ is represented by the Ecalle acceleration operator (6) of $v(s, z)$.

It will be convenient to introduce some special type of fine 1-summable series.

Definition 5. We say that a formal series $\hat{u}(\tau, z) \in \mathbb{E}_n[[\tau]]_1$ is 2-fine 1-summable in a direction θ if and only if its modified Borel transform $v(s, z)$ belongs to $\mathcal{O}(E_+^2(\theta, \varepsilon) \times D^n(r))$ (for some $\varepsilon > 0$ and $r > 0$) and is of exponential growth as $s \rightarrow \infty$ in $E_+^2(\theta, \varepsilon)$, where $E_+^2(\theta, \varepsilon) = \{s \in \mathbb{C}: s = \tau^2, \text{dist}(\tau, e^{i\theta/2}\mathbb{R}) < \sqrt{\varepsilon}\}$.

Observe that $E_+(\theta, \varepsilon) \subseteq E_+^2(\theta, \varepsilon)$, hence every 2-fine 1-summable formal series is also fine 1-summable.

At the end of this section we will establish a Gevrey estimate of the formal solution to the heat equation (1).

Lemma 1. Let $\varphi(z)$ be analytic in a complex neighbourhood of the origin and let $\hat{f}(\tau, z)$ be a 1-Gevrey formal power series. Then the formal solution (3) of the Cauchy problem (1) is 1-Gevrey formal power series. Moreover, if the Cauchy data $\varphi(z) \in \mathcal{O}(D^n(\tilde{r}))$ then for any $r \in (0, \tilde{r})$ the formal solution $\hat{u}(\tau, z) \in \mathbb{E}_n(r)[[\tau]]_1$.

Proof. Take $\tilde{r} > 0$ such that $\varphi(z) \in \mathcal{O}(D^n(\tilde{r}))$ and $\hat{f}(\tau, z) \in \mathbb{E}_n(\tilde{r})[[\tau]]_1$. We need to show that for any $r \in (0, \tilde{r})$ the formal solution $\hat{u}(\tau, z)$ belongs to $\mathbb{E}_n(r)[[\tau]]_1$. To this end take any $0 < r < r_1 < \tilde{r}$ and put $\varepsilon := \frac{r_1-r}{\sqrt{n+1}}$. Observe that for any $z \in D^n(r)$ the set $\{\zeta \in \mathbb{C}^n: |\zeta_i - z_i| = \varepsilon \text{ for } i = 1, \dots, n\}$ is contained in $D^n(r_1)$. Hence, by the Cauchy integral formula, the coefficients $(u_j(z))_{j=0}^\infty$ of the formal solution (3) satisfy

$$\begin{aligned} \max_{|z| \leq r} |u_j(z)| &= \max_{|z| \leq r} \left| \sum_{\substack{m, k \geq 0 \\ m+k=j}} \frac{m!(\partial_{z_1}^2 + \dots + \partial_{z_n}^2)^k g_m(z)}{j!} \right| \\ &\leq \max_{|z| \leq r} \sum_{\substack{m, k \geq 0 \\ m+k=j}} \sum_{\substack{i_1, \dots, i_n \geq 0 \\ i_1 + \dots + i_n = k}} \frac{k!m!}{i_1! \dots i_n! j!} |\partial_{z_1}^{2i_1} \dots \partial_{z_n}^{2i_n} g_m(z)| \\ &\leq \max_{|z| \leq r} \sum_{\substack{m, k \geq 0 \\ m+k=j}} \sum_{\substack{i_1, \dots, i_n \geq 0 \\ i_1 + \dots + i_n = k}} \frac{k!m!(2i_1)! \dots (2i_n)!}{i_1! \dots i_n! j! (2\pi)^n} \left| \int_{|\zeta_1 - z_1| = \varepsilon} \dots \int_{|\zeta_n - z_n| = \varepsilon} \frac{g_m(\zeta)}{(\zeta_1 - z_1)^{2i_1+1} \dots (\zeta_n - z_n)^{2i_n+1}} d\zeta \right| \\ &\leq \sum_{\substack{m, k \geq 0 \\ m+k=j}} \sum_{\substack{i_1, \dots, i_n \geq 0 \\ i_1 + \dots + i_n = k}} \frac{4^k k! m! i_1! \dots i_n!}{\varepsilon^{2k} j!} \max_{|z| \leq r_1} |g_m(z)| \\ &\leq \sum_{\substack{m, k \geq 0 \\ m+k=j}} \left(\frac{4n}{\varepsilon^2} \right)^k \frac{(k!)^2 m!}{j!} \max_{|z| \leq r_1} |g_m(z)| \leq \sum_{\substack{m, k \geq 0 \\ m+k=j}} \left(\frac{4n}{\varepsilon^2} \right)^k \frac{(k!m!)^2}{j!} A \tilde{B}^m \\ &\leq A \left(\max \left\{ 2\tilde{B}, \frac{8n}{\varepsilon^2} \right\} \right)^j j! = AB^j j! \end{aligned}$$

