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Weak solutions to the stationary quantum drift-diffusion model ☆

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ABSTRACT

The existence of weak solutions to the stationary quantum drift-diffusion equations for semiconductor devices is investigated. The proof is based on minimization procedure of non-linear functional and Schauder fixed-point theorem. Furthermore, the semiclassical limit $\varepsilon \to 0$ from the quantum drift-diffusion model to the classical drift-diffusion model is discussed.

1. Introduction

Due to the ongoing miniaturization of electronic devices, mathematical models of ultra small semiconductors have to incorporate the quantum mechanical effects [1,2,6]. This paper is concerned with quantum drift-diffusion models. In Ref. [3], Jüngel and Pinnau have proved the existence of solution in one dimension, the proof is finished by introducing a positivity-preserving numerical scheme. Then the conclusion is extended to multi-dimensions in Ref. [4]. The solutions they have got are all strong solutions, but the assumptions they imposed are somewhat strict.

The scaled equations of the quantum drift-diffusion model read:

$$\begin{aligned} & n_t - \nabla \cdot J = 0, \\ & n\nabla V + \nabla r(n) - \varepsilon^2 n \nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) = J, \\ & -\lambda^2 \Delta V = n - C \end{aligned} \tag{1}$$

where $n(x,t)$ is the electron density, $J(x,t)$ the current density, and $V(x,t)$ the electrostatic potential. The dimensionless constants ε and λ are the scaled Planck constant and the scaled Debye length, respectively. The doping profile $C(x)$ models fixed background charges, $r(n)$ is the pressure function.

We assume $\Omega \subset \mathbb{R}^N$ $(N = 1, 2, 3)$ is a bounded domain, and the boundary $\partial\Omega$ splits into two disjoint parts Γ_D and Γ_N, where Γ_D models the Ohmic contacts of the device and Γ_N represents the insulating parts of the boundary. Assuming $\partial\Omega \in C^{0,1}$, Γ_D has nonvanishing $(N-1)$-dimensional Lebesgue measure. Let γ denote the unit outward normal vector along $\partial\Omega$.

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The objective of this paper is to analyze the stationary version

$$
\begin{aligned}
&\nabla\cdot J=0,\quad J=n\nabla F,\\
&F=V+h(n)-\varepsilon^2\frac{\Delta\sqrt{n}}{\sqrt{n}},\\
&-\lambda^2\Delta V=n-C
\end{aligned}
\tag{2}
$$

subject to the boundary conditions

$$
\begin{aligned}
&n=n_D,\ V=V_D,\ F=F_D\quad\text{on }\Gamma_D,\\
&\frac{\partial n}{\partial\gamma}=\frac{\partial V}{\partial\gamma}=\frac{\partial F}{\partial\gamma}=0\quad\text{on }\Gamma_N.
\end{aligned}
$$

We shall make use of the following assumptions.

(A1) $n_D, V_D, F_D, C\in L^\infty(\Omega)\cap H^1(\Omega)$, $\inf n_D>0$.
(A2) The enthalpy function $h(s)$ $(s\geqslant 0)$ is strictly monotone increasing, locally Lipschitz continuous, and

$$\lim_{s\to 0^+}h(s)=-\infty,\qquad \lim_{s\to+\infty}h(s)=+\infty.$$

We introduce a new variable $\rho=\sqrt{n}$, then from (2) we obtain

$$
\begin{aligned}
&\nabla\cdot\big(\rho^2\nabla F\big)=0,\\
&\varepsilon^2\Delta\rho=\rho\big(V+h\big(\rho^2\big)-F\big),\\
&-\lambda^2\Delta V=\rho^2-C
\end{aligned}
\tag{3}
$$

subject to the boundary conditions

$$
\begin{aligned}
&\rho=\rho_D,\ V=V_D,\ F=F_D\quad\text{on }\Gamma_D,\\
&\frac{\partial\rho}{\partial\gamma}=\frac{\partial V}{\partial\gamma}=\frac{\partial F}{\partial\gamma}=0\quad\text{on }\Gamma_N,
\end{aligned}
\tag{4}
$$

where $\rho_D=\sqrt{n_D}$.

2. Preliminaries

Given $f\in L^2(\Omega)$, assume $\Phi\in L^\infty(\Omega)\cap H^1(\Omega)$ is the unique weak solution of

$$-\lambda^2\Delta\Phi=f,\qquad \Phi=0\quad\text{on }\Gamma_D,\qquad \frac{\partial\Phi}{\partial\gamma}=0\quad\text{on }\Gamma_N.$$

Define $\tilde\Phi[f]=\Phi$, then $\tilde\Phi[\cdot]$ is a continuous linear mapping [7].

