J. Math. Anal. Appl. 398 (2013) 362–371

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Nonresonant Hopf–Hopf bifurcation and a chaotic attractor in neutral functional differential equations☆

Ben Niu [a,b], Weihua Jiang [a,*]

[a] Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, China

[b] Department of Applied Mathematics, Harbin University of Science and Technology, Harbin, 150080, China

ARTICLE INFO

Article history:
Received 5 December 2011
Available online 4 September 2012
Submitted by Junping Shi

Keywords:
Neutral functional differential equation
Nonresonant Hopf–Hopf bifurcation
Van der Pol's equation
Quasi-periodic solution
Chaos

ABSTRACT

An algorithm for calculating the third-order normal form of a nonresonant Hopf–Hopf singularity in a neutral functional differential equation (NFDE) is established. The van der Pol equation with extended delay feedback is investigated as an NFDE of second order. The existence of Hopf–Hopf bifurcation is studied and the unfolding near these critical points is given by applying this algorithm. Periodic solutions and quasi-periodic solutions are found with the aid of the bifurcation diagram, and corresponding numerical illustrations are presented. With the breaking down of the 3-torus, a chaotic attractor appears in this NFDE of second order, following the Ruelle–Takens–Newhouse scenario which usually arises for an ordinary differential equation of order at least 4. This transition is shown via both theoretical and numerical approaches.

1. Introduction

The investigation of codimension 2 bifurcations in functional differential equations (FDE) has been the subject of much recent activity; among these, the Hopf–Hopf bifurcations are particularly complicated because the oscillations have at least two distinct frequencies, and thus quite rich dynamics may arise. For more details on codimension 2 bifurcations we refer the reader to [1–5] and the references cited therein.

An important approach to bifurcation analysis is the normal form method; see [6–8]. The normal form method for NFDEs was developed recently by Weedermann [9]; the author gave a computation procedure by employing the method introduced in [8]. In [10], the normal form for NFDEs with parameters was established and applied to study the Hopf bifurcation in the lossless transmission line, which is the most famous equation of neutral type [11–13]. In [14], the authors found the system of NFDEs naturally arising from the extended delay feedback control problem. Thus studying the codimension 2 bifurcation in NFDEs is necessary for revealing the rich dynamics in those models, such as quasi-periodic oscillations, connecting orbits and even chaos. As far as we know, these results are almost all for the retarded functional differential equations (RFDEs; e.g. [1–5,8] and the references therein). However, there are very rare references concerning codimension 2 bifurcations of NFDEs.

This paper is dedicated to extending the idea given in [8–10] to the nonresonant Hopf–Hopf singularity in an NFDE with parameters:

$$\frac{\mathrm{d}}{\mathrm{d}t}[Dx_t - G(x_t)] = L(\alpha)x_t + F(\alpha, x_t) \tag{1}$$

☆ Supported in part by the National Natural Science Foundation of China (No. 11031002), by the Fund of the Education Department of Heilongjiang Province (No. 12521085) and by the Heilongjiang Provincial Natural Science Foundation (No. A200806).

* Corresponding author.
E-mail address: jiangwh@hit.edu.cn (W. Jiang).

0022-247X/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2012.08.051

where $x_t \in C := C([-\tau, 0], \mathbb{R}^n)$, $x_t(\theta) := x(t+\theta)$. D and $L(\alpha)$ are bounded linear operators from C to $\mathbb{R}^n$ for any $\alpha \in \mathbb{R}^p$, with $D\phi = \phi(0) - \int_{-\tau}^{0} \mathrm{d}[\mu(\theta)]\phi(\theta)$ and $L(\alpha)\phi = \int_{-\tau}^{0} \mathrm{d}[\eta(\theta,\alpha)]\phi(\theta)$ for $\phi \in C$, where $\mu(\theta)$ and $\eta(\theta,\alpha)$ are matrix-valued functions of bounded variation which are continuous from the left on $(-\tau, 0)$ and such that $\eta(0,\alpha) = \mu(0) = 0$ and μ is non-atomic at zero. Note that Eq. (1) degenerates to an RFDE when $D\phi = \phi(0)$ and $G(\phi) \equiv 0$. Thus the method that we establish below is an extension to the RFDE case. Recall that at a nonresonant Hopf–Hopf bifurcation point, the corresponding characteristic equation has two pairs of pure imaginary roots $\pm i\omega_+$ and $\pm i\omega_-$, and the ratio $\omega_- : \omega_+$ is not a rational number.

For later use, we give an explicit and complete algorithm for dealing with the Hopf–Hopf singularity. Firstly, the center manifold reduction and the normal form derivation in the parameterized NFDE are presented. Secondly, after calculating the normal form near the Hopf–Hopf bifurcation point, we show that the dynamical behavior of (1) near the critical point of the Hopf–Hopf bifurcation is governed by a four-dimensional system up to the third order with unfolding parameters restricted to the center manifold. Finally, it can be further reduced to a two-dimensional amplitude system, where these unfolding parameters can be expressed in terms of the perturbation parameters in the original system (1). Our algorithm is a formulated procedure for studying the dynamical behavior near a nonresonant Hopf–Hopf bifurcation.

The van der Pol equation has a long history of being used in both the physical and biological sciences [15,16] and has been widely studied by many authors since it was first formulated for an electrical circuit with a triode valve; see, e.g., [3,6,17–19]. Using these theoretical results, we study the nonresonant Hopf–Hopf bifurcation in van der Pol's equation with extended delay feedback. This method is actually the so-called extended time delay autosynchronization; see [20–22]. The equation is equivalent to a system of NFDEs. The universal unfoldings near the Hopf–Hopf point are obtained and the detailed bifurcation sets indicate the existence of a stable periodic solution and a quasi-periodic solution on the torus. We find that Hopf–Hopf bifurcation admits a three-dimensional torus which vanishes via a saddle connection bifurcation through perturbations. According to the famous result of Newhouse et al. [23], a strange attractor might exist. Thus the strange attractor in van der Pol's equation with extended delay feedback is investigated with the aid of the Hopf–Hopf bifurcation set. The transition from the three-dimensional torus to the strange attractor is also illustrated by a numerical method.

The paper is organized as follows. Section 2 is devoted to calculating the normal forms for system (1) near the nonresonant Hopf–Hopf singularity. Section 3 focuses on van der Pol's equation. The conditions for the existence of Hopf–Hopf bifurcation are obtained. The corresponding normal form is calculated, and the detailed bifurcation sets are drawn. Appropriate simulations are carried out to illustrate the theoretical results, e.g., the existence of a 2-torus or 3-torus and the transition from quasi-periodic oscillations to chaos. Finally a conclusion section is provided, in which we also give some discussion.

