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Hopf-pitchfork bifurcation in van der Pol's oscillator with nonlinear delayed feedback ☆

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ABSTRACT

First, we identify the critical values for Hopf-pitchfork bifurcation. Second, we derive the normal forms up to third order and their unfolding with original parameters in the system near the bifurcation point, by the normal form method and center manifold theory. Then we give a complete bifurcation diagram for original parameters of the system and obtain complete classifications of dynamics for the system. Furthermore, we find some interesting phenomena, such as the coexistence of two asymptotically stable states, two stable periodic orbits, and two attractive quasi-periodic motions, which are verified both theoretically and numerically.

1. Introduction

In the research of nonlinear dynamical system, van der Pol equation is one of the most intensely studied equation (see [12,15] and the references therein). This celebrated equation has a nonlinear damping

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = f(x), \quad x \in \mathrm{R},\ \varepsilon > 0, \tag{1}$$

which originally was a model for an electrical circuit with a triode valve, and was extensively studied as a host of a rich class of dynamical behavior, including relaxation oscillations, quasi-periodicity, elementary bifurcations and chaos [3]. Noting that most practical implementations of feedback have inherent delays, some researchers have considered the effect of time delay in van der Pol's oscillator [8,13,14,18,20,23,24]. It is shown that the presence of time delay can change the amplitude of limit cycle oscillations.

Although the van der Pol equation has been studied over wide parameter regimes, from perturbations of harmonic motion to relaxation oscillations, bifurcations and high-codimensional singularities of the system with or without delay have been discussed little, such as Hopf-pitchfork bifurcation, double Hopf bifurcation and Bogdanov–Takens singularity etc. [1,11,14,16]. Particularly, by our existing knowledge, there is no study in Hopf-pitchfork bifurcation of van der Pol's equation with delayed feedback. Moreover, there are only a few articles on Hopf-pitchfork bifurcation in delay differential equations (see [9,17,19,25]).

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In this study we consider the forcing f of being a delayed feedback of the position x. Since the limit cycle disappears when $\varepsilon = 0$, it is convenient to scale the parameters by ε. Hence, Eq. (1) will be considered with $f(x) = \varepsilon g(x(t-\tau))$ and we assume that $g \in C^3$ is an odd function and satisfies

$$g(0) = g''(0) = 0, \qquad g'(0) = k \neq 0, \qquad g'''(0) = 3!b \neq 0.$$

The equilibrium at the origin exhibits a diversity of local bifurcations. Among them, a codimension-1 bifurcation with a single zero eigenvalue (see reference [13]), Hopf bifurcation [20], Bogdanov–Takens bifurcation (analyzed in reference [14]) and also a Hopf-pitchfork bifurcation occur at the corresponding critical values, respectively. The present analysis will focus on the study of the Hopf-pitchfork bifurcation of the origin, which occurs when the parameters satisfy $k = \frac{1}{\varepsilon}$,

$$\tau = \tau_0 = \begin{cases} \frac{\arcsin(\varepsilon\sqrt{2-\varepsilon^2})}{\sqrt{2-\varepsilon^2}}, & 1 \leqslant \varepsilon < \sqrt{2}, \\ \frac{\pi - \arcsin(\varepsilon\sqrt{2-\varepsilon^2})}{\sqrt{2-\varepsilon^2}}, & 0 < \varepsilon < 1, \end{cases}$$

and $0 < \varepsilon < \sqrt{2}$. In fact, the characteristic equation associated with Eq. (1) with the above parameters has a single zero root and a pair of purely imaginary roots $\pm \mathrm{i}\omega_0$ with

$$\omega_0 = \sqrt{2 - \varepsilon^2},$$

and the remaining roots have negative real parts (see reference [13]).

This paper can be regarded as a further study of [13,14,20]. We shall use the normal form method introduced by Faria and Magalhaes [10] to investigate the stability of the fixed point and the dynamics near the zero solution under the Hopf-pitchfork bifurcation. The normal form method has been applied effectively in the study of singularities of vector fields and in bifurcation theory (e.g., see [2,5–7,21,22,26–28]). It provides a convenient tool to compute a simple form of the original differential equation, which can be used to analyze the dynamic behavior of the system, such as periodic solutions, quasi-periodic motions, and more complex bifurcation solutions.

The paper is organized as follows. In Section 2, we perform the center manifold reduction and normal form computation, and derive the normal forms with the Hopf-pitchfork singularity for the van der Pol's equation (1); in Section 3, we give a complete bifurcation analysis; in Section 4 the numerical simulation results are shown to demonstrate the theoretical predictions; and in Section 5, we summarize our results.

2. Computation of normal form with original parameters

We rewrite van der Pol equation (1) in the following form:

$$\begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= -x(t) + \varepsilon g\big(x(t-\tau)\big) - \varepsilon\big(x(t)^2 - 1\big)y(t). \end{aligned} \tag{2}$$

Then the characteristic equation of the linearization equation at the trivial equilibrium of (2) is given by

$$\lambda^2 - \varepsilon\lambda - \varepsilon k e^{-\lambda\tau} + 1 = 0. \tag{3}$$

We can check that when $0 < \varepsilon < \sqrt{2}$, $k = \frac{1}{\varepsilon}$ and $\tau = \tau_0$ Eq. (3) has a single zero root and a pair of purely imaginary roots $\pm \mathrm{i}\omega_0$. Moreover, all the other eigenvalues have negative real parts [13]. This implies that Eq. (2) undergoes a Hopf-pitchfork bifurcation at the origin when $0 < \varepsilon < \sqrt{2}$, $k = \frac{1}{\varepsilon}$ and $\tau = \tau_0$.

Rescaling the time by $t \mapsto \frac{t}{\tau}$ to normalize the delay, and expanding the function g in Eq. (2), we get

$$\begin{aligned} \dot{x}(t) &= \tau y(t), \\ \dot{y}(t) &= -\tau x(t) + \varepsilon\tau\big(kx(t-1) + bx^3(t-1)\big) - \varepsilon\tau\big(x^2(t) - 1\big)y(t) + \text{h.o.t.} \end{aligned} \tag{4}$$