Let $\Phi_e\in L^\infty(\Omega)\cap H^1(\Omega)$ be the unique weak solution of

$$-\Delta\Phi_e=0,\qquad \Phi_e=V_D\quad\text{on }\Gamma_D,\qquad \frac{\partial\Phi_e}{\partial\gamma}=0\quad\text{on }\Gamma_N.$$

Given $\rho\in L^2(\Omega)$, then $V=\tilde\Phi[\rho^2-C]+\Phi_e$ is the unique weak solution of

$$-\lambda^2\Delta V=\rho^2-C,\qquad V=V_D\quad\text{on }\Gamma_D,\qquad \frac{\partial V}{\partial\gamma}=0\quad\text{on }\Gamma_N.$$

Let $B=\{f\in L^2(\Omega)\mid \inf F_D\leqslant f\leqslant\sup F_D\}$. Given $F\in B$, $\delta\in(0,\infty)$, for all $\rho\in L^2(\Omega)$, define

$$E_\delta(\rho)=\varepsilon^2\int_\Omega|\nabla\rho|^2\,dx+\int_\Omega H_\delta\big(\rho^2\big)\,dx+\frac{\lambda^2}{2}\int_\Omega\big|\nabla\tilde\Phi\big[\rho^2-C\big]\big|^2\,dx+\int_\Omega\rho^2\Phi_e\,dx-\int_\Omega F\rho^2\,dx,$$

where $H_\delta(s)=\int_1^s h_\delta(u)\,du$, $h_\delta(u)=\max\{h(u),h(\delta)\}$.

Assuming $\rho_c\in L^\infty(\Omega)\cap H^1(\Omega)$ and $\rho_c=\rho_D$ on Γ_D, we set $\mathcal{X}=\rho_c+H_0^1(\Omega)$.

Lemma 1. *Given $F \in B$, if there exists a minimizer ρ_δ of $E_\delta(\rho)$ in $\mathcal{X}$, we set $V = \tilde{\Phi}[\rho^2 - C] + \Phi_e$. Then ρ_δ is a weak solution of*

$$\varepsilon^2 \Delta\rho = \rho\big(V + h_\delta\big(\rho^2\big) - F\big), \qquad \rho = \rho_D \quad \text{on } \Gamma_D, \qquad \frac{\partial\rho}{\partial\gamma} = 0 \quad \text{on } \Gamma_N.$$

Proof. For all $\varphi \in H_0^1(\Omega)$, $\rho \in \mathcal{X}$, $s \in \mathbb{R}$ satisfying $\rho + s\varphi \in \mathcal{X}$, it holds

$$\begin{aligned}
E_\delta(\rho + s\varphi) - E_\delta(\rho) &= \varepsilon^2\bigg(\int_\Omega |\nabla\rho + s\nabla\varphi|^2\,\mathrm{d}x - \int_\Omega |\nabla\rho|^2\,\mathrm{d}x\bigg) + \int_\Omega \big(H_\delta\big((\rho + s\varphi)^2\big) - H_\delta\big(\rho^2\big)\big)\,\mathrm{d}x \\
&\quad + \frac{\lambda^2}{2}\int_\Omega \big(\big|\nabla\tilde{\Phi}\big[(\rho + s\varphi)^2 - C\big]\big|^2 - \big|\nabla\tilde{\Phi}\big[\rho^2 - C\big]\big|^2\big)\,\mathrm{d}x \\
&\quad + \bigg(\int_\Omega (\rho + s\varphi)^2\Phi_e\,\mathrm{d}x - \int_\Omega \rho^2\Phi_e\,\mathrm{d}x\bigg) + \bigg(\int_\Omega F(\rho + s\varphi)^2\,\mathrm{d}x - \int_\Omega F\rho^2\,\mathrm{d}x\bigg) \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

According to mean value theorem of integral, we obtain

$$I_2 = \int_\Omega \bigg(\int_{\rho^2}^{(\rho+s\varphi)^2} h_\delta(u)\,\mathrm{d}u\bigg)\,\mathrm{d}x = \int_\Omega 2\rho s\varphi h_\delta\big(\rho^2\big)\,\mathrm{d}x + o(s).$$