for $j = 0, 1, \dots$, with some positive constants A, B, \tilde{B} satisfying $\max_{|z| \leq r_1} |g_m(z)| \leq A \tilde{B}^m m!$ ($m = 0, 1, \dots$) and $B = \max\{2\tilde{B}, \frac{8n}{\varepsilon^2}\}$. \square

4. Reduction to the wave equation

The next lemma reduces question about summability of $\hat{u}(\tau, z)$ to the investigation of the function $\mathcal{B}\hat{u}(\tau^2, z)$ and shows that this function satisfies the wave equation.

Lemma 2. Let $\hat{u}(\tau, z)$ be a formal solution (3) to the Cauchy problem (1) for the inhomogeneous complex n -dimensional heat equation. Then $w(\tau, z) := \mathcal{B}\hat{u}(\tau^2, z)$ is a solution to the Cauchy problem for the inhomogeneous complex n -dimensional wave equation

$$\partial_\tau^2 w(\tau, z) - \Delta_z w(\tau, z) = \partial_\tau^2 h(\tau, z), \quad w(0, z) = h(0, z), \quad \partial_\tau w(0, z) = \partial_\tau h(0, z) = 0, \quad (7)$$

where

$$h(\tau, z) := \mathcal{B}\hat{g}(\tau^2, z) = \sum_{j=0}^{\infty} \frac{j!g_j(z)}{(2j)!} \tau^{2j}. \quad (8)$$

Moreover, $\hat{u}(\tau, z)$ is fine 1-summable (resp. 2-fine 1-summable) in a direction θ if and only if $w(\tau, z)$ is analytically continued to $\Delta(\theta/2, \delta) \times D^n(r)$ (resp. to $\Omega(\theta/2, \delta) \times D^n(r)$) for some $\delta > 0$, $r > 0$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Delta(\theta/2, \delta)$ (resp. in $\Omega(\theta/2, \delta)$).

Proof. Notice that by (3) and (5)

$$v(s, z) = \sum_{j=0}^{\infty} \frac{j!u_j(z)}{(2j)!} s^j = \sum_{j=0}^{\infty} \sum_{\substack{m,k \geq 0 \\ m+k=j}} \frac{m!\Delta^k g_m(z)}{(2j)!} s^j,$$

hence

$$w(\tau, z) = v(\tau^2, z) = \sum_{j=0}^{\infty} \sum_{\substack{m,k \geq 0 \\ m+k=j}} \frac{m!\Delta^k g_m(z)}{(2j)!} \tau^{2j}.$$

A direct verification shows that $w(\tau, z)$ is a solution to the Cauchy problem for the wave equation (7).

If $\hat{u}(\tau, z)$ is fine 1-summable (resp. 2-fine 1-summable) in a direction θ then $v(s, z) = \mathcal{B}\hat{u}(s, z)$ is analytic on $E_+(\theta, \delta^2) \times D^n(r)$ (resp. on $E_+^2(\theta, \delta^2) \times D^n(r)$) for some $\delta > 0$, $r > 0$ and is of exponential growth as $s \rightarrow \infty$ in $E_+(\theta, \delta^2)$ (resp. in $E_+^2(\theta, \delta^2)$). Hence the function $w(\tau, z) = v(\tau^2, z)$ is analytic on $\Delta(\theta/2, \delta) \times D^n(r)$ (resp. on $\Omega(\theta/2, \delta) \times D^n(r)$) and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Delta(\theta/2, \delta)$ (resp. in $\Omega(\theta/2, \delta)$).

