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Convergence of sequences of two-dimensional Fejér means of trigonometric Fourier series of integrable functions

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ABSTRACT

The aim of this paper is to prove the a.e. convergence of sequences of the Fejér means of the trigonometric Fourier series of two variable integrable functions. That is, let $a = (a_1, a_2) : \mathbb{N} \to \mathbb{N}^2$ such that $a_j(n+1) \geqslant \alpha \sup_{k \leqslant n} a_j(k)$ $(j = 1, 2,\ n \in \mathbb{N})$ for some $\alpha > 0$ and $a_1(+\infty) = a_2(+\infty) = +\infty$. Then for each integrable function $f \in L^1(\mathbb{T}^2)$ we have the a.e. relation $\lim_{n\to\infty} \sigma_{a(n)} f = f$. It will be a straightforward and easy consequence of this result the historical cone restricted a.e. convergence result with respect to the two-dimensional Fejér means of integrable functions due to Marcinkiewicz and Zygmund (1939) [7].

First, we give a brief introduction to the theory of the Fourier series. Let $\mathbb{N}$ denote the set of natural numbers, that is, $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{P} = \mathbb{N} \setminus \{0\}$.

The system of functions

$$e^{\imath nx} \quad (n = 0, \pm 1, \pm 2, \dots)$$

$(x \in \mathbb{R},\ \imath = \sqrt{-1})$ is called the trigonometric system. It is orthogonal over any interval of length 2π, specially over $\mathbb{T} := [-\pi, \pi)$. Let $f \in L^1(\mathbb{T})$, that is integrable on $\mathbb{T}$. The kth Fourier coefficient of f is

$$\hat{f}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-\imath kt}\, dt,$$

where k is any integer number. The nth $(n \in \mathbb{N})$ partial sum of the Fourier series of f is

$$S_n f(y) := \sum_{k=-n}^{n} \hat{f}(k) e^{\imath ky}.$$

The nth $(n \in \mathbb{N})$ Fejér or $(C, 1)$ mean of function f is defined in the following way:

$$\sigma_n f(y) := \frac{1}{n+1} \sum_{k=0}^{n} S_k f(y).$$

It is known that

$$\sigma_n f(y) = \frac{1}{\pi} \int_{T} f(x) K_n(y - x)\, dx,$$

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where the function K_n is known as the nth Fejér kernel; we find an appropriate expression for it e.g. in the book of Bary [1].

$$K_n(u) = \frac{1}{2(n+1)}\left(\frac{\sin(\frac{u}{2}(n+1))}{\sin(\frac{u}{2})}\right)^2.$$

From this expression one immediately derives the following properties of the kernel. They will play an essential role later.

$$K_n(u) \geqslant 0,$$
$$K_n(u) \leqslant \frac{\pi^2}{2(n+1)u^2} \quad \bigl(0 < |u| \leqslant \pi\bigr).$$

Let f be an integrable function, that is, let $f \in L^1(\mathbb{T}^2)$. The $k = (k_1, k_2)$th Fourier coefficient of f is

$$\hat{f}(k) = \hat{f}(k_1, k_2) := \frac{1}{4\pi^2}\int_{\mathbb{T}\times\mathbb{T}} f(x_1, x_2)e^{-\imath(k_1t_1+k_2t_2)}\,d(t_1, t_2),$$

where k_1, k_2 are integers. The nth ($n \in \mathbb{N}^2$) rectangular partial sum of the Fourier series of f is

$$S_n f(y) = S_{(n_1,n_2)} f(y_1, y_2) := \sum_{k_1=-n_1}^{n_1}\sum_{k_2=-n_2}^{n_2} \hat{f}(k_1, k_2)e^{\imath(k_1y_1+k_2y_2)}.$$

The nth ($n \in \mathbb{N}^2$) two-dimensional Fejér or $(C, 1)$ mean of function f is defined in the following way:

$$\sigma_n f(y) = \sigma_{(n_1,n_2)} f(y) := \frac{1}{(n_1+1)(n_2+1)}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} S_k f(y),$$

where $y \in \mathbb{T}^2$. In 1939 Marcinkiewicz and Zygmund [7] proved their celebrated theorem on the convergence of the two-dimensional restricted $(C, 1)$ means of trigonometric Fourier series. They proved for any integrable function $f \in L^1(\mathbb{T}^2)$ the a.e. convergence

$$\sigma_{(n_1,n_2)} f \to f$$

provided $n_1/\beta \leqslant n_2 \leqslant \beta n_1$, where $\beta > 1$ is a fixed constant. So, the set of indices (n_1, n_2) remains in some positive cone around the identical function. Actually, their proof is not a simple one. Among others, the main theorem of this paper, that is, Theorem 1 provides an easy proof for this celebrated result of Marcinkiewicz and Zygmund.

We also mention that Jessen, Marcinkiewicz and Zygmund [8] also proved the a.e. convergence $\sigma_n f \to f$ without any restriction on the indices (other than $\min\{n_1, n_2\} \to \infty$), but for functions in $L\log^+ L$. For a joint generalization of these results of Marcinkiewicz–Zygmund and Jessen–Marcinkiewicz–Zygmund see the paper of the author [4]. For another proof of the "cone restricted" convergence of the two-dimensional Fejér means see the paper of Weisz [10] and the result of Marcinkiewicz and Zygmund with respect to the multi-dimensional case was also proved with a different proof by Weisz [11]. With respect to this issue one can find some interesting and important read in [6] and [5].

For another modern treatise on the theory of Fourier series see for instance the book of Edwards [3].

We study the a.e. convergence of subsequences of the two-dimensional $(C, 1)$ means $\sigma_{a(n)} f$ of integrable functions, that is, $f \in L^1(\mathbb{T}^2)$, where $a : \mathbb{N} \to \mathbb{N}^2$.