By integration by parts and the linear property of $\tilde{\Phi}[\,\cdot\,]$, we get

$$\begin{aligned}
I_3 &= \frac{\lambda^2}{2}\int_\Omega \tilde{\Phi}\big[(\rho + s\varphi)^2 - C\big]\frac{(\rho + s\varphi)^2 - C}{\lambda^2}\,\mathrm{d}x - \frac{\lambda^2}{2}\int_\Omega \tilde{\Phi}\big[\rho^2 - C\big]\frac{\rho^2 - C}{\lambda^2}\,\mathrm{d}x \\
&= \frac{\lambda^2}{2}\int_\Omega \tilde{\Phi}\big[(\rho + s\varphi)^2 - C\big]\bigg(\frac{(\rho + s\varphi)^2 - C}{\lambda^2} - \frac{\rho^2 - C}{\lambda^2}\bigg)\,\mathrm{d}x + \frac{\lambda^2}{2}\int_\Omega \tilde{\Phi}\big[2\rho s\varphi + s^2\varphi^2\big]\frac{\rho^2 - C}{\lambda^2}\,\mathrm{d}x \\
&= \frac{\lambda^2}{2}\bigg(\int_\Omega \tilde{\Phi}\big[\rho^2 - C\big]\frac{2\rho s\varphi}{\lambda^2}\,\mathrm{d}x + o(s)\bigg) + \frac{\lambda^2}{2}\int_\Omega \tilde{\Phi}\big[\rho^2 - C\big]\frac{2\rho s\varphi + s^2\varphi^2}{\lambda^2}\,\mathrm{d}x \\
&= \frac{\lambda^2}{2}\bigg(\frac{4}{\lambda^2}\int_\Omega \tilde{\Phi}\big[\rho^2 - C\big]\rho s\varphi\,\mathrm{d}x + o(s)\bigg) \\
&= 2\int_\Omega \tilde{\Phi}\big[\rho^2 - C\big]\rho s\varphi\,\mathrm{d}x + o(s).
\end{aligned}$$

By simply calculating, we get

$$I_1 = \varepsilon^2\int_\Omega 2\nabla\rho \cdot s\nabla\varphi\,\mathrm{d}x + o(s), \qquad I_4 = \int_\Omega 2\rho s\varphi\Phi_e\,\mathrm{d}x + o(s), \qquad I_5 = \int_\Omega 2\rho s\varphi F\,\mathrm{d}x + o(s).$$

To sum up, we obtain

$$E_\delta(\rho + s\varphi) - E_\delta(\rho) = 2\bigg(\int_\Omega \varepsilon^2\nabla\rho \cdot s\nabla\varphi\,\mathrm{d}x + \int_\Omega \rho(h_\delta + V - F)s\varphi\,\mathrm{d}x\bigg) + o(s).$$

If ρ_δ is a minimizer of $E_\delta(\rho)$ in $\mathcal{X}$, then as $s \to 0$,

$$E_\delta(\rho_\delta + s\varphi) - E_\delta(\rho_\delta) = 2\bigg(\int_\Omega \varepsilon^2\nabla\rho_\delta \cdot s\nabla\varphi\,\mathrm{d}x + \int_\Omega \rho_\delta(h_\delta + V - F)s\varphi\,\mathrm{d}x\bigg) + o(s) \geqslant 0.$$

Let $s \to 0^+$ and $s \to 0^-$, respectively. Then

$$\int_\Omega \varepsilon^2\nabla\rho_\delta \cdot \nabla\varphi\,\mathrm{d}x + \int_\Omega \rho_\delta(h_\delta + V - F)\varphi\,\mathrm{d}x = 0. \qquad \square$$

Lemma 2 *(Pseudo-convexity inequality). For all $s \in (0,1)$, $\rho_1, \rho_2 \in L^2(\Omega)$ and $|\rho_1| \neq |\rho_2|$, one has*

$$E_\delta\bigl(\sqrt{s\rho_1^2 + (1-s)\rho_2^2}\bigr) < sE_\delta(\rho_1) + (1-s)E_\delta(\rho_2).$$