Now, let us suppose that the function $w(\tau, z) = v(\tau^2, z)$ is analytic on $\Delta(\theta/2, \delta) \times D^n(r)$ (resp. on $\Omega(\theta/2, \delta) \times D^n(r)$) and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Delta(\theta/2, \delta)$ (resp. in $\Omega(\theta/2, \delta)$). In particular, for any $z \in D^n(r)$ the function $\tau \mapsto w(\tau, z)$ is analytic on $D(\delta)$. Hence, by the Cauchy–Hadamard formula,

$$\frac{1}{\limsup_{k \rightarrow \infty} \sqrt[2k]{\frac{|\Delta^k \varphi(z)|}{(2k)!}}} \geq r \quad \text{and consequently} \quad \frac{1}{\limsup_{k \rightarrow \infty} \sqrt[k]{\frac{|\Delta^k \varphi(z)|}{(2k)!}}} \geq r^2,$$

which implies that the function $s \mapsto v(s, z)$ is analytic on $D(\delta^2)$.

On the other hand, since $w(\tau, z)$ is even with respect to τ , $v(s, z) = w(\sqrt{s}, z) = w(-\sqrt{s}, z)$. Therefore $v(s, z)$ is also analytic on $(E_+(\theta, \delta^2) \cap S(\theta, \pi)) \times D^n(r)$ (resp. on $(E_+^2(\theta, \delta^2) \cap S(\theta, \pi)) \times D^n(r)$). Finally, $v(s, z)$ is analytic on $E_+(\theta, \delta^2) \times D^n(r)$ (resp. on $E_+^2(\theta, \delta^2) \times D^n(r)$) and is of exponential growth as $s \rightarrow \infty$ in $E_+(\theta, \delta^2)$ (resp. in $E_+^2(\theta, \delta^2)$). It means that $\hat{u}(\tau, z)$ is fine 1-summable (resp. 2-fine 1-summable) in a direction θ . \square

In the proof of the main result we shall need the following auxiliary lemmas.

Lemma 3. Assume that a C^2 function $H(\tau, z)$ (for $\tau, z \in \mathbb{C}$) is even with respect to τ and let

$$\Phi(\tau) := \partial_\tau \int_0^\tau H(\tau - y, y) dy.$$

Furthermore, suppose that a function $W(\tau, z)$ is a solution to the Cauchy problem for the complex 1-dimensional inhomogeneous wave equation

$$\partial_\tau^2 W(\tau, z) - \partial_z^2 W(\tau, z) = \partial_\tau^2 H(\tau, z), \quad W(0, z) = H(0, z), \quad \partial_\tau W(0, z) = 0. \quad (9)$$

Then

$$W(\tau, z) = \frac{1}{2} \left[\Phi(\tau + z) + \Phi(-\tau + z) + \int_0^z (\partial_\tau H(\tau - z + y, y) - \partial_\tau H(\tau + z - y, y)) dy \right], \quad (10)$$

$$\partial_z W(\tau, z) = \partial_\tau \frac{1}{2} \left[\Phi(\tau + z) - \Phi(-\tau + z) - \int_0^z (\partial_\tau H(\tau - z + y, y) + \partial_\tau H(\tau + z - y, y)) dy \right] \quad (11)$$

and

$$\Phi(\tau + z) = W(\tau, z) + \int_0^\tau \partial_z W(t, z) dt + \int_0^z \partial_\tau H(\tau + z - y, y) dy. \quad (12)$$

Proof. By the d'Alembert formula and the Duhamel principle we have a solution to (9)

$$W(\tau, z) = \frac{1}{2} \left[H(0, z + \tau) + H(0, z - \tau) + \int_0^\tau \int_{z-\tau+s}^{z+\tau-s} \partial_s^2 H(s, y) dy ds \right].$$