Theorem 1. *Let $a = (a_1, a_2) : \mathbb{N} \to \mathbb{N}^2$ be a sequence with property $a_j(+\infty) = +\infty$ $(j = 1, 2)$. Suppose that there exists an $\alpha > 0$ such that $a_j(n+1) \geqslant \alpha \sup_{k\leqslant n} a_j(k)$ $(j = 1, 2,\ n \in \mathbb{N})$. Then for each integrable function $f \in L^1(\mathbb{T}^2)$ we have the a.e. relation*

$$\lim_{n\to\infty} \sigma_{a(n)} f = f.$$

This theorem, which is the main result of this paper is a consequence of the following lemma.

Lemma 2. *Let $a = (a_1, a_2) : \mathbb{N} \to \mathbb{N}^2$ be a sequence with property $a_j(+\infty) = +\infty$ $(j = 1, 2)$. Suppose that $\lfloor \log_2 a_j \rfloor$ ($\lfloor x \rfloor$ denotes the lower integer part of x) is monotone increasing $(j = 1, 2)$. Then for each integrable function $f \in L^1(\mathbb{T}^2)$ we have the a.e. relation*

$$\lim_{n\to\infty} \sigma_{a(n)} f = f.$$

A straightforward and easy consequence of Lemma 2 is the celebrated result of Marcinkiewicz and Zygmund [7] with respect to the "cone restricted" almost everywhere convergence of two-dimensional Fejér means of integrable function.

Corollary 3. *Let $\beta > 1$ and $f \in L^1(I^2)$. Then we have the a.e. relation*

$$\lim_{\substack{n_1,n_2\to\infty \\ 1/\beta\leqslant n_1/n_2\leqslant\beta}} \sigma_{(n_1,n_2)} f = f.$$

Proof. The proof of this corollary comes directly from Lemma 2. So, let $\gamma := \lceil \log_2 \beta \rceil$. For $k, l \in \mathbb{N}$ set $N_{\gamma,l,k} := \{(n_1, n_2) \in \mathbb{N}^2\colon 2^k \leqslant n_1 < 2^{k+1}, 2^{k-\gamma+l} \leqslant n_2 < 2^{k-\gamma+l+1}\}$. Let $N_{\gamma,l}$ be the union of the disjoint sets $N_{\gamma,l,k}$. It is easy to give a sequence $a : \mathbb{N} \to \mathbb{N}^2$ such that $\lfloor \log_2 a_1 \rfloor$, $\lfloor \log_2 a_2 \rfloor$ are monotone increasing (for $n \in N_{\gamma,l,k}$ we have $\lfloor \log_2 n_1 \rfloor = k$, $\lfloor \log_2 n_2 \rfloor = k - \gamma + l$) and $a(\mathbb{N}) = N_{\gamma,l}$. This by Lemma 2 gives that for each integrable function f

$$\sigma_{(n_1,n_2)} f \to f$$

a.e. provided by $n \in N_{\gamma,l}$ and $n_1, n_2 \to \infty$. Hence, we also have this a.e. relation for $n \in \bigcup_{l=0}^{2\gamma} N_{\gamma,l} =: N_\gamma$ and $n_1, n_2 \to \infty$. After then, let $n \in \mathbb{P}^2$ be such that $1/\beta \leqslant n_1/n_2 \leqslant \beta$. Denote by k the natural number for which $2^k \leqslant n_1 < 2^{k+1}$. Then, $2^{k-\gamma} \leqslant 2^k/\beta \leqslant n_2 < 2^{k+1}\beta \leqslant 2^{k+\gamma+1}$. Consequently, $n \in N_\gamma$. This completes the proof of this corollary. □

Extend the map $a : \mathbb{N} \to \mathbb{N}^2$ to $[1, +\infty)$ linearly in a way that for $n \leqslant x \leqslant n + 1$ set

$$\begin{aligned}
\tilde{a}(x) &= \bigl(\tilde{a}_1(x), \tilde{a}_2(x)\bigr) \\
&= \bigl((x-n)a_1(n+1) + (n+1-x)a_1(n), (x-n)a_2(n+1) + (n+1-x)a_2(n)\bigr) \\
&= (x-n)a(n+1) + (n+1-x)a(n).
\end{aligned}$$

Without the loss of generality we can suppose from now that $a_j(n) \geqslant 1$ for $n \in \mathbb{N}$ and $j = 1, 2$. Also set $\beta(x) = \lfloor \log_2(\tilde{a}(x)) \rfloor : [1, +\infty) \to \mathbb{N}^2$.

The following Calderon–Zygmund type decomposition lemma on $\mathbb{T}^2$ will play a fundamental role in the proof of Lemma 2.

The dyadic subintervals of $\mathbb{T}$ are defined in the following way:

$$\begin{aligned}
&\mathcal{I}_0 := \{\mathbb{T}\}, \qquad \mathcal{I}_1 := \bigl\{[-\pi, 0), [0, \pi)\bigr\}, \\
&\mathcal{I}_2 := \bigl\{[-\pi, -\pi/2), [-\pi/2, 0), [0, \pi/2), [\pi/2, \pi)\bigr\}, \quad \ldots \\
&\mathcal{I} := \bigcup_{n=0}^{\infty} \mathcal{I}_n.
\end{aligned}$$

The elements of $\mathcal{I}$ are said to be dyadic intervals. If $F \in \mathcal{I}$, then there exists a unique $n \in \mathbb{N}$ such that $F \in \mathcal{I}_n$, and consequently $\mathrm{mes}(F) = \frac{2\pi}{2^n}$ (the Lebesgue measure). Each $\mathcal{I}_n$ has 2^n disjoint elements ($n \in \mathbb{N}$). $\mathcal{I} \times \mathcal{I}$ is the set of dyadic rectangles. For $x \in \mathbb{T}$ denote by $I_n(x)$ the element of $\mathcal{I}_n$ for which $x \in I_n(x)$.

