J. Math. Anal. Appl. 400 (2013) 689–709

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

The large-time energy concentration in solutions to the Navier–Stokes equations in the frequency space

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ARTICLE INFO

Article history:
Received 4 July 2012
Available online 8 November 2012
Submitted by Pierre Lemarie-Rieusset

Keywords:
Navier–Stokes equations
Large-time behavior of solutions
Energy concentration

ABSTRACT

In the paper we study the large-time behavior of solutions to the Navier–Stokes equations in the frequency space. We describe in detail the large-time energy concentration which occurs in every (turbulent) solution. If the energy of the solution decreases exponentially then it concentrates in frequencies localized in an annulus in the frequency space. The annulus can be taken arbitrarily narrow and its diameter determines the rate of the exponential decay. All the other solutions are characterized by the concentration of the energy in the frequencies localized in a ball with an arbitrarily small diameter centered at the origin of the coordinates. It will follow from the presented results that the frequencies outside the annulus or the ball and especially the higher frequencies die out very quickly. We will further observe the concentration occurring in any time derivative of the solution or in the vorticity and its time derivatives with the same annulus or the ball for the particular solution.

1. Introduction

The large-time behavior of solutions to the Navier–Stokes equations has been intensively studied for several decades; see for example [1,2] or [3]. In this paper we are mostly interested in the large-time behavior of solutions in the frequency space. It was proved in [4] that if u is a nonzero turbulent solution of the Navier–Stokes equations (that is a global weak solution satisfying the strong energy inequality), then

$$\lim_{t\to\infty}\frac{\|(E_{a+\varepsilon}-E_{a-\varepsilon})u(t)\|}{\|u(t)\|}=1 \tag{1}$$

for every $\varepsilon>0$, where $a=\lim_{t\to\infty}\|A^{1/2}u(t)\|^2/\|u(t)\|^2$ is a well defined nonnegative finite number. Here A is the Stokes operator, $\{E_\lambda;\,\lambda\ge 0\}$ denotes the resolution of the identity of the Stokes operator and $\|\cdot\|$ is the L^2-norm (see Section 2 for other notation). We put $E_{a-\varepsilon}=0$ if $a-\varepsilon\le 0$. This result holds for the case of any sufficiently smooth three-dimensional domain. For the case of the whole three-dimensional space, the result can be formulated as follows: if $a>0$ and $\varepsilon\in(0,a)$ then

$$\lim_{t\to\infty}\frac{\int_{B_{\sqrt{a+\varepsilon}}(0)\setminus B_{\sqrt{a-\varepsilon}}(0)}|F(u(t))(\xi)|^2\,d\xi}{\int_{\mathbf{R}^3}|F(u(t))(\xi)|^2\,d\xi}=1. \tag{2}$$

If $a=0$ and $\varepsilon>0$ then

$$\lim_{t\to\infty}\frac{\int_{B_{\sqrt{\varepsilon}}(0)}|F(u(t))(\xi)|^2\,d\xi}{\int_{\mathbf{R}^3}|F(u(t))(\xi)|^2\,d\xi}=1. \tag{3}$$

Here F denotes the Fourier transform, $\chi_{B_{\sqrt{\lambda}}(0)}$ is the characteristic function of $B_{\sqrt{\lambda}}(0)=\{x\in\mathbf{R}^3;\,|x|\le\sqrt{\lambda}\}$.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.11.004

It follows from (2) and (3) that the energy of solutions concentrates in frequencies from an annulus if $a > 0$ (solutions with exponentially decreasing energy—see [4]) or from a ball if $a = 0$. This result holds for any turbulent solution and for any sufficiently smooth three-dimensional domain—in the sense of (1).

In the present paper we improve the mentioned results for the case of the three-dimensional space. For example, we will show instead of (2) and (3) that for any $\alpha \geq 0$

$$\lim_{t\to\infty} \frac{\int_{K_{a,\varepsilon}^C} |\xi|^{4\alpha} |F(u(t))(\xi)|^2\, d\xi}{\int_{K_{a,\varepsilon}} |\xi|^{4\alpha} |F(u(t))(\xi)|^2\, d\xi} = 0, \tag{4}$$

where $K_{a,\varepsilon} = B_{\sqrt{a+\varepsilon}}(0) \setminus B_{\sqrt{a-\varepsilon}}(0)$ for (2) and $K_{a,\varepsilon} = B_{\sqrt{\varepsilon}}(0)$ for (3). $K_{a,\varepsilon}^C = \mathbf{R}^3 \setminus K_{a,\varepsilon}$. It is clear that (4) offers a much better insight into the evolution of particular frequencies than (2) and (3).

Let us mention here that it is not difficult to find solutions with the large-time energy concentration in the low frequencies—the solutions satisfying (4) with $K_{a,\varepsilon} = B_{\sqrt{\varepsilon}}(0)$ for any $\varepsilon > 0$. Several classes of the initial conditions yielding such solutions were described in [5]. In fact, every solution with the energy not decreasing exponentially has this property. On the other hand, it is still an open problem to find a solution with exponentially decreasing energy and so satisfying (4) with some $K_{a,\varepsilon} = B_{\sqrt{a+\varepsilon}}(0) \setminus B_{\sqrt{a-\varepsilon}}(0)$ even though the existence of such solutions is very well known (see [6,7]).

In fact, we will prove that (4) remains true even if the solution u is replaced with the time derivative of u of any order or with the vorticity and its time derivatives. It is interesting here that for a fixed solution the number a does not change notwithstanding if we consider the solution itself, its time derivatives or the vorticity and its time derivatives.

The main results of the present paper are summed up in Theorem 1 and Corollary 1 in Section 3. The proofs and results concerning time derivatives of solutions and the vorticity are presented in Sections 4 and 5.

The topic of the present paper seems to be connected with the study of the Navier–Stokes solutions in the Besov spaces. The connection of the Besov spaces and the Navier–Stokes equations was studied in a series of papers; for example [8–11] or [12]. In [13] the author studied the large-time behavior of turbulent solutions with initial conditions $u_0 \in L^2_\sigma$ such that

$$\int (1 + |x|)|u_0(x)|dx < \infty.$$

Such solutions always lie in the homogeneous Besov space $B^{-1}_{1,\infty}$ and consequently in the homogeneous space $B^{-5/2}_{2,\infty}$. Due to the definition of $B^{-5/2}_{2,\infty}$, the presence of the solution in this space gives us information on the distribution of energy throughout the entire frequency spectrum. On the other hand, the results presented in the present paper seem to be different: they give information on the relative incidence of frequencies in the frequency spectrum of the particular solution and its evolution in time. Nevertheless it would be interesting to establish some deeper relation of the results presented in the present paper with the results from the literature mentioned above.

2. Notation and preliminaries

The notation and most of the results from this section comes from [14].

Let $q \in [1, \infty]$ and $k \in N$. Then $L^q = L^q(\mathbf{R}^3)$ and $W^{k,2} = W^{k,2}(\mathbf{R}^3)$ denote the Lebesgue and Sobolev spaces with the norms $\|\cdot\|_q$ and $\|\cdot\|_{k,2}$. We will often denote $\|\cdot\|$ instead of $\|\cdot\|_2$. Define L^2_σ, resp. $W^{1,2}_{0,\sigma}$, as the closure of $C^\infty_{0,\sigma} = \{\varphi \in C_0^\infty(\mathbf{R}^3)^3; \nabla\cdot\varphi = 0\}$ in $(L^2)^3$, resp. $(W^{1,2})^3$. P_σ denotes the Helmholtz projection from $(L^2)^3$ onto L^2_σ. The Stokes operator A is defined as $A = -\Delta = -\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 - \partial^2/\partial x_3^2$ with the domain $D(A) = (W^{2,2})^3 \cap L^2_\sigma$. A is a positive self-adjoint operator. $\{E_\lambda; \lambda \geq 0\}$ denotes the resolution of identity of A. If $\mu \in \mathbf{R}$, then the powers A^μ of A can be defined by the use of $\{E_\lambda; \lambda \geq 0\}$ (see [14, Chapters II.3.2 and III.2.3]). The domains of A^μ are denoted by $D(A^\mu)$. $\{e^{-At}; t \geq 0\}$ denotes the Stokes semigroup generated by $-A$. $|||\cdot|||_\alpha = \|\cdot\| + \|A^\alpha\cdot\|$ is the graph norm. F denotes the Fourier transform defined as $Fu(\xi) = \int e^{-ix\cdot\xi}u(x)\, dx$. The Navier–Stokes equations in $\mathbf{R}^3$ can be written as

$$\frac{\partial u}{\partial t} - \Delta u + u\cdot\nabla u + \nabla p = 0 \quad \text{in } \mathbf{R}^3 \times (0,\infty), \tag{5}$$

$$\nabla\cdot u = 0 \quad \text{in } \mathbf{R}^3 \times (0,\infty), \tag{6}$$

$$u|_{t=0} = u_0, \tag{7}$$

where $u = u(x,t)$ and $p = p(x,t)$ denote the unknown velocity and pressure and $u_0 = u_0(x)$ is a given initial velocity.

If $u_0 \in L^2_\sigma$, a measurable function u defined on $\mathbf{R}^3 \times (0,\infty)$ is called a global weak solution of (5)–(7) if

$$u \in L^\infty((0,\infty); L^2_\sigma) \cap L^2((0,T); W^{1,2}_{0,\sigma}) \quad \text{for every } T > 0$$

and the integral relation

$$\int_0^\infty [-(u(t), \partial_t\phi(t)) + (\nabla u(t), \nabla\phi(t)) + (u(t)\cdot\nabla u(t), \phi(t))]dt = (u_0, \phi(0))$$

holds for all $\phi \in C_0^\infty([0,\infty); C_{0,\sigma}^\infty)$. We say that a global weak solution satisfies the strong energy inequality if

$$\|u(t)\|^2 + 2\int_s^t \|\nabla u(\sigma)\|^2 d\sigma \le \|u(s)\|^2$$

for almost all $s \ge 0$ including $s = 0$ and all $t \ge s$. A global weak solution to (5)–(7) satisfying the strong energy inequality is called a turbulent solution.

Let $u_0 \in D(A)$. A function $u \in C([0,\infty); D(A)) \cap C^1((0,\infty); L^2_\sigma)$ is called a global strong solution of (5)–(7) if $u(0) = u_0$ and $du/dt + Au + P_\sigma(u\cdot\nabla u) = 0$ for every $t > 0$.

It is known that for $u_0 \in L^2_\sigma$ there exists at least one turbulent solution of (5)–(7). It is also known (see [14, Chapter V]) that this solution becomes strong after some transient time which means that there exists $T_0 \ge 0$ such that $u \in C([T_0,\infty); D(A)) \cap C^1((T_0,\infty); L^2_\sigma)$ and $du/dt + Au + P_\sigma(u\cdot\nabla u) = 0$ for every $t > T_0$.

We will now present several known results which we will use in the present paper.

(i) $\|\nabla u\| = \|A^{1/2}u\|$ for every $u \in D(A^{1/2}) = W^{1,2}_{0,\sigma}$ (see [14, Chapter III.2]).

(ii) $D(A^k) = (W^{2k,2})^3 \cap L^2_\sigma$ for any $k \in N$. If $g \in (W^{k,2})^3$, then $P_\sigma(g) \in (W^{k,2})^3$ and $P_\sigma(\nabla^k g) = \nabla^k P_\sigma g$ (see [14, Chapter III.2]).

(iii) If u is a global strong solution of (5)–(7), then $u(t) \in D(A^n)$ for any $n \in N$ and $t > 0$. Consequently, it follows from (ii) that $P_\sigma(u\cdot\nabla u(t)) \in D(A^n)$ for any $n \in N$.

(iv) If $\alpha \in [0, 3/4)$, $q \in [2,\infty)$ and $2\alpha + 3/q = 3/2$, then there exists $c = c(\alpha, q)$ such that $\|u\|_q \le c\|A^\alpha u\|$ for every $u \in D(A^\alpha)$ (see [14, Chapter III.2.4]).

(v) Let u be a global strong solution of (5)–(7). Then for any $t, \delta \ge 0$

$$u(t+\delta) = e^{-A\delta}u(t) - \int_0^\delta e^{-A(\delta - s)} P_\sigma(u\cdot\nabla u(t+s))\, ds \tag{8}$$

(see [14, Chapter V.1.3]).

(vi) If $0 \le z < y < x$ and $u \in D(A^x)$ then a so called moment inequality holds:

$$\|A^y u\| \le \|A^z u\|^{\frac{x-y}{x-z}} \|A^x u\|^{\frac{y-z}{x-z}}. \tag{9}$$

(vii) Let $k \in N$. There exists a positive constant $c(k)$ such that

$$\|A^{k/2}u\| \le \|\nabla^k u\| \le c(k)\|A^{k/2}u\|$$

for every $u \in D(A^{k/2})$ (it follows directly from [14, Lemma 2.3.2]).

(viii) If u is a global strong solution of (5)–(7) and $\|u(t)\| = O(t^{-\mu})$, $t \to \infty$, for some $\mu \ge 0$, then for any $\alpha \ge 0$

$$\|A^\alpha u(t)\| = O(t^{-\alpha-\mu}), \quad t \to \infty, \tag{10}$$

(see [15]).

The following result was proved in [16].

(ix) Let Assumption 1 be satisfied. Let $0 \le \alpha \le \beta < \infty$. Then there exist $C = C(\alpha,\beta) > 1$, $\delta_0 = \delta_0(\alpha,\beta) \in (0,1)$ and $t_0 = t_0(\alpha,\beta)$ such that

$$\frac{\|A^\beta u(t)\|}{\|A^\alpha u(t+\delta)\|} \le C$$

for every $t \ge t_0$ and every $\delta \in [0, \delta_0]$.

(x) The equality $F(E_\lambda u) = \chi_{B_{\sqrt{\lambda}}(0)} F(u)$ holds for every $u \in L^2$ and every $\lambda > 0$ (see [17]). χ_A denotes the characteristic function of A, $B_{\sqrt{\lambda}}(0) = \{x \in \mathbf{R}^3;\ |x|^2 \le \lambda\}$ and F is the Fourier transform.