2. The normal form for NFDEs with Hopf–Hopf singularity

In this section, we present the regular normal form method for system (1), and then calculate the normal form near a nonresonant Hopf–Hopf bifurcation point. Assume that x_t is differentiable, F and G are C^N-smooth, $\alpha \in \mathbb{R}^2$, $N \geq 3$, $F(\alpha, 0) = G(0) = 0$, $F'(\alpha, 0) = G'(0) = 0$ and G does not depend on $\phi(0)$ in (1), where the notation $'$ stands for the Fréchet derivative. Under these hypotheses, obviously $x_t = 0$ is an equilibrium of Eq. (1), whose equivalent form is

$$\frac{\mathrm{d}}{\mathrm{d}t} Dx_t = L(\alpha)x_t + F(\alpha, x_t) + G'(x_t)\dot{x}_t. \tag{2}$$

By introducing the enlarged phase space BC, in which the functions from $[-\tau, 0]$ to $\mathbb{R}^n$ are uniformly continuous on $[-\tau, 0)$ with a possible discontinuous jump at 0, Eq. (2) can be written as an abstract ODE on BC:

$$\frac{\mathrm{d}}{\mathrm{d}t} x_t = Ax_t + X_0[L(\alpha) - L(0)]x_t + X_0F(\alpha, x_t) + X_0G'(x_t)\dot{x}_t \tag{3}$$

where $A\phi := \phi' + X_0[L(0)\phi - D\phi']$ is the infinitesimal generator of the semigroup of solutions to the linear system $\frac{\mathrm{d}}{\mathrm{d}t}Dx_t = L(0)x_t$. $X_0(\theta) = 0$ for $-\tau \leq \theta < 0$ and $X_0(0) = Id_{n\times n}$. Regarding α as a new variable, we can consider Eq. (3) as an ODE with no parameters in the product space $\widetilde{BC} := BC \times \mathbb{R}^2$.

Generally, Hopf–Hopf bifurcation occurs in Eq. (1) when $\alpha = (\alpha_1, \alpha_2) = 0$ if $\Lambda = \{\pm i\omega_1, \pm i\omega_2\}$ belongs to the point set spectrum $\sigma(A)$ and the rest of the elements have nonzero real part. This is just to say that the characteristic equation of Eq. (2), $\det(\Delta(\lambda)) = \det(\lambda D(e^{\lambda\cdot}) - L(e^{\lambda\cdot})) = 0$, has two pairs of pure imaginary roots. Without loss of generality, we assume that $\omega_1 < \omega_2$. Following [8,9] we summarize the calculation of the normal forms for Eq. (3) as follows. Note that here we use $x_t \in C := C([-\tau, 0], \mathbb{C}^n)$. $\mathbb{C}$ is the space of the complex numbers. The operators L, D, F are extended to complex functions in the natural way.

Decompose $\widetilde{BC}$ by using $\widetilde{BC} = \widetilde{P} \bigoplus \mathrm{Ker}\, \widetilde{\pi}$, where $\widetilde{P} = P \times \mathbb{R}^2$, and P is the generalized eigenspace for A associated with Λ. $\widetilde{\pi}$ is the projection of $\widetilde{BC}$ onto $\widetilde{P}$. $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is a basis for P with $(\Psi, \Phi) = Id_{4\times 4}$ where $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ is a basis for P^*, the dual space of P, and $(\psi, \phi) = \psi(0)\phi(0) - \int_{-\tau}^{0} \mathrm{d}\left[\int_0^\theta \psi(\xi - \theta)\mathrm{d}\mu(\xi)\right]\phi(\theta) + \int_{-\tau}^{0}\int_0^\theta \psi(\xi - \theta)\mathrm{d}\eta(\theta, 0)\phi(\xi)\mathrm{d}\xi$.

Write $B = \mathrm{diag}\{i\omega_1, -i\omega_1, i\omega_2, -i\omega_2\}$, such that $A\Phi = \Phi B$. If we carry out the decompositions $x_t = \Phi z(t) + w_1$ and $\alpha_t = \alpha(t) + w_2$, where $z(t) = (z_1(t), z_2(t), z_3(t), z_4(t))^T \in \mathbb{C}^4$ and $(w_1, w_2) \in \mathrm{Ker}\widetilde{\pi}$, then (3) has a corresponding decomposition. Noting that $w_2(0) = 0$ because $w_2 \in \mathbb{R}^2$, and dropping some auxiliary equations, we get the equation

$$\begin{aligned}\dot{z} &= Bz + \Psi(0)[(L(\alpha) - L(0))(\Phi z + w_1) + F(\Phi z + w_1, \alpha) + G'(\phi z + w_1)(\Phi \dot{z} + \dot{w}_1)]\\ \dot{w}_1 &= A_{Q^1}w_1 + (Id - \pi)X_0[(L(\alpha) - L(0))(\Phi z + w_1) + F(\Phi z + w_1, \alpha)] + (Id - \pi)G'(\Phi z + w_1)(\Phi \dot{z} + \dot{w}_1)\end{aligned} \tag{4}$$

where the newly defined A_{Q^1} is the restriction of A to $Q^1 := Q \bigcap C^1$ with Q being the complementary space of P in C. Write the Taylor expansion

$$\begin{aligned}\dot{z} &= Bz + \sum_{j\geq 2} \frac{1}{j!} f_j^1(z, w_1, \alpha)\\ \dot{w}_1 &= A_{Q^1}w_1 + \sum_{j\geq 2} \frac{1}{j!} f_j^2(z, w_1, \alpha).\end{aligned} \tag{5}$$

To derive the normal form of jth order, we carry out transformations of variables for $j \geq 2$: $(z, w_1, \alpha) \mapsto (\hat{z}, \hat{w}_1, \hat{\alpha})$, given by $(z, w_1, \alpha) = (\hat{z}, \hat{w}_1, \hat{\alpha}) + \frac{1}{j!}\widetilde{U}_j(\hat{z}, \hat{\alpha})$ with $\widetilde{U}_j = (U_j^1, U_j^2, U_j^3) \in V_j^6(\mathbb{C}^4) \times V_j^6(Q^1) \times V_j^6(\mathbb{R}^2)$, $U_j = (U_j^1, U_j^2)$, where for a normed space X, we denote by $V_j^6(X)$ the linear space of homogeneous polynomials of degree j in six variables with coefficients in X. To compute the normal form we define the operator M_j on $V_j^6(\mathbb{C}^4 \times \mathrm{Ker}\pi)$ by $M_j(q, h) = (M_j^1 q, M_j^2 h)$, where $(M_j^1 q)(z, \alpha) = D_z q(z, \alpha)Bz - Bq(z, \alpha)$, $(M_j^2 h)(z, \alpha) = D_z h(z, \alpha)Bz - A_{Q^1}h(z, \alpha)$, with $q(z, \alpha) \in V_j^6(\mathbb{C}^4)$, $h(z, \alpha)(\theta) \in V_j^6(Q^1)$ and the operator D_z stands for the derivative with respect to z. Then we have the following decompositions:

$$\begin{aligned}&V_j^6(\mathbb{C}^4) = \mathrm{Im}(M_j^1) \bigoplus \mathrm{Im}(M_j^1)^c, \qquad V_j^6(\mathbb{C}^4) = \mathrm{Ker}(M_j^1) \bigoplus \mathrm{Ker}(M_j^1)^c,\\ &V_j^6(Ker\pi) = \mathrm{Im}(M_j^2) \bigoplus \mathrm{Im}(M_j^2)^c, \qquad V_j^6(Q^1) = \mathrm{Ker}(M_j^2) \bigoplus \mathrm{Ker}(M_j^2)^c.\end{aligned}$$

Denote the projections associated with the above decompositions of $V_j^6(\mathbb{C}^4) \times V_j^6(\mathrm{Ker}\pi)$ over $\mathrm{Im}(M_j^1) \times \mathrm{Im}(M_j^2)$ and of $V_j^6(\mathbb{C}^4) \times V_j^6(Q^1)$ over $\mathrm{Ker}(M_j^1)^c \times \mathrm{Ker}(M_j^2)^c$ by, respectively, $P_{I,j} = (P_{I,j}^1, P_{I,j}^2)$ and $P_{K,j} = (P_{K,j}^1, P_{K,j}^2)$. The jth-order term in the normal form becomes $g_j = \bar{f}_j - M_j U_j$, where $\bar{f}_j$ denotes the terms of order j obtained after computation of the normal form up to order $j - 1$. Following [8] we have an adequate choice of U_j given by $U_j(z, \alpha) = M_j^{-1}P_{I,j}\bar{f}_j(z, 0, \alpha)$ and thus $g_j(z, 0, \alpha) = (I - P_{I,j})\bar{f}_j(z, 0, \alpha)$.

