Skew exactness and range-kernel orthogonality II

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The Fuglede property extends to *-hyponormal Banach algebra elements, and certain Banach algebra elements are approximated by self-commutators.

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Introduction

In this note we show that the *-hyponormal operators of Kirsti Mattila have the Fuglede property of Victor Shulman, and make an extension to Banach algebra elements of a self-commutator approximation result of Duggal and Maher: unlike them we have no need to assume that the Palmer subspace is closed under multiplication.

Suppose throughout that A is a complex Banach algebra, with identity 1, and dual space $A^\dagger$: then the *numerical range* $V_A : A \to 2^C$ is given by

$$V_A(a) = \{\varphi(a): \ \varphi \in \text{State}(A)\} \quad (a \in A), \tag{0.1}$$

where

$$\text{State}(A) = \{\varphi \in A^\dagger: \ \|\varphi\| = 1 = \varphi(1)\}; \tag{0.2}$$

now the *hermitian* elements of A are given by

$$\text{Re}(A) = \{a \in A: \ V_A(a) \subseteq \mathbf{R}\} = \{a \in A: \ \forall t \in \mathbf{R}, \ \|e^{ita}\| = 1\}. \tag{0.3}$$

Here the equivalence [1, Lemma 5.2] is Vidav's lemma. The "Palmer subspace"

$$\text{Reim}(A) = \text{Re}(A) + i\,\text{Re}(A), \tag{0.4}$$

is a closed complex-linear subspace of A, not necessarily closed under multiplication, and the condition

$$\text{Reim}(A) = A \tag{0.5}$$

characterizes C^* algebras [7]. Since [1, Lemma 5.7]

$$\mathrm{Re}(A) \cap i\,\mathrm{Re}(A) = \{0\}, \tag{0.6}$$

the "real and imaginary parts" of elements of Reim(A) are well-defined, and hence we can successfully define ([1, Lemma 5.8], [4]) a mapping $a \mapsto a^*$ on Reim(A) by setting

$$(h + ik)^* = h - ik, \quad h, k \in \mathrm{Re}(A). \tag{0.7}$$

We claim that the Palmer subspace is actually a Lie subalgebra of A, relative to the usual bracket

$$[a, b] = ab - ba \quad (a, b \in A). \tag{0.8}$$

Theorem 1. *If A is arbitrary then*

$$\big[\mathrm{Reim}(A), \mathrm{Reim}(A)\big] \subseteq \mathrm{Reim}(A), \tag{1.1}$$

and for arbitrary $a, b \in \mathrm{Reim}(A)$

$$[a, b]^* = [b^*, a^*]. \tag{1.2}$$

Proof. This all follows easily from inclusion [1, Lemma 5.4]

$$\big[\mathrm{Re}(A), \mathrm{Re}(A)\big] \subseteq i\,\mathrm{Re}(A). \quad \square \tag{1.3}$$

Self-adjoint operators on Banach spaces have a certain "range kernel orthogonality": we write for subspaces $E, F \subseteq X$

$$E \perp F \iff \forall x \in E, \ \|x\| \leqslant \mathrm{dist}(x, F), \tag{1.4}$$

and [4] for operators $S, T \in B(X)$

$$S \perp T \iff S^{-1}(0) \perp T(X). \tag{1.5}$$

When (1.5) holds with $S = T$ we shall describe the operator $T \in B(X)$ as *self-orthogonal*. It is Sinclair's theorem [9, Proposition 1] that T is self-orthogonal if zero is not in the interior of its numerical range:

$$0 \notin \mathrm{int}\,V_{B(X)}(T) \implies T \perp T; \tag{1.6}$$

this at once applies to hermitian T. We shall describe $T \in \mathrm{Reim}\,B(X)$ as "Fuglede" ([8]; [4, Definition 6]) provided

$$T^{-1}(0) \subseteq T^{*-1}(0); \tag{1.7}$$

now a nice proof of Sinclair's theorem is given [3, Corollary 7] by Fong, who uses the same argument to show [3, Lemma 3, Theorem A] that normal operators are both self-orthogonal and Fuglede:

$$T \ \mathrm{normal} \implies T^{-1}(0) \subseteq T^{*-1}(0) \perp T(X). \tag{1.8}$$

Here $a \in A$ is said to be normal iff $a = h + ik$ with mutually commuting $h, k \in \mathrm{Re}(A)$; equivalently a and a^* commute. Since "generalized inner derivations" $L_a - R_b \in B(A)$ induced by hermitian or normal Banach algebra elements $a, b \in A$ become hermitian or normal operators on the underlying Banach space, a "Putnam–Fuglede theorem" for normal Banach algebra elements follows. More generally, if A and B are Banach algebras, and M is a "Banach (left A, right B) bimodule," in the sense [4] of a Banach space on which left and right multiplications by elements of A and B act as bounded operators, then to $a \in A$ and $b \in B$ we can associate the generalized derivation

$$L_a - R_b : x \mapsto ax - xb \quad (M \to M). \tag{1.9}$$

The self-orthogonality of $T = L_a - R_a$ on $M = A$ is [5] in some sense a quantitative version of the Wielandt–Wintner result that no commutator of bounded operators can ever be the identity, consigning all discussion of Heisenberg uncertainty to the realm of unbounded operators.