Assumption 1. We suppose that u is a turbulent solution to (5)–(7) such that $u(t) \ne 0$ for every $t \in [T,\infty)$ for some $T \ge 0$.

Remark 1. It is not clear if Assumption 1 is satisfied for any turbulent solution to (5)–(7) with a nonzero initial condition. However, if Assumption 1 was not satisfied for some solution, it would mean that this solution is equal to zero for every sufficiently large time. So such a solution is not interesting from the point of view of the large-time dynamics.

3. Large-time localization of $A^\alpha u$ in the frequency space

We will start this section with a precise formulation of the main result.

Theorem 1. *Let $0 \le \alpha < \beta < \infty$. Let Assumption 1 be satisfied. Then $a = \lim_{t\to\infty}\left(\frac{\|A^\beta u(t)\|}{\|A^\alpha u(t)\|}\right)^{1/(\beta-\alpha)}$ is a well defined finite nonnegative number independent of α and β. Further, if $\varepsilon > 0$ then*

$$\lim_{t\to\infty} \frac{|||E_{a,\varepsilon}A^\alpha u(t)|||_{\beta-\alpha}}{|||A^\alpha u(t)|||_{\beta-\alpha}} = \lim_{t\to\infty} \frac{\|E_{a,\varepsilon}A^\alpha u(t)\|}{\|A^\alpha u(t)\|} = 1, \tag{11}$$

where $E_{a,\varepsilon} = E_{a+\varepsilon} - E_{a-\varepsilon}$ if $a > 0$ (we put $E_{a-\varepsilon} = 0$ if $a - \varepsilon \le 0$) and $E_{a,\varepsilon} = E_\varepsilon$ if $a = 0$. Moreover,

$$\lim_{t\to\infty} \frac{\int_{K_{a,\varepsilon}} |\xi|^{4\alpha}|F(u(t))(\xi)|^2\, d\xi}{\int_{\mathbf{R}^3} |\xi|^{4\alpha}|F(u(t))(\xi)|^2\, d\xi} = 1, \tag{12}$$

where $K_{a,\varepsilon} = B_{\sqrt{a+\varepsilon}}(0) \setminus B_{\sqrt{a-\varepsilon}}(0)$ if $a > \varepsilon > 0$, $K_{a,\varepsilon} = B_{\sqrt{a+\varepsilon}}(0)$ if $\varepsilon \ge a > 0$ and $K_{a,\varepsilon} = B_{\sqrt{\varepsilon}}(0)$ if $a = 0$, $B_r(0) = \{x \in \mathbf{R}^3;\, |x| < r\}$ and F denotes the Fourier transform.

Corollary 1. *It follows immediately from (11) that*

$$\lim_{t\to\infty} \frac{\|(I - E_{a,\varepsilon})A^\alpha u(t)\|}{\|E_{a,\varepsilon}A^\alpha u(t)\|} = 0 \tag{13}$$

and

$$\lim_{t\to\infty} \frac{\int_{K^C_{a,\varepsilon}} |\xi|^{4\alpha}|F(u(t))(\xi)|^2\, d\xi}{\int_{K_{a,\varepsilon}} |\xi|^{4\alpha}|F(u(t))(\xi)|^2\, d\xi} = 0, \tag{14}$$

where $K^C_{a,\varepsilon} = \mathbf{R}^3 \setminus K_{a,\varepsilon}$.

The following lemma is a key result for the proof of Theorem 1.

Lemma 1. *Let Assumption 1 be satisfied and $0 \le \alpha < \beta < \infty$. Then*

$$\lim_{t\to\infty} \frac{\|A^\beta u(t)\|^2}{\|A^\alpha u(t)\|^2} \tag{15}$$

is well defined and it is a nonnegative finite number.

Proof. We can suppose without loss of generality that $k/2 + 1/2 \le \alpha < \beta \le k/2 + 1$ for some $k \in N_0 \cup \{-1\}$. There exists $t_0 \ge 2$ such that u is strong on $[t_0 - 1, \infty)$. Let $\kappa(t) = \|A^\beta u(t)\|^2/\|A^\alpha u(t)\|^2$ for $t \ge t_0$. First, it follows from (5) that

$$\frac{d}{dt}\|A^\alpha u(t)\|^2 = -2\|A^{\alpha+1/2}u(t)\|^2 - 2(P_\sigma(u\cdot\nabla u(t)), A^{2\alpha}u(t)).$$

Therefore,

$$\begin{aligned}
\kappa'(t) &= \frac{-2(\|A^{\beta+1/2}u(t)\|^2 + (P_\sigma(u\cdot\nabla u(t)), A^{2\beta}u(t)))\|A^\alpha u(t)\|^2}{\|A^\alpha u(t)\|^4} \\
&\quad + \frac{2\|A^\beta u(t)\|^2(\|A^{\alpha+1/2}u(t)\|^2 + (P_\sigma(u\cdot\nabla u(t)), A^{2\alpha}u(t)))}{\|A^\alpha u(t)\|^4} \\
&= \frac{-2(A^{\beta+1/2}u(t), A^{\beta+1/2}u(t) + A^{\beta-1/2}P_\sigma(u\cdot\nabla u(t)))}{\|A^\alpha u(t)\|^2} \\
&\quad + \kappa(t)\frac{2(A^{\beta+1/2}u(t), A^{2\alpha-\beta+1/2}u(t))}{\|A^\alpha u(t)\|^2} + \kappa(t)\frac{2(P_\sigma(u\cdot\nabla u(t)), A^{2\alpha}u(t))}{\|A^\alpha u(t)\|^2} \\
&= \frac{-2(A^{\beta+1/2}u(t), A^{\beta+1/2}u(t) + A^{\beta-1/2}P_\sigma(u\cdot\nabla u(t)) - \kappa(t)A^{2\alpha-\beta+1/2}u(t))}{\|A^\alpha u(t)\|^2} + \kappa(t)\frac{2(P_\sigma(u\cdot\nabla u(t)), A^{2\alpha}u(t))}{\|A^\alpha u(t)\|^2}.
\end{aligned}$$

Since

$$\begin{aligned}
&(\kappa(t)A^{2\alpha-\beta+1/2}u(t), A^{\beta+1/2}u(t) + A^{\beta-1/2}P_\sigma(u\cdot\nabla u(t)) - \kappa(t)A^{2\alpha-\beta+1/2}u(t)) \\
&\quad = \frac{\|A^\beta u(t)\|^2}{\|A^\alpha u(t)\|^2}\,\|A^{\alpha+1/2}u(t)\|^2 + \kappa(t)(A^{2\alpha}u(t), P_\sigma(u\cdot\nabla u(t))) - \frac{\|A^\beta u(t)\|^4}{\|A^\alpha u(t)\|^4}\,\|A^{2\alpha-\beta+1/2}u(t)\|^2,
\end{aligned}$$

we have

$$\kappa'(t) = \frac{-2(A^{\beta+1/2}u(t) - \kappa(t)A^{2\alpha-\beta+1/2}u(t), A^{\beta+1/2}u(t) + A^{\beta-1/2}P_\sigma(u\cdot\nabla u(t)) - \kappa(t)A^{2\alpha-\beta+1/2}u(t))}{\|A^\alpha u(t)\|^2}$$
$$-2\frac{\|A^\beta u(t)\|^2}{\|A^\alpha u(t)\|^4}\,\|A^{\alpha+1/2}u(t)\|^2 + 2\frac{\|A^\beta u(t)\|^4}{\|A^\alpha u(t)\|^6}\,\|A^{2\alpha-\beta+1/2}u(t)\|^2.$$

The sum of the last two terms is smaller than or equal to zero, so

$$\kappa'(t) \le \frac{-2\|A^{\beta+1/2}u(t) - \kappa(t)A^{2\alpha-\beta+1/2}u(t)\|^2}{\|A^\alpha u(t)\|^2} + \frac{-2(A^{\beta+1/2}u(t) - \kappa(t)A^{2\alpha-\beta+1/2}u(t), A^{\beta-1/2}P_\sigma(u\cdot\nabla u(t)))}{\|A^\alpha u(t)\|^2}$$
$$\le \frac{\|A^{\beta-1/2}P_\sigma(u\cdot\nabla u(t))\|^2}{\|A^\alpha u(t)\|^2}.$$

If $0 \le \alpha < \beta < 1/2$, then

$$\|A^{\beta-1/2}P_\sigma(u\cdot\nabla u(t))\|^2 \le c\|A^{\beta+1/4}u(t)\|^2\|A^{1/2}u(t)\|^2 \le c\|A^\alpha u(t)\|^2\|A^{2\beta-\alpha+1/2}u(t)\|\,\|A^{1-\alpha}u(t)\|$$

and

$$\kappa'(t) \le c\|A^{2\beta-\alpha+1/2}u(t)\|\,\|A^{1-\alpha}u(t)\| \le c(1+t)^{-3/2}.$$

If $0 \le \alpha < \beta = 1/2$, then

$$\|P_\sigma(u\cdot\nabla u(t))\|^2 \le c\|u(t)\|_6^2\|\nabla u(t)\|_3^2 \le c\|A^{1/2}u(t)\|^2\|\nabla u(t)\|\,\|\nabla u(t)\|_6$$
$$\le c\|A^{1/2}u(t)\|^3\|Au(t)\| \le c\|A^\alpha u(t)\|^{3/2}\|A^{1-\alpha}u(t)\|^{3/2}\|A^\alpha u(t)\|^{1/2}\|A^{2-\alpha}u(t)\|^{1/2}$$

and

$$\kappa'(t) \le c\|A^{1-\alpha}u(t)\|^{3/2}\,\|A^{2-\alpha}u(t)\|^{1/2} \le c(1+t)^{-3/2}.$$

Suppose now that $k/2 + 1/2 \le \alpha < \beta \le k/2 + 1$ for some $k \in N_0$. Then

$$\|A^{\beta-1/2}P_\sigma(u(t)\cdot\nabla u(t))\|^2 \le \|A^{k/2}P_\sigma(u\cdot\nabla u)\|^{2(k+2-2\beta)}\|A^{k/2+1/2}P_\sigma(u\cdot\nabla u)\|^{2(2\beta-k-1)}$$
$$\le \left(\sum_{j=0}^{k}\|\nabla^j u\cdot\nabla^{k-j+1}u\|\right)^{2(k+2-2\beta)}\left(\sum_{j=0}^{k+1}\|\nabla^j u\cdot\nabla^{k-j+2}u\|\right)^{2(2\beta-k-1)}$$
$$\times\ \|\nabla^j u\cdot\nabla^{k-j+1}u\| \le \|\nabla^j u\|_6\|\nabla^{k-j+1}u\|_3$$
$$\le c\|A^{j/2+1/2}u\|\cdot\|A^{k/2-j/2+1/2}u\|^{1/2}\|A^{k/2-j/2+1}u\|^{1/2} \le c\|A^{k/2+1}u\|^{(2k+5)/(2k+4)}$$
$$\le c\|A^\alpha u\|^{[(2k+5)(2\tilde\beta-k-2)]/[2(2k+4)(\tilde\beta-\alpha)]}\|A^{\tilde\beta}u\|^{[(2k+5)(k-2\alpha+2)]/[2(2k+4)(\tilde\beta-\alpha)]},$$

where $\tilde\beta$ is sufficiently large. Analogically,

$$\|\nabla^j u\cdot\nabla^{k-j+2}u\| \le c\|A^\alpha u\|^{[(2k+7)(2\tilde\beta-k-3)]/[2(2k+6)(\tilde\beta-\alpha)]}\|A^{\tilde\beta}u\|^{[(2k+7)(k-2\alpha+3)]/[2(2k+6)(\tilde\beta-\alpha)]}.$$

So,

$$\|A^{\beta-1/2}P_\sigma(u(t)\cdot\nabla u(t))\|^2 \le c\|A^\alpha u\|^{\gamma_1}\|A^{\tilde\beta}u\|^{\gamma_2},$$

where

$$\gamma_1 = \frac{2k+5}{k+2}\,\frac{2\tilde\beta-k-2}{2(\tilde\beta-\alpha)}\,(k+2-2\beta) + \frac{2k+7}{k+3}\,\frac{2\tilde\beta-k-3}{2(\tilde\beta-\alpha)}\,(2\beta-k-1)$$

and

$$\gamma_2 = \frac{2k+5}{k+2}\,\frac{k-2\alpha+2}{2(\tilde\beta-\alpha)}\,(k+2-2\beta) + \frac{2k+7}{k+3}\,\frac{k-2\alpha+3}{2(\tilde\beta-\alpha)}\,(2\beta-k-1).$$

It is possible to verify elementarily that $\gamma_1 \ge 2$ and $\alpha(\gamma_1 - 2) + \tilde\beta\gamma_2 > 3/2$ if $\tilde\beta$ is sufficiently large. Using (10) we then have

$$\|A^{\beta-1/2}P_\sigma(u(t)\cdot\nabla u(t))\|^2 \le c\|A^\alpha u\|^2\|A^\alpha u\|^{\gamma_1-2}\|A^{\tilde\beta}u\|^{\gamma_2} \le c\|A^\alpha u\|^2(1+t)^{-3/2}.$$

Consequently, $\kappa'(t) \le c(1+t)^{-3/2}$ and a nonnegative $\lim_{t\to\infty}\kappa(t)$ exists. □

Definition 1. Let $0 \le \alpha < \beta < \infty$. Let u be a global strong solution of (5)–(7), $u \ne 0$. We define (dropping for simplicity the index α)

$$
\begin{aligned}
&C(\beta) = \lim_{t\to\infty} \frac{\|A^\beta u(t)\|}{\|A^\alpha u(t)\|},\\
&M_\beta = \left\{\lambda \ge 0;\ \liminf_{t\to\infty} \frac{\|\!|E_\lambda A^\alpha u(t)|\!\|_{\beta-\alpha}}{\|\!|A^\alpha u(t)|\!\|_{\beta-\alpha}} > 0\right\},\\
&L_\beta = \left\{\lambda \ge 0;\ \limsup_{t\to\infty} \frac{\|\!|E_\lambda A^\alpha u(t)|\!\|_{\beta-\alpha}}{\|\!|A^\alpha u(t)|\!\|_{\beta-\alpha}} > 0\right\},\\
&M = \left\{\lambda \ge 0;\ \liminf_{t\to\infty} \frac{\|E_\lambda A^\alpha u(t)\|}{\|A^\alpha u(t)\|} > 0\right\},\\
&L = \left\{\lambda \ge 0;\ \limsup_{t\to\infty} \frac{\|E_\lambda A^\alpha u(t)\|}{\|A^\alpha u(t)\|} > 0\right\},\\
&a_\beta = \inf M_\beta, \qquad d_\beta = \inf L_\beta, \quad a = \inf M,\ d = \inf L.
\end{aligned}
$$

Theorem 1 will be proved as an immediate consequence of lemmas presented in the rest of this section. We suppose throughout the rest of the section that Assumption 1 is satisfied. We also suppose that $0 \le \alpha < \beta < \infty$. First, we will get as the main consequence of Lemmas 2–11 that $a = a_\beta = d = d_\beta = C(\beta)^{1/(\beta-\alpha)}$—see Remark 2.