We let $k = \frac{1}{\varepsilon}$ and $\tau = \tau_0$, and choose

$$\eta(\theta) = \begin{cases} \tau_0 B_0, & \theta = 0, \\ \tau_0 B_1, & \theta \in (-1, 0), \\ B_2, & \theta = -1 \end{cases}$$

with

$$B_0 = \begin{pmatrix} 0 & 1 \\ 0 & \varepsilon \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the linearization equation at the trivial equilibrium of (4) is

$$\dot{X}(t) = L_0 X_t,$$

where $L_0\varphi = \int_{-1}^{0} \mathrm{d}\eta(\theta)\varphi(\xi)\,\mathrm{d}\xi$, $\varphi \in C = C([-1,0], R^2)$, and the bilinear form on $C^* \times C$ is

$$(\psi, \varphi) = \psi(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi-\theta)\,\mathrm{d}\eta(\theta)\varphi(\xi)\,\mathrm{d}\xi$$
$$= \psi(0)\varphi(0) - \tau_0 \int_{0}^{-1} \psi_2(\xi+1)\varphi_1(\xi)\,\mathrm{d}\xi,$$

where $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta)) \in C$, $\psi(s) = \binom{\psi_1(s)}{\psi_2(s)} \in C^*$. Then the phase space C is decomposed by $\Lambda = \{0, \pm i\tau_0\omega_0\}$ as $C = P \oplus Q$, where $Q = \{\varphi \in C\colon (\psi,\varphi) = 0, \text{ for all } \psi \in P^*\}$, and the bases for P and its adjoint P^* are

$$\Phi(\theta) = \begin{pmatrix} 1 & e^{\mathrm{i}\tau_0\omega_0\theta} & e^{-\mathrm{i}\tau_0\omega_0\theta} \\ 0 & \mathrm{i}\omega_0 e^{\mathrm{i}\tau_0\omega_0\theta} & -\mathrm{i}\omega_0 e^{-\mathrm{i}\tau_0\omega_0\theta} \end{pmatrix}, \quad -1 \leqslant \theta \leqslant 0 \tag{5}$$

and

$$\Psi(s) = \begin{pmatrix} \frac{\varepsilon}{\varepsilon-\tau_0} & -\frac{1}{\varepsilon-\tau_0} \\ \overline{D}(\varepsilon - \mathrm{i}\omega_0)e^{-\mathrm{i}\tau_0\omega_0 s} & -De^{-\mathrm{i}\tau_0\omega_0 s} \\ D(\varepsilon + \mathrm{i}\omega_0)e^{\mathrm{i}\tau_0\omega_0 s} & -De^{\mathrm{i}\tau_0\omega_0 s} \end{pmatrix}, \quad 0 \leqslant s \leqslant 1,$$

respectively, where $(\Psi, \Phi) = I$ and $D = (\varepsilon + 2\mathrm{i}\omega_0 - \tau_0 e^{\mathrm{i}\tau_0\omega_0})^{-1}$. Thus the dual bases satisfy

$$\dot{\Phi} = \Phi B \quad \text{and} \quad -\dot{\Psi} = B\Psi \quad \text{with } B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{i}\tau_0\omega_0 & 0 \\ 0 & 0 & -\mathrm{i}\tau_0\omega_0 \end{pmatrix}.$$

We now introduce two bifurcation parameters by $k = \frac{1}{\varepsilon} + \mu_1$ and $\tau = \tau_0 + \mu_2$ in Eq. (4), and denote $\mu = (\mu_1, \mu_2)$. Then Eq. (4) can be written as

$$\dot{X}(t) = L(\mu)X_t + F(X_t, \mu), \tag{6}$$

where

$$L(\mu)X_t = \begin{pmatrix} (\tau_0 + \mu_2)y_t(0) \\ -(\tau_0+\mu_2)x_t(0) + \varepsilon(\tau_0+\mu_2)(\frac{1}{\varepsilon}+\mu_1)x_t(-1) + \varepsilon(\tau_0+\mu_2)y_t(0) \end{pmatrix},$$

and

$$F(X_t, \mu) = \begin{pmatrix} 0 \\ \varepsilon(\tau_0+\mu_2)bx_t^3(-1) - \varepsilon(\tau_0+\mu_2)x_t^2(0)y_t(0) \end{pmatrix} + \text{h.o.t.},$$

where h.o.t. stands for higher order terms.

As in Faria and Magalhães [10], we consider the enlarged phase space BC of functions from $[-1,0]$ to R^2, which are continuous on $[-1,0)$ and with a possible jump discontinuity at zero. This space can be identified with $C \times R^2$. Thus its elements can be written in the form $\phi = \varphi + X_0 c$, where $\varphi \in C$, $c \in R^2$ and X_0 is the 2×2 matrix-valued function defined by $X_0(\theta) = 0$ for $\theta \in [-1,0)$ and $X_0(0) = I$. In BC, Eq. (6) becomes an abstract ODE,

$$\frac{d}{dt}u = Au + X_0\tilde{F}(u, \mu), \tag{7}$$

where $u \in C$, and A is defined by

$$A: C^1 \to BC, \qquad Au = \dot{u} + X_0\big[L_0 u - \dot{u}(0)\big],$$

and

$$\tilde{F}(u, \mu) = \big[L(\mu) - L_0\big]u + F(u, \mu).$$

By the continuous projection $\pi: BC \longmapsto P, \pi(\varphi + X_0 c) = \Phi[(\Psi, \varphi) + \Psi(0)c]$, we can decompose the enlarged phase space by $\Lambda = \{0, \pm i\tau_0\omega_0\}$ as $BC = P \oplus Ker\,\pi$. Let $u_t = \Phi x(t) + y$, Eq. (7) is therefore decomposed as the system

$$\begin{aligned} \dot{x} &= Bx + \Psi(0)\tilde{F}(\Phi x + y, \mu), \\ \dot{y} &= A_{Q^1}y + (I - \pi)X_0\tilde{F}(\Phi x + y, \mu), \end{aligned} \tag{8}$$

where $y \in Q^1 := Q \cap C^1 \subset Ker\,\pi$, A_{Q^1} is the restriction of A as an operator from Q^1 to the Banach space $Ker\,\pi$. Neglecting higher order terms with respect to parameters μ_1 and μ_2, Eq. (8) can be written as

$$\begin{aligned}
\dot{x}_1 &= \frac{\varepsilon}{\varepsilon-\tau_0}F_2^1 - \frac{1}{\varepsilon-\tau_0}F_2^2 - \frac{1}{\varepsilon-\tau_0}F_3^2 + \text{h.o.t.},\\
\dot{x}_2 &= \mathrm{i}\tau_0\omega_0 x_2 + \overline{D}(\varepsilon - \mathrm{i}\omega_0)F_2^1 - \overline{D}F_2^2 - \overline{D}F_3^2 + \text{h.o.t.},\\
\dot{x}_3 &= -\mathrm{i}\tau_0\omega_0 x_3 + D(\varepsilon + \mathrm{i}\omega_0)F_2^1 - DF_2^2 - DF_3^2 + \text{h.o.t.},\\
\dot{y} &= A_{Q^1}y + (I-\pi)X_0\tilde{F}(\Phi x + y, \mu),
\end{aligned}$$

where

$$\begin{aligned}
F_2^1 &= \mu_2\big(\mathrm{i}\omega_0 x_2 - \mathrm{i}\omega_0 x_3 + y_2(0)\big),\\
F_2^2 &= \big(-1 + \mathrm{i}\omega_0\varepsilon + e^{-\mathrm{i}\tau_0\omega_0}\big)\mu_2 x_2 + \big(-1 - \mathrm{i}\omega_0\varepsilon + e^{\mathrm{i}\tau_0\omega_0}\big)\mu_2 x_3 + \mu_2\big(-y_1(0) + y_1(-1) + \varepsilon y_2(0)\big)\\
&\quad + \varepsilon\tau_0\mu_1\big(x_1 + x_2 e^{-\mathrm{i}\tau_0\omega_0} + x_3 e^{\mathrm{i}\tau_0\omega_0} + y_1(-1)\big),\\
F_3^2 &= \varepsilon\tau_0\big[b\big(x_1 + x_2 e^{-\mathrm{i}\tau_0\omega_0} + x_3 e^{\mathrm{i}\tau_0\omega_0} + y_1(-1)\big)^3 - \big(x_1 + x_2 + x_3 + y_1(0)\big)^2\big(\mathrm{i}\omega_0 x_2 - \mathrm{i}\omega_0 x_3 + y_2(0)\big)\big]\\
&\quad + \varepsilon\mu_1\mu_2\big(x_1 + x_2 e^{-\mathrm{i}\tau_0\omega_0} + x_3 e^{\mathrm{i}\tau_0\omega_0} + y_1(-1)\big).
\end{aligned}$$