Proof. Set $\rho = \sqrt{s\rho_1^2 + (1-s)\rho_2^2}$, then

$$\nabla\rho = \frac{s\rho_1\nabla\rho_1 + (1-s)\rho_2\nabla\rho_2}{\rho},$$

$$\begin{aligned}
&s|\nabla\rho_1|^2 + (1-s)|\nabla\rho_2|^2 - \left|\frac{s\rho_1}{\rho}\nabla\rho_1 + \frac{(1-s)\rho_2}{\rho}\nabla\rho_2\right|^2 \\
&\quad= \frac{s(1-s)\rho_2^2}{\rho^2}|\nabla\rho_1|^2 + \frac{s(1-s)\rho_1^2}{\rho^2}|\nabla\rho_2|^2 - 2s(1-s)\frac{\rho_1\rho_2}{\rho^2}\nabla\rho_1\cdot\nabla\rho_2 \\
&\quad= s(1-s)\left|\frac{\rho_2}{\rho}\nabla\rho_1 - \frac{\rho_1}{\rho}\nabla\rho_2\right|^2 \geqslant 0.
\end{aligned}$$

So we get

$$|\nabla\rho|^2 \leqslant s|\nabla\rho_1|^2 + (1-s)|\nabla\rho_2|^2.$$

Next, we prove that for all $x_0 \in \Omega$, $H_\delta(\rho^2(x_0)) < H_\delta(\rho_1^2(x_0)) + H_\delta(\rho_2^2(x_0))$.

We just need to prove

$$\int_1^{\rho^2(x_0)} h_\delta(u)\,\mathrm{d}u \leqslant s\int_1^{\rho_1^2(x_0)} h_\delta(u)\,\mathrm{d}u + (1-s)\int_1^{\rho_2^2(x_0)} h_\delta(u)\,\mathrm{d}u$$

or

$$\int_{\rho_1^2(x_0)}^{\rho^2(x_0)} h_\delta(u)\,\mathrm{d}u \leqslant (1-s)\int_{\rho_1^2(x_0)}^{\rho_2^2(x_0)} h_\delta(u)\,\mathrm{d}u.$$

There is no loss in generality in assuming that $\rho_1(x_0) < \rho_2(x_0)$. According to the mean value theorem, there exist $\xi_1 \in (\rho_1^2(x_0), \rho^2(x_0))$, $\xi_2 \in (\rho^2(x_0), \rho_2^2(x_0))$ satisfying

$$\begin{aligned}
(1-s)\int_{\rho_1^2(x_0)}^{\rho_2^2(x_0)} h_\delta(u)\,\mathrm{d}u &= (1-s)\bigl(\rho^2(x_0) - \rho_1^2(x_0)\bigr)h_\delta(\xi_1) + (1-s)\bigl(\rho_2^2(x_0) - \rho^2(x_0)\bigr)h_\delta(\xi_2) \\
&\geqslant (1-s)\bigl(\rho_2^2(x_0) - \rho_1^2(x_0)\bigr)h_\delta(\xi_1) \\
&= \bigl(\rho^2(x_0) - \rho_1^2(x_0)\bigr)h_\delta(\xi_1) \\
&= \int_{\rho_1^2(x_0)}^{\rho^2(x_0)} h_\delta(u)\,\mathrm{d}u.
\end{aligned}$$

Then from $|\rho_1| \neq |\rho_2|$, we get $H_\delta(\rho^2(x_0)) < H_\delta(\rho_1^2(x_0)) + H_\delta(\rho_2^2(x_0))$.

According to the linear property of $\tilde{\Phi}[\,\cdot\,]$, we obtain

$$\nabla\tilde{\Phi}\bigl[\rho^2 - C\bigr] = s\nabla\tilde{\Phi}\bigl[\rho_1^2 - C\bigr] + (1-s)\nabla\tilde{\Phi}\bigl[\rho_2^2 - C\bigr].$$

So we conclude

$$s\bigl|\nabla\tilde{\Phi}\bigl[\rho_1^2 - C\bigr]\bigr|^2 + (1-s)\bigl|\nabla\tilde{\Phi}\bigl[\rho_2^2 - C\bigr]\bigr|^2 - \bigl|\nabla\tilde{\Phi}\bigl[\rho^2 - C\bigr]\bigr|^2 = s(1-s)\bigl|\nabla\tilde{\Phi}\bigl[\rho_1^2 - C\bigr] - \nabla\tilde{\Phi}\bigl[\rho_2^2 - C\bigr]\bigr|^2 \geqslant 0. \qquad \Box$$

Lemma 3. *Given $F \in B$, there exists a unique nonnegative minimizer of $E_\delta(\rho)$ in $\mathcal{X}$.*

Proof. From $\lim_{u\to\infty} h_\delta(u) = \infty$, we know that the functional E_δ is coercive with respect to the $L^2(\Omega)$ norm. $\mathcal{X}$ is a translate of a Hilbert space, thus the existence of minimizers of $E_\delta(\rho)$ follows from the $H^1(\Omega)$-weakly sequentially lower semicontinuity of $E_\delta(\rho)$ (which is easy to see).