Lemma 2. *Let $\lambda > C(\beta)^{1/(\beta-\alpha)}$. Then*

$$\liminf_{t\to\infty} \frac{\|E_\lambda A^\alpha u(t)\|}{\|A^\alpha u(t)\|} \ge \frac{(\lambda^{2(\beta-\alpha)} - C(\beta)^2)^{1/2}}{\lambda^{\beta-\alpha}} > 0, \tag{16}$$

$$\liminf_{t\to\infty} \frac{\|\!|E_\lambda A^\alpha u(t)|\!\|_{\beta-\alpha}}{\|\!|A^\alpha u(t)|\!\|_{\beta-\alpha}} \ge \frac{(\lambda^{2(\beta-\alpha)} - C(\beta)^2)^{1/2}}{\lambda^{\beta-\alpha}(1+C(\beta))} > 0. \tag{17}$$

Proof.

$$\frac{\|A^\beta u(t)\|^2}{\|A^\alpha u(t)\|^2} \ge \frac{\lambda^{2(\beta-\alpha)}\|(I-E_\lambda)A^\alpha u(t)\|^2}{\|A^\alpha u(t)\|^2} = \lambda^{2(\beta-\alpha)} \frac{\|A^\alpha u(t)\|^2 - \|E_\lambda A^\alpha u(t)\|^2}{\|A^\alpha u(t)\|^2},$$

thus

$$C(\beta)^2 \ge \limsup_{t\to\infty} \lambda^{2(\beta-\alpha)} \left(1 - \frac{\|E_\lambda A^\alpha u(t)\|^2}{\|A^\alpha u(t)\|^2}\right) = \lambda^{2(\beta-\alpha)}(1-\gamma),$$

where $\gamma = \liminf_{t\to\infty} \|E_\lambda A^\alpha u(t)\|^2/\|A^\alpha u(t)\|^2$. It gives that $\gamma \ge (\lambda^{2(\beta-\alpha)} - C(\beta)^2)/\lambda^{2(\beta-\alpha)}$ and (16) follows. Further,

$$\frac{\|\!|E_\lambda A^\alpha u(t)|\!\|_{\beta-\alpha}}{\|\!|A^\alpha u(t)|\!\|_{\beta-\alpha}} \ge \frac{\|E_\lambda A^\alpha u(t)\|}{\|A^\alpha u(t)\| + \|A^\beta u(t)\|} \ge \frac{\|E_\lambda A^\alpha u(t)\|}{\|A^\alpha u(t)\|}\left(1 + \frac{\|A^\beta u(t)\|}{\|A^\alpha u(t)\|}\right)^{-1}. \tag{18}$$

So, it follows from (16) and Definition 1 that

$$\liminf_{t\to\infty} \frac{\|\!|E_\lambda A^\alpha u(t)|\!\|_{\beta-\alpha}}{\|\!|A^\alpha u(t)|\!\|_{\beta-\alpha}} \ge \frac{(\lambda^{2(\beta-\alpha)} - C(\beta)^2)^{1/2}}{\lambda^{\beta-\alpha}(1+C(\beta))}$$

and (17) is also proved. □

Lemma 3. *$a \le a_\beta \le C(\beta)^{1/(\beta-\alpha)}$.*

Proof. The second inequality is an immediate consequence of (17). To prove the first inequality we suppose, by contradiction, that $a_\beta < \lambda < a$. Then

$$
\begin{aligned}
\liminf_{t\to\infty} \frac{\|\!|A^\alpha E_\lambda u(t)|\!\|_{\beta-\alpha}}{\|\!|A^\alpha u(t)|\!\|_{\beta-\alpha}} &\le \liminf_{t\to\infty} \frac{\|A^\alpha E_\lambda u(t)\| + \|A^\beta E_\lambda u(t)\|}{\|A^\alpha u(t)\|}\\
&\le \liminf_{t\to\infty} \frac{\|E_\lambda A^\alpha u(t)\|(1+\lambda^{\beta-\alpha})}{\|A^\alpha u(t)\|} = 0
\end{aligned}
$$

and it is the contradiction with the fact that $\lambda > a_\beta$. □

Lemma 4. *Let $\lambda > a$. Then*

$$\lim_{t\to\infty} \frac{\|E_\lambda A^\alpha u(t)\|}{\|A^\alpha u(t)\|} = 1. \tag{19}$$

Proof. Let $a < \lambda_1 < \lambda$ and $\delta \in (0, 1)$. Put

$$g(t) = \frac{\|(I - E_\lambda)A^\alpha u(t)\|}{\|E_{\lambda_1} A^\alpha u(t)\|}.$$

Since $\lambda_1 > a$, the function g is well defined and continuous on an interval $[t_0, \infty)$ for some non-negative t_0. It is easy to show that

$$\frac{\|(I - E_\lambda)e^{-A\delta} A^\alpha u(t)\|}{\|E_{\lambda_1} e^{-A\delta} A^\alpha u(t)\|} \le e^{-(\lambda-\lambda_1)\delta} \frac{\|(I - E_\lambda)A^\alpha u(t)\|}{\|E_{\lambda_1} A^\alpha u(t)\|}. \tag{20}$$

Using the point (v) from Section 2 we have

$$g(t+\delta) \le \frac{\|(I - E_\lambda)e^{-A\delta} A^\alpha u(t)\| + J}{\|E_{\lambda_1} e^{-A\delta} A^\alpha u(t)\| - J},$$

where J can be estimated in the following way: let $\alpha \in [k/2, k/2 + 1/2)$, $k \in N_0$. Then

$$\begin{aligned}
J &\le \int_0^\delta \|A^\alpha e^{-A(\delta-s)} P_\sigma(u \cdot \nabla u(t+s))\|\, ds \le \int_0^\delta \|A^\alpha P_\sigma(u \cdot \nabla u(t+s))\|\, ds \\
&\le \int_0^\delta \|A^{k/2} P_\sigma(u \cdot \nabla u(t+s))\|^{k-2\alpha+1} \|A^{k/2+1/2} P_\sigma(u \cdot \nabla u(t+s))\|^{2\alpha-k}\, ds \\
&\quad \times \|A^{k/2} P_\sigma(u \cdot \nabla u(t+s))\| \le c\|\nabla^k P_\sigma(u \cdot \nabla u(t+s))\| = c\|P_\sigma \nabla^k(u \cdot \nabla u(t+s))\| \\
&\le c\|\nabla^k(u \cdot \nabla u(t+s))\| \le c \left\| \sum_{\gamma=0}^k \nabla^\gamma u \cdot \nabla^{k+1-\gamma} u(t+s) \right\| \\
&\le c \sum_{\gamma=0}^k \|\nabla^\gamma u \cdot \nabla^{k+1-\gamma} u(t+s)\| \le c \sum_{\gamma=0}^k \|\nabla^\gamma u(t+s)\|_6\, \|\nabla^{k+1-\gamma} u(t+s)\|_3 \\
&\le c \sum_{\gamma=0}^k \|\nabla^\gamma u(t+s)\|_6 \|\nabla^{k+1-\gamma} u(t+s)\|^{1/2} \|\nabla^{k+1-\gamma} u(t+s)\|_6^{1/2} \\
&\le c \sum_{\gamma=0}^k (\|A^{(\gamma+1)/2} u(t+s)\|\, \|A^{k/2-\gamma/2+1/2} u(t+s)\|^{1/2} \|A^{k/2-\gamma/2+1} u(t+s)\|^{1/2}) \\
&\le c\|u(t+s)\|^{2-(2k+5)/(2k+4)} \|A^{k/2+1} u(t+s)\|^{(2k+5)/(2k+4)} \\
&\le c\|u(t+s)\|^{2-(2k+5)/(2k+4)} \|A^{k^2+(9/2)k+5-2\alpha k-4\alpha} u(t+s)\|^{1/(2k+4)} \|A^\alpha u(t+s)\| \\
&\le r(t)\|A^\alpha u(t+\delta)\|,
\end{aligned}$$

where $\lim_{t\to\infty} r(t) = 0$. In the last inequality we used the point (ix) from Section 2. Similarly,

$$\begin{aligned}
\|A^{k/2+1/2} P_\sigma(u \cdot \nabla u(t+s))\| &\le c \sum_{\gamma=0}^{k+1} \|\nabla^\gamma u(t+s)\|_6\, \|\nabla^{k+2-\gamma} u(t+s)\|_3 \\
&\le c \sum_{\gamma=0}^{k+1} (\|A^{(\gamma+1)/2} u(t+s)\|\, \|A^{k/2-\gamma/2+1} u(t+s)\|^{1/2} \|A^{k/2-\gamma/2+3/2} u(t+s)\|^{1/2}) \\
&\le c\|u(t+s)\|^{2-(2k+7)/(2k+6)} \|A^{k/2+3/2} u(t+s)\|^{(2k+7)/(2k+6)} \\
&\le c\|u(t+s)\|^{2-(2k+7)/(2k+6)} \|A^{k^2+(13/2)k+21/2-2\alpha k-6\alpha} u(t+s)\|^{1/(2k+7)} \|A^\alpha u(t+s)\| \\
&\le r(t)\|A^\alpha u(t+\delta)\|.
\end{aligned}$$

So

$$J \le \int_0^\delta r(t)\|A^\alpha u(t+\delta)\|\, ds = r(t)\delta\|A^\alpha u(t+\delta)\|. \tag{21}$$

Since

$$\begin{aligned}\|A^\alpha u(t+\delta)\| &\le \|A^\alpha u(t)\| + \int_0^\delta \|A^\alpha e^{-A(\delta-s)}P_\sigma(u\cdot\nabla u(t+s))\|\,ds\\ &\le \|A^\alpha u(t)\| + r(t)\delta\|A^\alpha u(t+\delta)\|,\end{aligned}$$

we have for all sufficiently large t that

$$\|A^\alpha u(t+\delta)\| \le 2\|A^\alpha u(t)\|. \tag{22}$$

So, it follows from (21) that

$$J \le r(t)\delta\|A^\alpha u(t)\|.$$

Further, since $\lambda_1 > a$, we can suppose that for a sufficiently large t_0

$$\|A^\alpha u(t)\| \le c\|E_{\lambda_1}A^\alpha u(t)\| \le ce^{\lambda_1\delta}\|E_{\lambda_1}e^{-A\delta}A^\alpha u(t)\|$$

for every $t \ge t_0$ and so we have

$$g(t+\delta) \le \frac{\|(I-E_\lambda)e^{-A\delta}A^\alpha u(t)\| + c\delta\|E_{\lambda_1}e^{-A\delta}A^\alpha u(t)\|r(t)}{\|E_{\lambda_1}e^{-A\delta}A^\alpha u(t)\| - c\delta\|E_{\lambda_1}e^{-A\delta}A^\alpha u(t)\|r(t)}.$$

Using (20), we arrive at the inequality

$$g(t+\delta) \le \frac{e^{-(\lambda-\lambda_1)\delta}}{1-c\delta r(t)}g(t) + \frac{c\delta r(t)}{1-c\delta r(t)}.$$

If t_0 is sufficiently large then there exist α_0 such that

$$\frac{e^{-(\lambda-\lambda_1)\delta}}{1-c\delta r(t)} \le \alpha_0 < 1 \quad \text{for all } t \ge t_0,$$

which gives

$$g(t+\delta) \le \alpha_0 g(t) + \frac{c\delta r(t)}{1-c\delta r(t)} \quad \text{for all } t \ge t_0.$$

It follows immediately that $\limsup_{t\to\infty} g(t) < \infty$ and subsequently $\lim_{t\to\infty} g(t) = 0$. (19) now follows from the following inequalities:

$$\begin{aligned}1 &\ge \lim_{t\to\infty}\frac{\|E_\lambda A^\alpha u(t)\|}{\|A^\alpha u(t)\|} = \lim_{t\to\infty}\left(1-\frac{\|(I-E_\lambda)A^\alpha u(t)\|^2}{\|A^\alpha u(t)\|^2}\right)^{1/2}\\ &\ge \lim_{t\to\infty}\left(1-\frac{\|(I-E_\lambda)u(t)\|^2}{\|E_{\lambda_1}u(t)\|^2}\right)^{1/2} = \lim_{t\to\infty}(1-g(t))^{1/2} = 1. \quad \square\end{aligned}$$

Lemma 5. $a = a_\beta$.