Let M_2 denote the operator defined in $V_2^5(\mathbb{C}^3 \times \mathit{Ker}\,\pi)$, with

$$M_2^1 : V_2^5\big(\mathbb{C}^3\big) \longmapsto V_2^5\big(\mathbb{C}^3\big), \quad \text{and} \quad \big(M_2^1 p\big)(x,\mu) = D_x p(x,\mu)Bx - Bp(x,\mu),$$

where $V_2^5(\mathbb{C}^3)$ denotes the linear space of the second order homogeneous polynomials in five variables $(x_1, x_2, x_3, \mu_1, \mu_2)$, and with coefficients in $\mathbb{C}^3$. Then it is easy to check that one may choose the decomposition

$$V_2^5\big(\mathbb{C}^3\big) = \mathit{Im}\big(M_2^1\big) \oplus \mathit{Im}\big(M_2^1\big)^c$$

with complementary space $(\mathit{Im}(M_2^1))^c$ spanned by the elements

$$\begin{pmatrix} x_1^2\\ 0\\ 0\end{pmatrix}; \quad \begin{pmatrix} x_2x_3\\ 0\\ 0\end{pmatrix}; \quad \begin{pmatrix} x_1\mu_i\\ 0\\ 0\end{pmatrix}; \quad \begin{pmatrix} \mu_1^2\\ 0\\ 0\end{pmatrix}; \quad \begin{pmatrix} \mu_2^2\\ 0\\ 0\end{pmatrix}; \quad \begin{pmatrix} \mu_1\mu_2\\ 0\\ 0\end{pmatrix};$$
$$\begin{pmatrix} 0\\ x_1x_2\\ 0\end{pmatrix}; \quad \begin{pmatrix} 0\\ x_2\mu_i\\ 0\end{pmatrix}; \quad \begin{pmatrix} 0\\ 0\\ x_1x_3\end{pmatrix}; \quad \begin{pmatrix} 0\\ 0\\ x_3\mu_i\end{pmatrix}, \quad i = 1, 2.$$

Then the normal form of Eq. (6) on the center manifold of the origin near $\mu = 0$ has the form (see [10])

$$\dot{x} = Bx + \frac{1}{2}g_2^1(x, 0, \mu) + \text{h.o.t.}, \tag{9}$$

where g_2^1 is the function giving the quadratic terms in (x,μ) for $y = 0$, and is determined by $g_2^1(x,0,\mu) = \mathit{Proj}_{(I_m(M_2^1))^c} \times f_2^1(x, k0, \mu)$, where $f_2^1(x,0,\mu)$ is the function giving the quadratic terms in (x,μ) for $y = 0$ defined by the first equation of (8). Then, the normal form in Eq. (2) is truncated to the second order, as

$$\begin{aligned}
\dot{x}_1 &= -\frac{\varepsilon\tau_0}{\varepsilon-\tau_0}\mu_1 x_1 + \text{h.o.t.},\\
\dot{x}_2 &= \mathrm{i}\tau_0\omega_0 x_2 + \overline{D}(\mathrm{i}\varepsilon + 2\omega_0)\omega_0\mu_2 x_2 - \overline{D}\varepsilon\tau_0 e^{-\mathrm{i}\tau_0\omega_0}\mu_1 x_2 + \text{h.o.t.},\\
\dot{x}_3 &= -\mathrm{i}\tau_0\omega_0 x_3 + D(-\mathrm{i}\varepsilon + 2\omega_0)\omega_0\mu_2 x_3 - D\varepsilon\tau_0 e^{\mathrm{i}\tau_0\omega_0}\mu_1 x_3 + \text{h.o.t.}
\end{aligned} \tag{10}$$

Since Eq. (10) is degenerate, we need calculate the higher order normal form. To find the third-order normal form, let M_3 denote the operator defined in $V_3^3(\mathbb{C}^3 \times \mathit{Ker}\,\pi)$, with

$$M_3^1 : V_3^3\big(\mathbb{C}^3\big) \longmapsto V_3^3\big(\mathbb{C}^3\big), \quad \text{and} \quad \big(M_3^1 p\big)(x,\mu) = D_x p(x,\mu)Bx - Bp(x,\mu),$$

where $V_3^3(\mathbb{C}^3)$ denotes the linear space of the third order homogeneous polynomials in three variables (x_1, x_2, x_3), and with coefficients in $\mathbb{C}^3$. Then it is easy to check that one may choose the decomposition

$$V_3^3\big(\mathbb{C}^3\big) = \mathit{Im}\big(M_3^1\big) \oplus \mathit{Im}\big(M_3^1\big)^c$$

with complementary space $(\mathit{Im}(M_3^1))^c$ spanned by the elements

$$\begin{pmatrix} x_1^3\\ 0\\ 0\end{pmatrix}; \quad \begin{pmatrix} x_1x_2x_3\\ 0\\ 0\end{pmatrix}; \quad \begin{pmatrix} 0\\ x_1^2x_2\\ 0\end{pmatrix}; \quad \begin{pmatrix} 0\\ x_2^2x_3\\ 0\end{pmatrix}; \quad \begin{pmatrix} 0\\ 0\\ x_1^2x_3\end{pmatrix}; \quad \begin{pmatrix} 0\\ 0\\ x_2x_3^2\end{pmatrix}.$$

Then we can derive the normal form up to the third order

$$\dot{x} = Bx + \frac{1}{2!}g_2^1(x,0,\mu) + \frac{1}{3!}g_3^1(x,0,0) + \text{h.o.t.}, \tag{11}$$

where

$$\frac{1}{3!}g_3^1(x,0,0) = \frac{1}{3!}\big(I - P_{I,3}^1\big)f_3^1(x,0,0),$$

and $f_3^1(x,0,0)$ is the function giving the cubic terms in (x,μ,y) for $\mu = 0$, $y = 0$ defined by the first equation of (8). Then, Eq. (11) can be written as

$$\begin{aligned}
\dot{x}_1 &= -\frac{\varepsilon\tau_0}{\varepsilon-\tau_0}\mu_1 x_1 - \frac{b\varepsilon\tau_0}{\varepsilon-\tau_0}\big(x_1^3 + 6x_1x_2x_3\big) + \text{h.o.t.},\\
\dot{x}_2 &= \mathrm{i}\tau_0\omega_0 x_2 + \overline{D}(\mathrm{i}\varepsilon + 2\omega_0)\omega_0\mu_2 x_2 - \overline{D}\varepsilon\tau_0 e^{-\mathrm{i}\tau_0\omega_0}\mu_1 x_2 - \overline{D}\varepsilon\tau_0\big[\big(3be^{-\mathrm{i}\tau_0\omega_0} - \mathrm{i}\omega_0\big)\big(x_1^2x_2 + x_2^2x_3\big)\big] + \text{h.o.t.},\\
\dot{x}_3 &= -\mathrm{i}\tau_0\omega_0 x_3 + D(-\mathrm{i}\varepsilon + 2\omega_0)\omega_0\mu_2 x_3 - D\varepsilon\tau_0 e^{\mathrm{i}\tau_0\omega_0}\mu_1 x_3\\
&\quad - D\varepsilon\tau_0\big[\big(3be^{\mathrm{i}\tau_0\omega_0} + \mathrm{i}\omega_0\big)\big(x_1^2x_3 + x_2x_3^2\big)\big] + \text{h.o.t.}
\end{aligned} \tag{12}$$