Lemma 4. *Let $f \in L^1(\mathbb{T}^2)$, and $\lambda > \|f\|_1/(2\pi)^2$. Suppose that the functions $\beta_j(x) = \lfloor \log_2 \tilde{a}_j(x) \rfloor : [1, +\infty) \to \mathbb{N}$ are monotone increasing, where functions $\tilde{a}_j$ are continuous ($j = 1, 2$). Then there exists a sequence of integrable functions (f_i) and disjoint rectangles $I_{\beta_1(s_i)}(u_{i,1}) \times I_{\beta_2(s_i)}(u_{i,2}) \in \mathcal{I}_{\beta_1(s_i)} \times \mathcal{I}_{\beta_2(s_i)}$ such that*

$$\begin{aligned}
&f = \sum_{i=0}^{\infty} f_i, \\
&\|f_0\|_\infty \leqslant C_\beta \lambda, \qquad \|f_0\|_1 \leqslant 3\|f\|_1, \quad \textit{and} \\
&\operatorname{supp} f_i \subset I_{\beta_1(s_i)}(u_{i,1}) \times I_{\beta_2(s_i)}(u_{i,2})
\end{aligned}$$

for some $s_i \geqslant 1$, $u_i \in \mathbb{T}^2$ ($i \in \mathbb{P}$). Moreover, $\int_{\mathbb{T}^2} f_i(x)\,dx = 0$ ($i \geqslant 1$) and for

$$F := \bigcup_{i=1}^{\infty} \bigl(I_{\beta_1(s_i)}(u_{i,1}) \times I_{\beta_2(s_i)}(u_{i,2})\bigr) \quad \textit{we have} \quad \mathrm{mes}(F) \leqslant \|f\|_1/\lambda.$$

Proof. Let $s_1 := 1$ and

$$\Omega_1 := \left\{ J = J_1 \times J_2 \in \mathcal{I}_{\beta_1(s_1)} \times \mathcal{I}_{\beta_2(s_1)}\colon \mathrm{mes}(J)^{-1} \int_J \bigl|f(x)\bigr|\,dx > \lambda \right\}.$$

Since, for each $J \in \Omega_1$, we have

$$\mathrm{mes}(J)^{-1} = \frac{2^{\beta_1(1)+\beta_2(1)}}{4\pi^2},$$

then we also have

$$\lambda < \mathrm{mes}(J)^{-1} \int_J \left|f(x)\right| dx \leqslant 2^{\beta_1(1)+\beta_2(1)} \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \left|f(x)\right| dx < 2^{\beta_1(1)+\beta_2(1)} \lambda \leqslant C_\beta \lambda.$$

Let $s_2 := \inf\{s \in [s_1, +\infty)\colon \sum_{j=1}^2 |\beta_j(s) - \beta_j(s_1)| \geqslant 1\}$. Recall that the function $\tilde{a}$ is continuous, and $\beta(x) = \lfloor \log_2 \tilde{a}(x) \rfloor$ is monotone increasing and continuous from the right with respect to its both variables. Then we have the following three cases:

Case 1. $\beta_1(s_2) = \beta_1(s_1) + 1$ and $\beta_2(s_2) = \beta_1(s_1)$,
Case 2. $\beta_1(s_2) = \beta_1(s_1)$ and $\beta_2(s_2) = \beta_1(s_1) + 1$,
Case 3. $\beta_1(s_2) = \beta_1(s_1) + 1$ and $\beta_2(s_2) = \beta_1(s_1) + 1$.

We decompose the dyadic rectangles contained in

$$[\mathcal{I}_{\beta_1(s_1)} \times \mathcal{I}_{\beta_2(s_1)}] \setminus \{J\colon J \in \Omega_1\}.$$

That is,

$$\Omega_2 := \left\{ J \in \mathcal{I}_{\beta_1(s_2)} \times \mathcal{I}_{\beta_2(s_2)}\colon \mathrm{mes}(J)^{-1} \int_J \left|f(x)\right| dx > \lambda \text{ and } \nexists K \in \Omega_1 \text{ such as } J \subset K \right\}.$$

Consequently, for all $J \in \Omega_2$ we get

$$\lambda < \mathrm{mes}(J)^{-1} \int_J \left|f(x)\right| dx \leqslant 4\lambda.$$

(In Cases 1 and 2 we even have 2λ, but it makes no problem to take 4λ, instead.) Generally, for $\mathbb{N} \ni n \geqslant 3$

$$s_n := \inf\left\{ s \in [s_{n-1}, +\infty)\colon \sum_{j=1}^2 \left|\beta_j(s) - \beta_j(s_{n-1})\right| \geqslant 1 \right\}.$$

That is, $\beta_j(s_n) = \beta_j(s_{n-1}) + 1$ for at least one j ($j = 1, 2$). If for a j this is not valid, then $\beta_j(s_n) = \beta_j(s_{n-1})$. Also take

$$\Omega_n := \left\{ J \in \mathcal{I}_{\beta_1(s_n)} \times \mathcal{I}_{\beta_2(s_n)}\colon \mathrm{mes}(J)^{-1} \int_J \left|f(x)\right| dx > \lambda \text{ and } \nexists K \in \bigcup_{i=1}^{n-1} \Omega_i \text{ such as } J \subset K \right\}.$$