Proof. Due to Lemma 3 it is sufficient to prove that $a_\beta \le a$. Suppose, by contradiction, that $a < \lambda_0 < a_\beta < \lambda$. Then

$$\begin{aligned}\lim_{t\to\infty}\frac{\|\!|(E_\lambda-E_{\lambda_0})A^\alpha u(t)|\!\|_{\beta-\alpha}}{\|\!|A^\alpha u(t)|\!\|_{\beta-\alpha}} &\le \lim_{t\to\infty}\frac{\|(E_\lambda-E_{\lambda_0})A^\alpha u(t)\| + \lambda^{\beta-\alpha}\|(E_\lambda-E_{\lambda_0})A^\alpha u(t)\|}{\|A^\alpha u(t)\|}\\ &\le \lim_{t\to\infty}(1+\lambda^{\beta-\alpha})\frac{\|(E_\lambda-E_{\lambda_0})A^\alpha u(t)\|}{\|A^\alpha u(t)\|} = 0,\end{aligned}$$

where we used Lemma 4 for the last equality. So,

$$\begin{aligned}\liminf_{t\to\infty}\frac{\|\!|E_{\lambda_0}A^\alpha u(t)|\!\|^2_{\beta-\alpha}}{\|\!|A^\alpha u(t)|\!\|^2_{\beta-\alpha}} &\ge \liminf_{t\to\infty}\frac{\|E_{\lambda_0}A^\alpha u(t)\|^2 + \|A^\beta E_{\lambda_0}u(t)\|^2}{\|\!|A^\alpha u(t)|\!\|^2_{\beta-\alpha}}\\ &= \liminf_{t\to\infty}\frac{\|E_\lambda A^\alpha u(t)\|^2 + \|A^\beta E_\lambda u(t)\|^2}{\|\!|A^\alpha u(t)|\!\|^2_{\beta-\alpha}} \ge \frac{1}{2}\liminf_{t\to\infty}\frac{\|\!|E_\lambda A^\alpha u(t)|\!\|^2_{\beta-\alpha}}{\|\!|A^\alpha u(t)|\!\|^2_{\beta-\alpha}} > 0\end{aligned}$$

and it is a contradiction, since $\lambda_0 < a_\beta$. $\square$

Lemma 6. *Let $\lambda > a_\beta$. Then*

$$\lim_{t\to\infty} \frac{|||E_\lambda A^\alpha u(t)|||_{\beta-\alpha}}{|||A^\alpha u(t)|||_{\beta-\alpha}} = 1. \tag{23}$$

Proof. We proceed analogically as in the proof of Lemma 4. Let $a_\beta < \lambda_1 < \lambda$ and $\delta \in (0, 1)$. Put

$$g(t) = \frac{|||(I - E_\lambda)A^\alpha u(t)|||_{\beta-\alpha}}{|||E_{\lambda_1} A^\alpha u(t)|||_{\beta-\alpha}}.$$

Since $\lambda_1 > a_\beta$, the function g is defined and continuous on an interval $[t_0, \infty)$ for some non-negative t_0. It is easy to show that

$$\frac{|||(I - E_\lambda)e^{-A\delta}A^\alpha u(t)|||_{\beta-\alpha}}{|||E_{\lambda_1} e^{-A\delta}A^\alpha u(t)|||_{\beta-\alpha}} \le e^{-(\lambda-\lambda_1)\delta} \frac{|||(I - E_\lambda)A^\alpha u(t)|||_{\beta-\alpha}}{|||E_{\lambda_1} A^\alpha u(t)|||_{\beta-\alpha}}. \tag{24}$$

Using the point (v) from Section 2 we get:

$$g(t+\delta) \le \frac{|||(I - E_\lambda)e^{-A\delta}A^\alpha u(t)|||_{\beta-\alpha} + J}{|||E_{\lambda_1} e^{-A\delta}A^\alpha u(t)|||_{\beta-\alpha} - J},$$

where

$$\begin{aligned} J &\le \int_0^\delta \|A^\alpha e^{-A(\delta-s)} P_\sigma(u\cdot\nabla u(t+s))\| + \|A^\beta e^{-A(\delta-s)} P_\sigma(u\cdot\nabla u(t+s))\|\, ds \\ &\le r(t)\delta(\|A^\alpha u(t)\| + \|A^\beta u(t)\|) = r(t)\delta|||A^\alpha u(t)|||_{\beta-\alpha} \le cr(t)\delta|||E_{\lambda_1} e^{-A\delta} A^\alpha u(t)|||_{\beta-\alpha}. \end{aligned}$$

In the last inequality we used the fact that $\lambda_1 > a_\beta = a$. We have

$$g(t+\delta) \le \frac{|||(I - E_\lambda)e^{-A\delta}A^\alpha u(t)|||_{\beta-\alpha} + cr(t)\delta|||E_{\lambda_1} e^{-A\delta}A^\alpha u(t)|||_{\beta-\alpha}}{|||E_{\lambda_1} e^{-A\delta}A^\alpha u(t)|||_{\beta-\alpha} - cr(t)\delta|||E_{\lambda_1} e^{-A\delta}A^\alpha u(t)|||_{\beta-\alpha}}.$$

Using (24), we arrive at the inequality

$$g(t+\delta) \le \frac{e^{-(\lambda-\lambda_1)\delta}}{1 - c\delta r(t)} g(t) + \frac{c\delta r(t)}{1 - c\delta r(t)}$$

and we can derive exactly in the same way as in the proof of Lemma 4 that

$$\lim_{t\to\infty} g(t) = 0. \tag{25}$$

We now have

$$\begin{aligned} \frac{|||E_\lambda A^\alpha u(t)|||_{\beta-\alpha}}{|||A^\alpha u(t)|||_{\beta-\alpha}} &= \frac{(\|A^\alpha u(t)\|^2 - \|(I - E_\lambda)A^\alpha u(t)\|^2)^{1/2} + (\|A^\beta u(t)\|^2 - \|(I - E_\lambda)A^\beta u(t)\|^2)^{1/2}}{|||A^\alpha u(t)|||_{\beta-\alpha}} \\ &= \left(\frac{\|A^\alpha u(t)\|^2}{|||A^\alpha u(t)|||^2_{\beta-\alpha}} - \frac{\|(I - E_\lambda)A^\alpha u(t)\|^2}{|||A^\alpha u(t)|||^2_{\beta-\alpha}}\right)^{1/2} + \left(\frac{\|A^\beta u(t)\|^2}{|||A^\alpha u(t)|||^2_{\beta-\alpha}} - \frac{\|A^\beta (I - E_\lambda) u(t)\|^2}{|||A^\alpha u(t)|||^2_{\beta-\alpha}}\right)^{1/2}. \end{aligned} \tag{26}$$

Now, by contradiction, if (23) does not hold then there would exist $\alpha_0 \in [0, 1)$ and a sequence $\{t_j\}_{j=1}^\infty$, $\lim_{j\to\infty} t_j = \infty$, such that

$$\lim_{j\to\infty} \frac{|||E_\lambda A^\alpha u(t_j)|||_{\beta-\alpha}}{|||A^\alpha u(t_j)|||_{\beta-\alpha}} = \alpha_0. \tag{27}$$

We can suppose, without loss of generality, that there exist $\alpha_1, \alpha_2 \in [0, 1]$ such that

$$\lim_{j\to\infty} \frac{\|A^\alpha u(t_j)\|^2}{|||A^\alpha u(t_j)|||^2_{\beta-\alpha}} = \alpha_1,$$

$$\lim_{j\to\infty} \frac{\|A^\beta u(t_j)\|^2}{|||A^\alpha u(t_j)|||^2_{\beta-\alpha}} = \alpha_2$$

and $\sqrt{\alpha_1} + \sqrt{\alpha_2} = 1$. Returning to (26) and using (25) we obtain that

$$\lim_{j\to\infty} \frac{|||E_\lambda A^\alpha u(t_j)|||_{\beta-\alpha}}{|||A^\alpha u(t_j)|||_{\beta-\alpha}} = \sqrt{\alpha_1 - 0} + \sqrt{\alpha_2 - 0} = 1,$$

which is a contradiction with (27). So, (23) holds and Lemma 6 is proved. □

Lemma 7. *$a_\beta = C(\beta)^{1/(\beta-\alpha)}$ for every $\beta > \alpha$.*

Proof. Due to Lemma 3 it is sufficient to prove that $a_\beta \geq C(\beta)^{1/(\beta-\alpha)}$. Suppose, by contradiction, that $a_\beta < \lambda < C(\beta)^{1/(\beta-\alpha)}$. Using Lemmas 6, 5, 4 and the definition of $C(\beta)$ we get

$$\begin{aligned} 1 &= \lim_{t\to\infty} \frac{|||E_\lambda A^\alpha u(t)|||_{\beta-\alpha}}{|||A^\alpha u(t)|||_{\beta-\alpha}} = \lim_{t\to\infty} \frac{\|E_\lambda A^\alpha u(t)\| + \|E_\lambda A^\beta u(t)\|}{\|A^\alpha u(t)\| + \|A^\beta u(t)\|} \\ &\leq \lim_{t\to\infty} \frac{(1+\lambda^{\beta-\alpha})\|E_\lambda A^\alpha u(t)\|}{\|A^\alpha u(t)\| + \|A^\beta u(t)\|} = \lim_{t\to\infty} \frac{(1+\lambda^{\beta-\alpha})\frac{\|E_\lambda A^\alpha u(t)\|}{\|A^\alpha u(t)\|}}{1 + \frac{\|A^\beta u(t)\|}{\|A^\alpha u(t)\|}} = \frac{1+\lambda^{\beta-\alpha}}{1+C(\beta)}. \end{aligned}$$

So $\lambda^{\beta-\alpha} \geq C(\beta)$ and $\lambda \geq C(\beta)^{1/(\beta-\alpha)}$, which is the contradiction with the assumption above and Lemma 7 is proved. □

Lemma 8. *$d \leq d_\beta \leq C(\beta)^{1/(\beta-\alpha)}$.*

Proof. The second inequality is an immediate consequence of the previous lemma and the fact that $d_\beta \leq a_\beta$. To prove the first inequality we suppose, by contradiction, that $d_\beta < \lambda < d$. Then

$$\begin{aligned} \limsup_{t\to\infty} \frac{|||E_\lambda A^\alpha u(t)|||_{\beta-\alpha}}{|||A^\alpha u(t)|||_{\beta-\alpha}} &= \limsup_{t\to\infty} \frac{\|E_\lambda A^\alpha u(t)\| + \|E_\lambda A^\beta u(t)\|}{|||A^\alpha u(t)|||_{\beta-\alpha}} \\ &\leq \limsup_{t\to\infty} \frac{\|E_\lambda A^\alpha u(t)\|(1+\lambda^{\beta-\alpha})}{\|A^\alpha u(t)\|} = 0 \end{aligned}$$

and it is the contradiction with the fact that $\lambda > d_\beta$. □

Lemma 9. *$d = d_\beta$.*

Proof. Due to Lemma 8 it is sufficient to prove that $d_\beta \leq d$. Suppose, by contradiction, that $d < \lambda_0 < d_\beta \leq a < \lambda$. Then

$$\limsup_{t\to\infty} \frac{|||E_{\lambda_0} A^\alpha u(t)|||_{\beta-\alpha}}{|||A^\alpha u(t)|||_{\beta-\alpha}} \geq \limsup_{t\to\infty} \left(\frac{\|E_{\lambda_0} A^\alpha u(t)\|}{\|A^\alpha u(t)\|} \frac{\|A^\alpha u(t)\|}{|||E_\lambda A^\alpha u(t)|||_{\beta-\alpha}} \frac{|||E_\lambda A^\alpha u(t)|||_{\beta-\alpha}}{|||A^\alpha u(t)|||_{\beta-\alpha}} \right).$$

Since

$$\frac{\|A^\alpha u(t)\|}{|||E_\lambda A^\alpha u(t)|||_{\beta-\alpha}} = \frac{\|A^\alpha u(t)\|}{\|E_\lambda A^\alpha u(t)\| + \|E_\lambda A^\beta u(t)\|} \geq \frac{\|A^\alpha u(t)\|}{(1+\lambda^{\beta-\alpha})\|E_\lambda A^\alpha u(t)\|}$$

we have by the use of (19) that

$$\liminf_{t\to\infty} \frac{\|A^\alpha u(t)\|}{|||E_\lambda A^\alpha u(t)|||_{\beta-\alpha}} \geq \frac{1}{(1+\lambda^{\beta-\alpha})} > 0.$$

Further, it follows from (23) and Lemma 5 that

$$\lim_{t\to\infty} \frac{|||E_\lambda A^\alpha u(t)|||_{\beta-\alpha}}{|||A^\alpha u(t)|||_{\beta-\alpha}} = 1.$$

Since $\lambda_0 > d$, it leads to the conclusion that

$$\limsup_{t\to\infty} \frac{|||E_{\lambda_0} A^\alpha u(t)|||_{\beta-\alpha}}{|||A^\alpha u(t)|||_{\beta-\alpha}} \geq \limsup_{t\to\infty} \frac{\|E_{\lambda_0} A^\alpha u(t)\|}{\|A^\alpha u(t)\|} \frac{1}{(1+\lambda^{\beta-\alpha})} > 0.$$

The preceding inequalities are in contradiction with the fact that $\lambda_0 < d_\beta$. Therefore, $d = d_\beta$ and Lemma 9 is proved. □

Lemma 10. *Let $\lambda \in (d, a)$ and $\varepsilon \in (0, \lambda)$. Then*

$$\limsup_{t\to\infty} \frac{\|(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon})A^\alpha u(t)\|}{\|A^\alpha u(t)\|} = 1, \tag{28}$$

$$\limsup_{t\to\infty} \frac{|||(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon})A^\alpha u(t)|||_{\beta-\alpha}}{|||A^\alpha u(t)|||_{\beta-\alpha}} = 1. \tag{29}$$

Proof. Let $\lambda \in (d, a)$. Put

$$g(t) = \frac{\|(I - E_\lambda)A^\alpha u(t)\|}{\|E_\lambda A^\alpha u(t)\|}.$$

Here it is not excluded that g can be equal, for some $t \geq 0$, to infinity. It follows from the definition of d and a (see Definition 1) that

$$\limsup_{t\to\infty} g(t) = \infty \tag{30}$$

and

$$\liminf_{t\to\infty} g(t) < K < \infty \tag{31}$$

for some $K > 1$. At first, we will show (28). Suppose, by contradiction, that $\varepsilon < \min(\lambda - d, a - \lambda)$ and

$$\limsup_{t\to\infty} \frac{\|(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon})A^\alpha u(t)\|}{\|A^\alpha u(t)\|} = \alpha_0 < 1. \tag{32}$$