In the above expressions, the higher order terms in the parameter μ have been omitted.

3. Bifurcation analysis

Let $x_1 = z$, $x_2 = r\cos\theta + \mathrm{i}r\sin\theta$, and $x_3 = r\cos\theta - \mathrm{i}r\sin\theta$. Then Eq. (12) becomes

$$\begin{aligned}
\dot{z} &= -\frac{\varepsilon\tau_0}{\varepsilon-\tau_0}\mu_1 z - \frac{b\varepsilon\tau_0}{\varepsilon-\tau_0}\big(z^3 + 6zr^2\big) + \text{h.o.t.},\\
\dot{r} &= \left[\left(\frac{2d_1}{G}\omega_0^2 - \frac{d_2}{G}\varepsilon\omega_0\right)\mu_2 - \frac{d_3}{G}\varepsilon\tau_0\mu_1\right]r - \varepsilon\tau_0\left(\frac{d_2}{G}\omega_0 + \frac{3d_3}{G}b\right)\big(z^2 + r^2\big)r + \text{h.o.t.},\\
\dot{\theta} &= \tau_0\omega_0 + \left(\frac{d_1}{G}\varepsilon\omega_0 + \frac{2d_2}{G}\omega_0^2\right)\mu_2 - \frac{d_4}{G}\varepsilon\tau_0\mu_1 - \varepsilon\tau_0\left(-\frac{d_1}{G}\omega_0 + \frac{3d_4}{G}b\right)\big(z^2 + r^2\big) + \text{h.o.t.},
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
d_1 &= \varepsilon - \tau_0 + \tau_0\omega_0^2,\\
d_2 &= \omega_0(2 - \tau_0\omega_0),\\
d_3 &= \big(1 - \omega_0^2\big)d_1 + \varepsilon\omega_0 d_2,\\
d_4 &= -\varepsilon\omega_0 d_1 + \big(1 - \omega_0^2\big)d_2,\\
G &= d_1^2 + d_2^2.
\end{aligned}$$

Truncating higher order terms and removing the azimuthal term, we obtain the planar system (see [12])

$$\begin{aligned}
\dot{r} &= r\big(\varepsilon_1 + r^2 + b_0 z^2\big),\\
\dot{z} &= z\big(\varepsilon_2 + c_0 r^2 + d_0 z^2\big),
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\varepsilon_1 &= -\left[\left(\frac{2d_1}{G}\omega_0^2 - \frac{d_2}{G}\varepsilon\omega_0\right)\mu_2 - \frac{d_3}{G}\varepsilon\tau_0\mu_1\right]\operatorname{sign}(d_2\omega_0 + 3d_3 b),\\
\varepsilon_2 &= \frac{\varepsilon\tau_0}{\varepsilon-\tau_0}\mu_1\operatorname{sign}(d_2\omega_0 + 3d_3 b),\\
b_0 &= \left|\frac{(\varepsilon-\tau_0)(d_2\omega_0 + 3d_3 b)}{Gb}\right| = \frac{(\tau_0-\varepsilon)}{G}\left|\frac{1}{b}(d_2\omega_0 + 3d_3 b)\right|,\\
c_0 &= \frac{6bG}{(\varepsilon-\tau_0)(d_2\omega_0 + 3d_3 b)},\\
d_0 &= \operatorname{sign}\left(\frac{b}{\varepsilon-\tau_0}\right)\operatorname{sign}(d_2\omega_0 + 3d_3 b) = -\operatorname{sign}\big[b(d_2\omega_0 + 3d_3 b)\big].
\end{aligned} \tag{15}$$

Based on [12, §7.5], by the different signs of $b_0, c_0, d_0, d_0 - b_0c_0$ in Table 1 Eq. (14) has twelve distinct types of unfoldings, which mean twelve essentially distinct types of phase portraits and bifurcation diagrams.

Table 1
The twelve unfoldings [12].

Case	I*a*	I*b*	II	III	IV*a*	IV*b*	V	VI*a*	VI*b*	VII*a*	VII*b*	VIII
d_0	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	-1	-1	-1
b_0	$+$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$-$	$-$	$-$
c_0	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$-$
$d_0 - b_0c_0$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$-$

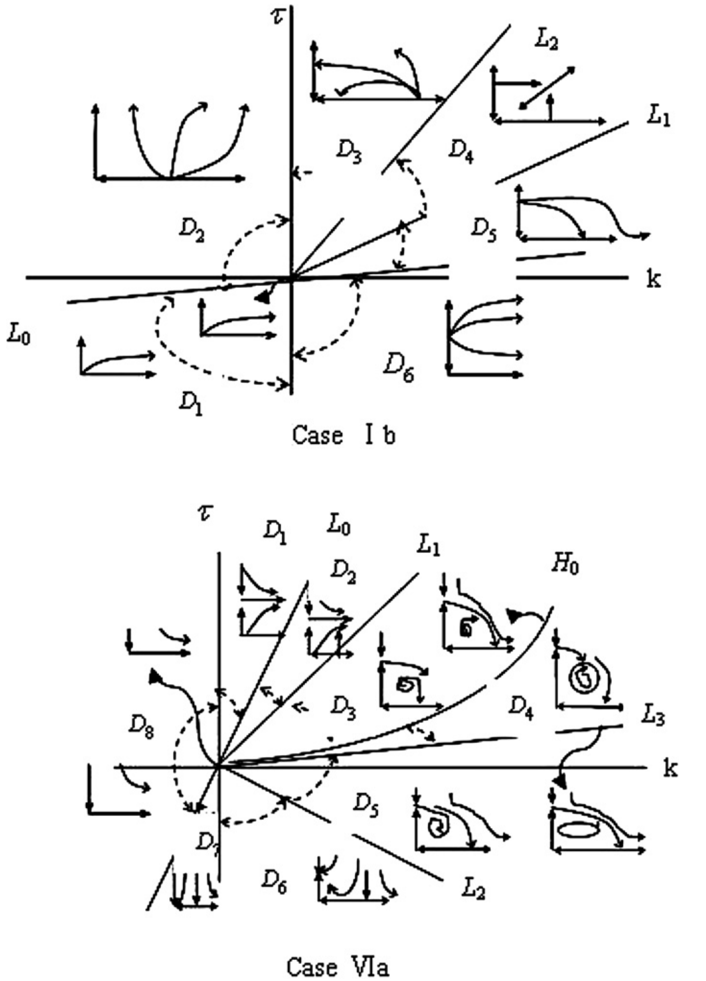

Fig. 1. The bifurcation diagram and phase portraits for Eq. (2) with parameter (k,τ) near the coordinate origin $(\frac{1}{\varepsilon},\tau_0)$.