By the using of Lemma 2, we get that when $|\rho_1| \neq |\rho_2|$, one has

$$E_\delta\left(\sqrt{\frac{1}{2}\rho_1^2 + \frac{1}{2}\rho_2^2}\right) < \frac{1}{2}E_\delta(\rho_1) + \frac{1}{2}E_\delta(\rho_2).$$

So we obtain the uniqueness of the nonnegative minimizer.

Due to the above analysis, we obtain that given $F \in L^\infty(\Omega)$, $\delta \in (0,\infty)$, there exists a unique solution (V_δ, ρ_δ) satisfying $V_\delta \in L^\infty(\Omega) \cap H^1(\Omega)$, $\rho_\delta \in H^1(\Omega)$, and the equations

$$\varepsilon^2 \Delta\rho = \rho\big(V + h_\delta\big(\rho^2\big) - F\big), \qquad -\lambda^2 \Delta V = \rho^2 - C, \tag{5}$$

with the boundary condition

$$\rho = \rho_D,\ V = V_D \quad \text{on } \Gamma_D, \qquad \frac{\partial\rho}{\partial\gamma} = \frac{\partial V}{\partial\gamma} = 0 \quad \text{on } \Gamma_N. \qquad \square \tag{6}$$

Lemma 4. *Let* (A1), (A2) *hold. Given $F \in L^\infty(\Omega)$, then there exists $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$, there exists a unique solution (V_δ, ρ_δ) to* (5)–(6)*, which satisfies $V_\delta \in L^\infty(\Omega) \cap H^1(\Omega)$, $\rho_\delta \in H^1(\Omega)$, $\rho_\delta > c$, $c > 0$ is independent of δ.*

Proof. Assuming $h(s_0) = 0$, we obtain from the monotonicity of the enthalpy function, that for all $\delta \in (0, s_0)$, one has

$$\int_{s_0}^{\rho^2} h_\delta(u)\,\mathrm{d}u \geqslant 0.$$

So we get

$$H_\delta\big(\rho^2\big) = \int_1^{\rho^2} h_\delta(u)\,\mathrm{d}u \geqslant \int_1^{s_0} h_\delta(u)\,\mathrm{d}u \geqslant (s_0 - 1)h_\delta(1) \geqslant (s_0 - 1)h(1).$$

Then from the definition of $E_\delta(\rho)$, we know E_δ is bounded from below uniformly for δ in $\mathcal{X}$. So $\|\rho_\delta\|_{H^1(\Omega)} \leqslant c_1$, with $c_1 > 0$ independent of δ.

From the second equation of (5), we get $\|V_\delta\|_{L^\infty(\Omega)} \leqslant c_2$, and $c_2 > 0$ is independent of δ.

Using $(\rho_\delta - c)^- = \min\{0, \rho_\delta - c\}$ as a test function for the first equation of (5) for $0 < c \leqslant \inf \rho_D$, we get

$$\varepsilon^2 \int_\Omega \big|\nabla(\rho_\delta - c)^-\big|^2\,\mathrm{d}x = \int_\Omega \rho_\delta\big(V_\delta + h_\delta\big(\rho_\delta^2\big) - F\big)\big(-(\rho_\delta - c)^-\big)\,\mathrm{d}x \leqslant \int_\Omega \rho_\delta\big(V_\delta + h_\delta\big(c^2\big) - F\big)\big(-(\rho_\delta - c)^-\big)\,\mathrm{d}x.$$

From $\lim_{s\to 0^+} h(s) = -\infty$, we get that there exists $(c, \bar\delta)$, with $c > 0$, $\bar\delta > 0$, such that for $0 < \delta \leqslant \bar\delta$, one has

$$\varepsilon^2 \int_\Omega \big|\nabla(\rho_\delta - c)^-\big|^2\,\mathrm{d}x \leqslant 0.$$

Hence $\rho_\delta \geqslant c$ when $0 < \delta \leqslant \bar\delta$. Set $\delta_0 = \min\{\bar\delta, c^2\}$. Then when $\delta < \delta_0$, it holds $h(\rho_\delta^2) \geqslant h(\delta)$. This gives $h_\delta(\rho_\delta^2) = h(\rho_\delta^2)$. $\square$

Lemma 4 immediately implies the following lemma.