We have been unable to extend (1.8) to "hyponormal" elements, but Kirsti Mattila has extended the first part to what she calls "*-hyponormal" operators:

Definition 2. $a \in \mathrm{Reim}(A)$ is said to be positive if it has positive numerical range,

$$V_A(a) \subseteq [0, \infty), \tag{2.1}$$

hyponormal if it has positive self-commutator,

$$V_A(a^*a - aa^*) \subseteq [0, \infty),$$ (2.2)

and *-hyponormal if

$$\left\| e^{za^*} e^{-\bar{z}a} \right\| \leqslant 1 \quad \text{on } \mathbf{C}.$$ (2.3)

Here we write $z : \mathbf{C} \to \mathbf{C}$ for the complex coordinate, or identity mapping. In this note we claim that *-hyponormal Banach algebra elements induce derivations with the Fuglede property, which in turn leads to approximation by self-commutators. As a piece of book keeping, we record

Theorem 3. *If* $a \in \text{Reim}(A)$ *there is implication*

$$a \text{ *-hyponormal} \quad \Longrightarrow \quad a \text{ hyponormal.}$$ (3.1)

Proof. This is just the argument of Mattila [6, Proposition 2] from the case $A = B(X)$. If $a \in A$ is *-hyponormal then, by (2.2),

$$1 \geqslant \left\| e^{za^*} e^{-\bar{z}a} e^{-za^*} e^{\bar{z}a} \right\| = \left\| 1 - |z|^2 (a^*a - aa^*) \right\| + O\left(|z|^3\right) \quad \text{on } \mathbf{D} = \left\{ |z| \leqslant 1 \right\}.$$

If $\varphi \in A^\dagger$ with $\|\varphi\| = \varphi(1) = 1$ then

$$\left| 1 - |z|^2 \varphi(a^*a - aa^*) \right| = O\left(|z|^3\right) \quad \text{on } \mathbf{D}.$$

By (1.3) $\varphi(a^*a - aa^*) \in \mathbf{R}$ and hence

$$-\varphi(a^*a - aa^*) = O(t), \quad 0 < t \leqslant 1.$$ (3.2)

This means $\varphi(a^*a - aa^*) \geqslant 0$, giving (3.1). □

Specialising to $A = B(X)$, *-hyponormal operators have [6, Theorem 3] the Fuglede property (1.7): if $T \in B(X)$ then

$$T \text{ *-hyponormal} \quad \Longrightarrow \quad T \text{ Fuglede.}$$ (3.3)

Like hyponormality [4, (10.3)], *-hyponormality is transmitted to multiplication operators:

Theorem 4. *If* M *is a Banach* (A, B) *bimodule, and if* $a \in \text{Reim}(A)$ *and* $b \in \text{Reim}(B)$*, then*

$$a, b^* \text{ *-hyponormal} \quad \Longrightarrow \quad L_a - R_b \text{ *-hyponormal,}$$ (4.1)

and hence for arbitrary $m \in M$ *there is implication*

$$am = mb \quad \Longrightarrow \quad a^*m = mb^*.$$ (4.2)

Proof. Observe

$$e^{z(L_a - R_b)^*} e^{-\bar{z}(L_a - R_b)} \equiv L_{e^{za^*} e^{-\bar{z}a}} R_{e^{\bar{z}b} e^{-zb^*}},$$

giving (4.1). This with (3.2) gives (4.2). □

Specialising to the case $A = B = M$ and $a = b$, the Fuglede property for $L_a - R_a$ gives, if not the orthogonality we were looking for, another kind of approximability:

Theorem 5. *If the derivation induced by* $a \in \text{Reim}(A)$ *has the Fuglede property,*

$$(L_a - R_a)^{-1}(0) \subseteq (L_{a^*} - R_{a^*})^{-1}(0),$$ (5.1)

then for arbitrary $b \in \text{Reim}(A)$ *there is implication*

$$(L_a - R_a)(b) = 0 \quad \Longrightarrow \quad \|a\| \leqslant \left\| a - [b^*, b] \right\|.$$ (5.2)