For sufficiently large t, either

$$\frac{\|E_{\lambda-\varepsilon}A^\alpha u(t)\|}{\|E_\lambda A^\alpha u(t)\|} \geq \frac{\sqrt{1-\alpha_0^2}}{2}, \tag{33}$$

or

$$\frac{\|(I - E_{\lambda+\varepsilon})A^\alpha u(t)\|}{\|(I - E_\lambda)A^\alpha u(t)\|} \geq \frac{\sqrt{1-\alpha_0^2}}{2}. \tag{34}$$

Indeed, if both (33) and (34) did not hold then we would get the following contradiction:

$$\begin{aligned} 1 &= \frac{\|E_{\lambda-\varepsilon}A^\alpha u(t)\|^2 + \|(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon})A^\alpha u(t)\|^2 + \|(I - E_{\lambda+\varepsilon})A^\alpha u(t)\|^2}{\|A^\alpha u(t)\|^2} \\ &< \frac{1-\alpha_0^2}{4} + \left(\frac{\alpha_0 + 1}{2}\right)^2 + \frac{1-\alpha_0^2}{4} < 1. \end{aligned}$$

Suppose now that $g(t) = K$ for some t sufficiently large. Let $\delta \in (0, 1)$. We get from the point (v) from Section 2 that

$$g(t+\delta) \leq \frac{\|(I - E_\lambda)e^{-A\delta}A^\alpha u(t)\| + J}{\|E_\lambda e^{-A\delta}A^\alpha u(t)\| - J},$$

where the inequality $J \leq r(t)\delta\|A^\alpha u(t)\|$ can be derived exactly in the same way as in Lemma 4. Since

$$K = g(t) = \frac{\|(I - E_\lambda)A^\alpha u(t)\|}{\|E_\lambda A^\alpha u(t)\|} = \frac{\left(\|A^\alpha u(t)\|^2 - \|E_\lambda A^\alpha u(t)\|^2\right)^{1/2}}{\|E_\lambda A^\alpha u(t)\|},$$

we have $\|A^\alpha u(t)\| = (1 + K^2)^{1/2}\|E_\lambda A^\alpha u(t)\| \leq 2K\|E_\lambda A^\alpha u(t)\| \leq 2Ke^{\lambda\delta}\|E_\lambda e^{-A\delta}A^\alpha u(t)\|$.

First, suppose that (33) holds. Then

$$\begin{aligned} \frac{\|(I - E_\lambda)e^{-A\delta}A^\alpha u(t)\|}{\|E_\lambda e^{-A\delta}A^\alpha u(t)\|} &= \frac{\|(I - E_\lambda)e^{-A\delta}A^\alpha u(t)\|}{(\|(E_\lambda - E_{\lambda-\varepsilon})e^{-A\delta}A^\alpha u(t)\|^2 + \|E_{\lambda-\varepsilon}e^{-A\delta}A^\alpha u(t)\|^2)^{1/2}} \\ &\leq \frac{e^{-\lambda\delta}\|(I - E_\lambda)A^\alpha u(t)\|}{(e^{-2\lambda\delta}\|(E_\lambda - E_{\lambda-\varepsilon})A^\alpha u(t)\|^2 + e^{-2(\lambda-\varepsilon)\delta}\|E_{\lambda-\varepsilon}A^\alpha u(t)\|^2)^{1/2}} \\ &= \frac{\|(I - E_\lambda)A^\alpha u(t)\|}{(\|(E_\lambda - E_{\lambda-\varepsilon})A^\alpha u(t)\|^2 + e^{2\varepsilon\delta}\|E_{\lambda-\varepsilon}A^\alpha u(t)\|^2)^{1/2}} \\ &= g(t)\left(\frac{\|(E_\lambda - E_{\lambda-\varepsilon})A^\alpha u(t)\|^2}{\|E_\lambda A^\alpha u(t)\|^2} + e^{2\varepsilon\delta}\frac{\|E_{\lambda-\varepsilon}A^\alpha u(t)\|^2}{\|E_\lambda A^\alpha u(t)\|^2}\right)^{-1/2} =: g(t)H_1. \end{aligned}$$

Thus,

$$\begin{aligned} g(t+\delta) &\leq \frac{\|(I - E_\lambda)e^{-A\delta}A^\alpha u(t)\| + cr(t)\delta\|E_\lambda e^{-A\delta}A^\alpha u(t)\|}{\|E_\lambda e^{-A\delta}A^\alpha u(t)\| - cr(t)\delta\|E_\lambda e^{-A\delta}A^\alpha u(t)\|} \\ &\leq \frac{H_1 g(t)}{1 - cr(t)\delta} + \frac{cr(t)\delta}{1 - cr(t)\delta}. \end{aligned} \tag{35}$$

If we put for simplicity

$$F = \frac{\|(E_\lambda - E_{\lambda-\varepsilon})A^\alpha u(t)\|^2}{\|E_\lambda A^\alpha u(t)\|^2}, \qquad B = \frac{\|E_{\lambda-\varepsilon}A^\alpha u(t)\|^2}{\|E_\lambda A^\alpha u(t)\|^2}, \qquad C = cr(t),$$

we can write the right hand side of (35) as

$$f(\delta) = \frac{K}{(F + e^{2\varepsilon\delta}B)^{1/2}} \frac{1}{1 - C\delta} + \frac{C\delta}{1 - C\delta}.$$

We use $F + B = 1$ and compute elementarily that

$$f'(0) = K(-\varepsilon B + C) + C.$$

Since $B \geq (1 - \alpha_0^2)/4$, we have

$$f'(0) \leq K(-\varepsilon(1 - \alpha_0^2)/4 + cr(t)) + cr(t)$$

and since $\lim_{t\to\infty} r(t) = 0$, we conclude, that for t sufficiently large $f'(0) < 0$. Thus, $g(t + \delta') < g(t) = K$ for all $\delta' > 0$ sufficiently small. This result holds for every t sufficiently large such that $g(t) = K$. It is in contradiction with (30). Thus, (32) does not hold and (28) follows immediately.

Suppose, second, that (34) holds. Then

$$\begin{aligned}
\frac{\|(I - E_\lambda)e^{-A\delta}A^\alpha u(t)\|}{\|E_\lambda e^{-A\delta}A^\alpha u(t)\|} &= \frac{(\|(I - E_{\lambda+\varepsilon})e^{-A\delta}A^\alpha u(t)\|^2 + \|(E_{\lambda+\varepsilon} - E_\lambda)e^{-A\delta}A^\alpha u(t)\|^2)^{1/2}}{\|E_\lambda e^{-A\delta}A^\alpha u(t)\|} \\
&\leq \frac{(e^{-2(\lambda+\varepsilon)\delta}\|(I - E_{\lambda+\varepsilon})A^\alpha u(t)\|^2 + e^{-2\lambda\delta}\|(E_{\lambda+\varepsilon} - E_\lambda)A^\alpha u(t)\|^2)^{1/2}}{e^{-\lambda\delta}\|E_\lambda A^\alpha u(t)\|} \\
&= \frac{(e^{-2\varepsilon\delta}\|(I - E_{\lambda+\varepsilon})A^\alpha u(t)\|^2 + \|(E_{\lambda+\varepsilon} - E_\lambda)A^\alpha u(t)\|^2)^{1/2}}{\|E_\lambda A^\alpha u(t)\|} \\
&= g(t)\left(e^{-2\varepsilon\delta}\frac{\|(I - E_{\lambda+\varepsilon})A^\alpha u(t)\|^2}{\|(I - E_\lambda)A^\alpha u(t)\|^2} + \frac{\|(E_{\lambda+\varepsilon} - E_\lambda)A^\alpha u(t)\|^2}{\|(I - E_\lambda)A^\alpha u(t)\|^2}\right)^{1/2} =: g(t)H_2.
\end{aligned}$$

Thus,

$$g(t + \delta) \leq \frac{H_2 g(t)}{1 - cr(t)\delta} + \frac{cr(t)\delta}{1 - cr(t)\delta}. \tag{36}$$

If we put

$$F = \frac{\|(I - E_{\lambda+\varepsilon})A^\alpha u(t)\|^2}{\|(I - E_\lambda)A^\alpha u(t)\|^2}, \qquad B = \frac{\|(E_{\lambda+\varepsilon} - E_\lambda)A^\alpha u(t)\|^2}{\|(I - E_\lambda)A^\alpha u(t)\|^2}, \qquad C = cr(t),$$

we can write the right hand side of (36) as

$$f(\delta) = \frac{K(Fe^{-2\varepsilon\delta} + B)^{1/2}}{1 - C\delta} + \frac{C\delta}{1 - C\delta}.$$

We use $F + B = 1$ and compute elementarily that

$$f'(0) = K(-\varepsilon F + C) + C.$$

Since $F \geq (1 - \alpha_0^2)/4$, we have

$$f'(0) \leq K(-\varepsilon(1 - \alpha_0^2)/4 + cr(t)) + cr(t)$$

and so for t sufficiently large $f'(0) < 0$. As above, (28) follows immediately.

Prove now (29). Let $\lambda_0 > a$. Then

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\|(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon})A^\alpha u(t)g\|_{\beta-\alpha}}{\|A^\alpha u(t)\|_{\beta-\alpha}} &= \limsup_{t\to\infty}\left[\frac{(\|E_{\lambda_0}A^\alpha u(t)\|^2 - \|E_{\lambda-\varepsilon}A^\alpha u(t)\|^2 - \|(E_{\lambda_0} - E_{\lambda+\varepsilon})A^\alpha u(t)\|^2)^{1/2}}{\|E_{\lambda_0}A^\alpha u(t)\| + \|E_{\lambda_0}A^\beta u(t)\|}\right. \\
&\quad \left. + \frac{(\|A^\beta E_{\lambda_0}u(t)\|^2 - \|A^\beta E_{\lambda-\varepsilon}u(t)\|^2 - \|A^\beta(E_{\lambda_0} - E_{\lambda+\varepsilon})u(t)\|^2)^{1/2}}{\|E_{\lambda_0}A^\alpha u(t)\| + \|E_{\lambda_0}A^\beta u(t)\|}\right] \\
&= \limsup_{t\to\infty}\left[\left(1 - \frac{\|E_{\lambda-\varepsilon}A^\alpha u(t)\|^2}{\|E_{\lambda_0}A^\alpha u(t)\|^2} - \frac{\|(E_{\lambda_0} - E_{\lambda+\varepsilon})A^\alpha u(t)\|^2}{\|E_{\lambda_0}A^\alpha u(t)\|^2}\right)^{1/2}\right.
\end{aligned}$$

$$+\left(\frac{\|A^\beta E_{\lambda_0}u(t)\|^2}{\|E_{\lambda_0}A^\alpha u(t)\|^2}-\frac{\|A^\beta E_{\lambda-\varepsilon}u(t)\|^2}{\|E_{\lambda_0}A^\alpha u(t)\|^2}-\frac{\|A^\beta (E_{\lambda_0}-E_{\lambda+\varepsilon})u(t)\|^2}{\|E_{\lambda_0}A^\alpha u(t)\|^2}\right)^{1/2}\Bigg]$$
$$\times\left(1+\frac{\|A^\beta E_{\lambda_0}u(t)\|}{\|E_{\lambda_0}A^\alpha u(t)\|}\right)^{-1}.$$

If we put

$$G(t)=\frac{\|E_{\lambda-\varepsilon}A^\alpha u(t)\|^2}{\|E_{\lambda_0}A^\alpha u(t)\|^2}+\frac{\|(E_{\lambda_0}-E_{\lambda+\varepsilon})A^\alpha u(t)\|^2}{\|E_{\lambda_0}A^\alpha u(t)\|^2},$$
$$H(t)=\frac{\|A^\beta E_{\lambda-\varepsilon}u(t)\|^2}{\|E_{\lambda_0}A^\alpha u(t)\|^2}+\frac{\|A^\beta (E_{\lambda_0}-E_{\lambda+\varepsilon})u(t)\|^2}{\|E_{\lambda_0}A^\alpha u(t)\|^2},$$
$$F(t)=\frac{\|A^\beta E_{\lambda_0}u(t)\|}{\|E_{\lambda_0}A^\alpha u(t)\|},$$

it follows from the equality (28) and Lemma 4 that there exists an increasing sequence $\{t_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty} t_j = \infty$, $\lim_{j\to\infty} G(t_j)=\lim_{j\to\infty} H(t_j)=0$ and $\lim_{j\to\infty} F(t_j)=F\ge 0$. So we can write

$$\lim_{j\to\infty}\frac{|||(E_{\lambda+\varepsilon}-E_{\lambda-\varepsilon})A^\alpha u(t_j)|||_{\beta-\alpha}}{|||A^\alpha u(t_j)|||_{\beta-\alpha}}=\lim_{j\to\infty}\frac{(1-G(t_j))^{1/2}+(F^2(t_j)-H(t_j))^{1/2}}{1+F(t_j)}=1.$$

Thus, (29) follows immediately and Lemma 10 is proved. □

Corollary 2. *Let $\lambda\in(d,a)$. Then*

$$\limsup_{t\to\infty}\frac{\|E_\lambda A^\alpha u(t)\|}{\|A^\alpha u(t)\|}=\limsup_{t\to\infty}\frac{|||E_\lambda A^\alpha u(t)|||_{\beta-\alpha}}{|||A^\alpha u(t)|||_{\beta-\alpha}}=1.$$

Lemma 11. $d_\beta=C(\beta)^{1/(\beta-\alpha)}$.