Recall that $\omega_1 : \omega_2 \neq 1 : 2$ or 1:3; we have that $(\mathrm{Im}(M_2^1))^c$ is spanned by the elements $\{z_1\alpha_i e_1,\ z_2\alpha_i e_2,\ z_3\alpha_i e_3,\ z_4\alpha_i e_4\}$, $i = 1, 2$, with $e_1 = (1, 0, 0, 0)^T$, $e_2 = (0, 1, 0, 0)^T$, $e_3 = (0, 0, 1, 0)^T$, $e_4 = (0, 0, 0, 1)^T$. Thus the normal form of (1) on the center manifold of the origin near $(\alpha_1, \alpha_2) = 0$ has the form

$$\dot{z} = Bz + \frac{1}{2}g_2^1(z, 0, \alpha) + h.o.t., \tag{6}$$

with $g_2^1(z, 0, \alpha) = Proj_{(\mathrm{Im}(M_2^1))^c} f_2^1(z, 0, \alpha)$.

To find the third-order normal form of the Hopf–Hopf singularity, let M_3 denote the operator defined in $V_3^4(\mathbb{C}^4 \times \mathrm{Ker}(\pi))$. Here we neglect the high order terms of the perturbation parameters. $(\mathrm{Im}(M_3^1))^c$ is spanned by

$$\{z_1^2 z_2 e_1, z_2^2 z_1 e_2, z_3^2 z_4 e_3, z_4^2 z_3 e_4, z_1 z_3 z_4 e_1, z_2 z_3 z_4 e_2, z_1 z_2 z_3 e_3, z_1 z_3 z_4 e_4\}.$$

The normal form of (1) up to the third order is

$$\dot{z} = Bz + \frac{1}{2!}g_2^1(z, 0, \alpha) + \frac{1}{3!}g_3^1(z, 0, 0) + h.o.t., \tag{7}$$

where $g_3^1(z, 0, 0) = Proj_{(\mathrm{Im}(M_3^1))^c}\bar{f}_3^1(z, 0, 0)$, with $(\bar{f}_3^1, \bar{f}_3^2)^T = (f_3^1, f_3^2)^T + \frac{3}{2}[(D_{z,w}(f_2^1, f_2^2)^T U_2 - (D_{z,w}U_2)(g_2^1, g_2^2))]$ and $U_2(z, \alpha) = (U_2^1, U_2^2)^T = M_2^{-1}P_{I,2}f_2(z, 0, \alpha)$.

Remark 2.1. Generally, we need to compute the normal form up to the third order to investigate a Hopf–Hopf point. In the absence of the quadratic terms of x_t in NFDE (1), the calculations are quite simple because $(\bar{f}_3^1, \bar{f}_3^2)^T = (f_3^1, f_3^2)^T$. In van der Pol's equation with extended feedback, the situation is of this type (see Section 3). In the presence of the quadratic terms in x_t in NFDE (1), the calculation is complicated, as we need to calculate $U_2(z, \alpha)$. In any case, we can obtain the normal form by using the method above.

Now, we are in a position to give the normal form of Eq. (1) near the Hopf–Hopf bifurcation.

Table 1
The twelve unfoldings of system (9).

Case	Ia	Ib	II	III	IVa	IVb	V	VIa	VIb	VIIa	VIIb	VIII
d_0	+1	+1	+1	+1	+1	+1	−1	−1	−1	−1	−1	−1
b_0	+	+	+	−	−	−	+	+	+	−	−	−
c_0	+	+	−	+	−	−	+	−	−	+	+	−
$d_0 - b_0c_0$	+	−	+	+	+	−	−	+	−	+	−	−

Theorem 2.2. *Suppose that in system (1) the infinitesimal generator A has two pairs of nonresonant pure imaginary eigenvalues $\Lambda = \{\pm i\omega_1, \pm i\omega_2\}$, and all the other eigenvalues have negative real part. Then the dynamics in (1) near $x_t = 0$, $\alpha = 0$ is governed by the normal form $\dot{z} = Bz + \sum_{j\geq 2}\frac{1}{j!}g_j(z, 0, \alpha)$, whose exact form is, truncated to the third order,*

$$\begin{aligned}
\dot{z}_1 &= i\omega_1 z_1 + a_{11}\alpha_1 z_1 + a_{12}\alpha_2 z_1 + c_{11}z_1^2 z_2 + c_{12}z_1z_3z_4,\\
\dot{z}_2 &= -i\omega_1 z_2 + \bar{a}_{11}\alpha_1 z_2 + \bar{a}_{12}\alpha_2 z_2 + \bar{c}_{11}z_1 z_2^2 + \bar{c}_{12}z_2z_3z_4,\\
\dot{z}_3 &= i\omega_2 z_3 + a_{21}\alpha_1 z_3 + a_{22}\alpha_2 z_3 + c_{21}z_1z_2z_3 + c_{22}z_3^2 z_4,\\
\dot{z}_4 &= -i\omega_2 z_4 + \bar{a}_{21}\alpha_1 z_4 + \bar{a}_{22}\alpha_2 z_4 + \bar{c}_{21}z_1z_2z_4 + \bar{c}_{22}z_3z_4^2.
\end{aligned} \tag{8}$$

Proof. The nonresonance conditions (see [8]) are naturally satisfied since all the roots except those in Λ have negative real part. By the center manifold theory given in [24,25] and the general work in [8], the theorem is proved. □

Remark 2.3. In the normal form (8), the coefficients c_{11}, c_{12}, c_{21} and c_{22} are important for determining the dynamics near the bifurcation point. If none of these four coefficients has zero real part, we can use the discussion below to investigate the dynamical behavior; otherwise, the normal form is degenerate, and we need to calculate a fifth-order normal form.

Eq. (8) is complicated. Make the transformation $z_1 = r_1\cos\theta_1 + \mathrm{i}r_1\sin\theta_1$, $z_2 = r_1\cos\theta_1 - \mathrm{i}r_1\sin\theta_1$, $z_3 = r_2\cos\theta_2 + \mathrm{i}r_2\sin\theta_2$, $z_4 = r_2\cos\theta_2 - \mathrm{i}r_2\sin\theta_2$, $r_1, r_2 > 0$, and define $\epsilon_1 = \mathrm{Sign}(\mathrm{Re}c_{11})$, $\epsilon_2 = \mathrm{Sign}(\mathrm{Re}c_{22})$. After carrying out the rescaling $\hat{r}_1 = r_1\sqrt{|\mathrm{Re}c_{11}|}$, $\hat{r}_2 = r_2\sqrt{|\mathrm{Re}c_{22}|}$ and $\hat{t} = t\epsilon_1$, then Eq. (8) becomes, after dropping the hats,

$$\begin{aligned}
\dot{r}_1 &= r_1(c_1 + r_1^2 + b_0 r_2^2),\\
\dot{r}_2 &= r_2(c_2 + c_0 r_1^2 + d_0 r_2^2),
\end{aligned} \tag{9}$$

where $c_1 = \epsilon_1\mathrm{Re}a_{11}\alpha_1 + \epsilon_1\mathrm{Re}a_{12}\alpha_2$, $c_2 = \epsilon_1\mathrm{Re}a_{21}\alpha_1 + \epsilon_1\mathrm{Re}a_{22}\alpha_2$, $b_0 = \frac{\epsilon_1\epsilon_2\mathrm{Re}c_{12}}{\mathrm{Re}c_{22}}$, $c_0 = \frac{\mathrm{Re}c_{21}}{\mathrm{Re}c_{11}}$, and $d_0 = \epsilon_1\epsilon_2$. Eq. (9) has twelve distinct kinds of unfoldings (see Table 1). The detailed phase portraits can be found in Section 7.5 of [6].