Lemma 5. *Let* (A1), (A2) *hold, given $F \in L^\infty(\Omega)$, there exists a unique solution (V, ρ), with $V \in L^\infty(\Omega) \cap H^1(\Omega)$, $\rho \in H^1(\Omega)$, to the equations*

$$\varepsilon^2 \Delta\rho = \rho\big(V + h\big(\rho^2\big) - F\big), \qquad -\lambda^2 \Delta V = \rho^2 - C, \tag{7}$$

with the boundary conditions

$$\rho = \rho_D,\ V = V_D \quad \text{on } \Gamma_D, \qquad \frac{\partial\rho}{\partial\gamma} = \frac{\partial V}{\partial\gamma} = 0 \quad \text{on } \Gamma_N.$$

Lemma 6. *Let ρ be the solution in Lemma* 5*. Then $\rho \in L^\infty(\Omega)$.*

Proof. We just need to prove ρ is bounded from above.

Using $(\rho - a)^+ = \max\{0, \rho - a\}$, for some $a \geqslant \sup_{\Gamma_D} \rho_D > 0$ to be determined, as a test function in the first equation of (7), we get

$$\varepsilon^2 \int_\Omega \left|\nabla(\rho - a)^+\right|^2 \mathrm{d}x = \int_\Omega \rho\big(F - V - h\big(\rho^2\big)\big)(\rho - a)^+ \,\mathrm{d}x.$$

Following from $\lim_{s\to+\infty} h(s) = +\infty$, we know that there exists a positive constant $a \geqslant \sup_{\Gamma_D} \rho_D$ such that when $\rho > a$, one has $F - V - h(\rho^2) < 0$. Hence we get

$$\varepsilon^2 \int_\Omega \left|\nabla(\rho - a)^+\right|^2 \mathrm{d}x \leqslant 0.$$

This gives $\rho \leqslant a$. $\square$

3. Existence of weak solutions

Theorem 1. *Let* (A1), (A2) *hold. Then there exists a solution* $(\rho, V, F) \in L^\infty(\Omega) \cap H^1(\Omega)$ *to* (3)–(4).

Proof. We set $F = f \in B$ in the second equation of (3) to get

$$\begin{aligned}
&\nabla \cdot \big(\rho^2 \nabla F\big) = 0,\\
&\varepsilon^2 \Delta\rho = \rho\big(V + h\big(\rho^2\big) - f\big),\\
&-\lambda^2 \Delta V = \rho^2 - C.
\end{aligned} \tag{8}$$

By Lemmas 5, 6, there exists a unique solution $(\rho, V, F) \in (L^\infty(\Omega) \cap H^1(\Omega))^3$ of (8) satisfying the boundary condition

$$\begin{aligned}
&\rho = \rho_D,\ V = V_D,\ F = F_D \quad \text{on } \Gamma_D,\\
&\frac{\partial\rho}{\partial\gamma} = \frac{\partial V}{\partial\gamma} = \frac{\partial F}{\partial\gamma} = 0 \quad \text{on } \Gamma_N.
\end{aligned} \tag{9}$$

Using the maximum principle we get

$$\inf F_D \leqslant F \leqslant \sup F_D.$$

Hence $F \in B$.

Thus the mapping $T : B \to B$, $T(f) = F$, is well defined. Moreover, it is not difficult to check that T is compact, noting the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$. Next we prove T is continuous.

Assume $f_n \in B$ is convergent to f as $n \to \infty$ in $L^2(\Omega)$, $(\rho, V, F) \in (L^\infty(\Omega) \cap H^1(\Omega))^3$ is a solution to (8)–(9), and $(\rho^{(n)}, V^{(n)}, F^{(n)}) \in (L^\infty(\Omega) \cap H^1(\Omega))^3$ is a solution to (8)–(9) in which we have substituted f_n for f.

Replace F with f_n in the definition of $E_\delta(\rho)$ and define it as $E_\delta^{(n)}(\rho)$. It is not difficult to conclude

$$\lim_{n\to\infty} E_\delta^{(n)}(\rho) = E_\delta(\rho).$$

Thus one has

$$\lim_{n\to\infty} \inf_{\rho\in\mathcal{X}} E_\delta^{(n)}(\rho) = \inf_{\rho\in\mathcal{X}} E_\delta(\rho).$$

$E_\delta(\rho)$ is continuous, so we get from Lemma 3

$$\lim_{n\to\infty} \left\|\rho^{(n)} - \rho\right\| = 0.$$

From the first equation of (8) we obtain

$$\lim_{n\to\infty} \left\|F^{(n)} - F\right\| = 0.$$

Hence T is continuous.