Proof. Suppose that $\lambda\in(d_\beta, C(\beta)^{1/(\beta-\alpha)})$. Then, by the use of Lemma 9, Corollary 2 and the definition of $C(\beta)$ we obtain the following contradiction

$$\begin{aligned}1&=\limsup_{t\to\infty}\frac{|||E_\lambda A^\alpha u(t)|||_{\beta-\alpha}}{|||A^\alpha u(t)|||_{\beta-\alpha}}=\limsup_{t\to\infty}\left(\frac{\|E_\lambda A^\alpha u(t)\|}{\|A^\alpha u(t)\|}+\frac{\|A^\beta E_\lambda u(t)\|}{\|A^\alpha u(t)\|}\right)\\&\times\left(1+\frac{\|A^\beta u(t)\|}{\|A^\alpha u(t)\|}\right)^{-1}\le\limsup_{t\to\infty}(1+\lambda^{\beta-\alpha})\frac{\|E_\lambda A^\alpha u(t)\|}{\|A^\alpha u(t)\|}\left(1+\frac{\|A^\beta u(t)\|}{\|A^\alpha u(t)\|}\right)^{-1}\\&\le\frac{1+\lambda^{\beta-\alpha}}{1+C(\beta)}<\frac{1+C(\beta)}{1+C(\beta)}=1.\end{aligned}$$

The proof of Lemma 11 now follows from Lemma 8. □

Remark 2. If we fix $\alpha\ge 0$, it follows immediately from Lemmas 2–11 that $a=a_\beta=d=d_\beta=C(\beta)^{1/(\beta-\alpha)}$ for any $\beta>\alpha$. So $C(\beta)^{1/(\beta-\alpha)}$ is independent of β. The validity of (11) follows from the definition of a, a_β, d and d_β in Definition 1. To finish the proof of Theorem 1, it suffices to prove that the number a from Theorem 1 is independent of α. It is our goal in the rest of this section.

Remark 3. Up until now we have not stressed (because of simplicity of notation) the possible dependence of the number a on α. From now on we will write a_α instead of a, meaning that a_α possibly depends on the chosen $\alpha\ge 0$.

Lemma 12. *Let $0\le\alpha_1<\alpha_2<\infty$. Then $a_{\alpha_1}=0$ or $a_{\alpha_1}=a_{\alpha_2}$.*

Proof.

$$a_{\alpha_1}=\lim_{t\to\infty}\left(\frac{\|A^{\alpha_2}u(t)\|}{\|A^{\alpha_1}u(t)\|}\right)^{1/(\alpha_2-\alpha_1)}\le\lim_{t\to\infty}\left(\frac{\|A^{2\alpha_2-\alpha_1}u(t)\|}{\|A^{\alpha_2}u(t)\|}\right)^{1/(\alpha_2-\alpha_1)}=a_{\alpha_2}$$

and so $a_{\alpha_1}\in[0,a_{\alpha_2}]$. Thus, if $a_{\alpha_2}=0$ then $a_{\alpha_1}=0$. Let $a_{\alpha_2}>0$. Then for every $\varepsilon\in(0,a_{\alpha_2}/2)$

$$\lim_{t\to\infty}\frac{\|(E_{a_{\alpha_2}-\varepsilon}-E_\varepsilon)A^{\alpha_1}u(t)\|}{\|(E_{a_{\alpha_2}+\varepsilon}-E_{a_{\alpha_2}-\varepsilon})A^{\alpha_1}u(t)\|}\le\lim_{t\to\infty}\frac{\varepsilon^{\alpha_1-\alpha_2}\|(E_{a_{\alpha_2}-\varepsilon}-E_\varepsilon)A^{\alpha_2}u(t)\|}{(a_{\alpha_2}+\varepsilon)^{\alpha_1-\alpha_2}\|(E_{a_{\alpha_2}+\varepsilon}-E_{a_{\alpha_2}-\varepsilon})A^{\alpha_2}u(t)\|}=0$$

and so $a_{\alpha_1}\notin(0,a_{\alpha_2})$ due to (11) (see also Remark 2). □

Lemma 13. *Let $\alpha \geq 0$. Then $\|A^\alpha u(t)\|$ decreases exponentially for $t \to \infty$ if and only if $a_\alpha > 0$.*

Proof. Let $\alpha \geq 0$ and $a_\alpha > 0$. Take $\varepsilon \in (0, a_\alpha)$ and $\delta \in (0, 1)$. It was proved in Lemma 4 that

$$\left\| A^\alpha \int_0^\delta e^{-A(\delta-s)} P_\sigma(u \cdot \nabla u(t+s))\, ds \right\| \leq r(t)\delta \|A^\alpha u(t)\|,$$

where $\lim_{t\to\infty} r(t) = 0$. We have

$$\begin{aligned}(E_{a_\alpha+\varepsilon} - E_{a_\alpha-\varepsilon})A^\alpha u(t+\delta) \;=\; & (E_{a_\alpha+\varepsilon} - E_{a_\alpha-\varepsilon})A^\alpha e^{-A\delta} u(t) \\ & - \int_0^\delta (E_{a_\alpha+\varepsilon} - E_{a_\alpha-\varepsilon})A^\alpha e^{-A(\delta-s)} P_\sigma(u \cdot \nabla u(t+s))\, ds,\end{aligned}$$

$$\|(E_{a_\alpha+\varepsilon} - E_{a_\alpha-\varepsilon})A^\alpha e^{-A\delta} u(t)\| \leq e^{-(a_\alpha-\varepsilon)\delta} \|(E_{a_\alpha+\varepsilon} - E_{a_\alpha-\varepsilon})A^\alpha u(t)\|$$

and

$$\|(E_{a_\alpha+\varepsilon} - E_{a_\alpha-\varepsilon})A^\alpha e^{-A\delta} u(t)\| \geq e^{-(a_\alpha+\varepsilon)\delta} \|(E_{a_\alpha+\varepsilon} - E_{a_\alpha-\varepsilon})A^\alpha u(t)\|.$$

If we put $f(t) = \|(E_{a_\alpha+\varepsilon} - E_{a_\alpha-\varepsilon})A^\alpha u(t)\|$ and use (11), we get for all sufficiently large t

$$e^{-(a_\alpha+\varepsilon)\delta} f(t) - 2\delta r(t) f(t) \leq f(t+\delta) \leq e^{-(a_\alpha-\varepsilon)\delta} f(t) + 2\delta r(t) f(t).$$

It gives

$$(-a_\alpha - \varepsilon - 2r(t)) f(t) \leq f'(t) \leq (-a_\alpha + \varepsilon + 2r(t)) f(t)$$

and there exists $c > 0$ so that

$$ce^{-(a_\alpha+\varepsilon)t} \leq f(t) \leq ce^{-(a_\alpha-\varepsilon)t}$$

for t sufficiently large. Consequently,

$$ce^{-(a_\alpha+\varepsilon)t} \leq \|A^\alpha u(t)\| \leq ce^{-(a_\alpha-\varepsilon)t}$$

and so $\|A^\alpha u(t)\|$ decreases exponentially.

Let $a_\alpha = 0$. Let $\varepsilon > 0$. Then

$$\|E_\varepsilon A^\alpha u(t+\delta)\| \geq \|E_\varepsilon A^\alpha e^{-A\delta} u(t)\| - \delta r(t) \|A^\alpha u(t)\| \geq e^{-\varepsilon\delta} \|E_\varepsilon A^\alpha u(t)\| - \delta r(t) \|A^\alpha u(t)\|.$$

If $f(t) = \|E_\varepsilon A^\alpha u(t)\|$, then $f'(t) \geq -(\varepsilon + 2r(t)) f(t)$, $f(t) \geq ce^{-2\varepsilon t}$ and by the use of (11)

$$\|A^\alpha u(t)\| \geq ce^{-2\varepsilon t}$$

for some $c > 0$ and all sufficiently large t. Since $\varepsilon > 0$ has been chosen arbitrarily, $\|A^\alpha u(t)\|$ does not decrease exponentially and lemma is proved. □

Lemma 14. *a_α is independent of $\alpha \in [0, \infty)$.*

Suppose that $a_\alpha > 0$ for some $\alpha > 0$. It means according to Lemma 13 that $\|A^\alpha u(\cdot)\|$ decreases exponentially and since

$$\|A^{\alpha'} u(t)\| \leq \|u(t)\|^{1-\alpha'/\alpha} \|A^\alpha u(t)\|^{\alpha'/\alpha},$$

$\|A^{\alpha'} u(\cdot)\|$ also decreases exponentially and consequently, due to Lemma 12, $a_\alpha = a_{\alpha'}$ for any $\alpha' \in (0, \alpha)$. To finish the proof it suffices to exclude the possibility that $a_\alpha = a > 0$ for every $\alpha > 0$ and $a_0 = 0$.

Thus, suppose that $a_\alpha = a_{1/2} > 0$ for every $\alpha > 0$. Then according to the proof of Lemma 13 for every $\lambda \in (0, a_{1/2})$

$$\lim_{t\to\infty} \|A^{1/2} u(t)\| e^{\lambda t} = 0. \tag{37}$$

Fix $\lambda \in (0, a_{1/2})$. It suffices to show that

$$\limsup_{t\to\infty} \|u(t)\| e^{\lambda t} < \infty. \tag{38}$$

Indeed, it then follows from Lemma 13 that $a_0 > 0$. We can write

$$\begin{aligned} u(t) \;=\; & e^{-At}\left(u_0 - \int_0^\infty e^{As} E_\lambda P_\sigma(u \cdot \nabla u(s))\, ds\right) - \int_0^t e^{-A(t-s)} (I - E_\lambda) P_\sigma(u \cdot \nabla u(s))\, ds \\ & + \int_t^\infty e^{-A(t-s)} E_\lambda P_\sigma(u \cdot \nabla u(s))\, ds. \end{aligned} \tag{39}$$

Let us remark that all the integrals presented here have sense due to (37). Applying the operator $e^{At}A^{1/2}E_\lambda$ on the equality $u(t) = e^{-At}u_0 - \int_0^t e^{-A(t-s)}P_\sigma(u \cdot \nabla u(s))\, ds$ we get

$$e^{At}A^{1/2}E_\lambda u(t) = A^{1/2}E_\lambda u(0) - \int_0^t A^{1/2}e^{As}E_\lambda P_\sigma(u \cdot \nabla u(s))\, ds.$$

Applying $\lim_{t\to\infty}$ on both sides of the last equality, we get

$$0 = A^{1/2}E_\lambda u(0) - \int_0^\infty A^{1/2}e^{As}E_\lambda P_\sigma(u \cdot \nabla u(s))\, ds = A^{1/2}\left(E_\lambda u(0) - \int_0^\infty e^{As}E_\lambda P_\sigma(u \cdot \nabla u(s))\, ds\right)$$

and

$$E_\lambda u(0) = \int_0^\infty e^{As}E_\lambda P_\sigma(u \cdot \nabla u(s))\, ds.$$

If we put $\eta = u(0) - \int_0^\infty e^{As}E_\lambda P_\sigma(u \cdot \nabla u(s))\, ds$ then $\eta = (I - E_\lambda)u(0)$ and it follows from (39)

$$u(t) = e^{-At}\eta - \int_0^t e^{-A(t-s)}(I - E_\lambda)P_\sigma(u \cdot \nabla u(s))\, ds + \int_t^\infty e^{-A(t-s)}E_\lambda P_\sigma(u \cdot \nabla u(s))\, ds. \tag{40}$$

We will now finish the proof by estimating the three terms on the right hand side of (40). Since $\eta = (I - E_\lambda)\eta$, the treatment of the first term is clear. Let $t > 1/(4\lambda)$. Then

$$\begin{aligned}
\left\| \int_0^t e^{-A(t-s)}(I - E_\lambda)P_\sigma(u \cdot \nabla u(s))\, ds \right\| &\le \left\| \int_0^{t-1/(4\lambda)} A^{1/4}e^{-A(t-s)}(I - E_\lambda)A^{-1/4}P_\sigma(u \cdot \nabla u(s))\, ds \right\| \\
&\quad + \left\| \int_{t-1/(4\lambda)}^t A^{1/4}e^{-A(t-s)}(I - E_\lambda)A^{-1/4}P_\sigma(u \cdot \nabla u(s))\, ds \right\| \\
&\le \int_0^{t-1/(4\lambda)} \lambda^{1/4}e^{-\lambda(t-s)}\|A^{1/2}u(s)\|^2\, ds \\
&\quad + \int_{t-1/(4\lambda)}^t c(t-s)^{-1/4}\|A^{1/2}u(s)\|^2\, ds \\
&\le c\int_0^\infty e^{-\lambda(t-s)}e^{-2\lambda s}\, ds \\
&\quad + ce^{-2\lambda(t-1/(4\lambda))}\int_{t-1/(4\lambda)}^t (t-s)^{-1/4}\, ds \le ce^{-\lambda t}
\end{aligned}$$

and

$$\begin{aligned}
\left\| \int_t^\infty e^{-A(t-s)}E_\lambda P_\sigma(u \cdot \nabla u(s))\, ds \right\| &= \left\| \int_t^\infty A^{1/4}e^{-A(t-s)}E_\lambda A^{-1/4}P_\sigma(u \cdot \nabla u(s))\, ds \right\| \\
&\le \int_t^\infty \lambda^{1/4}e^{\lambda(s-t)}\|A^{1/2}u(s)\|^2\, ds \le ce^{-\lambda t}.
\end{aligned}$$

Therefore, (38) follows from the previous estimates and Lemma 14 is proved.

Proof of Theorem 1. It follows immediately from Remark 2 and Lemma 14. (12) follows from the point (x) in Section 2. □

4. Large-time localization of time derivatives of $A^\alpha u$

Theorem 1 in the previous section was proved for every solution satisfying Assumption 1. We will now prove that the same results can also be derived for the time derivatives of u of any order. We again suppose in this section that Assumption 1 is satisfied and we use the notation from Definition 1 – especially the number a – and the results from the previous section.