Similarly, as in the case of Ω_2 we have that for each $J \in \Omega_n$ the inequalities

$$\lambda < \mathrm{mes}(J)^{-1} \int_J \left|f(x)\right| dx \leqslant 4\lambda$$

hold. Denote by $l_n \in \mathbb{N}$ the number of elements of Ω_n, and the elements of Ω_n by $J_{n,k}$ ($k = 1, \ldots, l_n$, $n \in \mathbb{N}$). Since $\mathcal{I}_{\beta_1(s_n)} \times \mathcal{I}_{\beta_2(s_n)}$ has $2^{\beta_1(s_n)+\beta_2(s_n)}$ (disjoint) elements, then $l_n \leqslant 2^{\beta_1(s_n)+\beta_2(s_n)}$ ($n \in \mathbb{N}$). For an arbitrary set $B \subset \mathbb{T}^2$ the characteristic function of B is denoted by 1_B. Let

$$f_{n,k} := \left(f - \mathrm{mes}(J_{n,k})^{-1} \int_{J_{n,k}} f(x)\, dx \right) 1_{J_{n,k}},$$

$k = 1, \ldots, l_n$, $n \in \mathbb{N}$ and $F := \bigcup_{n=1}^\infty \bigcup_{k=1}^{l_n} J_{n,k}$. Since the dyadic rectangles $J_{n,k}$ are disjoint, then we have the following decomposition of the function f:

$$
\begin{aligned}
f &= \sum_{n=1}^{\infty}\sum_{k=1}^{l_n} f 1_{J_{n,k}} + f 1_{\mathbb{T}^2\setminus F} \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{l_n}\Bigl(f - \operatorname{mes}(J_{n,k})^{-1}\int_{J_{n,k}} f(x)\,dx\Bigr)1_{J_{n,k}} + \sum_{n=1}^{\infty}\sum_{k=1}^{l_n}\Bigl[\operatorname{mes}(J_{n,k})^{-1}\int_{J_{n,k}} f(x)\,dx\Bigr]1_{J_{n,k}} + f 1_{\mathbb{T}^2\setminus F} \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{l_n} f_{n,k} + f_0.
\end{aligned}
$$

This means that $f_0 = \sum_{n=1}^{\infty}\sum_{k=1}^{l_n}[\operatorname{mes}(J_{n,k})^{-1}\int_{J_{n,k}} f(x)\,dx]1_{J_{n,k}} + f 1_{\mathbb{T}^2\setminus F}$ and the functions f_i $(i = 1, 2, \ldots)$ in the statement of Lemma 4 will be the functions $f_{n,k}$ $(k = 1, \ldots, l_n,\ n \in \mathbb{N})$. $\operatorname{supp} f_{n,k} \subset J_{n,k}$ are disjoint dyadic rectangles,

$$
\begin{aligned}
&\operatorname{mes}(J_{n,k}) = \frac{4\pi^2}{2^{\beta_1(s_n)+\beta_2(s_n)}}, \\
&\int_{\mathbb{T}^2} f_{n,k}(x)\,dx = \int_{J_{n,k}} f(x)\,dx - \operatorname{mes}(J_{n,k})^{-1}\int_{J_{n,k}} f(x)\,dx \cdot \operatorname{mes}(J_{n,k}) = 0, \\
&\|f_{n,k}\|_1 \leqslant \|f 1_{J_{n,k}}\|_1 + \operatorname{mes}(J_{n,k})^{-1}\int_{J_{n,k}} \bigl|f(x)\bigr|\,dx\,\|1_{J_{n,k}}\|_1 = 2\|f 1_{J_{n,k}}\|_1.
\end{aligned}
$$

Consequently,

$$
\Biggl\|\sum_{n=1}^{\infty}\sum_{k=1}^{l_n} f_{n,k}\Biggr\|_1 \leqslant 2\sum_{n=1}^{\infty}\sum_{k=1}^{l_n}\|f 1_{J_{n,k}}\|_1 = 2\int_F \bigl|f(x)\bigr|\,dx \leqslant 2\|f\|_1.
$$

This immediately gives

$$
\|f_0\|_1 = \Biggl\|f - \sum_{n=1}^{\infty}\sum_{k=1}^{l_n} f_{n,k}\Biggr\|_1 \leqslant 3\|f\|_1.
$$

Since F is the disjoint union of the dyadic rectangles $J_{n,k}$, then for the two-dimensional Lebesgue measure of F we get

$$
\begin{aligned}
\operatorname{mes}(F) &= \sum_{n=1}^{\infty}\sum_{k=1}^{l_n}\operatorname{mes}(J_{n,k}) \\
&< \sum_{n=1}^{\infty}\sum_{k=1}^{l_n}\frac{1}{\lambda}\int_{J_{n,k}}\bigl|f(x)\bigr|\,dx \\
&= \frac{1}{\lambda}\int_F \bigl|f(x)\bigr|\,dx \leqslant \frac{1}{\lambda}\|f\|_1.
\end{aligned}
$$

It remains to prove $\|f_0\| \leqslant C_\beta\lambda$. The construction of Ω_n gives the inequality

$$
\operatorname{mes}(J_{n,k})^{-1}\int_{J_{n,k}}\bigl|f(x)\bigr|\,dx \leqslant C_\beta\lambda
$$

(in the case of $n = 1$ we have $2^{\beta_1(1)+\beta_2(1)}$, and in the case of $n \geqslant 2$ we have number 4 as constant C). That is,

$$
\begin{aligned}
\|f_0\|_\infty &\leqslant C_\beta\lambda\Biggl\|\sum_{n=1}^{\infty}\sum_{k=1}^{l_n} 1_{J_{n,k}}\Biggr\|_\infty + \|f 1_{\mathbb{T}^2\setminus F}\|_\infty \\
&= C_\beta\lambda\|1_F\|_\infty + \|f 1_{\mathbb{T}^2\setminus F}\|_\infty \\
&\leqslant C_\beta\lambda + \|f 1_{\mathbb{T}^2\setminus F}\|_\infty.
\end{aligned}
$$