3. Hopf–Hopf bifurcation in van der Pol's equation with extended delay feedback

In this section, van der Pol's equation with extended delay feedback is studied. Hopf–Hopf points are detected by analyzing the associated characteristic equation. Near these points, we calculate the normal form by using the algorithm given in Section 2, and all the key values are obtained. Thus the bifurcation sets are drawn on the plane of parameters; by this means, the periodic solutions, the quasi-periodic solutions on a 2-torus, the quasi-periodic solution on a 3-torus and the strange attractor are obtained.

3.1. The existence and the normal form derivation

Consider the following van der Pol equation:

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = \varepsilon k\vartheta(t) \tag{10}$$

where $\varepsilon > 0$. k is the strength of the extended delay feedback (see [20–22]) $\vartheta(t)$. $\vartheta(t)$ depends on the current state and a sequence of the past states, which is defined by

$$\vartheta(t) = (1-\mu)x(t) + \mu\vartheta(t-\tau), \tag{11}$$

with $0 < \mu < 1$. Eq. (10) together with the extended delay feedback (11) is equivalent to

$$\ddot{x} - \mu\ddot{x}(t-\tau) + \varepsilon(x^2-1)\dot{x} - \mu\varepsilon(x^2(t-\tau)-1)\dot{x}(t-\tau) + x - \mu x(t-\tau) = \varepsilon k(1-\mu)x(t). \tag{12}$$

Introduce a new variable $y(t) = \dot{x}(t)$; then (12) becomes a system of NFDEs

$$\begin{cases}\dot{x} = y\\ \dot{y} - \mu\dot{y}(t-\tau) = [-1 + \varepsilon k(1-\mu)]x + \varepsilon y + \mu x(t-\tau) - \varepsilon\mu y(t-\tau) - \varepsilon x^2 y + \varepsilon\mu x^2(t-\tau)y(t-\tau).\end{cases} \tag{13}$$

The characteristic equation of the corresponding linearized equation at the trivial equilibrium $E_0 = (0, 0)^T$ of (13) is

$$\lambda^2 - \mu\lambda^2 e^{-\lambda\tau} - \varepsilon\lambda + \varepsilon\mu\lambda e^{-\lambda\tau} - \mu e^{-\lambda\tau} + 1 - \varepsilon k(1-\mu) = 0. \tag{14}$$

We investigate Hopf–Hopf bifurcation by detecting the intersection of the Hopf bifurcation curves in Eq. (13). By substituting $\lambda = i\omega, \omega > 0$ into (14) and separating the real and imaginary parts, we have

$$\begin{cases}(\mu\omega^2 - \mu)\cos\omega\tau + \varepsilon\mu\omega\sin\omega\tau = \omega^2 - 1 + \varepsilon k(1-\mu)\\ -(\mu\omega^2 - \mu)\sin\omega\tau + \varepsilon\mu\omega\cos\omega\tau = \varepsilon\omega\end{cases} \tag{15}$$

which is solved by

$$\begin{cases}\cos(\omega\tau) = \dfrac{(\mu\omega^2 - \mu)(\omega^2 - 1 + \varepsilon k(1-\mu)) + (\varepsilon\mu\omega)(\varepsilon\omega)}{(\mu\omega^2-\mu)^2 + (\varepsilon\mu\omega)^2}\\ \sin(\omega\tau) = \dfrac{-(\mu\omega^2 - \mu)(\varepsilon\omega) + (\varepsilon\mu\omega)(\omega^2 - 1 + \varepsilon k(1-\mu))}{(\mu\omega^2-\mu)^2 + (\varepsilon\mu\omega)^2}.\end{cases} \tag{16}$$

Hence, we have

$$W(\rho) = a\rho^2 + b\rho + c = 0 \tag{17}$$

where $\rho = \omega^2$, $a = (1+\mu)$, $b = [2\varepsilon k - 2(1+\mu) + \varepsilon^2(1+\mu)]$, $c = \varepsilon^2k^2(1-\mu) - 2\varepsilon k + 1 + \mu$. Assume that (H1) : $k < \min\left\{\frac{1}{\varepsilon}, \frac{1+\mu}{\varepsilon} - \frac{\varepsilon(1+\mu)}{2}\right\}$; then we have $c > 0, b < 0$. Furthermore if (H2) : $\Delta = (b^2 - 4ac) > 0$ holds, then (17) is solved by two positive roots $\omega_\pm = \sqrt{\rho_\pm}$, where $\rho_\pm = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Denote by τ_0^+ (or τ_0^-) the unique root of Eq. (16) when $\omega = \omega_+$ (or $\omega = \omega_-$), such that $\omega\tau_0^\pm \in [0, 2\pi)$. Furthermore, we define

$$\tau_j^\pm = \tau_0^\pm + \frac{2j\pi}{\omega_\pm}, \quad j = 0, 1, 2, \ldots \tag{18}$$

Take the derivative with respect to τ in Eq. (14), and use Eq. (16). After a few straightforward calculations, we have

$$\operatorname{Sign}\left(\operatorname{Re}\frac{d\lambda}{d\tau}\bigg|_{\tau=\tau_j^\pm}\right) = \operatorname{Sign}(W'(\rho))|_{\rho=\omega_\pm^2}. \tag{19}$$

On the basis of the above preparation, together with the Hopf bifurcation theorem in [13], we can give the conclusions about the Hopf bifurcation in (13).

Theorem 3.1. *Assume* (H1) , (H2) *hold. If* $\tau_0^- > \tau_0^+$, *then* E_0 *in system* (13) *is unstable for any* $\tau \geq 0$. *If* $\tau_0^- < \tau_0^+$, *then there exists an integer* $m \geq 0$ *such that* E_0 *is stable when* $\tau \in (\tau_0^-, \tau_0^+) \cup (\tau_1^-, \tau_1^+) \cup \cdots \cup (\tau_m^-, \tau_m^+)$, *and is unstable when* $\tau \in (0, \tau_0^-) \cup (\tau_0^+, \tau_1^-) \cup \cdots \cup (\tau_m^+, +\infty)$. *Moreover, system* (13) *undergoes a Hopf bifurcation at every* τ_j^+ *(or* τ_j^-*),* $j = 0, 1, 2, \ldots$.