From (15) we know that $b_0 > 0$, $\mathrm{sign}(c_0) = \mathrm{sign}(d_0)$, $\mathrm{sign}(d_0 - b_0c_0) = -\mathrm{sign}(d_0)$. So, only the cases I*b* and VI*a* arise. Hence, noting that $k = \frac{1}{\varepsilon} + \mu_1$ and $\tau = \tau_0 + \mu_2$, and by Figs. 7.5.2, 7.5.5 and 7.5.7 in [12, §7.5], the phase portraits and bifurcation diagrams of the two cases can be given out and are shown in Fig. 1.

We note that $M_0 = (r,z) = (0,0)$ is always an equilibrium and the other equilibria are

$$M_1 = (\sqrt{-\varepsilon_1}, 0) \quad \text{for } \varepsilon_1 < 0,$$
$$M_2^{\pm} = (0, \pm\sqrt{-\varepsilon_2/d_0}) \quad \text{for } \varepsilon_2 d_0 < 0,$$
$$M_3^{\pm} = \left(\sqrt{\frac{b_0\varepsilon_2 - d_0\varepsilon_1}{d_0 - b_0c_0}}, \pm\sqrt{\frac{c_0\varepsilon_1 - \varepsilon_2}{d_0 - b_0c_0}}\right) \quad \text{for } \frac{b_0\varepsilon_2 - d_0\varepsilon_1}{d_0 - b_0c_0}, \frac{c_0\varepsilon_1 - \varepsilon_2}{d_0 - b_0c_0} > 0.$$

For the τ coordinate axis: $k = \frac{1}{\varepsilon}$ and lines L_0: $\tau = \tau_0 + \frac{d_3\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)}(k - \frac{1}{\varepsilon})$, L_1: $\tau = \tau_0 + \frac{\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)}(d_3 - \frac{G}{c_0(\varepsilon - \tau_0)}) \times (k - \frac{1}{\varepsilon})$, L_2: $\tau = \tau_0 + \frac{\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)}(d_3 - \frac{b_0G}{d_0(\varepsilon - \tau_0)})(k - \frac{1}{\varepsilon})$ are pitchfork bifurcation critical lines; the line L_3: $\tau = \tau_0 + \frac{\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)}(d_3 - \frac{(b_0+1)G}{(c_0-1)(\varepsilon - \tau_0)})(k - \frac{1}{\varepsilon})$ is Hopf bifurcation critical line; on the curve H_0: $\tau = \tau_0 + \frac{\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)} \times$

$(d_3 - \frac{(b_0+1)G}{(c_0-1)(\varepsilon-\tau_0)})(k - \frac{1}{\varepsilon}) + \mathcal{O}((k - \frac{1}{\varepsilon})^2)$ (see [12]), the system undergoes a saddle connection bifurcation, i.e. there is a pair of symmetric heterclinic orbits connecting the two nontrivial saddle points.

Remark. In these phase portraits of Fig. 1, the horizontal axis is the r coordinate, and the vertical axis is the z coordinate. We only draw the orbits in the first quadrant, since the orbits are symmetrical with respect to r coordinate.

According to the center manifold theory [4], Eqs. (13) on the center manifold determine the asymptotic behavior of solutions of the full equations (2) when there exists no unstable manifold containing the trivial solution. And the bifurcation analysis for the three-dimensional system (13) is based on the rotational symmetry. Rotating around the z-axis, correspondences between 2-dimensional flows for (14) and 3-dimensional flows for (13) can be established. So for (13), equilibria on the z-axis in Fig. 1 remain equilibria, while equilibria outside the z-axis become periodic orbits (period $\approx 2\pi/(\tau_0\omega_0)$). Periodic solutions turn into quasi-periodic solutions with two basic periods ($\approx 2\pi/(\tau_0\omega_0)$ and $\mathcal{O}(1/\varepsilon_i)$) which constitutes an invariant two torus (see [12, p. 410]) and the heterclinic orbits turn into the heterclinic orbits which is called the sphere-like surface (see [12, Fig. 7.4.11]).

Furthermore, considering that $u_t = \Phi x$, where u_t is the flow on center manifold of (7), x is the solution of (12) and Φ is expressed as in (5), and by rescaling the time $t \to \frac{t}{\tau}$, the above equilibria, periodic orbits, quasi-periodic solutions and heterclinic orbits of (13) are corresponding with equilibria, periodic solutions (period $\approx 2\pi/\omega_0$) and quasi-periodic solutions with two basic periods ($\approx 2\pi/\omega_0$ and $\mathcal{O}(1/\varepsilon_i)$) of the original system (2), respectively.

From the above discussion, we know that an equilibrium outside the z-axis in Fig. 1 is corresponding to a periodic solution of the original system (2). So, we shall call the periodic solution the source (respectively, saddle, sink) periodic solution of (2) when the equilibrium is a source (respectively, saddle, sink) in Fig. 1.

Hence, for the original system (2), in case Ib the above bifurcation criteria divide the parameter plane (k,τ) into six regions (see Fig. 1). In region D_1, there is only one trivial equilibrium which is unstable; when the parameters vary across the line L_0 from region D_1 to D_2, the trivial equilibrium becomes a saddle point, and an unstable periodic solution (source) is bifurcated; with the variation of the parameters from region D_2 to D_3, the trivial equilibrium becomes a sink, and two nontrivial saddle points are bifurcated; in region D_4, the two nontrivial saddle points become sources, and two unstable periodic solutions (saddle) are bifurcated; when the parameters vary across the line L_1 from region D_4 to D_5, three periodic solutions overlap and become a periodic solution (saddle); from region D_5 to D_6, the periodic solution disappears, while the stable trivial equilibrium becomes a saddle point.

In case VIa the above bifurcation criteria divide the parameter plane (k,τ) into eight regions (see Fig. 1). In region D_1, the trivial equilibrium is a source, and two nontrivial equilibria are saddle points; when the parameters vary across the line L_0 from region D_1 to D_2, the trivial equilibrium becomes a saddle point, and an unstable limit cycle (source) is bifurcated; in region D_3 the above unstable periodic solution becomes a saddle from a source, and two unstable periodic solutions (source) are bifurcated; on H_0, there is a pair of symmetric heterclinic orbits, each of which connects a nontrivial saddle point and the saddle periodic solution; in region D_4, there are two quasi-periodic motions, attractors, which are bifurcated from the two source periodic solutions, respectively; on L_3, there is also a pair of symmetric heterclinic orbits, each of which connects a nontrivial saddle point and the saddle periodic solution; in region D_5, two stable periodic solutions appear, while the above quasi-periodic motions disappear; from region D_5 to D_6, two stable periodic solutions disappear and the trivial equilibrium becomes a sink; in region D_7, two stable nontrivial equilibria overlap and become the origin which is a sink; next, in region D_8 the unstable periodic solution disappears and the trivial equilibrium becomes a saddle point.