Let $\mathcal{A}_n$ be the σ-algebra generated by the elements of $\mathcal{I}_{\beta_1(s_n)} \times \mathcal{I}_{\beta_2(s_n)}$ ($n \in \mathbb{N}$). Then we have an increasing sequence of σ algebras

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots.$$

The conditional expectation operator of the function f with respect to $\mathcal{A}_n$ at a given point $x \in \mathbb{T}^2$ is

$$\operatorname{mes}(J)^{-1} \int\limits_J f(t)\,dt,$$

where J is the unique element of $\mathcal{I}_{\beta_1(s_n)} \times \mathcal{I}_{\beta_2(s_n)}$ such that $x \in J$. Since $\lim_{+\infty} \beta_1 = \lim_{+\infty} \beta_2 = +\infty$, then the martingale convergence theorem (see e.g. the book of Neveau [9]) gives that this integral mean value converges to $f(x)$ for almost all x in $\mathbb{T}^2$.

Now let $x \in \mathbb{T}^2 \setminus F$. Then the construction of the set Ω_n gives for each $J \in \mathcal{I}_{\beta_1(s_n)} \times \mathcal{I}_{\beta_2(s_n)}$ that $\operatorname{mes}(J)^{-1} \int_J |f(t)|\,dt \leqslant \lambda$ (for all $n \in \mathbb{N}$). From the lines above there follows

$$\left|f(x)\right| \leqslant \lambda$$

for almost all $x \in \mathbb{T}^2 \setminus F$, so

$$\|f 1_{\mathbb{T}^2 \setminus F}\|_\infty \leqslant \lambda, \qquad \|f_0\|_\infty \leqslant C_\beta \lambda.$$

With this the proof of Lemma 4 is complete. $\square$

For $A \in \mathbb{N}$, $x \in \mathbb{T}$ denote by $I_A^1(x) = I_A(x) + \frac{2\pi}{2^A} := \{y + \frac{2\pi}{2^A}\colon\ y \in I_A(x)\} \in \mathcal{I}_A$, $I_A^{-1}(x) = I_A(x) - \frac{2\pi}{2^A} := \{y - \frac{2\pi}{2^A}\colon\ y \in I_A(x)\} \in \mathcal{I}_A$ the two adjacent intervals of the one-dimensional interval $I_A(x)$. Remark that we mean the addition $y + \frac{2\pi}{2^A}$ and the subtraction $y - \frac{2\pi}{2^A}$ by modulo 2π. That is, for $y \in \mathbb{T}$ we also have $y + \frac{2\pi}{2^A}, y - \frac{2\pi}{2^A} \in \mathbb{T}$. Also use the notation $I_A^0(x) = I_A(x)$.

Also define the integral mean values of the function $f \in L^1(\mathbb{T})$ at $x \in \mathbb{T}$

$$E_{A,\delta} f(x) := \frac{2^A}{2\pi} \int\limits_{I_A^\delta(x)} f(t)\,dt,$$

where $\delta \in \{-1, 0, 1\}$. For $A = (A_1, A_2) \in \mathbb{N}^2$, $f \in L^1(\mathbb{T}^2)$, $\delta = (\delta_1, \delta_2) \in \{-1, 0, 1\} \times \{-1, 0, 1\}$, $I_A^\delta(x) = I_{A_1}^{\delta_1}(x_1) \times I_{A_2}^{\delta_2}(x_2)$, $x \in \mathbb{T}^2$ set the two-dimensional integral mean values

$$E_{A,\delta} f(x) := \frac{2^{A_1+A_2}}{4\pi^2} \int\limits_{I_A^\delta(x)} f(t)\,dt.$$

In the sequel we suppose that the functions $\beta_j(x) = \lfloor \log_2 \tilde{a}_j(x) \rfloor$ are monotone increasing and the functions $\tilde{a}_j(x)$ are continuous ($j = 1, 2$). Let δ be as above and f an integrable two variable function. Define the maximal operator $E_{\beta,\delta}^* f := \sup_{t \in [1,+\infty)} |E_{\beta(t),\delta} f|$. With the application of some lemmas below we prove that the maximal operator $\sigma^* f = \sup_n |\sigma_{a(n)} f|$ is of weak type (L^1, L^1). In order to have this the first lemma is:

Lemma 5. *$E_{\beta,\delta}^*$ is of weak type (L^1, L^1). That is, $\operatorname{mes}\{x \in \mathbb{T}^2\colon\ E_{\beta,\delta}^* f(x) > \lambda\} \leqslant C_\beta \|f\|_1/\lambda$ for every positive λ.*

Proof. Apply Lemma 4. Recall its notation. $\operatorname{supp} f_i \subset I_{\beta(s_i)}(u_i) = I_{\beta_1(s_i)}(u_{i,1}) \times I_{\beta_2(s_i)}(u_{i,2})$ ($i \in \mathbb{P}$) and $F = \bigcup_{i=1}^\infty I_{\beta(s_i)}(u_i)$. Set

$$9F = \bigcup_{\epsilon_1, \epsilon_2 \in \{-1,0,1\}} \bigcup_{i=1}^\infty I_{\beta(s_i)}^\epsilon(u_i).$$

Obviously, $\operatorname{mes}(9F) \leqslant 9 \operatorname{mes}(F)$. Fix a $\delta \in \{-1, 0, 1\}^2$ and let $y \in \mathbb{T}^2 \setminus 9F = \overline{9F}$ and $i \in \mathbb{P}$. Then we prove $E_{\beta,\delta}^* f_i(y) = 0$.

Suppose that $1 \leqslant t \leqslant s_i$.