Now, we are in a position to give the condition under which a Hopf–Hopf bifurcation occurs. If we fix ε and μ, then the exactly critical value can be obtained by the following process. The first step is to obtain $\omega_\pm$ as a function of k from Eq. (17). The second one is to substitute $\omega_\pm$ and $\tau_j^\pm$ into Eq. (16). For $j_0, j_1 \in \mathbb{N}$, obtain $k = k_0$ from

$$\begin{aligned}&\left(\arccos\frac{(\mu\omega_+^2 - \mu)(\omega_+^2 - 1 + \varepsilon k(1-\mu)) + (\varepsilon\mu\omega_+)(\varepsilon\omega_+)}{(\mu\omega_+^2-\mu)^2 + (\varepsilon\mu\omega_+)^2} + 2j_0\pi\right)\Big/\omega_+\\ &\quad = \left(\arccos\frac{(\mu\omega_-^2 - \mu)(\omega_-^2 - 1 + \varepsilon k(1-\mu)) + (\varepsilon\mu\omega_-)(\varepsilon\omega_-)}{(\mu\omega_-^2-\mu)^2 + (\varepsilon\mu\omega_-)^2} + 2j_1\pi\right)\Big/\omega_-.\end{aligned} \tag{20}$$

Finally we compute $\tau_0 = \tau_{j_0}^+$ from Eq. (18). Then we have that when $k = k_0$, $\tau = \tau_0$, system (13) undergoes a Hopf–Hopf bifurcation. By estimating the ratio of $\omega_\pm$ we can check whether this point is a nonresonant Hopf–Hopf point. Another algorithm for detecting a $k_1 : k_2$ resonant Hopf–Hopf point can be found in [26]. Here we do not use their approach because of the complexity of the characteristic equation (14).

Now we will use the algorithm in Section 2 to calculate the normal form of (13) for when a Hopf–Hopf bifurcation occurs at $(k, \tau) = (k_0, \tau_0)$.

When $\tau > 0$, carry out the rescaling $t \to t/\tau$, and define $(k, \tau) = (k_0 + \alpha_1, \tau_0 + \alpha_2)$; we have an equivalent form of (13):

$$\begin{cases}\dot{x} = (\tau_0 + \alpha_2)\, y\\ \dot{y} - \mu\dot{y}(t-1) = (\tau_0 + \alpha_2)\,\{[-1 + \varepsilon\,(k_0 + \alpha_1)\,(1-\mu)]\,x + \varepsilon y\\ \quad + \mu x(t-1) - \varepsilon\mu y(t-1) - \varepsilon x^2 y + \varepsilon\mu x^2(t-1)y(t-1)\}\end{cases}. \tag{21}$$

When $\alpha_1 = \alpha_2 = 0$, the corresponding characteristic equation has four roots with zero real part, $\pm i\omega_1\tau_0$, $\pm i\omega_2\tau_0$. Following the procedure in Section 2, we choose $B = \mathrm{diag}\{i\omega_1\tau_0, -i\omega_1\tau_0, i\omega_2\tau_0, -i\omega_2\tau_0\}$,

$$\eta(\theta,\alpha) = \begin{cases} 0, & \theta = 0; \\ -B_1, & \theta \in (-1, 0); \\ -B_1 - B_2, & \theta = -1, \end{cases}$$

with

$$B_1 = \begin{pmatrix} 0 & \tau_0 + \alpha_2 \\ (\tau_0+\alpha_2)[-1+\varepsilon(k_0+\alpha_1)(1-\mu)] & \varepsilon(\tau_0+\alpha_2) \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} 0 & 0 \\ (\tau_0+\alpha_2)\mu & -(\tau_0+\alpha_2)\varepsilon\mu \end{pmatrix},$$

and then (13) can be considered as an abstract ODE such as Eq. (3). After quite a few calculations, we have that the bases of P and P^* are, respectively,

$$\Phi(\theta) = \begin{pmatrix} e^{i\theta\tau_0\omega_1} & e^{-i\theta\tau_0\omega_1} & e^{i\theta\omega_2\tau_0} & e^{-i\theta\omega_2\tau_0} \\ ie^{i\theta\tau_0\omega_1}\omega_1 & -ie^{-i\theta\tau_0\omega_1}\omega_1 & ie^{i\theta\omega_2\tau_0}\omega_2 & -ie^{-i\theta\omega_2\tau_0}\omega_2 \end{pmatrix},$$

$$\Psi(s) = \begin{pmatrix} D_1 e^{-is\tau_0\omega_1}\left(-e^{-i\tau_0\omega_1}\mu\epsilon + \epsilon + ie^{-i\tau_0\omega_1}\mu\omega_1 - i\omega_1\right) & -D_1 e^{-is\tau_0\omega_1} \\ \bar{D}_1 e^{is\tau_0\omega_1}\left(-e^{i\tau_0\omega_1}\mu\epsilon + \epsilon - ie^{i\tau_0\omega_1}\mu\omega_1 + i\omega_1\right) & -\bar{D}_1 e^{is\tau_0\omega_1} \\ D_2 e^{-is\omega_2\tau_0}\left(-e^{-i\omega_2\tau_0}\mu\epsilon + \epsilon - i\omega_2 + ie^{-i\omega_2\tau_0}\omega_2\mu\right) & -D_2 e^{-is\omega_2\tau_0} \\ \bar{D}_2 e^{is\omega_2\tau_0}\left(-e^{i\omega_2\tau_0}\mu\epsilon + \epsilon + i\omega_2 - ie^{i\omega_2\tau_0}\omega_2\mu\right) & -\bar{D}_2 e^{is\omega_2\tau_0} \end{pmatrix}$$

where $D_1 = (e^{-i\tau_0\omega_1}\left(i\mu\omega_1(\varepsilon\tau_0+2) - \mu(\varepsilon+\tau_0) + \mu\tau_0\omega_1^2\right) + \varepsilon - 2i\omega_1)^{-1}$ and $D_2 = (e^{-i\tau_0\omega_2}(i\mu\omega_2(\varepsilon\tau_0+2) - \mu(\varepsilon+\tau_0) + \mu\tau_0\omega_2^2) + \varepsilon - 2i\omega_2)^{-1}$.

Decomposing Eq. (21) as Eq. (3), following the algorithm in Section 2, recalling Remark 2.1 and carrying out the projection onto $(\mathrm{Im}(M_2^1))^c$ and $(\mathrm{Im}(M_3^1))^c$, we then have the following coefficients:

$$a_{11} = -D_1\varepsilon(1-\mu)\tau_0, \qquad a_{12} = D_1\left(k_0\varepsilon(\mu-1) - \mu\left(\omega_1^2+1\right)e^{-i\tau_0\omega_1} + \omega_1^2 + 1\right),$$

$$c_{11} = -\frac{1}{2}D_1\left(2i\varepsilon\mu\tau_0\omega_1 e^{-i\tau_0\omega_1} - 2i\varepsilon\tau_0\omega_1\right), \qquad c_{12} = -D_1\left(2i\varepsilon\mu\tau_0\omega_1 e^{-i\tau_0\omega_1} - 2i\varepsilon\tau_0\omega_1\right),$$

$$a_{21} = -D_2\varepsilon(1-\mu)\tau_0, \qquad a_{22} = D_2\left(k_0\varepsilon(\mu-1) - \mu\left(\omega_2^2+1\right)e^{-i\tau_0\omega_2} + \omega_2^2 + 1\right),$$

$$c_{21} = -D_2\left(2i\varepsilon\mu\tau_0\omega_2 e^{-i\tau_0\omega_2} - 2i\varepsilon\tau_0\omega_2\right), \qquad c_{22} = -\frac{1}{2}D_2\left(2i\varepsilon\mu\tau_0\omega_2 e^{-i\tau_0\omega_2} - 2i\varepsilon\tau_0\omega_2\right).$$

It is quite difficult to estimate the signs of b_0, c_0, d_0 and $d_0 - b_0c_0$; thus we give a numerical example in the following section.