Lemma 15. *Let $\beta \ge 0$ and $l \in N_0$. Then*

$$\lim_{t\to\infty} \frac{\left\| A^\beta \frac{d^l u}{dt^l}(t) \right\|}{\|A^{\beta+l}u(t)\|} = 1. \tag{41}$$

Proof. We proceed by the mathematical induction. (41) holds if $l = 0$. We suppose that

$$\lim_{t\to\infty} \frac{\left\| A^\beta \frac{d^j u}{dt^j}(t) \right\|}{\|A^{\beta+j}u(t)\|} = 1$$

for every $\beta \geq 0$ and $j = 0, 1, \dots, l,\ l \in N_0$, and we will show that

$$\lim_{t\to\infty} \frac{\left\| A^{\beta} \frac{d^{l+1}u}{dt^{l+1}}(t) \right\|}{\|A^{\beta+l+1}u(t)\|} = 1 \tag{42}$$

for every $\beta \geq 0$. Let $\beta \in [n/2, n/2 + 1/2)$ for some $n \in N_0$. We start with the equality

$$A^{\beta} \frac{d^{l+1}u}{dt^{l+1}}(t) + A^{\beta+1} \frac{d^{l}u}{dt^{l}}(t) + \frac{d^{l}}{dt^{l}} A^{\beta} P_{\sigma}(u \cdot \nabla u(t)) = 0. \tag{43}$$

Then

$$\left\| \frac{d^{l}}{dt^{l}} A^{\beta} P_{\sigma}(u \cdot \nabla u(t)) \right\| \leq \left\| A^{n/2} \frac{d^{l}}{dt^{l}} P_{\sigma}(u \cdot \nabla u(t)) \right\|^{n+1-2\beta} \left\| A^{n/2+1/2} \frac{d^{l}}{dt^{l}} P_{\sigma}(u \cdot \nabla u(t)) \right\|^{2\beta-n},$$

$$\left\| A^{n/2} \frac{d^{l}}{dt^{l}} P_{\sigma}(u \cdot \nabla u(t)) \right\| \leq c \sum_{\gamma=0}^{n} \sum_{j=0}^{l} \left\| \frac{d^{j}}{dt^{j}} \nabla^{\gamma} u(t) \cdot \frac{d^{l-j}}{dt^{l-j}} \nabla^{n+1-\gamma} u(t) \right\|$$

and

$$\begin{aligned}
\left\| \frac{d^{j}}{dt^{j}} \nabla^{\gamma} u(t) \cdot \frac{d^{l-j}}{dt^{l-j}} \nabla^{n+1-\gamma} u(t) \right\| &\leq \left\| \nabla^{\gamma} \frac{d^{j}u}{dt^{j}}(t) \right\|_{6} \cdot \left\| \nabla^{n+1-\gamma} \frac{d^{l-j}u}{dt^{l-j}}(t) \right\|_{3} \\
&\leq \left\| A^{(\gamma+1)/2} \frac{d^{j}u}{dt^{j}}(t) \right\| \cdot \left\| A^{(n+1-\gamma)/2} \frac{d^{l-j}u}{dt^{l-j}}(t) \right\|^{1/2} \cdot \left\| A^{(n+2-\gamma)/2} \frac{d^{l-j}u}{dt^{l-j}}(t) \right\|^{1/2} \\
&\leq c \|A^{(\gamma+1+2j)/2} u(t)\| \cdot \|A^{(n+1-\gamma+2l-2j)/2} u(t)\|^{1/2} \cdot \|A^{(n+2-\gamma+2l-2j)/2} u(t)\|^{1/2} \\
&\leq c \|u(t)\|^{1-1/(2n+4l+4)} \|A^{n/2+1+l} u(t)\|^{(2n+4l+5)/(2n+4l+4)}.
\end{aligned}$$

So,

$$\left\| A^{n/2} \frac{d^{l}}{dt^{l}} P_{\sigma}(u \cdot \nabla u(t)) \right\| \leq c \|u(t)\|^{1-1/(2n+4l+4)} \|A^{n/2+1+l} u(t)\|^{(2n+4l+5)/(2n+4l+4)}.$$

Analogically,

$$\left\| A^{n/2+1/2} \frac{d^{l}}{dt^{l}} P_{\sigma}(u \cdot \nabla u(t)) \right\| \leq c \|u(t)\|^{1-1/(2n+4l+6)} \|A^{n/2+3/2+l} u(t)\|^{(2n+4l+7)/(2n+4l+6)}.$$

We further estimate

$$\begin{aligned}
&\|A^{n/2+1+l} u(t)\| \leq \|u(t)\|^{(\beta-n/2)/(\beta+1+l)} \|A^{\beta+1+l} u(t)\|^{(n/2+1+l)/(\beta+1+l)}, \\
&\|A^{n/2+3/2+l} u(t)\| \leq \|A^{\beta+1+l} u(t)\|^{(\tilde{\beta}-n/2-3/2-l)/(\tilde{\beta}-\beta-1-l)} \|A^{\tilde{\beta}} u(t)\|^{(n/2+1/2-\beta)/(\tilde{\beta}-\beta-1-l)},
\end{aligned}$$

where $\tilde{\beta}$ is sufficiently large, and get

$$\left\| \frac{d^{l}}{dt^{l}} A^{\beta} P_{\sigma}(u \cdot \nabla u(t)) \right\| \leq c \|u(t)\|^{k_1} \|A^{\beta+1+l} u(t)\|^{k_2} \|A^{\tilde{\beta}} u(t)\|^{k_3}, \tag{44}$$

where $k_1, k_3 > 0$ and

$$k_2 = \frac{2n+4l+5}{2n+4l+4} \cdot \frac{2n+2+l}{2\beta+2+2l} \cdot (n+1-2\beta) + \frac{2n+4l+7}{2n+4l+6} \cdot \frac{2\tilde{\beta}-n-3-2l}{2\tilde{\beta}-2\beta-2-2l} \cdot (2\beta-n).$$

It is possible to verify that $k_2 > 1$ for $\tilde{\beta}$ sufficiently large. It follows from (43), (44) and (10) and by the use of the induction assumption that

$$\|A^{\beta+1+l} u(t)\|(1-\xi(t)) \leq \left\| A^{\beta} \frac{d^{l+1}u}{dt^{l+1}}(t) \right\| \leq \|A^{\beta+1+l} u(t)\|(1+\xi(t)),$$

where $\lim_{t\to\infty} \xi(t) = 0$. (42) immediately follows and Lemma 15 is proved. □

Corollary 3. *Let $\alpha, \beta \geq 0$ and $k, l \in N_0$. Then*

$$\lim_{t\to\infty} \frac{\left\| A^{\beta} \frac{d^k u}{dt^k}(t) \right\|}{\left\| A^{\alpha} \frac{d^l u}{dt^l}(t) \right\|} = a^{(\beta-\alpha)+(k-l)}. \tag{45}$$

If $a = 0$ and $(\beta - \alpha) + (k - l) < 0$ then the right hand side of (45) is equal to infinity.

Proof. The proof follows immediately from Lemma 15, Theorem 1 and the equality

$$\frac{\left\| A^{\beta} \frac{d^k u}{dt^k}(t) \right\|}{\left\| A^{\alpha} \frac{d^l u}{dt^l}(t) \right\|} = \frac{\left\| A^{\beta} \frac{d^k u}{dt^k}(t) \right\|}{\|A^{\beta+k} u(t)\|} \cdot \frac{\|A^{\beta+k} u(t)\|}{\|A^{\alpha+l} u(t)\|} \cdot \frac{\|A^{\alpha+l} u(t)\|}{\left\| A^{\alpha} \frac{d^l u}{dt^l}(t) \right\|}.$$

The following corollary is a consequence of Lemma 15 and the point (ix) from Preliminaries. The space–time derivatives of the solution have bounded decays in the L^2-norm on small time intervals. □

Corollary 4. *Let $\alpha, \beta \geq 0$ and $k, l \in N_0$ and $\beta + k \geq \alpha + l$. Then there exist $C > 0$, $\delta_0 \in (0, 1)$ and $t_0 \geq 0$ so that*

$$\frac{\left\| A^{\beta} \frac{d^k u}{dt^k}(t) \right\|}{\left\| A^{\alpha} \frac{d^l u}{dt^l}(t+\delta) \right\|} \leq C$$

for every $t \geq t_0$ and $\delta \in [0, \delta_0]$.

Theorem 2. *Let $\alpha \geq 0$ and $k \in N_0$. We suppose that Assumption 1 is satisfied. Let $\varepsilon > 0$. Then*

$$\lim_{t\to\infty} \frac{\left\| E_{a,\varepsilon} A^{\alpha} \frac{d^k u}{dt^k}(t) \right\|}{\left\| A^{\alpha} \frac{d^k u}{dt^k}(t) \right\|} = 1 \tag{46}$$

and

$$\lim_{t\to\infty} \frac{\int_{K_{a,\varepsilon}} |\xi|^{4\alpha} \left| F\left(\frac{d^k u}{dt^k}(t) \right)(\xi) \right|^2 d\xi}{\int_{\mathbf{R}^3} |\xi|^{4\alpha} \left| F\left(\frac{d^k u}{dt^k} u(t) \right)(\xi) \right|^2 d\xi} = 1, \tag{47}$$

where a, $E_{a,\varepsilon}$ and $K_{a,\varepsilon}$ were defined in Theorem 1.

Proof. We proceed by the mathematical induction. It follows from Theorem 1 that (46) holds for $k = 0$. Suppose that (46) holds for every $\alpha \geq 0$ and some $k \in N_0$. We will prove its validity for every $\alpha \geq 0$ and $k + 1$. We start with

$$E_{a,\varepsilon} A^{\alpha} \frac{d^{k+1} u}{dt^{k+1}}(t) + E_{a,\varepsilon} A^{\alpha+1} \frac{d^k u}{dt^k}(t) + E_{a,\varepsilon} A^{\alpha} \frac{d^k}{dt^k} P_{\sigma}(u \cdot \nabla u(t)) = 0.$$

We know that (see (44))

$$\left\| E_{a,\varepsilon} A^{\alpha} \frac{d^k}{dt^k} P_{\sigma}(u \cdot \nabla u(t)) \right\| \leq \left\| A^{\alpha} \frac{d^k}{dt^k} P_{\sigma}(u \cdot \nabla u(t)) \right\| \leq \xi(t) \|A^{\alpha+k+1} u(t)\|,$$

where $\lim_{t\to\infty} \xi(t) = 0$. So

$$\begin{aligned} \left\| E_{a,\varepsilon} A^{\alpha+1} \frac{d^k u}{dt^k}(t) \right\| - \xi(t) \|A^{\alpha+k+1} u(t)\| &\leq \left\| E_{a,\varepsilon} A^{\alpha} \frac{d^{k+1} u}{dt^{k+1}}(t) \right\| \\ &\leq \left\| E_{a,\varepsilon} A^{\alpha+1} \frac{d^k u}{dt^k}(t) \right\| + \xi(t) \|A^{\alpha+k+1} u(t)\|. \end{aligned}$$

Dividing the previous inequalities by $\|A^{\alpha+1} \frac{d^k u}{dt^k}(t)\|$, applying $\lim_{t\to\infty}$ and using the induction assumption and (41), we get

$$1 \leq \lim_{t\to\infty} \frac{\left\| E_{a,\varepsilon} A^{\alpha} \frac{d^{k+1} u}{dt^{k+1}}(t) \right\|}{\left\| A^{\alpha} \frac{d^{k+1} u}{dt^{k+1}}(t) \right\|} \leq 1.$$

(47) follows from (46) using the point (x) from Section 2. Theorem 2 is proved. □

Corollary 5. *It follows immediately from (46) that*

$$\lim_{t\to\infty} \frac{\left\|(I-E_{a,\varepsilon})A^\alpha \frac{d^k u}{dt^k}(t)\right\|}{\left\|E_{a,\varepsilon}A^\alpha \frac{d^k u}{dt^k}(t)\right\|} = 0 \tag{48}$$

and

$$\lim_{t\to\infty} \frac{\int_{K_{a,\varepsilon}^C} |\xi|^{4\alpha} \left|F\left(\frac{d^k u}{dt^k}(t)\right)(\xi)\right|^2 d\xi}{\int_{K_{a,\varepsilon}} |\xi|^{4\alpha} |F\left(\frac{d^k u}{dt^k}(t)\right)(\xi)|^2 d\xi} = 0, \tag{49}$$

where $K_{a,\varepsilon}^C = \mathbf{R}^3 \setminus K_{a,\varepsilon}$.

5. Large-time localization of vorticity

Up until now we have dealt with the solution u satisfying Assumption 1 and its time derivatives. We will now prove that the same results can also be derived for the vorticity and its time derivatives. It is interesting that for a fixed solution the number a does not change and plays the same role as in the previous sections. As before we suppose Assumption 1.

Define $\omega = (\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}, \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}, \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1})$.