Summarizing the above analysis we obtain the following conclusions.

Proposition. *System* (2) *undergoes a Hopf-pitchfork bifurcation at the origin when* $0 < \varepsilon < \sqrt{2}$, $k = \frac{1}{\varepsilon}$ *and* $\tau = \tau_0$. *And for a given odd function* $g(x)$ *with* $g(0) = g''(0) = 0$, $g'(0) = k$, $g'''(0) = 3!b \neq 0$, *we have*

(1) *If* $0 < \varepsilon < \sqrt{2}$ *and* $b(d_2\omega_0 + 3d_3b) < 0$, *then bifurcation phenomena of case* Ib *occur near* $(x, y, k, \tau) = (0, 0, \frac{1}{\varepsilon}, \tau_0)$.

(2) *If* $0 < \varepsilon < \sqrt{2}$ *and* $b(d_2\omega_0 + 3d_3b) > 0$, *then bifurcation phenomena of case* VIa *occur near* $(x, y, k, \tau) = (0, 0, \frac{1}{\varepsilon}, \tau_0)$. *Particularly, some interesting phenomena are as follows.*
   - (a) *The trivial equilibrium is asymptotically stable when* $\tau < \tau_0 + \frac{d_3\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)}(k - \frac{1}{\varepsilon})$ *and* $k < 0$ (*that is* $(k,\tau) \in D_7$);
   - (b) *There are two asymptotically stable nontrivial equilibria which are coexisted when* $\tau < \tau_0 + \frac{\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)}(d_3 - \frac{b_0G}{d_0(\varepsilon-\tau_0)}) \times (k - \frac{1}{\varepsilon})$ *and* $k > 0$ (*that is* $(k,\tau) \in D_6$);
   - (c) *There are two stable nontrivial periodic orbits which are coexisted when* $\tau_0 + \frac{\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)}(d_3 - \frac{b_0G}{d_0(\varepsilon-\tau_0)})(k - \frac{1}{\varepsilon}) < \tau < \tau_0 + \frac{\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)}(d_3 - \frac{(b_0+1)G}{(c_0-1)(\varepsilon-\tau_0)})(k - \frac{1}{\varepsilon})$ *and* $k > 0$ (*that is* $(k,\tau) \in D_5$);
   - (d) *The two nontrivial periodic orbits undergo a secondary Hopf bifurcation, giving rise to the appearance of quasi-periodic motions when* $\tau = \tau_0 + \frac{\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)}(d_3 - \frac{(b_0+1)G}{(c_0-1)(\varepsilon-\tau_0)})(k - \frac{1}{\varepsilon})$ *and* $k > 0$ (*that is* $(k,\tau) \in L_3$), *respectively*;
   - (e) *There are two attractive quasi-periodic motions which are coexisted when* $\tau_0 + \frac{\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)}(d_3 - \frac{(b_0+1)G}{(c_0-1)(\varepsilon-\tau_0)})(k - \frac{1}{\varepsilon}) < \tau < \tau_0 + \frac{\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)}(d_3 - \frac{(b_0+1)G}{(c_0-1)(\varepsilon-\tau_0)})(k - \frac{1}{\varepsilon}) + \mathcal{O}((k - \frac{1}{\varepsilon})^2)$ *and* $k > 0$ (*that is* $(k,\tau) \in D_4$);
   - (f) *There is a pair of symmetric heterclinic orbits, each of which connects a nontrivial saddle point and the saddle periodic solution when* $\tau = \tau_0 + \frac{\varepsilon\tau_0}{\omega_0(2d_1\omega_0 - d_2\varepsilon)}(d_3 - \frac{(b_0+1)G}{(c_0-1)(\varepsilon-\tau_0)})(k - \frac{1}{\varepsilon}) + \mathcal{O}((k - \frac{1}{\varepsilon})^2)$ *and* $k > 0$ (*that is* $(k,\tau) \in H_0$).

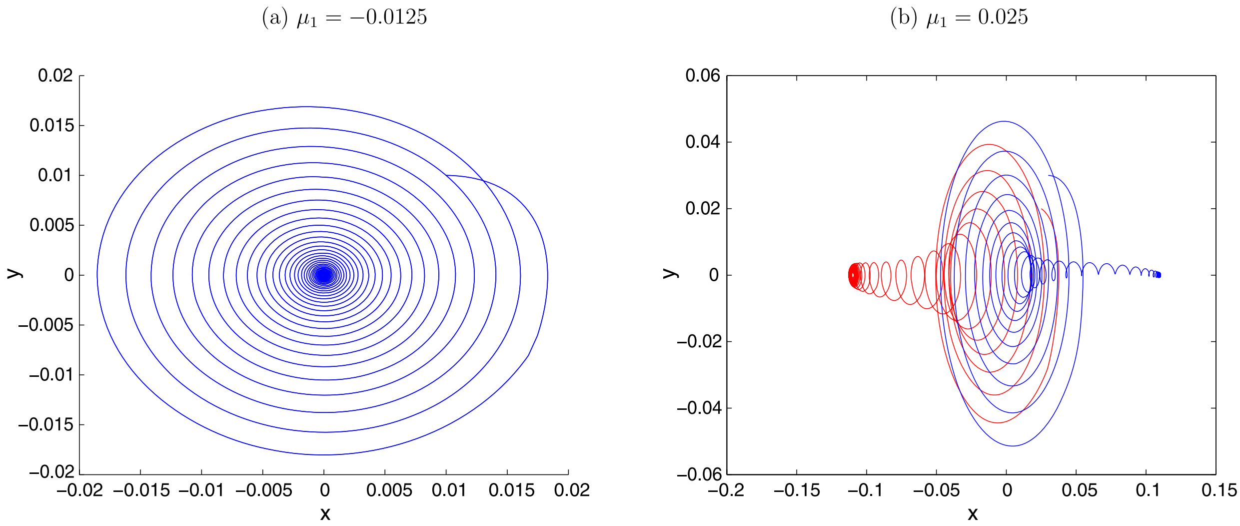

Fig. 2. (a) The stable trivial equilibrium. (b) Two asymptotically stable nontrivial equilibria are coexisted.