On the other hand

$$\begin{aligned} & \frac{1}{2} [\Phi(\tau + z) + \Phi(-\tau + z)] \\ &= \frac{1}{2} \left[H(0, \tau + z) + H(0, -\tau + z) + \int_0^{\tau+z} \partial_\tau H(\tau + z - y, y) dy - \int_0^{-\tau+z} \partial_\tau H(-\tau + z - y, y) dy \right]. \end{aligned}$$

Hence, to see the formula (10), it is sufficient to show that

$$L := \int_0^\tau \int_{z-\tau+s}^{z+\tau-s} \partial_s^2 H(s, y) dy ds$$

is equal to

$$R := \int_0^{\tau+z} \partial_\tau H(\tau + z - y, y) dy - \int_0^{-\tau+z} \partial_\tau H(-\tau + z - y, y) dy + \int_0^z (\partial_\tau H(\tau - z + y, y) - \partial_\tau H(\tau + z - y, y)) dy.$$

To this end let us denote by D the set $\{(s, y) \in \mathbb{C}^2: s \in [0, \tau], y \in [-\tau + z + s, \tau + z - s]\}$. Since $H(\tau, z)$ is even with respect to τ and by the Fubini theorem we have

$$\begin{aligned} L &= \iint_D \partial_s^2 H(s, y) dy ds \\ &= \int_{-\tau+z}^z \int_0^{\tau-z+y} \partial_s^2 H(s, y) ds dy + \int_z^{\tau+z} \int_0^{\tau+z-y} \partial_s^2 H(s, y) ds dy \\ &= \int_{-\tau+z}^z \partial_\tau H(\tau - z + y, y) dy + \int_z^{\tau+z} \partial_\tau H(\tau + z - y, y) dy \\ &= \int_{-\tau+z}^0 \partial_\tau H(-\tau + z - y, y) dy + \int_0^z \partial_\tau H(\tau - z + y, y) dy + \int_0^{\tau+z} \partial_\tau H(\tau + z - y, y) dy - \int_0^z \partial_\tau H(\tau + z - y, y) dy = R. \end{aligned}$$

Differentiating (10) with respect to z we get (11). Finally, combining (10) and (11), we obtain (12). \square

Using spherical means and the Duhamel principle (see Evans [6]) we can reduce (7) to the 1-dimensional case as follows.

Lemma 4. Let $w(\tau, z)$ will be a solution of the n -dimensional Cauchy problem (7).
If $n = 2k + 1$ ($n \geq 3$) then

$$\tilde{w}(\tau, \rho, z) := (\rho^{-1} \partial \rho)^{k-1} \rho^{2k-1} \oint_{\partial B^n(1)} w(\tau, z + \rho x) dS(x) \quad (13)$$

satisfies

$$\begin{cases} \partial_\tau^2 \tilde{w}(\tau, \rho, z) - \partial_\rho^2 \tilde{w}(\tau, \rho, z) = (\rho^{-1} \partial \rho)^{k-1} \rho^{2k-1} \oint_{\partial B^n(1)} \partial_\tau^2 h(\tau, z + \rho x) dS(x), \\ \tilde{w}(0, \rho, z) = (\rho^{-1} \partial \rho)^{k-1} \rho^{2k-1} \oint_{\partial B^n(1)} h(0, z + \rho x) dS(x), \\ \partial_\tau \tilde{w}(0, \rho, z) = 0. \end{cases}$$

If $n = 2k$ ($n \geq 2$) then

$$\bar{w}(\tau, \rho, z) := (\rho^{-1} \partial \rho)^{k-1} \rho^{2k-1} \oint_{B^n(1)} \frac{w(\tau, z + \rho x)}{\sqrt{1 - |x|^2}} dx \quad (14)$$

satisfies

$$\begin{cases} \partial_\tau^2 \bar{w}(\tau, \rho, z) - \partial_\rho^2 \bar{w}(\tau, \rho, z) = (\rho^{-1} \partial \rho)^{k-1} \rho^{2k-1} \oint_{B^n(1)} \frac{\partial_\tau^2 h(\tau, z + \rho x)}{\sqrt{1 - |x|^2}} dx, \\ \bar{w}(0, \rho, z) = (\rho^{-1} \partial \rho)^{k-1} \rho^{2k-1} \oint_{B^n(1)} \frac{h(0, z + \rho x)}{\sqrt{1 - |x|^2}} dx, \\ \partial_\tau \bar{w}(0, \rho, z) = 0. \end{cases}$$

In Lemma 2 the 2-fine 1-summability of $\hat{u}(\tau, z)$ has been expressed by some assumptions $w(\tau, z)$. In the next main lemma we will show that these assumptions can be formulated in terms of inhomogeneity and the Cauchy data of the wave equation (7).