Proof. The trick [2] is to note

$$(L_a - R_a)(b) = 0 \quad \Longleftrightarrow \quad (L_b - R_b)(a) = 0.$$

From Lemma 1, (5.1) and the left-hand side of (5.2) we have

$$(L_b - R_b)(a) = (L_{b^*} - R_{b^*})(a) = 0$$

and hence, with $b = h + ik$ with $h, k \in \mathrm{Re}(A)$,

$$(L_h - R_h)(a) = (L_k - R_k)(a) = 0.$$

By either Sinclair's theorem (1.6) or Fong's theorem (1.8) it follows

$$\forall x \in A, \quad \|a\| \leqslant \|a - [h, x]\|;$$

taking in particular $x = 2ik$ gives (5.2). $\square$

Theorem 5 has been obtained by Maher [5] for normal $a \in A = B(X)$ for a Hilbert space X, and extended by Duggal [2, Theorem 2.1] to a Banach space X, under the assumption that the subspace $\mathrm{Reim}(A)$ is closed under multiplication. The same argument extends to the situation ([5, Theorem 4.1, 4.2], [2, Theorem 2.3]) in which $J \subseteq A$ is a two-sided ideal and also a continuously embedded Banach space: if $a \in J \cap \mathrm{Reim}(A)$ and if $b \in A$ with $[b^*, b] \in J$ then the implication (5.2) continues to hold, but in the J norm.

If we make the Duggal assumption

$$\mathrm{Re}(A)^2 \subseteq \mathrm{Reim}(A) \tag{5.3}$$

that $\mathrm{Reim}(A)$ is a subalgebra of A, then equivalently [1, Theorem 5.3]

$$h, k \in \mathrm{Re}(A) \quad \Longrightarrow \quad hk + kh \in \mathrm{Re}(A), \tag{5.4}$$

which together with (1.3) gives

$$(ba)^* = a^* b^* \quad \left(a, b \in \mathrm{Reim}(A)\right). \tag{5.5}$$

Now (cf. [2, Theorem 2.6]) we can also say something about $L_a R_a - I$:

Theorem 6. *Suppose* $\mathrm{Reim}(A) \subseteq A$ *is a subalgebra: if* $a \in \mathrm{Reim}(A)$ *with*

$$(L_a R_a - I)^{-1}(0) \subseteq (L_{a^*} R_{a^*} - I)^{-1}(0), \tag{6.1}$$

and if also

$$b \in \mathrm{Reim}(A) \cap (L_a R_a - I)^{-1}(0), \tag{6.2}$$

then it follows

$$a \in (L_{b^*b} R_{b^*b} - I)^{-1}(0) \tag{6.3}$$

and hence

$$\forall x \in A, \quad \|a\| \leqslant \|a - [b^*b, x]\|. \tag{6.4}$$

Proof. The first part (6.3) follows the argument of Theorem 2, and then (6.4) uses the fact that b^*b is hermitian, which follows from (5.5). $\square$

We have been unable to settle

Problem 7. If $T \in B(X)$ is $*$-hyponormal then T is self-orthogonal?

By (3.2) it would be sufficient to show that

$$T^{-1}(0) \cap T^{*-1}(0) \perp T(X), \tag{7.1}$$

an interesting property in its own right. Following the argument [3, Lemma 1] of Fong, we might define $E : X^\dagger \to X^\dagger$ by setting

$$\forall x \in X, \ \forall \varphi \in X^\dagger : \quad (E\varphi)(x) = \mathrm{glim}_n \varphi\left(e^{nT^*} e^{-nT} x\right), \tag{7.2}$$

where

$$\mathrm{glim} : \ell_\infty \to \mathbf{C} \tag{7.3}$$

is a "Banach limit": then we would have

$$ST - TS = ST^* - T^* = 0 \quad \Longrightarrow \quad ES^\dagger = S^\dagger E; \tag{7.4}$$

$$\forall x \in X: \ Tx = T^*x = 0 \quad \Longrightarrow \quad \forall \varphi \in X^\dagger, \ (E\varphi)(x) = \varphi(x); \tag{7.5}$$

$$\forall \varphi \in X^\dagger: \ \varphi T = \varphi T^* = 0 \quad \Longrightarrow \quad E\varphi = \varphi. \tag{7.6}$$

It is not however immediately obvious that

$$e^{T^*} e^{-T} E = E = E^2. \tag{7.7}$$

If $T^*T = TT^*$ then of course (7.7) is clear, which suggests a slightly simpler proof of the normal case (1.8). An affirmative solution to the following would [4, Definition 6] enhance the significance of $*$-hyponormality:

Problem 8. If $T \in B(X)$ is $*$-hyponormal does it follow that there is $k > 0$ for which

$$\left\| T^*(\cdot) \right\| \leqslant k \left\| T(\cdot) \right\| \quad \text{on } X? \tag{8.1}$$

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