Lemma 16. *Let* $u \in W^{1,2}_{0,\sigma}$. *Then*

$$|F(\omega)(\xi)|^2 = |\xi|^2 |F(u)(\xi)|^2, \quad \text{almost all } \xi$$
$$|F(E_{a,\varepsilon}\omega)(\xi)|^2 = \chi_{K_{a,\varepsilon}}(\xi)|\xi|^2|F(u)(\xi)|^2, \quad \text{almost all } \xi.$$

Proof. At first for $u \in C^\infty_{0,\sigma}$ and then using the fact that $C^\infty_{0,\sigma}$ is dense in $W^{1,2}_{0,\sigma}$. □

Lemma 17. *Let* $0 \le \alpha < \beta < \infty$. *Let* $\varepsilon > 0$. *Then*

$$\lim_{t\to\infty} \frac{\|A^\beta \omega(t)\|}{\|A^\alpha \omega(t)\|} = a^{\beta-\alpha}$$

and

$$\lim_{t\to\infty} \frac{\|E_{a,\varepsilon}A^\alpha \omega(t)\|}{\|A^\alpha \omega(t)\|} = 1,$$

where a *and* $E_{a,\varepsilon}$ *are the numbers from Theorem 1.*

Proof. We have $F(A^\alpha u)(\xi) = |\xi|^{2\alpha}F(u)(\xi)$. Therefore, by the use of Lemma 16

$$\begin{aligned}\|A^\alpha\omega\|^2 &= \|F(A^\alpha\omega)\|^2 = \int |\xi|^{4\alpha}|F(\omega)(\xi)|^2\, d\xi \\ &= \int |\xi|^{4\alpha+2}|F(u)(\xi)|^2\, d\xi = \|F(A^{\alpha+1/2}u)\|^2 = \|A^{\alpha+1/2}u\|^2\end{aligned} \tag{50}$$

and

$$\begin{aligned}\|E_{a,\varepsilon}A^\alpha\omega\|^2 &= \|F(A^\alpha E_{a,\varepsilon}\omega)\|^2 = \int |\xi|^{4\alpha}|F(E_{a,\varepsilon}\omega)(\xi)|^2\, d\xi \\ &= \int_{K_{a,\varepsilon}} |\xi|^{4\alpha+2}|F(u)(\xi)|^2\, d\xi = \|F(E_{a,\varepsilon}A^{\alpha+1/2}u)\|^2 = \|E_{a,\varepsilon}A^{\alpha+1/2}u\|^2.\end{aligned}$$

The proof now follows from Theorem 1. □

Lemma 18. *Let* $\beta \ge 0$ *and* $l \in N_0$. *Then*

$$\lim_{t\to\infty} \frac{\left\|A^\beta \frac{d^l\omega}{dt^l}(t)\right\|}{\|A^{\beta+l}\omega(t)\|} = 1. \tag{51}$$

Proof. We proceed by the mathematical induction. (51) holds if $l = 0$. We suppose that

$$\lim_{t\to\infty} \frac{\left\|A^\beta \frac{d^j\omega}{dt^j}(t)\right\|}{\|A^{\beta+j}\omega(t)\|} = 1 \tag{52}$$

for every $\beta \geq 0$ and $j = 0, 1, \dots, l$, $l \in N_0$, and we will show that

$$\lim_{t\to\infty} \frac{\left\| A^{\beta} \frac{d^{l+1}\omega}{dt^{l+1}}(t)\right\|}{\|A^{\beta+l+1}\omega(t)\|} = 1 \tag{53}$$

for every $\beta \geq 0$. We start with the equation

$$A^{\beta}\frac{d^{l+1}\omega}{dt^{l+1}}(t) + A^{\beta+1}\frac{d^{l}\omega}{dt^{l}}(t) + \frac{d^l}{dt^l}A^{\beta}P_{\sigma}(u\cdot\nabla\omega(t)) - \frac{d^l}{dt^l}A^{\beta}P_{\sigma}(\omega\cdot\nabla u(t)) = 0. \tag{54}$$

Let $\beta \in [n/2, n/2+1/2)$ for some $n \in N_0$. Then

$$\left\|\frac{d^l}{dt^l}A^{\beta}P_{\sigma}(u\cdot\nabla\omega(t))\right\| \leq \left\|A^{n/2}\frac{d^l}{dt^l}P_{\sigma}(u\cdot\nabla\omega(t))\right\|^{n+1-2\beta}\left\|A^{n/2+1/2}\frac{d^l}{dt^l}P_{\sigma}(u\cdot\nabla\omega(t))\right\|^{2\beta-n}, \tag{55}$$

$$\left\|A^{n/2}\frac{d^l}{dt^l}P_{\sigma}(u\cdot\nabla\omega(t))\right\| \leq c\sum_{\gamma=0}^{n}\sum_{j=0}^{l}\left\|\frac{d^j}{dt^j}\nabla^{\gamma}u\cdot\frac{d^{l-j}}{dt^{l-j}}\nabla^{n+1-\gamma}\omega\right\| \tag{56}$$

and

$$\begin{aligned}\left\|\frac{d^j}{dt^j}\nabla^{\gamma}u\cdot\frac{d^{l-j}}{dt^{l-j}}\nabla^{n+1-\gamma}\omega\right\| &\leq \|\nabla^{\gamma}\frac{d^j u}{dt^j}\|_6\cdot\left\|\nabla^{n+1-\gamma}\frac{d^{l-j}\omega}{dt^{l-j}}\right\|^{1/2}\left\|\nabla^{n+1-\gamma}\frac{d^{l-j}\omega}{dt^{l-j}}\right\|_6^{1/2}\\ &\leq c\left\|A^{(\gamma+1)/2}\frac{d^j u}{dt^j}\right\|\cdot\left\|A^{(n+1-\gamma)/2}\frac{d^{l-j}\omega}{dt^{l-j}}\right\|^{1/2}\left\|A^{(n+2-\gamma)/2}\frac{d^{l-j}\omega}{dt^{l-j}}\right\|^{1/2}.\end{aligned} \tag{57}$$

If we now use (41), the induction assumption (52), (50) and the moment inequality, we obtain

$$\begin{aligned}\left\|\frac{d^j}{dt^j}\nabla^{\gamma}u\cdot\frac{d^{l-j}}{dt^{l-j}}\nabla^{n+1-\gamma}\omega\right\| &\leq c\|A^{(\gamma+2j)/2}\omega\|\cdot\|A^{(n+1-\gamma+2l-2j)/2}\omega\|^{1/2}\|A^{(n+2-\gamma+2l-2j)/2}\omega\|^{1/2}\\ &\leq c\|A^{\beta+1+l}\omega\|^{(2n+4l+3)/(4\beta+4+4l)}\|\omega\|^{2-(2n+4l+3)/(4\beta+4+4l)}.\end{aligned} \tag{58}$$

Similarly,

$$\left\|A^{n/2+1/2}\frac{d^l}{dt^l}P_{\sigma}(u\cdot\nabla\omega(t))\right\| \leq c\sum_{\gamma=0}^{n+1}\sum_{j=0}^{l}\left\|\frac{d^j}{dt^j}\nabla^{\gamma}u\cdot\frac{d^{l-j}}{dt^{l-j}}\nabla^{n+2-\gamma}\omega\right\| \tag{59}$$

and if $(\gamma, j) \neq (0,0)$ then as above

$$\left\|\frac{d^j}{dt^j}\nabla^{\gamma}u\cdot\frac{d^{l-j}}{dt^{l-j}}\nabla^{n+2-\gamma}\omega\right\| \leq c\|A^{(\beta+1+l)}\omega\|^{(2n+4l+5)/(4\beta+4+4l)}\|\omega\|^{2-(2n+4l+5)/(4\beta+4+4l)}. \tag{60}$$

If $(\gamma, j) = (0,0)$ then by the induction assumption and by the use of the inequality $\|u\|_{\infty} \leq \|\omega\|^{1/2}\|\omega\|_6^{1/2}$ (see [18, Lemma 2.1])

$$\begin{aligned}\left\|u\cdot\frac{d^l}{dt^l}\nabla^{n+2}\omega\right\| &\leq \|u\|_{\infty}\left\|\frac{d^l}{dt^l}\nabla^{n+2}\omega\right\|\\ &\leq \|\omega\|^{1/2}\|\omega\|_6^{1/2}\left\|A^{(n+2)/2}\frac{d^l\omega}{dt^l}\right\| \leq \|\omega\|^{1/2}\|A^{1/2}\omega\|^{1/2}\|A^{(n+2+2l)/2}\omega\|\\ &\leq \|A^{\beta+1+l}\omega\|^{(2n+5+4l)/(4\beta+4+4l)}\|\omega\|^{2-(2n+5+4l)/(4\beta+4+4l)}.\end{aligned} \tag{61}$$

We still estimate

$$\|\omega\| = \|A^{1/2}u\| \leq \|A^{\beta+3/2+l}u\|^{1/(2\beta+3+2l)} \leq c\|A^{\beta+1+l}\omega\|^{1/(2\beta+3+2l)}. \tag{62}$$

It now follows from (55)–(62) that

$$\left\|\frac{d^l}{dt^l}A^{\beta}P_{\sigma}(u\cdot\nabla\omega(t))\right\| \leq c\|A^{\beta+1+l}u\|^{k}, \tag{63}$$

where it is possible to compute elementarily that

$$k = \frac{4\beta+3+4l}{4\beta+4+4l} + \frac{1}{4\beta+4+4l}\,\frac{4\beta+5+4l}{2\beta+3+2l} > 1.$$

In the same way as above one can get the same estimate for $\| \frac{d^l}{dt^l} A^\beta P_\sigma(\omega \cdot \nabla u(t))\|$. It now follows from (54) and the induction assumption that

$$\|A^{\beta+1+l}\omega\|(1-\xi(t)) \le \left\| A^\beta \frac{d^{l+1}\omega}{dt^{l+1}} \right\| \le \|A^{\beta+1+l}\omega\|(1+\xi(t)),$$

where $\lim_{t\to\infty} \xi(t) = 0$ and Lemma 18 is proved. □

Corollary 6. *Let $\alpha, \beta \ge 0$ and $k, l \in N_0$. Then*

$$\lim_{t\to\infty} \frac{\left\| A^\beta \frac{d^k\omega}{dt^k}(t) \right\|}{\left\| A^\alpha \frac{d^l\omega}{dt^l}(t) \right\|} = a^{(\beta-\alpha)+(k-l)}. \tag{64}$$

If $a = 0$ and $(\beta - \alpha) + (k - l) < 0$ then the right hand side of (64) is equal to infinity.

Proof. The proof follows immediately from the previous lemma and the equality

$$\frac{\left\| A^\beta \frac{d^k\omega}{dt^k}(t) \right\|}{\left\| A^\alpha \frac{d^l\omega}{dt^l}(t) \right\|} = \frac{\left\| A^\beta \frac{d^k\omega}{dt^k}(t) \right\|}{\|A^{\beta+k}\omega(t)\|} \cdot \frac{\|A^{\beta+k}\omega(t)\|}{\|A^{\alpha+l}\omega(t)\|} \cdot \frac{\|A^{\alpha+l}\omega(t)\|}{\left\| A^\alpha \frac{d^l\omega}{dt^l}(t) \right\|}. \quad \square$$

Corollary 7. *Let $\alpha, \beta \ge 0$ and $k, l \in N_0$ and $\beta + k \ge \alpha + l$. Then there exist $C > 0$, $\delta_0 \in (0, 1)$ and $t_0 \ge 0$ so that*

$$\frac{\left\| A^\beta \frac{d^k\omega}{dt^k}(t) \right\|}{\left\| A^\alpha \frac{d^l\omega}{dt^l}(t+\delta) \right\|} \le C$$

for every $t \ge t_0$ and $\delta \in [0, \delta_0]$.

Theorem 3. *Let $\alpha \ge 0$ and $k \in N_0$. We suppose that Assumption 1 is satisfied. Let $\varepsilon > 0$. Then*

$$\lim_{t\to\infty} \frac{\left\| E_{a,\varepsilon} A^\alpha \frac{d^k\omega}{dt^k}(t) \right\|}{\left\| A^\alpha \frac{d^k\omega}{dt^k}(t) \right\|} = 1 \tag{65}$$

and

$$\lim_{t\to\infty} \frac{\int_{K_{a,\varepsilon}} |\xi|^{4\alpha} \left| F\left(\frac{d^k\omega}{dt^k}(t)\right)(\xi) \right|^2 d\xi}{\int_{\mathbf{R}^3} |\xi|^{4\alpha} \left| F\left(\frac{d^k\omega}{dt^k}(t)\right)(\xi) \right|^2 d\xi} = 1,$$

where a, $E_{a,\varepsilon}$ and $K_{a,\varepsilon}$ were defined in Theorem 1.

Proof. We proceed by the mathematical induction. Suppose that (65) holds for every $\alpha \ge 0$ and some $k \in N_0$. We will prove its validity also for every $\alpha \ge 0$ and $k + 1$. We start with

$$E_{a,\varepsilon} A^\alpha \frac{d^{k+1}\omega}{dt^{k+1}}(t) + E_{a,\varepsilon} A^{\alpha+1} \frac{d^k\omega}{dt^k}(t) + E_{a,\varepsilon} A^\alpha \frac{d^k}{dt^k} P_\sigma(u \cdot \nabla\omega(t)) - E_{a,\varepsilon} A^\alpha \frac{d^k}{dt^k} P_\sigma(\omega \cdot \nabla u(t)) = 0.$$

We know that (see (63))

$$\left\| E_{a,\varepsilon} A^\alpha \frac{d^k}{dt^k} P_\sigma(u \cdot \nabla\omega(t)) \right\| \le \left\| A^\alpha \frac{d^k}{dt^k} P_\sigma(u \cdot \nabla\omega(t)) \right\| \le \xi(t) \|A^{\alpha+k+1}\omega(t)\|,$$

where $\lim_{t\to\infty} \xi(t) = 0$. So

$$\begin{aligned} \left\| E_{a,\varepsilon} A^{\alpha+1} \frac{d^k\omega}{dt^k}(t) \right\| - \xi(t)\|A^{\alpha+k+1}\omega(t)\| &\le \left\| E_{a,\varepsilon} A^\alpha \frac{d^{k+1}\omega}{dt^{k+1}}(t) \right\| \\ &\le \left\| E_{a,\varepsilon} A^{\alpha+1} \frac{d^k\omega}{dt^k}(t) \right\| + \xi(t)\|A^{\alpha+k+1}\omega(t)\|. \end{aligned}$$

Dividing the previous inequalities by $\|A^{\alpha+1}\frac{d^k\omega}{dt^k}(t)\|$, applying $\lim_{t\to\infty}$ and using the induction assumption and (51), we get

$$1 \leq \lim_{t\to\infty} \frac{\left\| E_{a,\varepsilon} A^{\alpha} \frac{d^{k+1}\omega}{dt^{k+1}}(t) \right\|}{\left\| A^{\alpha+1} \frac{d^k\omega}{dt^k}(t) \right\|} \leq 1.$$

Theorem 3 is proved. □

Corollary 8. *It follows immediately from (65) that*

$$\lim_{t\to\infty} \frac{\left\| (I - E_{a,\varepsilon}) A^{\alpha} \frac{d^k\omega}{dt^k}(t) \right\|}{\left\| E_{a,\varepsilon} A^{\alpha} \frac{d^k\omega}{dt^k}(t) \right\|} = 0$$

and

$$\lim_{t\to\infty} \frac{\int_{K_{a,\varepsilon}^C} |\xi|^{4\alpha} \left| F\left(\frac{d^k\omega}{dt^k}(t)\right)(\xi) \right|^2 \, d\xi}{\int_{K_{a,\varepsilon}} |\xi|^{4\alpha} |F\left(\frac{d^k\omega}{dt^k}(t)\right)(\xi)|^2 \, d\xi} = 0,$$

where $K_{a,\varepsilon}^C = \mathbf{R}^3 \setminus K_{a,\varepsilon}$.

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