If there exists a $j \in \{1, 2\}$ such that the one-dimensional intervals $I_{\beta_j(t)}^{\delta_j}(y_j)$ and $I_{\beta_j(s_i)}(u_{i,j})$ are disjoint, then we have $I_{\beta(t)}^\delta(y) \cap I_{\beta(s_i)}(u_i) = \emptyset$ and consequently

$$E_{\beta(t),\delta} f_i(y) = \frac{2^{\beta_1(t)+\beta_2(t)}}{4\pi^2} \int\limits_{I_{\beta(t)}^\delta(y)} f_i(x)\,dx = \frac{2^{\beta_1(t)+\beta_2(t)}}{4\pi^2} \int\limits_{I_{\beta(t)}^\delta(y) \cap I_{\beta(s_i)}(u_i)} f_i(x)\,dx = 0.$$

If, for both $j \in \{1,2\}$, $I^{\delta_j}_{\beta_j(t)}(y_j) \cap I_{\beta_j(s_i)}(u_{i,j}) \neq \emptyset$, then since two one-dimensional intervals are disjoint or one of them contains the other, then $\beta_j(t) \leqslant \beta_j(s_i)$ (recall that β_j is monotone increasing) gives $I^{\delta_j}_{\beta_j(t)}(y_j) \supset I_{\beta_j(s_i)}(u_{i,j})$ $(j=1,2)$. That is, $I^{\delta}_{\beta(t)}(y) \supset I_{\beta(s_i)}(u_i)$ and this immediately gives

$$E_{\beta(t),\delta} f_i(y) = \frac{2^{\beta_1(t)+\beta_2(t)}}{4\pi^2} \int\limits_{I^{\delta}_{\beta(t)}(y)} f_i(x)\,dx = \frac{2^{\beta_1(t)+\beta_2(t)}}{4\pi^2} \int\limits_{I_{\beta(s_i)}(u_i)} f_i(x)\,dx = 0.$$

Now, we turn our attention to the other case:

Suppose that $t > s_i$.

$E_{\beta(t),\delta} f_i(y)$ is the integral mean value on the two-dimensional rectangle $I^{\delta}_{\beta(t)}(y)$ and consequently we integrate function f_i on the set $I^{\delta}_{\beta(t)}(y) \cap I_{\beta(s_i)}(u_i)$. Therefore, if, for either $j=1$ or $j=2$, $I^{\delta_j}_{\beta_j(t)}(y_j) \cap I_{\beta_j(s_i)}(u_{i,j}) = \emptyset$, then $E_{\beta(t),\delta} f_i(y) = 0$. That is, we can suppose that the intersection is not the empty set. Since β_j is monotone increasing, then $\beta_j(t) \geqslant \beta_j(s_i)$ and this implies $I^{\delta_j}_{\beta_j(t)}(y_j) \subset I_{\beta_j(s_i)}(u_{i,j})$ $(j=1,2)$. Thus, $I^{\delta_j}_{\beta_j(t)}(y_j) = I_{\beta_j(t)}(y_j) + \frac{2\pi\delta_j}{2^{\beta_j(t)}}$ gives $y_j + \frac{2\pi\delta_j}{2^{\beta_j(t)}} \in I_{\beta_j(s_i)}(u_{i,j})$. Consequently, if we add $-\frac{2\pi\delta_j}{2^{\beta_j(t)}}$ to $y_j + \frac{2\pi\delta_j}{2^{\beta_j(t)}}$ (modulo 2π), then applying the inequality $|\frac{2\pi\delta_j}{2^{\beta_j(t)}}| \leqslant \frac{2\pi}{2^{\beta_j(s_i)}}$ the result y_j will be an element of the union of the interval $I_{\beta_j(s_i)}(u_{i,j})$ and its two adjacent intervals. That is, $y_j \in I_{\beta_j(s_i)}(u_{i,j}) \cup I^{1}_{\beta_j(s_i)}(u_{i,j}) \cup I^{-1}_{\beta_j(s_i)}(u_{i,j})$ for $j=1,2$. This gives

$$y \in \bigcup_{\epsilon_1,\epsilon_2 \in \{-1,0,1\}} I^{\epsilon}_{\beta(s_i)}(u_i)$$

therefore $y \in 9F$ and this is a contradiction. That is, for all t we have $E_{\beta(t),\delta} f_i(y) = 0$ which gives $E^*_{\beta,\delta} f_i(y) = 0$ on $y \in \overline{9F}$. Now, turn back to the notation of Lemma 4.

$$\begin{aligned}
&\operatorname{mes}\{y \in \mathbb{T}^2\colon\ E^*_{\beta,\delta} f(y) > 2C_\beta\lambda\} \\
&\quad\leqslant \operatorname{mes}\{y \in \mathbb{T}^2\colon\ E^*_{\beta,\delta} f_0(y) > C_\beta\lambda\} + \operatorname{mes}\left\{y \in \mathbb{T}^2\colon\ E^*_{\beta,\delta}\left(\sum_{i=1}^{\infty} f_i\right)(y) > C_\beta\lambda\right\} \\
&\quad\leqslant \operatorname{mes}\{y \in \mathbb{T}^2\colon\ E^*_{\beta,\delta} f_0(y) > C_\beta\lambda\} + \operatorname{mes}(9F) + \operatorname{mes}\left\{y \in \overline{9F}\colon\ E^*_{\beta,\delta}\left(\sum_{i=1}^{\infty} f_i\right)(y) > C_\beta\lambda\right\} \\
&\quad=: I + II + III.
\end{aligned}$$

It is quite easy to have that $\|E^*_{\beta,\delta} f_0\|_\infty \leqslant \|f_0\|_\infty \leqslant C_\beta\lambda$ and consequently $I = 0$. For III by the Markov inequality and the σ-sublinearity of operator $E^*_{\beta,\delta}$ we have

$$\begin{aligned}
III &\leqslant \frac{C_\beta}{\lambda} \int\limits_{\overline{9F}} E^*_{\beta,\delta}\left(\sum_{i=1}^{\infty} f_i\right) \\
&\leqslant \frac{C_\beta}{\lambda} \int\limits_{\overline{9F}} \sum_{i=1}^{\infty} E^*_{\beta,\delta} f_i = \frac{C_\beta}{\lambda} \sum_{i=1}^{\infty} \int\limits_{\overline{9F}} E^*_{\beta,\delta} f_i = 0.
\end{aligned}$$

That is, $\operatorname{mes}\{E^*_{\beta,\delta} f > 2C_\beta\lambda\} \leqslant \operatorname{mes}(9F) \leqslant 9\|f\|_1/\lambda$. This completes the proof of Lemma 5. $\square$

The following lemma is concerned with an estimate of the one-dimensional Fejér kernel.