3.2. Illustrations

In this section we choose $\varepsilon = 0.1$ and $\mu = 0.5$. Following Theorem 3.1, we have the bifurcation diagram shown in Fig. 1(a). In Fig. 1(a), several colored Hopf bifurcation curves and a dotted fold bifurcation curve are presented. When $\tau = 0$ the zero solution is unstable and a stable region of the zero solution is marked "Stable Region". One Bogdanov–Takens point, three Hopf-fold points and two Hopf–Hopf points are marked as BT, HF1–HF3 and HH1–HH2, respectively. From Eqs. (16), (17) and (20) we have that when $k_0 = 4.834585$, $\tau_0 = 8.815987$, two different frequencies are obtained: $\omega_1 = 0.730796$, and $\omega_2 = 0.900735$. By the calculations in the previous section we have $c_1 = 0.242977\alpha_1 - 0.298185\alpha_2$, $c_2 = -0.200412\alpha_1 + 0.460212\alpha_2$. Using Remark 2.3, we know that the normal form is non-degenerate and $b_0 = 0.087454$, $c_0 = -45.7383$, $d_0 = -1$, and $d_0 - b_0c_0 = 3$. From Table 1, we have that the case VIa has arisen. From Guckenheimer and Holmes [6], we have that near the Hopf–Hopf point HH1 there are eight different kinds of phase diagrams in eight different regions, which are divided by lines L_1–L_8 as follows: L_1: $\alpha_2 = 0.435478\alpha_1$, $\alpha_1 > 0$, L_2: $\alpha_2 = 0.814854\alpha_1$, $\alpha_1 > 0$, L_3: $\alpha_2 = 0.828102\alpha_1$, $\alpha_1 > 0$, L_4: $\alpha_2 = 0.828985\alpha_1 + O(\alpha_1^2)$, $\alpha_1 > 0$, L_5: $\alpha_2 = 0.828985\alpha_1$, $\alpha_1 > 0$, L_6: $\alpha_2 = 0.874050\alpha_1$, $\alpha_1 > 0$, L_7: $\alpha_2 = 0.435478\alpha_1$, $\alpha_1 < 0$, L_8: $\alpha_2 = 0.814854\alpha_1$, $\alpha_1 < 0$. Recall that $\alpha_1 = k - k_0$, $\alpha_2 = \tau - \tau_0$; thus we give a bifurcation set on the plane of the original parameters in system (13) (see Fig. 1(b)). In Fig. 2, we draw these phase portraits and label the positions at which the corresponding parameters lie. In every portrait, a nontrivial equilibrium on the axis, an equilibrium with positive r_i $(i = 1, 2)$ and a cycle correspond to a nonconstant periodic solution, a quasi-periodic solution on the two-dimensional torus and a quasi-periodic solution on the three-dimensional torus of Eq. (13), respectively.

Fig. 3 illustrates the stable equilibrium and the periodic solution. When parameters are chosen in D_6, there exists a stable quasi-periodic solution on a 2-torus which is shown in Fig. 4. When parameters are chosen in D_5, there exists a quasi-periodic solution on a 3-torus which is shown in Fig. 6(a). Clearly, we find that the points on the Poincaré section exhibit quasi-periodic behavior, which indicates that the solution is a quasi-periodic solution on a 3-torus.

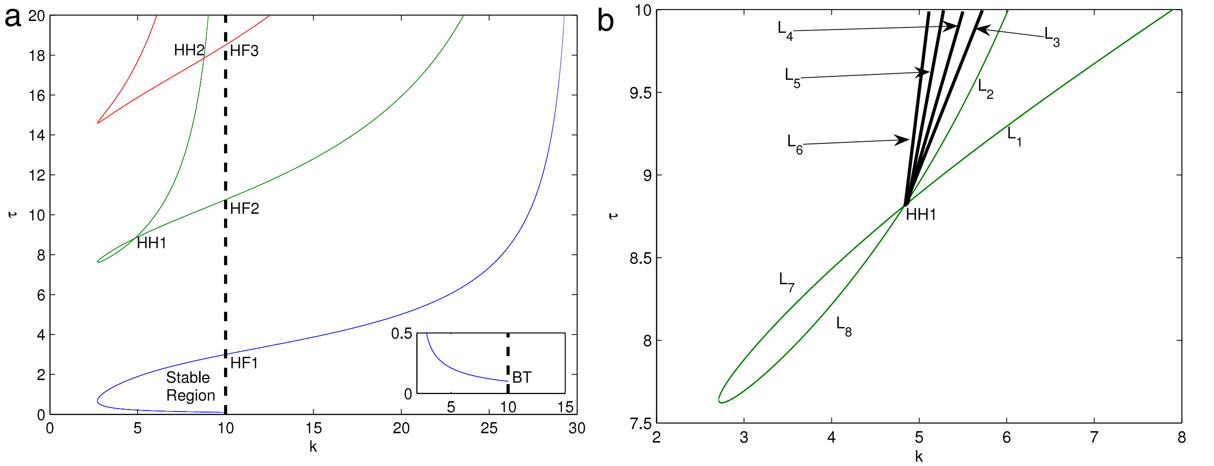

Fig. 1. (a) Partial bifurcation sets with parameters in the k–τ plane. The colored curves stand for Hopf bifurcation curves. (b) Complete bifurcation sets near HH1.

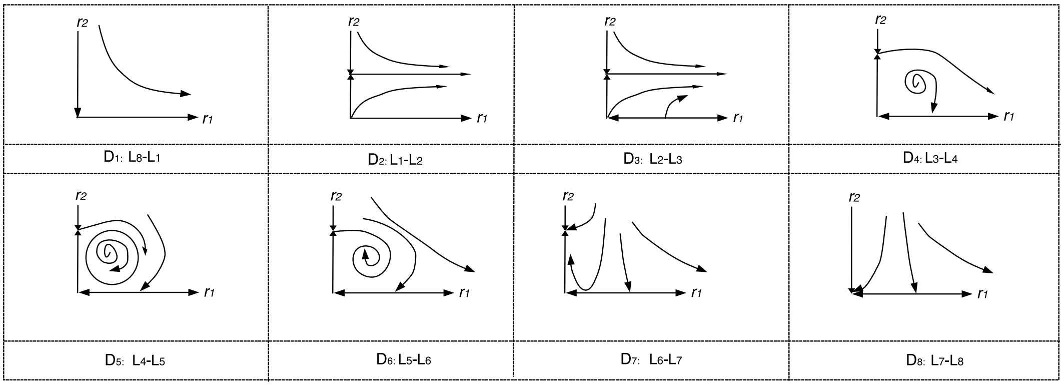

Fig. 2. The eight distinct phase portraits near HH1 in D_1–D_8. Below every figure, we mark the corresponding region in Fig. 1; e.g. the region D_1 is between L_8 and L_1.