Lemma 5. The solution $w(\tau, z)$ to the Cauchy problem (7) for the inhomogeneous complex n -dimensional wave equation is analytically continued to $\Omega(\theta/2, \delta) \times D^n(r)$ (for some $\delta > 0$ and $r > 0$) and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta)$ if and only if

1. the function $h(\tau, z)$ defined by (8) is analytically continued to $\Omega(\theta/2, \delta') \times D^n(r')$ (for some $\delta' > 0$ and $r' > 0$) and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta')$;
2. there exists $\tilde{\delta} > 0$ such that:
 - for $n = 1$ the function

$$\tilde{\varphi}(\tau) := \partial_\tau \int_0^\tau h(\tau - s, s) ds \quad (15)$$

is analytically continued to $\Omega(\theta/2, \tilde{\delta})$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \tilde{\delta})$,

- for $n > 1$ the function

$$\tilde{\Phi}_n(\tau, z) := \begin{cases} \partial_\tau \int_0^\tau (s^{-1} \partial_s)^{k-1} s^{2k-1} \oint_{\partial B^n(1)} h(\tau - s, z + sx) dS(x) ds & \text{for } n = 2k + 1, \\ \partial_\tau \int_0^\tau (s^{-1} \partial_s)^{k-1} s^{2k-1} \oint_{B^n(1)} \frac{h(\tau - s, z + sx)}{\sqrt{1 - |x|^2}} dx ds & \text{for } n = 2k \end{cases} \quad (16)$$

is analytically continued to $\Omega(\theta/2, \tilde{\delta}) \times D^n(\tilde{r})$ (for some $\tilde{r} > 0$) and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \tilde{\delta})$.

Proof.

- For $n = 1$.

(\Leftarrow) Observe, that $w(\tau, z)$ satisfies the initial problem

$$\begin{cases} \partial_z^2 w(\tau, z) - \partial_\tau^2 w(\tau, z) = -\partial_\tau^2 h(\tau, z), \\ w(\tau, 0) = \psi_0(\tau), \\ \partial_z w(\tau, 0) = \psi_1(\tau), \end{cases}$$

where, by Lemma 3, $\psi_0(\tau) = \frac{1}{2}[\tilde{\varphi}(\tau) + \tilde{\varphi}(-\tau)]$, $\psi_1(\tau) = \frac{1}{2}[\tilde{\varphi}'(\tau) + \tilde{\varphi}'(-\tau)]$ and, by the assumption, $\tilde{\varphi}(\tau)$ is analytically continued to $\Omega(\theta/2, \tilde{\delta})$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \tilde{\delta})$. Therefore the Cauchy data $\psi_0(\tau)$ and $\psi_1(\tau)$ also satisfy the same conditions. By the d'Alembert formula and the Duhamel principle we have

$$\begin{aligned} w(\tau, z) &= \frac{1}{2} \left[\psi_0(\tau + z) + \psi_0(\tau - z) + \int_{\tau-z}^{\tau+z} \psi_1(y) dy - \int_0^z \int_{\tau-z+y}^{\tau+z-y} \partial_s^2 h(s, y) ds dy \right] \\ &= \frac{1}{2} \left[\psi_0(\tau + z) + \psi_0(\tau - z) + \int_{\tau-z}^{\tau+z} \psi_1(y) dy - \int_0^z \partial_\tau h(\tau + z - y, y) dy + \int_0^z \partial_\tau h(\tau - z + y, y) dy \right]. \end{aligned}$$

In addition, since $h(\tau, z)$ is analytically continued to $\Omega(\theta/2, \delta') \times D(r')$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta')$, $w(\tau, z)$ is analytically continued to $\Omega(\theta/2, \delta) \times D(r)$, where $r \in (0, \min\{\tilde{\delta}, \delta', r'\})$ and $\delta := \min\{\tilde{\delta} - r, \delta' - r\}$. Moreover, $w(\tau, z)$ is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta)$.