Lemma 6. *Let $n, A \in \mathbb{N}$, $2^A \leqslant n < 2^{A+1}$, that is, $A = \lfloor \log_2 n \rfloor$, $x, y \in \mathbb{T}$. Then we have*

$$K_n(y - x) \leqslant 2 \sum_{k=0}^{A} 2^{A-2k} 1_{\bigcup_{\delta\in\{-1,0,1\}} I^{\delta}_{A-k}(y)}(x). \tag{1}$$

Proof. First, in order to prove Lemma 6 we prove the following inequality for $0 \leqslant x$.

$$K_n(x) \leqslant 2 \sum_{k=0}^{A} 2^{A-2k} 1_{I_{A-k}(0)}(x). \tag{2}$$

If $x \in I_A(0)$, then since the right-hand side of (2) for $k = 0$ is $2^A 1_{I_A(0)}(x) = 2^A$ and since the left-hand side $K_n(x) \leqslant \frac{n+1}{2} \leqslant 2^A$, then the inequality (2) is proved in this case.

If $x \notin I_A(0)$, then we have a $j \in \{0, \dots, A-1\}$ such that $0 \leqslant x \in I_j(0) \setminus I_{j+1}(0)$, which gives $\frac{\pi}{2^{j+1}} \leqslant x < \frac{\pi}{2^j}$. By the inequality (one can find it in Bary's book [1])

$$0 \leqslant K_n(x) \leqslant \frac{\pi^2}{2(n+1)x^2} \quad \bigl(0 < |x| \leqslant \pi,\ n \in \mathbb{N}\bigr)$$

we have $K_n(x) \leqslant \frac{2^{2j+1}}{2^A}$. What about the right-hand side of (2)? Since $1_{I_j(0)}(x) = 1$, then we have $2\sum_{k=0}^{A} 2^{A-2k} 1_{I_{A-k}(0)}(x) \geqslant 2^{2j-A+1} 1_{I_j(0)}(x) = 2^{2j-A+1}$. That is, inequality (2) is proved. Now, we turn our attention to (1). Since the function K_n is even, then $K_n(y-x) = K_n(|y-x|) \leqslant 2\sum_{k=0}^{A} 2^{A-2k} 1_{I_{A-k}(0)}(|y-x|)$. If $1_{I_{A-k}(0)}(|y-x|) = 1$, that is, $|y-x| \in I_{A-k}(0)$, then $|y-x| < \frac{\pi}{2^{A-k}}$ and this gives that x is an element one of the intervals $I^0_{A-k}(y)$, $I^{-1}_{A-k}(y)$, $I^1_{A-k}(y)$. That is, it is an element of $I_{A-k}(y)$ or one of its two adjacent intervals belonging to $\mathcal{I}_{A-k}$. Thus, $1_{\bigcup_{\delta\in\{-1,0,1\}} I^{\delta}_{A-k}(y)}(x) = 1$,

$$K_n(y-x) = K_n\bigl(|y-x|\bigr) \leqslant 2\sum_{k=0}^{A} 2^{A-2k} 1_{I_{A-k}(0)}\bigl(|y-x|\bigr) \leqslant 2\sum_{k=0}^{A} 2^{A-2k} 1_{\bigcup_{\delta\in\{-1,0,1\}} I^{\delta}_{A-k}(y)}(x).$$

This completes the proof of Lemma 6. □

Now, we are ready to prove Lemma 2.

Proof of Lemma 2. By (1) we prove the following two-dimensional inequality for $n \in \mathbb{P}^2$, $A = |n| := \lfloor \log_2 n \rfloor$, $f \in L^1(\mathbb{T}^2)$.

$$|\sigma_n f| \leqslant C \sum_{\delta_1,\delta_2\in\{-1,0,1\}} \sum_{k_1\leqslant A_1, k_2\leqslant A_2} 2^{-k_1-k_2} E_{A-k,\delta}|f|. \tag{3}$$

By the help of Lemma 6 we have

$$\begin{aligned}
\bigl|\sigma_n f(y)\bigr| &= \left|\frac{1}{\pi^2}\int_{\mathbb{T}^2} f(x) K_n(y-x)\,dx\right| \\
&\leqslant C \sum_{\delta_1,\delta_2\in\{-1,0,1\}} \sum_{k_1\leqslant A_1, k_2\leqslant A_2} 2^{A_1-2k_1} 2^{A_2-2k_2} \int_{\mathbb{T}^2} \bigl|f(x)\bigr| 1_{I^{\delta}_{A-k}(y)}(x)\,dx \\
&= C \sum_{\delta_1,\delta_2\in\{-1,0,1\}} \sum_{k_1\leqslant A_1, k_2\leqslant A_2} 2^{-k_1-k_2} E_{A-k,\delta}|f|(y).
\end{aligned}$$

That is, (3) is proved.