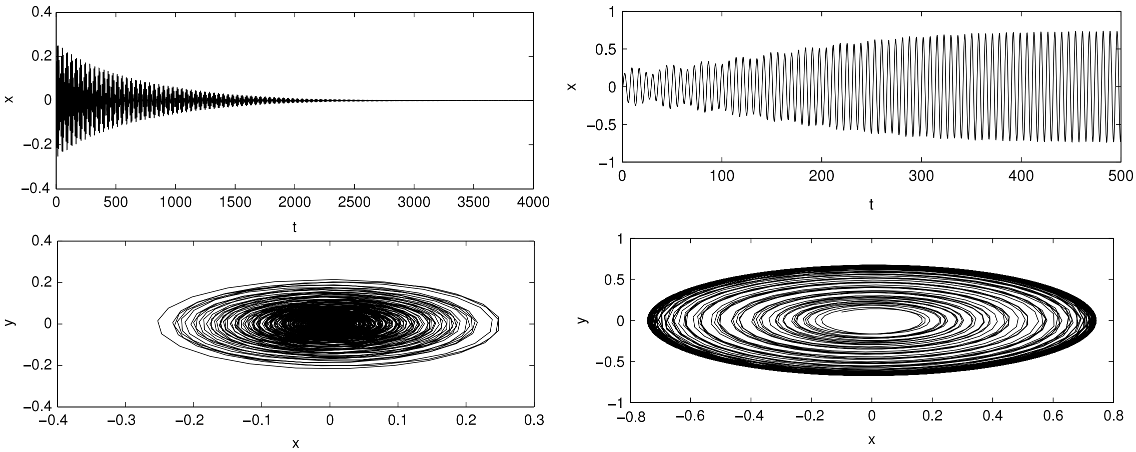

Fig. 3. Left: The trivial equilibrium of system (13) is stable when $\alpha_1 = -0.1, \alpha_2 = -0.08$ in D_8. Right: System (13) has a stable periodic solution when $\alpha_1 = -0.1, \alpha_2 = 0.1$ in D_7.

3.3. The strange attractor near a Hopf–Hopf bifurcation

Generally, a vanishing 3-torus might yield strange attractors. More precisely, Newhouse et al. [23] have shown that by means of a small C^2 perturbation of a quasi-periodic flow on the 3-torus, one can produce strange Axiom *A* attractors

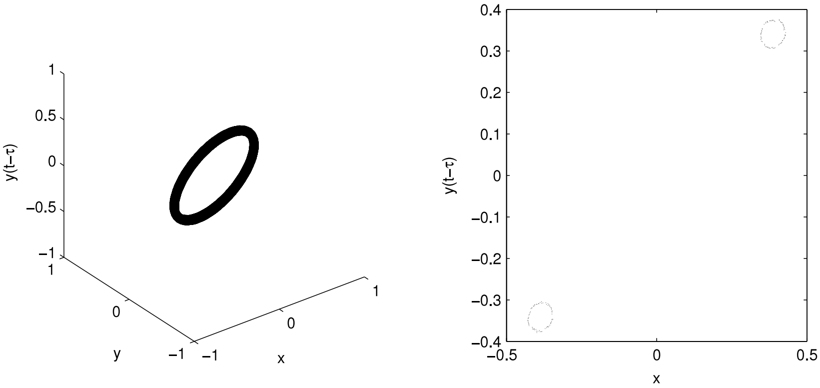

Fig. 4. $\alpha_1 = 0.1, \alpha_2 = 0.085$ in D_6; the bifurcated quasi-periodic solution of system (13) and the corresponding Poincaré map on the whole Poincaré section $y(t) = 0$.

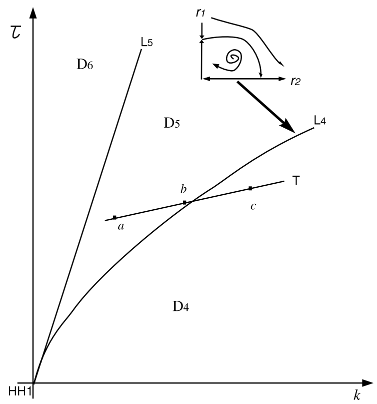

Fig. 5. The sketch of the saddle connection bifurcation curve (L_4) on the 3-torus. (a) $\iota = 2$; (b) $\iota = 2.4$; (c) $\iota = 2.5$.

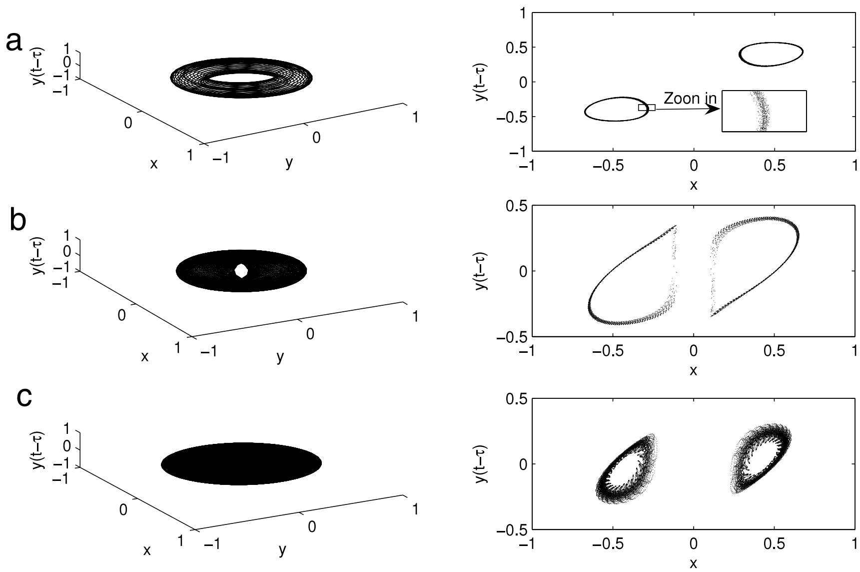

Fig. 6. The phase portraits in x–y–$y(\cdot-\tau)$ space and the corresponding Poincaré map on the whole Poincaré section $y(t) = 0$ for when parameters are chosen at (a), (b) and (c) in Fig. 5, respectively. Note that we delete the transient states for a clear expression.

(see Smale [27]). They claimed that when three pairs of complex conjugate eigenvalues of a ODE have crossed (that is, the oscillation has three frequencies), a motion asymptotic to a nontrivial Axiom A attractor may appear. The time dependence of the flow then becomes chaotic, a situation which one may call turbulent. This idea is also studied by [28,29]. This route

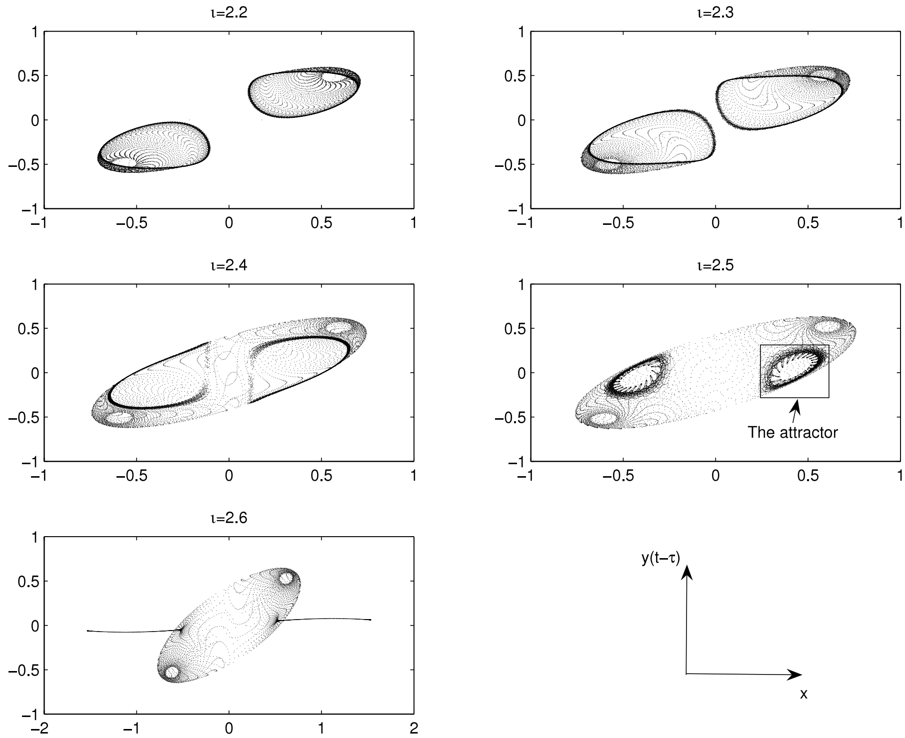

Fig. 7. The complete Poincaré maps near the saddle connection line L_4, for different values of ι on T in Fig. 5.

to chaos is often called "the Ruelle–Takens–Newhouse" scenario [30], which is the first insight into explaining a strange attractor.