Finally we can finish the proof by Schauder fixed-point theorem. $\square$

4. Semiclassical limit

We analyze the semiclassical limit in the isothermal condition. The isothermal quantum drift-diffusion equations read:

$$\begin{aligned}
&\nabla\cdot\big(\rho_\varepsilon^2\nabla F_\varepsilon\big)=0,\\
&\varepsilon^2\Delta\rho_\varepsilon=\rho_\varepsilon\big(V_\varepsilon+\ln\big(\rho_\varepsilon^2\big)-F_\varepsilon\big),\\
&-\lambda^2\Delta V_\varepsilon=\rho_\varepsilon^2-C
\end{aligned}\tag{10}$$

with the boundary condition

$$\begin{aligned}
&\rho_\varepsilon=\sqrt{n_D^\varepsilon},\ V_\varepsilon=V_D^\varepsilon,\ F_\varepsilon=F_D^\varepsilon\quad\text{on }\Gamma_D,\\
&\frac{\partial\rho_\varepsilon}{\partial\gamma}=\frac{\partial V_\varepsilon}{\partial\gamma}=\frac{\partial F_\varepsilon}{\partial\gamma}=0\quad\text{on }\Gamma_N.
\end{aligned}\tag{11}$$

We impose the following assumptions.

(A3) $n_D^\varepsilon, V_D^\varepsilon, F_D^\varepsilon\in L^\infty(\Omega)\cap H^1(\Omega)$, and there exists a $K>0$ such that $n_D^\varepsilon\geqslant K$ for all $\varepsilon>0$. F_D^ε is uniformly bounded.
(A4) $n_D^\varepsilon\to n_D$, $V_D^\varepsilon\to V_D$, $F_D^\varepsilon\to F_D$ in $H^1(\Omega)$ as $\varepsilon\to0$.

Theorem 2. *Let* (A1)–(A4) *hold, then there exist functions $n, F, V\in L^\infty(\Omega)\cap H^1(\Omega)$ satisfying*

$$\nabla\cdot(n\nabla F)=0,\qquad F=V+\ln(n),\qquad -\lambda^2\Delta V=n-C,\tag{12}$$

subject to the boundary condition

$$\begin{aligned}
&n=n_D,\ V=V_D,\ F=F_D\quad\text{on }\Gamma_D,\\
&\frac{\partial n}{\partial\gamma}=\frac{\partial V}{\partial\gamma}=\frac{\partial F}{\partial\gamma}=0\quad\text{on }\Gamma_N.
\end{aligned}\tag{13}$$

Assuming $\rho_\varepsilon, F_\varepsilon, V_\varepsilon\in L^\infty(\Omega)\cap H^1(\Omega)$ solve (10)–(11), *then there exists a subsequence of $(\rho_\varepsilon, F_\varepsilon, V_\varepsilon)$ (not relabeled) such that*

$$\rho_\varepsilon^2\to n,\ F_\varepsilon\to F,\ V_\varepsilon\to V\quad\textit{weakly in } H^1(\Omega),\textit{ strongly in } L^2(\Omega)\textit{ as }\varepsilon\to0.$$

Proof. Set $H(s)=\int_1^s\ln(u)\,du$, and define

$$\begin{aligned}
&I(n)\triangleq\int_\Omega H(n)\,dx+\frac{\lambda^2}{2}\int_\Omega\big|\nabla\tilde\Phi[n-C]\big|^2\,dx+\int_\Omega n\tilde\Phi_e\,dx-\int_\Omega Fn\,dx,\\
&E^\varepsilon(\rho)\triangleq\varepsilon^2\int_\Omega|\nabla\tilde\rho|^2\,dx+\int_\Omega H\big(\rho^2\big)\,dx+\frac{\lambda^2}{2}\int_\Omega\big|\nabla\tilde\Phi\big[\rho^2-C\big]\big|^2\,dx+\int_\Omega\rho^2\Phi_e\,dx-\int_\Omega F_\varepsilon\rho^2\,dx.
\end{aligned}$$

Similarly as in Section 3, we can prove that there exist functions $n, F, V\in L^\infty(\Omega)\cap H^1(\Omega)$ satisfying (12)–(13), where n is also the unique minimizer of $I(n)$ in $\mathcal{X}$. Furthermore, ρ_ε is the unique minimizer of $E^\varepsilon(\rho)$ in $\mathcal{Y}\triangleq\rho_D+H_0^1(\Omega)$.