For $j = 1, 2$ the functions $\beta_j(x) = \lfloor \log_2 \tilde{a}_j(x) \rfloor$ are monotone increasing on $[1, +\infty)$ $(j = 1, 2)$ and consequently so do the functions $\beta_j(x) - k_j$ for $j = 1, 2$. We apply Lemma 5.

$$\begin{aligned}
&\operatorname{mes}\Bigl\{y \in \mathbb{T}^2\colon \Bigl|\sup_n \sigma_{a(n)} f(y)\Bigr| > \lambda\Bigr\} \\
&\quad \leqslant \operatorname{mes}\left\{y \in \mathbb{T}^2\colon C \sup_{t\in[1,+\infty)} \sum_{\delta_1,\delta_2\in\{-1,0,1\}} \sum_{k_1\leqslant\beta_1(t), k_2\leqslant\beta_2(t)} 2^{-k_1-k_2} E_{\beta(t)-k,\delta}|f|(y) > \lambda\right\} \\
&\quad \leqslant \operatorname{mes}\left\{y \in \mathbb{T}^2\colon C \sum_{\delta_1,\delta_2\in\{-1,0,1\}} \sum_{k_1,k_2\in\mathbb{N}} 2^{-k_1-k_2} E^*_{\beta-k,\delta}|f|(y) > \lambda\right\} \\
&\quad \leqslant \operatorname{mes}\left(\bigcup_{\delta_1,\delta_2\in\{-1,0,1\}} \bigcup_{k_1,k_2\in\mathbb{N}} \left\{y \in \mathbb{T}^2\colon 2^{-k_1-k_2} E^*_{\beta-k,\delta}|f|(y) > \frac{C\lambda}{(|k_1|+1)^2(|k_2|+1)^2}\right\}\right) \\
&\quad \leqslant C \sum_{\delta_1,\delta_2\in\{-1,0,1\}} \sum_{k_1,k_2\in\mathbb{N}} \operatorname{mes}\left\{y \in \mathbb{T}^2\colon 2^{-k_1-k_2} E^*_{\beta-k,\delta}|f|(y) > \frac{C\lambda}{(|k_1|+1)^2(|k_2|+1)^2}\right\} \\
&\quad \leqslant C_\beta \sum_{\delta_1,\delta_2\in\{-1,0,1\}} \sum_{k_1,k_2\in\mathbb{N}} \frac{(|k_1|+1)^2(|k_2|+1)^2}{\lambda 2^{k_1+k_2}} \|f\|_1 \\
&\quad \leqslant C_\beta \|f\|_1/\lambda.
\end{aligned}$$

That is, we proved that the maximal operator $\sigma_a^* f = \sup_n |\sigma_{a(n)} f|$ is of weak type (L^1, L^1). Since for each trigonometric polynomial P we have the everywhere relation $\lim_{n\to\infty} \sigma_{(a_1(n),a_2(n))} P = P$, then by the standard density argument (see this principal for instance in [2]) the proof of Lemma 2 is complete. $\square$

Finally, we have to prove Theorem 1, that is, the main result of this paper. The proof comes from Lemma 2 with some easy calculations.

Proof of Theorem 1. Without the loss of generality $a_j(n) \geqslant 1$ ($j = 1, 2$, $n \in \mathbb{N}$) can be supposed. Let L be a positive integer discussed later. For $l, m = 0, 1, \ldots, L-1$ let some disjoint subsets of $\mathbb{N}$ be defined as

$$B_{l,m} = \left\{ n \in \mathbb{N}\colon \bigl(a_1(n), a_2(n)\bigr) \in \bigcup_{s,t=0}^{\infty} \bigl[2^{sL+l}, 2^{sL+l+1}\bigr) \times \bigl[2^{tL+m}, 2^{tL+m+1}\bigr) \right\}.$$

It is clear that these sets are pairwise disjoint and their union is $\mathbb{N}$. Denote the elements of $B_{l,m}$ by $n_1^{l,m} < n_2^{l,m} < \cdots$. We prove that $\lfloor \log_2 a_j(n_k^{l,m})\rfloor \leqslant \lfloor \log_2 a_j(n_{k+1}^{l,m})\rfloor$ for every $k \in \mathbb{N}, l, m \in \{0, 1, \ldots L-1\}$ and $j = 1, 2$. On the contrary, suppose that $\lfloor \log_2 a_j(n_{k+1}^{l,m})\rfloor < \lfloor \log_2 a_j(n_k^{l,m})\rfloor$ for some k, l, m and j. Then the definition of $B_{l,m}$ gives that $\lfloor \log_2 a_j(n_{k+1}^{l,m})\rfloor \leqslant \lfloor \log_2 a_j(n_k^{l,m})\rfloor - L$. Thus,

$$\frac{1}{2} a_j\bigl(n_{k+1}^{l,m}\bigr) \leqslant 2^{\lfloor \log_2 a_j(n_{k+1}^{l,m})\rfloor} \leqslant 2^{\lfloor \log_2 a_j(n_k^{l,m})\rfloor - L} \leqslant \frac{1}{2^L} a_j\bigl(n_k^{l,m}\bigr).$$

Since, $n_{k+1}^{l,m} > n_k^{l,m}$, then we have $a_j(n_{k+1}^{l,m}) \geqslant \alpha a_j(n_k^{l,m})$ and consequently, also have $\alpha \leqslant 2^{1-L}$. This is obviously not possible for an L large enough. That is, we proved that $\lfloor \log_2 a_j(n_k^{l,m})\rfloor$ is monotone increasing with respect to $k \in \mathbb{N}$. Lemma 2 gives the a.e. convergence

$$\lim_{k\to\infty} \sigma_{a(n_k^{l,m})} f = f$$

for each integrable function f and $l, m = 0, 1, \ldots, L-1$. Merging the L^2 pieces of subsequences of $\sigma_{a(n)} f$ the proof of Theorem 1 is complete. $\square$

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