(\Rightarrow) We have

$$h(\tau, z) = h(0, z) + \int_0^\tau \int_0^t \partial_s^2 h(s, z) ds dt.$$

On the other hand $\partial_\tau^2 h(\tau, z) = \partial_\tau^2 w(\tau, z) - \partial_z^2 w(\tau, z)$ and $h(0, z) = w(0, z)$. Therefore, similarly to $w(\tau, z)$, $h(\tau, z)$ is analytically continued to $\Omega(\theta/2, \delta) \times D(r)$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta)$.

Moreover, by Lemma 3, $\tilde{\varphi}(\tau) = w(\tau, 0) + \int_0^\tau \partial_z w(t, 0) dt$. Thus $\tilde{\varphi}(\tau)$ is analytically continued to $\Omega(\theta/2, \delta)$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta)$.

- For $n > 1$.

Using Lemma 4 we shall reduce the dimension of the wave equation. To this end fix $z \in D^n(r)$ and define

$$W_n(\tau, \rho) := \begin{cases} \tilde{w}(\tau, \rho, z) & \text{for } n = 2k + 1, \\ \bar{w}(\tau, \rho, z) & \text{for } n = 2k, \end{cases}$$

where $\tilde{w}(\tau, \rho, z)$ and $\bar{w}(\tau, \rho, z)$ are given by (13) and (14), respectively. Observe that, in particular, $W_n(\tau, 0) = 0$. Similarly we define

$$H_n(\tau, \rho) := \begin{cases} (\rho^{-1} \partial_\rho)^{k-1} \rho^{2k-1} \int_{\partial B^n(1)} h(\tau, z + \rho x) dS(x) ds & \text{for } n = 2k + 1, \\ (\rho^{-1} \partial_\rho)^{k-1} \rho^{2k-1} \int_{B^n(1)} \frac{h(\tau, z + \rho x)}{\sqrt{1 - |x|^2}} dx ds & \text{for } n = 2k. \end{cases}$$

By Lemma 4, $W_n(\tau, \rho)$ satisfies the 1-dimensional wave equation

$$\partial_\tau^2 W_n(\tau, \rho) - \partial_\rho^2 W_n(\tau, \rho) = \partial_\tau^2 H_n(\tau, \rho), \quad W_n(0, \rho) = H_n(0, \rho), \quad \partial_\tau W_n(0, \rho) = 0.$$

Since $h(\tau, z)$ is even with respect to τ , $H_n(\tau, \rho)$ has the same property. Hence, by Lemma 3, we have

$$0 = W_n(\tau, 0) = \frac{1}{2} [\tilde{\Phi}_n(\tau, z) + \tilde{\Phi}_n(-\tau, z)]$$

and consequently

$$w(\tau, z) = \partial_\rho W(\tau, 0) = \partial_\tau \frac{1}{2} [\tilde{\Phi}_n(\tau, z) - \tilde{\Phi}_n(-\tau, z)] = \partial_\tau \tilde{\Phi}_n(\tau, z)$$

or equivalently

$$\tilde{\Phi}_n(\tau, z) = \int_0^\tau w(t, z) dt.$$

It means that $\tilde{\Phi}_n(\tau, z)$ is analytically continued to $\Omega(\theta/2, \delta) \times D^n(r)$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta)$ if and only if $w(\tau, z)$ satisfies the same conditions.

Analogously to 1-dimensional case one can show that if $w(\tau, z)$ is analytically continued to $\Omega(\theta/2, \delta) \times D^n(r)$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta)$ then also $h(\tau, z)$ satisfies these conditions. \square

To obtain the analogous result to Lemma 5 for fine 1-summability we shall improve this lemma as follows.