By using maximum principle, we obtain from (A3) that F_ε is bounded uniformly for ε. This yields a uniform bound on $E^\varepsilon(\rho)$. Hence we get the uniform bound on ρ_ε. From the third equation of (10) we get the uniform bound on V_ε. So there exists a subsequence of $(\rho_\varepsilon, F_\varepsilon, V_\varepsilon)$ (not relabeled) such that

$$\rho_\varepsilon^2\to n^*,\ F_\varepsilon\to F^*,\ V_\varepsilon\to V^*\quad\text{weakly in }H^1(\Omega),\text{ strongly in }L^2(\Omega)$$

as $\varepsilon\to0$.

From $\sqrt{n^*}\in\mathcal{Y}$ we obtain $\lim_{\varepsilon\to0}\sup E^\varepsilon(\rho_\varepsilon)\leqslant\lim_{\varepsilon\to0}\sup E^\varepsilon(\sqrt{n^*})$. Hence $\lim_{\varepsilon\to0}\sup E^\varepsilon(\rho_\varepsilon)\leqslant I(n^*)$,

$$E^\varepsilon(\rho_\varepsilon)=I\big(\rho_\varepsilon^2\big)+\varepsilon^2\int_\Omega|\nabla\rho_\varepsilon|^2\,dx+\int_\Omega(F-F_\varepsilon)\rho_\varepsilon^2\,dx.$$

Then by the weakly lower semicontinuity of $I(n)$ (which is easy to see) we get

$$\lim_{\varepsilon\to0}\inf E^\varepsilon(\rho_\varepsilon)\geqslant I(n).$$

So $I(n^*)\geqslant I(n)$. n^* is the unique minimizer of $I(n)$, hence $I(n)=I(n^*)$, $n=n^*$.
Then it's not difficult to prove $F=F^*$, $V=V^*$. □

5. Conclusions

By the employing of the non-linear functional

$$E_\delta(\rho) = \varepsilon^2 \int_\Omega |\nabla\rho|^2 \,\mathrm{d}x + \int_\Omega H_\delta(\rho^2)\,\mathrm{d}x + \frac{\lambda^2}{2}\int_\Omega \left|\nabla\Phi\left[\rho^2 - C\right]\right|^2 \mathrm{d}x + \int_\Omega \rho^2\Phi_e\,\mathrm{d}x - \int_\Omega F\rho^2\,\mathrm{d}x,$$

we obtain the relations between its unique nonnegative minimizer and the weak solution to the stationary quantum drift-diffusion equations. Note the means we employ to remove the variable δ. Then the existence of weak solutions is proved by the Schauder fixed point theorem. This method works well for the stationary problem, but not for the transient equations. Therefore, seeking a new method for the transient quantum drift-diffusion equations will be our next work. Moreover, latest studies of quantum semiconductor models are all related to two kinds of particles, namely electrons and holes [5]. So we need to transfer our work focus to bipolar models in future.

References

[1] Abdallah N. Ben, A. Unterreiter, On the stationary quantum drift-diffusion model, Z. Angew. Math. Phys. 49 (2) (1998) 251–275.
[2] A. Jüngel, Nonlinear problems in quantum semiconductor modeling, Nonlinear Anal. 47 (2001) 5873–5884.
[3] A. Jüngel, R. Pinnau, A positivity-preserving numerical scheme for a nonlinear fourth order parabolic system, SIAM J. Numer. Anal. 39 (2001) 385–406.
[4] A. Jüngel, R. Pinnau, Convergent semidiscretization of a nonlinear fourth order parabolic system, M2AN Math. Model. Numer. Anal. 37 (2) (2003) 277–289.
[5] A. Unterreiter, The thermal equilibrium solution of a generic bipolar quantum hydrodynamic model, Comm. Math. Phys. 188 (1997) 69–88.
[6] Wei-Xiang Jiang, Ping Guan, Weak solutions to one-dimensional quantum drift-diffusion equations for semiconductors, J. Southeast Univ. (English Ed.) 22 (2006) 577–581.
[7] David Gilbarg, Neil S. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., Springer-Verlag, 1983.