4. Numerical simulation

The above analytical information is a useful starting point for the use of adequate numerical tools. We choose

$$g(x) = a_1\left(-1 + \frac{2}{1+e^{-a_2 x}}\right)$$

in Eq. (2), then $g(0) = g''(0) = 0$, $k = g'(0) = \frac{1}{2}a_1a_2$ and $g'''(0) = -\frac{1}{4}a_1a_2^3$. All the numerical work is considered only in the case VIa. The aim is to show some interesting behavior including attractive equilibria, periodic solutions and quasi-periodic behavior. If $\varepsilon = 1$, $a_1 = 0.4$ and $a_2 = 5$, then we get $k = 1 = \frac{1}{\varepsilon}$, $b = -2.0833$, $\tau_0 = 1.5708$, $\omega_0 = 1$. Furthermore we have $d_1 = 1$, $d_2 = d_3 = 0.4292$, $d_4 = -1$, $G = 1.1842$. Hence, it follows that

$$d_0 = -1,$$
$$b_0 = 0.5213 > 0,$$
$$c_0 = -11.5090 < 0,$$
$$d_0 - b_0c_0 = 5 > 0.$$

We know from Table 1 that bifurcation phenomena with the case VIa appear near $(k,\tau) = (\frac{1}{\varepsilon}, \tau_0)$ when $\varepsilon = 1$, $a_1 = 0.4$, $a_2 = 5$ and $\tau = \tau_0 = 1.5708$ in Eq. (2). Here, in Fig. 1 bifurcation critical lines are, respectively,

$$L_0\colon\ \tau = 1.5708 + 0.4292(k-1),$$
$$L_1\colon\ \tau = 1.5708 + 0.2489(k-1),$$
$$L_2\colon\ \tau = 1.5708 - 0.6524(k-1),$$
$$L_3\colon\ \tau = 1.5708 + 0.1769(k-1),$$
$$H_0\colon\ \tau = 1.5708 + 0.1769(k-1) + \mathcal{O}\big((k-1)^2\big).$$

In the following, fix $\varepsilon = 1$, $a_2 = 5$, and let $k = 1 + \mu_1$, $\tau = 1.5708 + \mu_2$.

(i) Fix $\mu_2 = -0.0308$. If $\mu_1 = -0.0125$, then $(k,\tau) \in D_7$. Fig. 2(a) shows that the zero solution of Eq. (2) is asymptotically stable, and the initial value is $(x_0, y_0) = (0.01, 0.01)$. If $\mu_1 = 0.025$, then $(k,\tau) \in D_6$, there are two asymptotically stable nontrivial equilibria in Eq. (2) which are coexisted (see Fig. 2(b)), and the initial values are $(x_0, y_0) = (0.025, 0.02)$ (red) and $(0.03, 0.03)$ (blue).

(ii) When $\mu_1 = 0.025$, $\mu_2 = 0.0044$, then $(k,\tau) \in D_5$. Fig. 3 shows that two stable nontrivial periodic solutions of Eq. (2) are coexisted. The initial values are $(x_0, y_0) = (0.05, 0.02)$ (red) and $(0.01, 0.02)$ (blue), and simulation time is from 0 to 500 and from 500 to 1000, respectively.

(iii) When $\mu_1 = 0.025$, $\mu_2 = 0.0062$, then $(k,\tau) \in D_4$. Fig. 4 shows that two attractive quasi-periodic motions of Eq. (2) are coexisted. The initial values are $(x_0, y_0) = (-0.001, -0.001)$ (red) and $(0.001, 0.001)$ (blue), and simulation time is $0 \sim 500$, $500 \sim 1000$, $1000 \sim 1500$ and $3000 \sim 3500$, respectively.

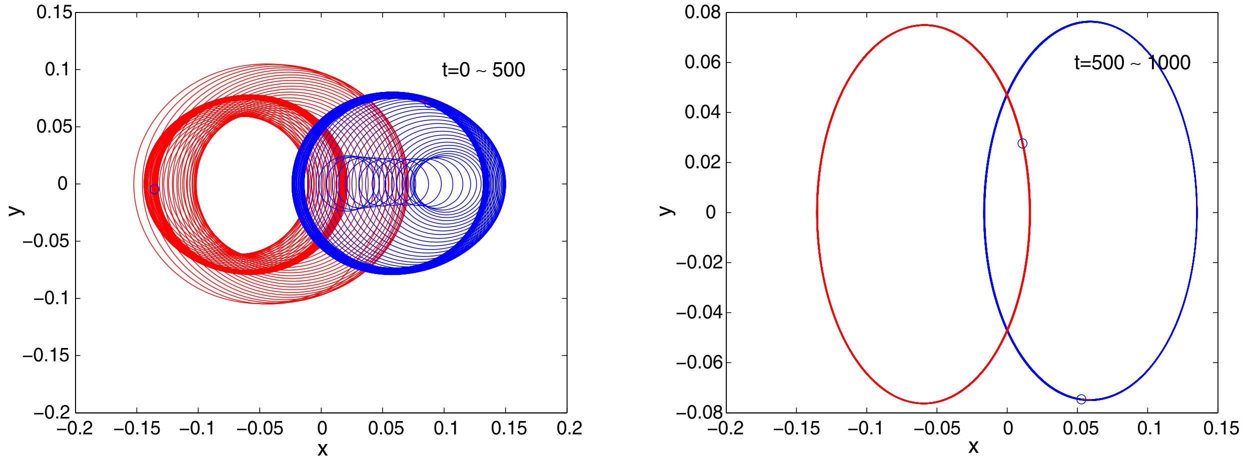

Fig. 3. Two stable nontrivial periodic solutions are coexisted.