Lemma 6. Let us suppose that the function $h(\tau, z)$ defined by (8) is analytically continued to $\Omega(\theta/2, \varepsilon') \times D^n(r')$ (for some $\varepsilon' > 0$ and $r' > 0$) and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \varepsilon')$. Then the solution $w(\tau, z)$ to the Cauchy problem (7) for the inhomogeneous complex n -dimensional wave equation is analytically continued to $\Delta(\theta/2, \varepsilon) \times D^n(r)$ (for some $\varepsilon > 0$ and $r > 0$) and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Delta(\theta/2, \varepsilon)$ if and only if there exists $\delta > 0$ such that:

- for $n = 1$ the function $\tilde{\varphi}(\tau)$ given by (15) is analytically continued to $\Omega(\theta/2, \delta)$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta)$;
- for $n > 1$ and any $\tilde{\delta} \in (0, \delta)$ the function $\tilde{\Phi}_n(\tau, z)$ given by (16) is analytically continued to $\Omega(\theta/2, \delta - \tilde{\delta}) \times D^n(\tilde{\delta})$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta - \tilde{\delta})$.

Proof. (\Leftarrow) Since $\Delta(\theta/2, \varepsilon) \subseteq \Omega(\theta/2, \varepsilon)$, by Lemma 5 we obtain our claim.

(\Rightarrow) By the assumption, $w(\tau, z)$ is analytic on $\Delta(\theta/2, \varepsilon) \times D^n(r)$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Delta(\theta/2, \varepsilon)$. Similarly, $h(\tau, z)$ is analytically continued to $\Omega(\theta/2, \varepsilon') \times D^n(r')$ and is of exponential growth of order at most 2. Now we shall consider two cases:

- For $n = 1$.

Fix $\delta \in (0, \min\{r, r', \varepsilon'\})$ and $z_0 \in D(\delta)$. By Lemma 3

$$\tilde{\varphi}(\tau + z_0) = w(\tau, z_0) + \int_0^\tau \partial_z w(t, z_0) dt + \int_0^{z_0} \partial_\tau h(\tau + z_0 - y, y) dy.$$

Thus $\tilde{\varphi}(\tau)$ is analytically continued to $-z_0 + \Delta(\theta/2, \tilde{\varepsilon})$, where $\tilde{\varepsilon} := \min\{\varepsilon, \varepsilon' - \delta\}$. Changing $z_0 \in D(\delta)$, we see that $\tilde{\varphi}(\tau)$ is analytically continued to the domain $\bigcup_{|z_0| < \delta} (-z_0 + \Delta(\theta/2, \tilde{\varepsilon}))$, which contains $\Omega(\theta/2, \delta)$. Moreover, $\tilde{\varphi}(\tau)$ is of exponential growth of order at most 2.

- For $n > 1$.

Let $\tilde{\delta}$ and δ satisfy $0 < \tilde{\delta} < \delta < \min\{r, r'\}$. Fix $z \in D^n(\tilde{\delta})$. Now, as in the proof of Lemma 5, we can define $W_n(\tau, \rho)$ and $H_n(\tau, \rho)$. By the assumption, $W_n(\tau, \rho)$ is analytic on $\Delta(\theta/2, \varepsilon) \times D(\delta - \tilde{\delta})$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Delta(\theta/2, \varepsilon)$. Similarly $H_n(\tau, \rho)$ is analytic on $\Omega(\theta/2, \varepsilon') \times D(\delta - \tilde{\delta})$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \varepsilon')$. Fix $\rho_0 \in D(\delta - \tilde{\delta})$. By Lemma 3

$$\tilde{\Phi}_n(\tau + \rho_0, z) = W_n(\tau, \rho_0) + \int_0^\tau \partial_\rho W_n(t, \rho_0) dt + \int_0^{\rho_0} \partial_\tau H_n(\tau + \rho_0 - y, y) dy.$$