A 3-torus appears in a system of ordinary differential equations of dimension at least 4. However, we find that in two-dimensional NFDEs, with the help of the Hopf–Hopf bifurcation sets, one can easily obtain the occurrence of a 3-torus and estimate where it appears and vanishes. Thus the existence of oscillations with a strange attractor can be traced theoretically.

For the rest of this section we devote our attention to checking for the existence of a strange attractor in system (13); we choose three points (a), (b) and (c) on the line T: $(\alpha_1, \alpha_2) = (0.1\iota, 0.081\iota)$, $\iota > 0$, which is shown in Fig. 5. In Fig. 6, the phase portraits are drawn. At (a), the system has a quasi-periodic solution on a 3-torus, which vanishes via the saddle connection bifurcation (b) on the 3-torus (the curve L_4). At point (c), system (13) exhibits chaotic behavior and the strange attractor is drawn on the Poincaré section $y(t) = 0$. Thus we confirm that in an NFDE the breaking down of a 3-torus yields a strange attractor. Fig. 7 is also an illustration of the transition where we give the complete Poincaré map for different values of ι on line T, from which we find the strange attractor vanishing when $\iota = 2.6$. After that, the system is stabilized to a periodic solution with large amplitude.

4. Conclusions

In this paper, we mainly investigate the nonresonant Hopf–Hopf bifurcation in an NFDE with parameters, Eq. (1). We compute the normal form near the bifurcation point. An explicit algorithm is given for calculating the four key variables: b_0, c_0, d_0 and $d_0 - b_0c_0$, by means of which the twelve unfoldings are distinguished.

As an illustration of this theory, van der Pol's equation with extended delay feedback is considered. Detailed dynamics near the critical point are obtained by drawing the corresponding bifurcation set. Both the theoretical bifurcation set and the simulations confirm the existence of stable periodic solutions and quasi-periodic solutions. With the guidance of the bifurcation sets we also find that in van der Pol's equation a strange attractor appears while the 3-torus vanishes via a saddle connection bifurcation. Strange attractors have been encountered a lot in the research into FDEs, most of which are illustrated by numerical methods. However, we find that the investigation of the normal form near a Hopf–Hopf point can sometimes give a theoretical explanation for the existence of chaos.

References

[1] W. Jiang, Y. Yuan, Bogdanov–Takens singularity in Van der Pol's oscillator with delayed feedback, Physica D 227 (2007) 149–161.
[2] H. Wang, W. Jiang, Hopf-pitchfork bifurcation in van der Pol's oscillator with nonlinear delayed feedback, J. Math. Anal. Appl. 368 (2010) 9–18.

[3] S. Ma, Q. Lu, Z. Feng, Double Hopf bifurcation for van der Pol–Duffing oscillator with parametric delay feedback control, J. Math. Anal. Appl. 338 (2008) 993–1007.
[4] P. Yu, Y. Yuan, J. Xu, Study of double Hopf bifurcation and chaos for an oscillator with time delayed feedback, Commun. Nonlinear Sci. Numer. Simul. 7 (2002) 69–91.
[5] P. Buono, J. Bélair, Restrictions and unfolding of double Hopf bifurcation in functional differential equations, J. Differential Equations 189 (2003) 234–266.
[6] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer, New York, 1983.
[7] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer, New York, 1980.
[8] T. Faria, L. Magalhaes, Normal forms for retarded functional differential equation with parameters and applications to Hopf bifurcation, J. Differential Equations 122 (1995) 181–200.
[9] M. Weedermann, Normal forms for neutral functional differential equations, in: T. Faria, P. Freitas (Eds.), Topics in Functional Differential and Difference Equations, Amer. Math. Soc., Providence, RI, 2001, pp. 361–368.
[10] C. Wang, J. Wei, Normal forms for NFDE with parameters and application to the lossless transmission line, Nonlinear Dynam. 52 (2008) 199–206.
[11] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer, New York, 1995.
[12] J. Hale, S. Lunel, Introduction to Functional Differential Equations, Springer, New York, 1993.
[13] J. Wei, S. Ruan, Stability and global Hopf bifurcation for neutral differential equations, Acta. Math. Sin. 45 (2002) 94–104.
[14] B. Niu, J. Wei, Bifurcation analysis of a NFDE arising from multiple delay feedback control, Internat. J. Bifur. Chaos 21 (2011) 759–774.
[15] R. FitzHugh, Impulses and physiological states in theoretical models of nerve membranes, J. Biophys. 1 (1961) 445–466.
[16] J. Nagumo, S. Arimoto, S. Yoshizawa, An active pulse transmission line simulating nerve axon, Proc. IRE 50 (1962) 2061–2070.
[17] F. Atay, Van der Pol's oscillator under delayed feedback, J. Sound Vib. 218 (1998) 333–339.
[18] A. Maccari, Vibration control for the primary resonance of the van der Pol oscillator by a time delay state feedback, Int. J. Non-Linear Mech. 38 (2003) 123–131.
[19] J. Wei, W. Jiang, Stability and bifurcation analysis in Van der Pol's oscillator with delayed feedback, J. Sound Vib. 283 (2005) 801–819.
[20] J.E.S. Socolar, D.W. Sukow, D.J. Gauthier, Stabilizing unstable periodic orbits in fast dynamical systems, Phys. Rev. E 50 (1994) 3245.
[21] D.W. Sukow, M.E. Bleich, D.J. Gauthier, J.E.S. Socolar, Controlling chaos in a fast diode resonator using extended time-delay autosynchronization: Experimental observations and theoretical analysis, Chaos 7 (1997) 560–576.
[22] K. Pyragas, Control of chaos via extended delay feedback, Phys. Lett. A 206 (1995) 323–330.
[23] S. Newhouse, D. Ruelle, F. Takens, Occurrence of strange Axiom A attractors near quasi-periodic flows on T^m, $m \geq 3$, Comm. Math. Phys. 64 (1978) 35–40.
[24] S.-N. Chow, K. Lu, C^k center unstable manifolds, Proc. Roy. Soc. Edinburgh 108 (1988) 303–320.
[25] J. Carr, Applications of Centre Manifold Theory, Springer, New York, 1981.
[26] J. Xu, K. Chung, C. Chan, An efficient method for studying weak resonant double Hopf bifurcation in nonlinear systems with delayed feedbacks, SIAM J. Appl. Dyn. Syst. 6 (2007) 29–60.
[27] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc. 73 (1967) 747–817.
[28] P. Battelino, C. Grebogi, E. Ott, J. Yorke, Chaotic attractors on a 3-torus, and torus break-up, Physica D 39 (1989) 299–314.
[29] D. Ruelle, F. Takens, On the nature of turbulence, Comm. Math. Phys. 20 (1971) 167–192.
[30] J.P. Eckmann, Roads to turbulence in dissipative dynamical systems, Rev. Modern Phys. 53 (1981) 643–654.