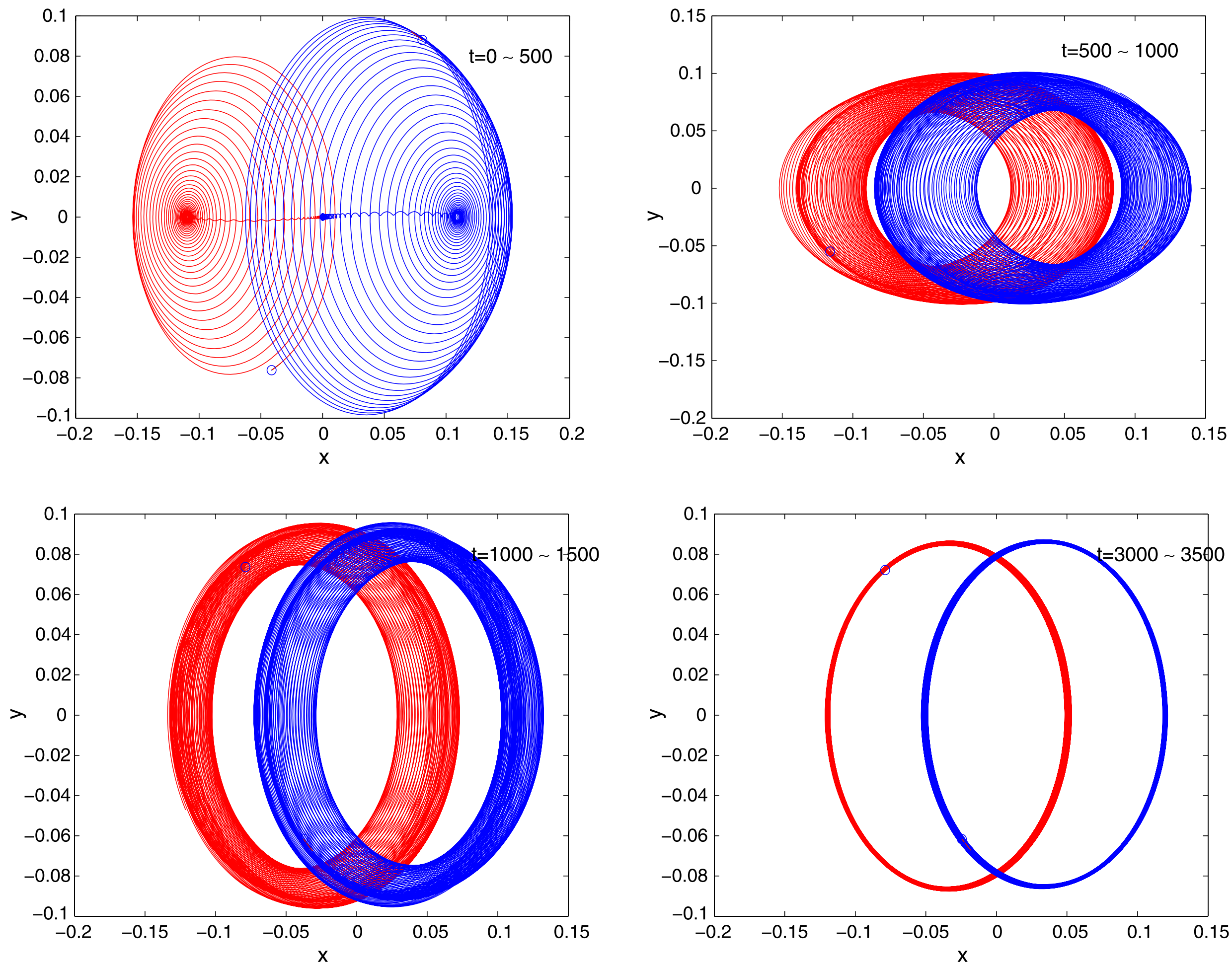

Fig. 4. Two attractive quasi-periodic motions are coexisted.

5. Conclusion

In this paper, we investigate the Hopf-pitchfork bifurcation of the zero solution for van der Pol's oscillator with delayed nonlinear feedback. Using the normal form method for FDE in Faria and Magalhaes [10], and the center manifold theory in [4], we have derived the normal form of the reduced system on the center manifold, discussed the Hopf-pitchfork bifurcation with the parameter perturbations in Eq. (2), and completely determined the stability and bifurcation of the zero solution near the critical value. We can obtain the coexistence of two asymptotically stable states, two stable periodic orbits and two attractive quasi-periodic motions by choosing suitable feedback control. Our work is a further study of [13,14,20], it is a helpful trying for study the complex phenomena caused by high co-dimensional bifurcation of delay differential equation.

References

[1] A. Algaba, F. Fernández-Sánchez, E. Freire, E. Gamero, A.J. Rodríguez-Luis, Oscillation-sliding in a modified van der Pol–Duffing electronic oscillator, J. Sound Vibration 249 (5) (2002) 899–907.
[2] M. Ashkenazi, S.N. Chow, Normal forms near critical points for differential equations and maps, IEEE Trans. Circuits Syst. 35 (1988) 850–862.
[3] F.M. Atay, Van der Pol's oscillator under delayed feedback, J. Sound Vibration 218 (22) (1998) 333–339.
[4] J. Carr, Applications of Centre Manifold Theory, Springer-Verlag, New York, 1981.
[5] S.N. Chow, C. Li, D. Wang, Normal Forms and Bifurcation of Planar Vector Fields, Cambridge University Press, New York, 1994.
[6] L.O. Chua, H. Kokubu, Normal forms for nonlinear vector fields – Part I: Theory and algorithm, IEEE Trans. Circuits Syst. 35 (1988) 863–880.
[7] L.O. Chua, H. Kokubu, Normal forms for nonlinear vector fields – Part II: Applications, IEEE Trans. Circuits Syst. 36 (1988) 51–70.
[8] T. Erneux, J. Grasman, Limit-cycle oscillators subject to a delayed feedback, Phys. Rev. E 78 (2) (2008) 026209.
[9] T. Faria, On a planar system modelling a neuron network with memory, J. Differential Equations 168 (2000) 129–149.
[10] T. Faria, L.T. Magalhães, Normal forms for retarded functional differential equations and applications to Bogdanov–Takens singularity, J. Differential Equations 122 (1995) 201–224.
[11] J. Guckenheimer, K. Hoffman, W. Weckesser, The forced van der Pol equation I: The slow flow and its bifurcation, SIAM J. Appl. Dyn. Syst. 2 (1) (2003) 1–35.
[12] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983.
[13] W. Jiang, J. Wei, Bifurcation analysis in van der Pol's oscillator with delayed feedback, J. Comput. Appl. Math. 213 (2) (2008) 604–615.
[14] W. Jiang, Y. Yuan, Bogdanov–Takens singularity in van der Pol's oscillator with delayed feedback, Phys. D 227 (2) (2007) 149–161.
[15] J. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.
[16] S. Ma, Q. Lu, Z. Feng, Double Hopf bifurcation for van der Pol–Duffing oscillator with parametric delay feedback control, J. Math. Anal. Appl. 338 (2008) 993–1007.
[17] L. Olien, J. Blair, Bifurcations, stability, and monotonicity properties of a delayed neural network model, Phys. D 102 (1997) 349–363.
[18] J.C.F. Oliveira, Oscillations in a van der Pol equation with delayed argument, J. Math. Anal. Appl. 275 (2002) 789–803.
[19] B.F. Redmond, V.G. LeBlanc, A. Longtin, Bifurcation analysis of a class of first-order nonlinear delay-differential equations with reflectional symmetry, Phys. D 166 (2002) 131–146.
[20] J. Wei, W. Jiang, Stability and bifurcation analysis in van der Pol's oscillator with delayed feedback, J. Sound Vibration 283 (2005) 801–819.
[21] D. Xiao, M. Han, Dynamics in a ratio-dependent predator-prey model with predator harvesting, J. Math. Anal. Appl. 324 (2006) 14–29.
[22] D. Xiao, S. Ruan, Multiple bifurcations in a delayed predator-prey system with non-monotonic functional response, J. Differential Equations 176 (2001) 494–510.
[23] J. Xu, K.W. Chung, Effects of time delayed position feedback on a van der Pol–Duffing oscillator, Phys. D 180 (2003) 17–39.
[24] X. Xu, H. Hu, H. Wang, Stability, bifurcation and chaos of a delayed oscillator with negative damping and delayed feedback control, Nonlinear Dynam. 49 (2007) 117–129.
[25] X. Yan, Bifurcation analysis in a simplified tri-neuron BAM network model with multiple delays, Nonlinear Anal. Real World Appl. 9 (3) (2008) 963–976.
[26] Y. Yuan, J. Wei, Multiple bifurcation analysis in a neural network model with delays, Internat. J. Bifur. Chaos 16 (2006) 2903–2913.
[27] Y. Yuan, J. Wei, Singularity analysis on a planar system with multiple delays, J. Dynam. Differential Equations 19 (2) (2007) 437–456.
[28] Y. Yuan, P. Yu, Computation of simplest normal forms of differential equations associated with a double-zero eigenvalues, Internat. J. Bifur. Chaos 11 (5) (2001) 1307–1330.