Therefore $\tilde{\Phi}_n(\tau, z)$ is analytically continued to $(-\rho_0 + \Delta(\theta/2, \tilde{\varepsilon})) \times D^n(\tilde{\delta})$, where $\tilde{\varepsilon}$ is defined as in the 1-dimensional case. Changing $\rho_0 \in D(\delta - \tilde{\delta})$, we see that $\tilde{\Phi}_n(t, z)$ is analytically continued to the domain

$$\bigcup_{\rho_0 \in D(\delta - \tilde{\delta})} (-\rho_0 + \Delta(\theta/2, \tilde{\varepsilon})) \times D^n(\tilde{\delta}),$$

which contains $\Omega(\theta/2, \delta - \tilde{\delta}) \times D^n(\tilde{\delta})$. Moreover, $\tilde{\Phi}_n(\tau, z)$ is of exponential growth of order at most 2 as $\tau \rightarrow \infty$. \square

5. The main results

Combining Lemmas 2 and 5 we obtain

Theorem 1. The formal solution $\hat{u}(\tau, z)$ to the inhomogeneous heat equation given by (3) is 2-fine 1-summable in a direction θ if and only if

1. the formal series $\hat{g}(\tau, z)$ defined by (2) is 2-fine 1-summable in a direction θ ;
2. there exists $\delta > 0$ such that:
 - for $n = 1$ the function

$$\tilde{\varphi}(\tau) := \partial_\tau \int_0^\tau \mathcal{B}\hat{g}((\tau - s)^2, s) ds \tag{17}$$

is analytically continued to $\Omega(\theta/2, \delta)$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta)$;

- for $n > 1$ the function

$$\tilde{\Phi}_n(\tau, z) := \begin{cases} \partial_\tau \int_0^\tau (s^{-1} \partial_s)^{k-1} s^{2k-1} f_{\partial B^n(1)} \mathcal{B}\hat{g}((\tau-s)^2, z+sx) dS(x) ds & \text{for } n = 2k+1, \\ \partial_\tau \int_0^\tau (s^{-1} \partial_s)^{k-1} s^{2k-1} f_{B^n(1)} \frac{\mathcal{B}\hat{g}((\tau-s)^2, z+sx)}{\sqrt{1-|x|^2}} dx ds & \text{for } n = 2k \end{cases} \quad (18)$$

is analytically continued to $\Omega(\theta/2, \delta) \times D^n(r)$ (for some $r > 0$) and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta)$.

Analogously, combining Lemmas 2 and 6 we have

Theorem 2. Let us suppose that the formal series $\hat{g}(\tau, z)$ defined by (2) is 2-fine 1-summable in a direction θ . Then the formal solution $\hat{u}(\tau, z)$ to the inhomogeneous heat equation given by (3) is fine 1-summable in a direction θ if and only if there exists $\delta > 0$ such that:

- for $n = 1$, the function $\tilde{\varphi}(\tau)$ given by (17) is analytically continued to $\Omega(\theta/2, \delta)$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta)$;
- for $n > 1$ and any $\tilde{\delta} \in (0, \delta)$ the function $\tilde{\Phi}_n(\tau, z)$ given by (18) is analytically continued to $\Omega(\theta/2, \delta - \tilde{\delta}) \times D^n(\tilde{\delta})$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\Omega(\theta/2, \delta - \tilde{\delta})$.

As a corollary to Theorem 1 we can characterise 1-summability of the formal solution (3) (see the proof of Theorem 3.2 in [7]).

Corollary 1. The formal solution $\hat{u}(\tau, z)$ to the inhomogeneous heat equation given by (3) is 1-summable in a sector $S(\theta, \alpha)$ if and only if

1. the formal series $\hat{g}(\tau, z)$ defined by (2) is 1-summable in a sector $S(\theta, \alpha)$;
2. • for $n = 1$, the function $\tilde{\varphi}(\tau)$ defined by (17) is analytically continued to a double sector $\tilde{S}(\theta, \alpha) := S(\theta/2, (\alpha - \pi)/2) \cup S(\pi + \theta/2, (\alpha - \pi)/2)$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\tilde{S}(\theta, \alpha)$,
 • for $n > 1$, the function $\tilde{\Phi}_n(\tau, z)$ defined by (18) is analytically continued to $\tilde{S}(\theta, \alpha) \times D^n(\delta)$ for some $\delta > 0$ and is of exponential growth of order at most 2 as $\tau \rightarrow \infty$ in $\tilde{S}(\theta, \alpha)$.

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